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Preliminaries

In this chapter, we recall some fundamental definitions and results that are needed for our work. The chapter is divided into three sections. In the first section, the basic definitions and results from the theory of fuzzy sets are collected. The second section recalls the basic definitions of BCK/BCI, B, d,....algebras while the third section recalls some basic definitions and results in Z-algebras.

1.1 Fuzzy Structures

In this section, we recall some fundamental definitions and results from fuzzy set theory are presented.

Definition 1.1.1:[73] Let X be a nonempty universal set. A fuzzy set A in X is characterized by a membership function μ_A which associates with each point x in X, a real number $\mu_A(x)$, in the interval $[0,1]$ with $\mu_A(x)$ representing the "grade of membership" of x in A.

That is, a fuzzy set A in X is an object $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ where $\mu_A : X \to [0,1]$ is the membership function of A and $\mu_A(x)$ is called membership value of x in A and $0 \le \mu_A(x) \le 1$.

Definition 1.1.2:[73] Let X be the universe of discourse. Let A and B be two fuzzy sets with membership functions μ_A and μ_B respectively. Then,

1. $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ for all $x \in X$
2. $A = B$ iff $\mu_A(x) = \mu_B(x)$ for all $x \in X$
3. $A^c = \{\langle x, \mu_{A^c}(x)\rangle \mid x \in X\} = \{\langle x, 1-\mu_A(x)\rangle \mid x \in X\}$
4. $A \cap B = \{\langle x, \mu_{A\cap B}(x)\rangle \mid x \in X\} = \{\langle x, \min(\mu_A(x), \mu_B(x))\rangle \mid x \in X\}$
5. $A \cup B = \{\langle x, \mu_{A\cup B}(x)\rangle \mid x \in X\} = \{\langle x, \max(\mu_A(x), \mu_B(x))\rangle \mid x \in X\}$
6. $\bigcap_{i\in\Omega} A_i = \left\{\left\langle x, \mu_{\underset{i\in\Omega}{\cap} A_i}(x)\right\rangle \mid x \in X\right\}$ where $\mu_{\underset{i\in\Omega}{\cap} A_i}(x) = \inf_{i\in\Omega}(\mu_{A_i}(x))$.

7. $\bigcup_{i\in\Omega} A_i = \left\{\left\langle x, \mu_{\bigcup_{i\in\Omega} A_i}(x)\right\rangle \middle| x \in X\right\}$ where $\mu_{\bigcup_{i\in\Omega} A_i}(x) = \sup_{i\in\Omega}\{(x, \mu_{A_i}(x)) | x \in X\}$

Definition 1.1.3:[20] Let A be a fuzzy set with membership function μ_A of a set X. For a fixed $t \in [0,1]$, the set $U(\mu_A; t) = \{x \in X \mid \mu_A(x) \geq t\}$ is called an **upper t-level subset** (upper level subset, upper level cut) of A and the set $L(\mu_A; t) = \{x \in X \mid \mu_A(x) \leq t\}$ is called a **lower t-level subset** (lower level subset, lower level cut) of A.

Note: (i) If $t_1 \leq t_2$, $U(\mu_A; t_2) \subseteq U(\mu_A; t_1)$ and $L(\mu_A; t_1) \subseteq L(\mu_A; t_2)$.

(ii) $U(\mu_A; t) \cup L(\mu_A; t) = X$ for all $t \in [0,1]$.

Definition 1.1.4:[65] A fuzzy set A in a set X with membership function μ_A is said to have the **sup property** if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that $\mu_A(x_0) = \sup_{t\in T} \mu_A(t)$.

Definition 1.1.5:[65] Let h be a mapping from a set X into a set Y.

(i) Let A be a fuzzy set in X with membership function μ_A. Then the image of A under h, denoted by h(A) is the fuzzy set in Y with membership function $\mu_{h(A)}$ defined by

$$\mu_{h(A)}(y) = \begin{cases} \sup\limits_{z\in h^{-1}(y)} \mu_A(z) & \text{if } h^{-1}(y) = \{x \mid h(x) = y\} \neq \phi \\ 0 & \text{, otherwise} \end{cases}$$

(ii) Let B be a fuzzy set in Y with membership function μ_B. The inverse image (or pre-image) of B under h, denoted by $h^{-1}(B)$ is the fuzzy set in X with membership function $\mu_{h^{-1}(B)}$ defined by $\mu_{h^{-1}(B)}(x) = \mu_B(h(x))$ for all $x \in X$.

Definition 1.1.6:[13] Let A and B be any two fuzzy sets in X with membership functions μ_A and μ_B respectively. Then, the Cartesian product $A \times B$ with membership function $\mu_{A\times B} : X \times X \to [0,1]$ is defined by $\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$ for all $x, y \in X$.

Definition 1.1.7:[13] A fuzzy relation A on a nonempty set X is a fuzzy set A with membership function $\mu_A : X \times X \to [0,1]$.

Definition 1.1.8:[13] If A is a fuzzy relation with membership function $\mu_A : X \times X \to [0,1]$ on a set X and B is a fuzzy set in X with membership function μ_B then A is a fuzzy relation on B if for all $x, y \in X$, $\mu_A(x, y) \leq \min\{\mu_B(x), \mu_B(y)\}$.

Definition 1.1.9:[13] Let B be a fuzzy set on a set X with membership function μ_B then the strongest fuzzy relation on X, that is, a fuzzy relation A on B is A_B whose membership function $\mu_{A_B} : X \times X \to [0,1]$ given by $\mu_{A_B}(x, y) = \min\{\mu_B(x), \mu_B(y)\}$ for all $x, y \in X$.

Now, we recall some fundamental definitions from intuitionistic fuzzy set theory. In the case of fuzzy sets, we consider only the membership values of the elements. Atanassov introduced the notion of intuitionistic fuzzy set by considering the values of non-membership together with membership values.

Definition 1.1.10: [8] An **Intuitionistic Fuzzy Set** A in a nonempty set X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ where $\mu_A : X \to [0,1]$ denote the degree of membership and $\nu_A : X \to [0,1]$ denote the degree of non-membership functions respectively such that for each $x \in X$ to the set A with $0 \le \mu_A(x) + \nu_A(x) \le 1$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

Definition 1.1.11:[8] If $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$ be any two intuitionistic fuzzy sets of a nonempty set X. Then,

1. $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in X$.
2. $A = B$ iff $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$.
3. $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$
4. $A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cup B}(x)\rangle \mid x \in X\} = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in X\}$
4. $A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cap B}(x)\rangle \mid x \in X\} = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in X\}$
6. $\oplus A = (\mu_A, (\mu_A)^c) = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$
7. $\otimes A = ((\nu_A)^c, \nu_A) = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in X\}$
8. $\bigcap_{i\in\Omega} A_i = \left\{\left\langle x, \mu_{\underset{i\in\Omega}{\cap} A_i}(x), \nu_{\underset{i\in\Omega}{\cup} A_i}(x)\right\rangle \middle| x \in X\right\}$ where $\mu_{\underset{i\in\Omega}{\cap} A_i}(x) = \inf_{i\in\Omega}(\mu_{A_i}(x))$ and

$\nu_{\underset{i\in\Omega}{\cup} A_i}(x) = \sup_{i\in\Omega}(\nu_{A_i}(x))$.

9. $\bigcup\limits_{i\in\Omega} A_i = \left\{\left\langle x, \mu_{\underset{i\in\Omega}{\cup} A_i}(x), \nu_{\underset{i\in\Omega}{\cap} A_i}(x)\right\rangle \middle| x \in X\right\}$ where $\mu_{\underset{i\in\Omega}{\cup} A_i}(x) = \sup\limits_{i\in\Omega}(\mu_{A_i}(x))$ and $\nu_{\underset{i\in\Omega}{\cap} A_i}(x) = \inf\limits_{i\in\Omega}(\nu_{A_i}(x))$.

Definition 1.1.12: [9] Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in a nonempty set X. For $s, t \in [0,1]$, $U(\mu_A; s) = \{x \in X \mid \mu_A(x) \geq s\}$ is called an **upper s-level subset of A** and $L(\nu_A; t) = \{x \in X \mid \nu_A(x) \leq t\}$ is called the **lower t-level subset of A**.

Definition 1.1.13:[9] An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in a set X with the degree of membership $\mu_A : X \to [0,1]$ and the degree of non-membership $\nu_A : X \to [0,1]$ is said to have **sup-inf property** if for any subset T of X there exists $x_0 \in T$ such that $\mu_A(x_0) = \sup\limits_{t\in T} \mu_A(t)$ and $\nu_A(x_0) = \inf\limits_{t\in T} \nu_A(t)$.

Definition 1.1.14:[45] Let h be any function from a set X into a set Y.

(i) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ be an intuitionistic fuzzy set in X. Then the image of A under h, denoted by $h(A) = \{\langle y, \mu_{h(A)}(y), \nu_{h(A)}(y)\rangle \mid y \in Y\}$ is an intuitionistic fuzzy set in Y, defined by:

$$\mu_{h(A)}(y) = \begin{cases} \sup\limits_{z\in h^{-1}(y)} \mu_A(z) & \text{if} \quad h^{-1}(y) = \{x \mid h(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{h(A)}(y) = \begin{cases} \inf\limits_{z\in h^{-1}(y)} \nu_A(z) & \text{if} \quad h^{-1}(y) = \{x \mid h(x) = y\} \neq \phi \\ 1 & \text{otherwise} \end{cases}$$

(ii) Let $B = \{\langle y, \mu_B(y), \nu_B(y)\rangle \mid y \in Y\}$ be an intuitionistic fuzzy set in Y. The pre-image of B under h, symbolized by $h^{-1}(B) = \{\langle x, \mu_{h^{-1}(B)}(x), \nu_{h^{-1}(B)}(x)\rangle \mid x \in X\}$ defined by: $\mu_{h^{-1}(B)}(x) = \mu_B(h(x))$ and $\nu_{h^{-1}(B)}(x) = \nu_B(h(x))$ for all $x \in X$ is an intuitionistic fuzzy set of X.

Definition 1.1.15:[8] Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be any two intuitionistic fuzzy sets of a nonempty set X. The Cartesian product $A\times B$ is given by $A\times B=(\mu_{A\times B},\nu_{A\times B})$ where membership function $\mu_{A\times B}:X\times X\to[0,1]$ and the non-membership function $\nu_{A\times B}:X\times X\to[0,1]$ are defined by $\mu_{A\times B}(x,y)=\min\{\mu_A(x),\mu_B(y)\}$ and $\nu_{A\times B}(x,y)=\max\{\nu_A(x),\nu_B(y)\}$ for all $x,y\in X$.

Notation: For our convenience, we consider $(L,\leq,\wedge,\vee)$ to be a complete lattice with least element $0\equiv\inf\{x\in L\}$ and greatest element $1\equiv\sup\{x\in L\}$.

Goguen [21] generalized the notion of fuzzy sets into the notion of L−fuzzy sets. A L−fuzzy set, A on X is a mapping from μ_A : X → L, where L is a transitive partial ordered set. After a long research, Atanassov and Stoeva [11] introduced the notion of intuitionistic L-fuzzy sets in 1984. Here they have defined both membership and non-membership function from the Universe of discourse X to the set L, where L is a complete lattice.

Now, we recall the fundamental definition of Intuitionistic L-fuzzy set.

Definition 1.1.16:[11] Let $(L,\leq,\wedge,\vee)$ be a complete lattice with least element 0 and greatest element 1 and an involutive order reversing operation $N:L\to L$. Then an **Intuitionistic L-fuzzy Set** A in a nonempty set X is defined as an object of the form $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in X\}$ where $\mu_A:X\to L$ is the degree of membership and $\nu_A:X\to L$ is the degree of non-membership of the element $x\in X$ satisfying $\mu_A(x)\leq N(\nu_A(x))$.

As an extension of the concept of a fuzzy set, Zadeh [74] introduced the concept of an interval-valued fuzzy set. That is, a fuzzy set with an interval-valued membership function. Biswas[15] described a method to find max/sup and min/inf between two intervals or a set of intervals.

Now, we recall some fundamental definitions from Interval-Valued fuzzy set theory.

Definition 1.1.17:[15] An interval number $D_1=[a_1,b_1]$ on [0,1] is a closed subinterval of [0,1] where $0\leq a_1\leq b_1\leq 1$. Let D[0,1] denote the family of all closed subintervals of [0,1]. The

refined minimum (briefly, rmin), **refined maximum** (briefly, rmax) and the operations "$\leq$", "$\geq$", "$=$" on the elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of D[0,1] are defined as follows:

1. $r\min(D_1, D_2) = r\min\{[a_1, b_1], [a_2, b_2]\} = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$
2. $r\max(D_1, D_2) = r\max\{[a_1, b_1], [a_2, b_2]\} = [\max\{a_1, a_2\}, \max\{b_1, b_2\}]$
3. $D_1 \leq D_2$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$.

   Similarly "$\geq$" and "$=$" can be defined.
4. $D_1 + D_2 = [a_1 + a_2, b_1 + b_2]$

   Let $D_i = [a_i, b_i] \in D[0,1]$ where $i \in \Omega$ and Ω is an index set.
5. $r\sup\limits_{i\in\Omega} D_i = [\sup\limits_{i\in\Omega} a_i, \sup\limits_{i\in\Omega} b_i]$
6. $r\inf\limits_{i\in\Omega} D_i = [\inf\limits_{i\in\Omega} a_i, \inf\limits_{i\in\Omega} b_i]$

Definition 1.1.18:[15] An **interval-valued fuzzy set** A defined on a nonempty set X is given by $A = \{\langle x, [\mu_A^L(x), \mu_A^U(x)]\rangle\}$ for all $x \in X$.

Briefly, denoted by $A = [\mu_A^L, \mu_A^U]$ where $\mu_A^L : X \to [0,1]$ and $\mu_A^U : X \to [0,1]$ are lower and upper fuzzy sets in X with $\mu_A^L(x) \leq \mu_A^U(x)$ for all $x \in X$.

Let $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$, for all $x \in X$. If $\mu_A^L(x) = \mu_A^U(x) = c$, where $0 \leq c \leq 1$ then $\tilde{\mu}_A(x) = [c, c]$ is in D[0,1]. Thus $\tilde{\mu}_A(x) \in D[0,1]$, for all $x \in X$. That is, the interval-valued fuzzy set A is an object having the form $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ where $\tilde{\mu}_A : X \to D[0,1]$ with $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$. The interval $\tilde{\mu}_A(x)$ denote the intervals of the degree of membership of the element x to the set A.

Definition 1.1.19:[15] (i) Let $A = \{\langle x, \tilde{\mu}_A(x)\rangle | x \in X\}$ where $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ be an interval-valued fuzzy set on a set X.

Then $A^c = \{\langle x, (\tilde{\mu}_A)^c(x)\rangle | x \in X\}$ where $(\tilde{\mu}_A)^c(x) = [1 - \mu_A^U(x), 1 - \mu_A^L(x)]$.

(ii) Let $A = \{\langle x, \tilde{\mu}_A(x)\rangle | x \in X\}$ and $B = \{\langle x, \tilde{\mu}_B(x)\rangle | x \in X\}$ be any two interval-valued fuzzy sets on a set X.

Then $A \cap B = \{\langle x, \tilde{\mu}_{A\cap B}(x)\rangle | x \in X\}$ where $\tilde{\mu}_{A\cap B}(x) = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\}$ and

$A \cup B = \{\langle x, \tilde{\mu}_{A\cup B}(x)\rangle | x \in X\}$ where $\tilde{\mu}_{A\cup B}(x) = r\max\{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\}$

Lemma 1.1.20:[15] Let h be a mapping from the set X into a set Y. Let $A = [\mu_A^L, \mu_A^U]$ and $B = [\mu_B^L, \mu_B^U]$ be interval-valued fuzzy sets in X and Y respectively. Then,

(i) $h(A) = [\mu_{h(A)}^L, \mu_{h(A)}^U]$

(ii) $h^{-1}(B) = [\mu_{h^{-1}(B)}^L, \mu_{h^{-1}(B)}^U]$

Definition 1.1.21:[15] Let $A = \{\langle x, \tilde{\mu}_A(x)\rangle | x \in X\}$ be an interval-valued fuzzy set in X. For a fixed $[\delta_1, \delta_2] \in D[0,1]$, the set $U(\tilde{\mu}_A; [\delta_1, \delta_2]) = \{x \in X \mid \tilde{\mu}_A(x) \geq [\delta_1, \delta_2]\}$ is called an **interval-valued upper** $[\delta_1, \delta_2]$ **-level subset** (interval-valued upper level subset or interval-valued upper level cut) of A.

Definition 1.1.22:[65] An interval-valued fuzzy set $A = \{\langle x, \tilde{\mu}_A(x)\rangle | x \in X\}$ in a set X is said to have the **rsup property** if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that $\tilde{\mu}_A(x_0) = r\sup_{t\in T} \tilde{\mu}_A(t)$.

Definition 1.1.23:[15] Let h be a mapping from a set X into a set Y.

(i) Let $A = \{\langle x, \tilde{\mu}_A(x)\rangle | x \in X\}$ be an interval-valued fuzzy set in X. Then the image of A under h, denoted by h(A) is the interval-valued fuzzy set in Y with membership function $\tilde{\mu}_{h(A)}$ defined by :

$$\tilde{\mu}_{h(A)}(y) = \begin{cases} r\sup\limits_{z\in h^{-1}(y)} \tilde{\mu}_A(z) & \text{if } h^{-1}(y) = \{x \mid h(x) = y\} \neq \phi \\ [0,0] & \text{, otherwise} \end{cases}$$

(ii) Let $B = \{\langle x, \tilde{\mu}_B(x)\rangle | x \in Y\}$ be an interval-valued fuzzy set in Y. Then the inverse image (or pre-image) of B under h, denoted by $h^{-1}(B)$ is the interval-valued fuzzy set in X membership function $\tilde{\mu}_{h^{-1}(B)}$ defined by $\tilde{\mu}_{h^{-1}(B)}(x) = \tilde{\mu}_B(h(x))$ for all $x \in X$.

Definition 1.1.24:[15] If $A=\{\langle x,\tilde{\mu}_A(x)\rangle | x\in X\}$ and $B=\{\langle x,\tilde{\mu}_B(x)\rangle | x\in X\}$ are interval-valued fuzzy sets in X, then the Cartesian product $A\times B$ of A and B with membership function $\tilde{\mu}_{A\times B}:X\times X\to D[0,1]$ is defined as $\tilde{\mu}_{A\times B}(x,y)=r\min\{\tilde{\mu}_A(x),\tilde{\mu}_B(y)\}$, for all $x,y\in X$.

Atanassov and Gargov introduced the concept of an interval-valued intuitionistic fuzzy sets as a generalization of the notion of an intuitionistic fuzzy set. Now, we recall some fundamental definitions from interval-valued intuitionistic fuzzy set theory.

Definition 1.1.25:[10] Let X be a nonempty set. An **interval-valued intuitionistic fuzzy set** A on X is an object $A=\{\langle x,[\mu_A^L(x),\mu_A^U(x)],[\nu_A^L(x),\nu_A^U(x)]\rangle | x\in X\}$, such that μ_A^L,μ_A^U and ν_A^L,ν_A^U are membership and non-membership fuzzy sets in X to A where $\mu_A^L\le\mu_A^U$ and $\nu_A^L\le\nu_A^U$ with $0\le\mu_A^L+\nu_A^L\le 1$ and $0\le\mu_A^U+\nu_A^U\le 1$ for all $x\in X$.

Equivalently, an interval-valued intuitionistic fuzzy set $A=\{\langle x,\tilde{\mu}_A(x),\tilde{\nu}_A(x)\rangle | x\in X\}$, where $\tilde{\mu}_A:X\to D[0,1]$ and $\tilde{\nu}_A:X\to D[0,1]$ denote the interval-valued membership and interval-valued non-membership functions in X to A, respectively such that $\tilde{\mu}_A(x)=[\mu_A^L(x),\mu_A^U(x)]$, $\tilde{\nu}_A(x)=[\nu_A^L(x),\nu_A^U(x)]$.

Definition 1.1.26:[10] Let $A_1=(\tilde{\mu}_{A_1},\tilde{\nu}_{A_1})$ and $A_2=(\tilde{\mu}_{A_2},\tilde{\nu}_{A_2})$ be interval-valued intuitionistic fuzzy sets on a nonempty set X. (i) The intersection of A_1 and A_2 is the interval-valued intuitionistic fuzzy set $A_1\cap A_2=(\tilde{\mu}_{A_1\cap A_2},\tilde{\nu}_{A_1\cup A_2})$ where, for all $x\in X$

$\tilde{\mu}_{A_1\cap A_2}(x)=r\min\{\tilde{\mu}_{A_1}(x),\tilde{\mu}_{A_2}(x)\}$ and $\tilde{\nu}_{A_1\cup A_2}(x)=r\max\{\tilde{\nu}_{A_1}(x),\tilde{\nu}_{A_2}(x)\}$.

(ii) $\bigcap_{i\in\Omega}A_i=\left\{\left\langle x,\tilde{\mu}_{\underset{i\in\Omega}{\cap}A_i}(x),\tilde{\nu}_{\underset{i\in\Omega}{\cup}A_i}(x)\right\rangle \middle| x\in X\right\}$

where $\tilde{\mu}_{\underset{i\in\Omega}{\cap}A_i}(x)=r\inf_{i\in\Omega}(\tilde{\mu}_{A_i}(x))$ and $\tilde{\nu}_{\underset{i\in\Omega}{\cup}A_i}(x)=r\sup_{i\in\Omega}(\nu_{A_i}(x))$

Definition 1.1.27:[10] Let $A=(\tilde{\mu}_A,\tilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy set of a set X. Then, the interval-valued intuitionistic fuzzy set $\oplus A$ and $\otimes A$, are defined as

$\oplus A = (\tilde{\mu}_A, (\tilde{\mu}_A)^c)$ and $\otimes A = ((\tilde{\nu}_A)^c, \tilde{\nu}_A)$ where $(\tilde{\mu}_A)^c(x) = [1 - \mu_A^U(x), 1 - \mu_A^L(x)]$ and $(\tilde{\nu}_A)^c(x) = [1 - \nu_A^U(x), 1 - \nu_A^L(x)]$ for all $x \in X$.

Definition 1.1.28:[10] Let $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy set of a nonempty set X. For $[s_1, s_2], [t_1, t_2] \in D[0,1]$, the set $U(\tilde{\mu}_A; [s_1, s_2]) = \{x \in X \mid \tilde{\mu}_A(x) \geq [s_1, s_2]\}$ is called the **interval-valued upper $[s_1, s_2]$-level subset (interval-valued upper level cut or interval-valued upper level subset) of A** and $L(\tilde{\nu}_A; [t_1, t_2]) = \{x \in X \mid \tilde{\nu}_A(x) \leq [t_1, t_2]\}$ is called the **interval-valued lower $[t_1, t_2]$-level subset (interval-valued lower level cut or interval-valued lower level subset) of A**.

Definition 1.1.29:[10] Let h be a mapping from a set X into a set Y.

(i) Let $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy set in X. Then the image of A under h, denoted by $h(A) = \{(x, \tilde{\mu}_{h(A)}(x), \tilde{\nu}_{h(A)}(x))\}$ for all $x \in X$, is defined by:

$$\tilde{\mu}_{h(A)}(y) = \begin{cases} \operatorname{rsup}_{z \in h^{-1}(y)} \tilde{\mu}_A(z) & \text{if } h^{-1}(y) = \{x \mid h(x) = y\} \neq \phi \\ [0,0] & \text{, otherwise} \end{cases}$$

and

$$\tilde{\nu}_{h(A)}(y) = \begin{cases} \operatorname{rinf}_{z \in h^{-1}(y)} \tilde{\nu}_A(z) & \text{if } h^{-1}(y) = \{x \mid h(x) = y\} \neq \phi \\ [1,1] & \text{, otherwise} \end{cases}$$

is an interval-valued intuitionistic fuzzy set in Y.

(ii) Let $B = (\tilde{\mu}_B, \tilde{\nu}_B)$ be an interval-valued intuitionistic fuzzy set in Y. Then the inverse image (or pre-image) of B under h, denoted by $h^{-1}(B) = \{(x, \tilde{\mu}_{h^{-1}(B)}(x), \tilde{\nu}_{h^{-1}(B)}(x))\}$ for all $x \in X$, is an interval-valued intuitionistic fuzzy set in X defined by $\tilde{\mu}_{h^{-1}(B)}(x) = \tilde{\mu}_B(h(x))$ and $\tilde{\nu}_{h^{-1}(B)}(x) = \tilde{\nu}_B(h(x))$ for all $x \in X$.

Definition 1.1.30:[10] An interval-valued intuitionistic fuzzy set $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ in a set X with the degree of membership $\tilde{\mu}_A : X \to D[0,1]$ and the degree of non-membership $\tilde{\nu}_A : X \to D[0,1]$ is said to have **rsup-rinf property** if for any subset T of X there exists $x_0 \in T$

such that $\tilde{\mu}_A(x_0) = r\sup_{t\in T}\tilde{\mu}_A(t)$ and $\tilde{\nu}_A(x_0) = r\inf_{t\in T}\tilde{\nu}_A(t)$.

Definition 1.1.31:[10] Let $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ and $B = (\tilde{\mu}_B, \tilde{\nu}_B)$ be any two interval-valued intuitionistic fuzzy sets in X. Then, the Cartesian product of A and B is given by $A \times B = (\tilde{\mu}_{A\times B}, \tilde{\nu}_{A\times B})$ where the membership function $\tilde{\mu}_{A\times B} : X \times X \to D[0,1]$ and the non-membership function $\tilde{\nu}_{A\times B} : X \times X \to D[0,1]$ are defined by $\tilde{\mu}_{A\times B}(x,y) = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\}$ and $\tilde{\nu}_{A\times B}(x,y) = r\max\{\tilde{\nu}_A(x), \tilde{\nu}_B(y)\}$ for all $x, y \in X$.

Combininig the notion of interval-valued fuzzy set and fuzzy set Jun et al. [36] introduced cubic set defined as follows:

Definition 1.1.32:[36] Let X be a nonempty set . A **cubic set** A in X is a structure $A = \{< x, \tilde{\mu}_A(x), \omega_A(x) > | x \in X\}$ briefly denoted by $A = (\tilde{\mu}_A, \omega_A)$ where $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)] : X \to D[0,1]$ is an interval-valued fuzzy set in X and $\omega_A : X \to [0,1]$ is a fuzzy set in X .

For two cubic sets $A = (\tilde{\mu}_A, \omega_A)$ and $B = (\tilde{\mu}_B, \omega_B)$ in X, we define

1. $A \subset B$ iff $\tilde{\mu}_A \le \tilde{\mu}_B$ and $\omega_A \ge \omega_B$
2. $A = B$ iff $A \subset B$ and $B \subset A$
3. $A^c = \{\langle x, \omega_A(x), \tilde{\mu}_A(x)\rangle \mid x \in X\}$
4. $A \cap B = \{\langle x, \tilde{\mu}_{A\cap B}(x), \omega_{A\cup B}(x)\rangle | x \in X\} = \{\langle x, r\min(\tilde{\mu}_A(x), \tilde{\mu}_B(x)), \max(\omega_A(x), \omega_B(x))\rangle | x \in X\}$
5. $A \cup B = \{\langle x, \tilde{\mu}_{A\cup B}(x), \omega_{A\cap B}(x)\rangle | x \in X\} = \{\langle x, r\max(\tilde{\mu}_A(x), \tilde{\mu}_B(x)), \min(\omega_A(x), \omega_B(x))\rangle | x \in X\}$

Definition 1.1.33:[34] Let $A = (\tilde{\mu}_A, \omega_A)$ be a cubic set of X. For $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$, the set $U(\tilde{\mu}_A; [s_1, s_2]) = \{x \in X | \tilde{\mu}_A(x) \ge [s_1, s_2]\}$ is called an **interval-valued upper** $[s_1, s_2]$**-level subset** of A and $L(\omega_A; t) = \{x \in X | \omega_A(x) \le t\}$ is called **lower t-level subset** of A.

Definition 1.1.34:[34] A cubic set $A = (\tilde{\mu}_A, \omega_A)$ in a nonempty set X is said to have the **rsup-inf property** if for any subset T of X there exists $t_0 \in T$ such that $\tilde{\mu}_A(t_0) = r\sup_{t\in T}\tilde{\mu}_A(t)$ and $\omega_A(t_0) = \inf_{t\in T}\omega_A(t)$ respectively.

Definition 1.1.35:[36] Consider a collection of cubic sets $A_i = \{< x, \tilde{\mu}_{A_i}(x), \omega_{A_i}(x) >| x \in X\}$ where $i \in \Omega$,

(i) P-union and P-intersection denoted by $P\left(\bigcup\limits_{i\in\Omega} A_i\right)$ and $P\left(\bigcap\limits_{i\in\Omega} A_i\right)$ are defined as follows.

$$P\left(\bigcup_{i\in\Omega} A_i\right)=\left\{\left\langle x,\tilde{\mu}_{\underset{i\in\Omega}{\cup} A_i}(x),\omega_{\underset{i\in\Omega}{\cup} A_i}(x)\right\rangle \middle| x\in X\right\}=\left\{\left\langle x, r\sup_{i\in\Omega}\tilde{\mu}_{A_i}(x),\sup_{i\in\Omega}\omega_{A_i}(x)\right\rangle \middle| x\in X\right\},$$

$$P\left(\bigcap_{i\in\Omega} A_i\right)=\left\{\left\langle x,\tilde{\mu}_{\underset{i\in\Omega}{\cap} A_i}(x),\omega_{\underset{i\in\Omega}{\cap} A_i}(x)\right\rangle \middle| x\in X\right\}=\left\{\left\langle x, r\inf_{i\in\Omega}\tilde{\mu}_{A_i}(x),\inf_{i\in\Omega}\omega_{A_i}(x)\right\rangle \middle| x\in X\right\}$$

(ii) Union and intersection denoted by $\bigcup\limits_{i\in\Omega} A_i$ and $\bigcap\limits_{i\in\Omega} A_i$ are defined as follows.

$$\bigcup_{i\in\Omega} A_i=\left\{\left\langle x,\tilde{\mu}_{\underset{i\in\Omega}{\cup} A_i}(x),\omega_{\underset{i\in\Omega}{\cap} A_i}(x)\right\rangle \middle| x\in X\right\}=\left\{\left\langle x, r\sup_{i\in\Omega}\tilde{\mu}_{A_i}(x),\inf_{i\in\Omega}\omega_{A_i}(x)\right\rangle \middle| x\in X\right\},$$

$$\bigcap_{i\in\Omega} A_i=\left\{\left\langle x,\tilde{\mu}_{\underset{i\in\Omega}{\cap} A_i}(x),\omega_{\underset{i\in\Omega}{\cup} A_i}(x)\right\rangle \middle| x\in X\right\}=\left\{\left\langle x, r\inf_{i\in\Omega}\tilde{\mu}_{A_i}(x),\sup_{i\in\Omega}\omega_{A_i}(x)\right\rangle \middle| x\in X\right\}$$

Definition 1.1.36:[34] Let h be a mapping from a set X into a set Y.

(i) Let $A = (\tilde{\mu}_A, \omega_A)$ be a cubic set in X. Then the image of A under h, denoted by $h(A) = \left\{\left\langle x, \tilde{\mu}_{h(A)}(x), \omega_{h(A)}(x)\right\rangle \middle| x \in X\right\}$, is defined by:

$$\tilde{\mu}_{h(A)}(y)=\begin{cases} r\sup\limits_{z\in h^{-1}(y)}\tilde{\mu}_A(z) & \text{if } h^{-1}(y)=\{x\mid h(x)=y\}\neq\phi \\ [0,0] & \text{, otherwise} \end{cases} \qquad \text{and}$$

$$\omega_{h(A)}(y)=\begin{cases} \inf\limits_{z\in h^{-1}(y)}\omega_A(z) \text{ if } h^{-1}(y)=\{x\mid h(x)=y\}\neq\phi \\ 1 \quad , \quad \text{otherwise} \end{cases} \qquad \text{is a cubic set in Y.}$$

(ii) Let $B = (\tilde{\mu}_B, \omega_B)$ be a cubic set in Y. Then the inverse image (or pre-image) of B under h, denoted by $h^{-1}(B) = \left\{\left\langle x, \tilde{\mu}_{h^{-1}(B)}(x), \omega_{h^{-1}(B)}(x)\right\rangle \middle| x \in X\right\}$ is a cubic set in X defined by

$\tilde{\mu}_{h^{-1}(B)}(x) = \tilde{\mu}_B(h(x))$ and $\omega_{h^{-1}(B)}(x) = \omega_B(h(x))$ for all $x \in X$.

Definition 1.1.37:[34] Let $A = (\tilde{\mu}_A, \omega_A)$ and $B = (\tilde{\mu}_B, \omega_B)$ be any two cubic sets in X. Then, the Cartesian product of cubic sets A and B is given by $A \times B = (\tilde{\mu}_{A\times B}, \omega_{A\times B})$ where $\tilde{\mu}_{A\times B} : X \times X \rightarrow D[0,1]$ and $\omega_{A\times B} : X \times X \rightarrow [0,1]$ are defined by $\tilde{\mu}_{A\times B}(x,y) = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\}$ and $\omega_{A\times B}(x,y) = \max\{\omega_A(x), \omega_B(y)\}$ for all $(x,y) \in X \times X$.

1.2 BCK/BCI-Algebras

In this section, we recall the definitions of different types of algebras that arise from propositional and implicational calculi.

Definition 1.2.1: [28] A BCK-algebra $(X,*,0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: For all $x, y, z \in X$,

1. $((x*y)*(x*z))*(z*y)=0$
2. $(x*(x*y))*y=0$
3. $x*x=0$
4. $x*y=0$ and $y*x=0$ implies $x=y$
5. $0*x=0$

Definition 1.2.2:[27] A BCI-algebra $(X,*,0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: For all $x, y, z \in X$,

1. $((x*y)*(x*z))*(z*y)=0$
2. $(x*(x*y))*y=0$
3. $x*x=0$
4. $x*y=0$ and $y*x=0$ implies $x=y$

Definition 1.2.3: [24] A BCH-algebra $(X,*,0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: For all $x, y, z \in X$,

1. $x*x=0$
2. $x*y=0$ and $y*x=0$ implies $x=y$
3. $(x*y)*z=(x*z)*y$

Definition 1.2.4: [60] A B-algebra $(X,*,0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: For all $x, y, z \in X$,

1. $x*x=0$
2. $x*0=x$
3. $(x*y)*z=x*(z*(0*y))$

Definition 1.2.5: [58] A Q-algebra $(X,*,0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x*x=0$
2. $x*0=x$
3. $(x*y)*z=(x*z)*y$ for all $x,y,z\in X$.

Definition 1.2.6: [59] A d-algebra $(X,*,0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: For all $x,y\in X$,

1. $x*x=0$
2. $0*x=0$
3. $x*y=0$ and $y*x=0$ implies $x=y$

Definition 1.2.7:[39] A SU-algebra $(X,*,0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: For any $x,y,z\in X$,

1. $((x*y)*(x*z))*(y*z)=0$
2. $x*0=x$
3. If $x*y=0$ implies $x=y$

Definition 1.2.8:[63] A KU-algebra $(X,*,0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: For any $x,y,z\in X$,

1. $(x*y)*((y*z)*(x*z))=0$
2. $x*0=0$
3. $0*x=x$
4. If $x*y=0=y*x$ implies $x=y$

Definition 1.2.9: [37] A BH-algebra $(X,*,0)$ is a nonempty set X with constant 0 and a binary operation $*$ satisfying the following conditions:

1. $x*0=x$
2. $x*x=0$
3. $x*y=0$ and $y*x=0$ implies $x=y$, for all $x,y\in X$.

Definition 1.2.10: [47] A TM-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * 0 = x$
2. $(x * y) * (x * z) = z * y$ for all $x, y, z \in X$.

Definition 1.2.11:[70] A BF-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * x = 0$,
2. $x * 0 = x$,
3. $0 * (x * y) = y * x$, for all $x, y \in X$.

Definition 1.2.12: [64] A nonempty set X with a constant 0 and a binary operation $*$ is called PS – algebra if it satisfies the following axioms.

1. $x * x = 0$
2. $x * 0 = 0$
3. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$, for all $x, y \in X$.

Definition 1.2.13:[41] A BM-algebra is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * 0 = x$
2. $(z * x) * (z * y) = y * x$, for any $x, y, z \in X$.

Definition 1.2.14: [42] A BO-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * x = 0$
2. $x * 0 = x$
3. $x * (y * z) = (x * y) * (0 * z)$ for any $x, y, z \in X$.

Definition 1.2.15:[61] An algebra $(X, *, 0)$ is called RG – algebra if the following axioms are satisfied: For any $x, y, z \in X$,

1. $x * 0 = x$
2. $x * y = (x * z) * (y * z)$
3. $x * y = y * x = 0$ imply $x = y$.

Definition 1.2.16:[12] A BRK-algebra is a nonempty set A with a constant 0 and a binary operation $*$ satisfying axioms:

1. $x * 0 = x$
2. $(x * y) * x = 0 * y$ for any $x, y \in A$.

Definition 1.2.17:[46] A BCL-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: For any $x, y, z \in X$.

1. $x * x = 0$;
2. $x * y = 0$ and $y * x = 0$ imply $x = y$
3. $((x * y) * z) * ((x * z) * y) * ((z * y) * x) = 0$

Definition 1.2.18:[6] A BP-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * x = 0$
2. $x * (x * y) = y$
3. $(x * z) * (y * z) = x * y$; for any $x, y, z \in X$.

Definition 1.2.19:[14] A CI-algebra $(X, *, 1)$ is a nonempty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:

1. $x * x = 1$
2. $1 * x = x$
3. $x * (y * z) = y * (x * z)$ for any $x, y, z \in X$.

Definition 1.2.20: [43] A BE-algebra $(X, *, 1)$ is a nonempty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:

1. $x * x = 1$ for all $x \in X$.
2. $x * 1 = 1$ for all $x \in X$.
3. $1 * x = x$ for all $x \in X$.
4. $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

1.3 Z-Algebras

In this section, we recall the definition of a new algebraic structure introduced by Chandramouleeswaran et al.[18], namely the Z-algebra which is an algebra based on propositional calculus and the relation between Z-algebra and other algebras.

Definition 1.3.1: [18] A Z-algebra $(X,*,0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

1. $x*0=0$
2. $0*x=x$
3. $x*x=x$
4. $x*y=y*x$, when $x\neq 0$ and $y\neq 0$ for all $x,y\in X$.

Note: Throughout our work, X means Z-algebra $(X,*,0)$.

Example 1.3.2: [18] Let X={0,1,2,3} be a set with constant 0 and a binary operation $*$ defined on X with the following Cayley table, is a Z-algebra.

$*$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Example 1.3.3: [18] Let $(X=R,*,0)$ where $x,y\in R$. Define a binary operation $*$ on X by,

$$x*y=\begin{cases} y & \text{if} \quad (x=0 \text{ and } y\neq 0) \quad \text{or} \quad (x\neq 0 \text{ and } y=0) \\ x & x=y \\ xy & x\neq y \end{cases}$$

Then, $(X,*,0)$ be a Z-algebra.

Definition 1.3.4: [18] Let S be a nonempty subset of a Z-algebra $(X,*,0)$. Then, S is called a **Z-Subalgebra** of $(X,*,0)$, if $x*y\in S$, for all $x,y\in S$.

Example 1.3.5:[18] Consider the Z-algebra in example 1.3.2. Then, the subset $A = \{1,3\} \subset X$ and $B = \{2,3\} \subset X$ are Z-Subalgebras of X, but the subset $C = \{1,2,3\} \subset X$ is not a Z-Subalgebra of X , for $1 * 2 = 0 \notin C$.

In **[18]** , the authors have proved that a Z-algebra $(X,*,0)$ is not a BCK-algebra, a Q-algebra, a BH-algebra, a SU-algebra, a BM-algebra, a B-algebra, a BF-algebra, a BRK-algebra, a RG-algebra and a TM-algebra.

Proposition 1.3.6:[18] Let $(X,*,0)$ be a Z-algebra. Then, it is not a d-algebra, a KU-algebra, a BE-algebra, a PS-algebra, a CI-algebra, a BCI-algebra, a BCH-algebra, a BP-algebra, a BO-algebra and a BCL-algebra.

Proof: In all the above algebras cited here except a Z-algebra, we have, $x * x = 0$. But by definition 1.3.1, for a Z-algebra we have, $x * x = x \neq 0$.

Hence the proof.

Proposition 1.3.7: [18] Let $(X,*,0)$ be a Z-algebra. Then, the following results hold for all $x, y, z \in X$.

1.$(x * (x * (y * x))) = x$, if $y = 0$

2. $(x * y) * 0 = (y * 0) * (x * 0)$

3.$(x * y) * [(y * x) * (x * y)] = y * x$

4.$0 * (x * y) = (0 * x) * (0 * y)$

5. $x * (0 * y) = x * y = y * x$, for all $x \neq y$.

Definition 1.3.8:[18] Let $(X,*,0)$ be a Z-algebra and I be a subset of X. Then, I is called an **Z-Ideal** of X, if it satisfies the following conditions: For all x , y in X,

1.$0 \in I$

2.$x * y \in I$ and $y \in I$ implies $x \in I$

Example 1.3.9:[18] Consider the Z-algebra in example 1.3.2. Then, $I = \{0,1,2\} \subset X$ is an Z-Ideal of X.

Proposition 1.3.10: [18] Let I be an Z-Ideal of a Z-algebra $(X,*,0)$. If $x \in I$ and $y * x = 0$, then $y \in I$.

Definition 1.3.11: [18] Let $(X,*,0)$ and $(Y,*',0')$ be two Z-algebras. A mapping $h:(X,*,0)\to(Y,*',0')$ is called a **Z-homomorphism of Z-algebras** if $h(x*y)=h(x)*'h(y)$ for all $x,y\in X$.

Definition 1.3.12:[18] Let $h:(X,*,0)\to(Y,*',0')$ be a Z-homomorphism of Z-algebras. Then

1. h is called a **Z-monomorphism** of Z-algebras if h is one to one.
2. h is called an **Z-epimorphism** (or surjective Z-homomorphism) of Z-algebras if h is onto.
3. Kernal of h defined by $Ker(h)=\{x\in X:h(x)=0'\}$ is a subset of X.

Definition 1.3.13:[18] If h is one-one and onto Z-homomorphism from a Z-algebra $(X,*,0)$ into itself then h is called an **Z-automorphism** of Z-algebras. The set of all Z-automorphisms of a Z-algebra X is denoted by Aut(X).

Definition 1.3.14: [18] If h is a Z-homomorphism from a Z-algebra $(X,*,0)$ into itself then h is called an **Z-endomorphism**.

Theorem 1.3.15:[18] Let $h:(X,*,0)\to(Y,*',0')$ be a homomorphism of Z-algebras. Then

(i) $h(0)=0'$.

(ii) Ker (h) is both Z-Subalgebra and Z-ideal of X.

(iii) If B is Z-Subalgebra (Z-ideal) of Y, then $h^{-1}(B)$ is Z-Subalgebra (Z-ideal) of X.

(iv) If A is Z-Subalgebra (Z-ideal) of X, then h(A) is Z-Subalgebra (Z-ideal) of Y if h is surjective Z-homomorphism.