

## Intuitionistic Fuzzy Structures in Z-Algebras

In 1986, the idea of “intuitionistic fuzzy set” was first published by Atanassov [8], as a generalization of the notion of fuzzy set. Using the Atanassov’s idea, in this chapter, we establish the intuitionistic fuzzification of Z-Subalgebras and Z-ideals in Z-algebras and explore some of their properties in detail. Further, we investigate some characterization theorems on these notions using the concepts of upper s-level and lower t-level subsets. This chapter is divided into two sections. The first section deals with the study of intuitionistic fuzzy Z-Subalgebras in Z-algebras. In section 2, we introduce the concept of intuitionistic fuzzy Z-Ideals in Z-algebras and prove some interesting results.

### 5.1 Intuitionistic Fuzzy Z-Subalgebras in Z-Algebras

In this section we introduce the notion of Intuitionistic Fuzzy Z-Subalgebra of a Z-algebra. Also we prove some interesting results.

**Definition 5.1.1:** An Intuitionistic Fuzzy Set  $A = (\mu_A, \nu_A)$  in a Z-algebra  $(X, *, 0)$  is called an **Intuitionistic fuzzy Z-Subalgebra** of X if it satisfies the following conditions:

- (i)  $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$
- (ii)  $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$  , for all  $x, y \in X$  .

**Example 5.1.2:** Consider a Z-algebra  $X = \{0, 1, 2, 3\}$  with the following Cayley table as in Example 2.1.2.

*	0	1	2	3
0	0	1	2	3
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

An intuitionistic fuzzy set  $A = (\mu_A, \nu_A)$  in  $X$  defined by  $\mu_A(x) = \begin{cases} 0.6 & \text{if } x = 0 \\ 0.4 & \text{if } x = 1 \\ 0.3 & \text{if } x = 2,3 \end{cases}$  and

$$\nu_A(x) = \begin{cases} 0.4 & \text{if } x = 0 \\ 0.5 & \text{if } x = 1 \\ 0.6 & \text{if } x = 2,3 \end{cases} \text{ is an intuitionistic fuzzy Z-Subalgebra of } X.$$

**Theorem 5.1.3:** Let  $A_1$  and  $A_2$  be two intuitionistic fuzzy Z-Subalgebras of a Z-algebra  $X$ .

Then  $A_1 \cap A_2$  is an intuitionistic fuzzy Z-Subalgebra of  $X$ .

**Proof:** Let  $x, y \in A_1 \cap A_2$  then  $x, y \in A_1$  and  $A_2$ .

Now,

$$\begin{aligned} \text{(i) } \mu_{A_1 \cap A_2}(x * y) &= \min\{\mu_{A_1}(x * y), \mu_{A_2}(x * y)\} \\ &\geq \min\{\min\{\mu_{A_1}(x), \mu_{A_1}(y)\}, \min\{\mu_{A_2}(x), \mu_{A_2}(y)\}\} \\ &= \min\{\min\{\mu_{A_1}(x), \mu_{A_2}(x)\}, \min\{\mu_{A_1}(y), \mu_{A_2}(y)\}\} \\ &= \min\{\mu_{A_1 \cap A_2}(x), \mu_{A_1 \cap A_2}(y)\} \end{aligned}$$

$$\begin{aligned} \text{(ii) } \nu_{A_1 \cap A_2}(x * y) &= \max\{\nu_{A_1}(x * y), \nu_{A_2}(x * y)\} \\ &\leq \max\{\max\{\nu_{A_1}(x), \nu_{A_1}(y)\}, \max\{\nu_{A_2}(x), \nu_{A_2}(y)\}\} \\ &= \max\{\max\{\nu_{A_1}(x), \nu_{A_2}(x)\}, \max\{\nu_{A_1}(y), \nu_{A_2}(y)\}\} \\ &= \max\{\nu_{A_1 \cap A_2}(x), \nu_{A_1 \cap A_2}(y)\} \end{aligned}$$

From (i) and (ii) we get  $A_1 \cap A_2$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$ .

The following example illustrates Theorem 5.1.3.

**Example 5.1.4:** Let  $X = \{0, 1, 2, 3\}$  be a Z-algebra with the following Cayley table:

*	0	1	2	3
<b>0</b>	0	1	2	3
<b>1</b>	0	1	2	3
<b>2</b>	0	2	2	1
<b>3</b>	0	3	1	3

Intuitionistic fuzzy sets  $A_1 = (\mu_{A_1}, \nu_{A_1})$  and  $A_2 = (\mu_{A_2}, \nu_{A_2})$  in  $X$  defined by

$$\mu_{A_1}(x) = \begin{cases} 0.6 & \text{if } x = 0, 2 \\ 0.3 & \text{otherwise} \end{cases}, \quad \nu_{A_1}(x) = \begin{cases} 0.4 & \text{if } x = 0, 2 \\ 0.5 & \text{otherwise} \end{cases}, \quad \mu_{A_2}(x) = \begin{cases} 0.7 & \text{if } x = 0 \\ 0.5 & \text{if } x = 1, 2 \\ 0.2 & \text{if } x = 3 \end{cases} \quad \text{and}$$

$$\nu_{A_2}(x) = \begin{cases} 0.3 & \text{if } x = 0 \\ 0.4 & \text{if } x = 1, 2 \\ 0.7 & \text{if } x = 3 \end{cases} \quad \text{are intuitionistic fuzzy Z-Subalgebras of } X.$$

$$\text{Then, } \mu_{A_1 \cap A_2}(x) = \begin{cases} 0.6 & \text{if } x = 0 \\ 0.3 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2 \\ 0.2 & \text{if } x = 3 \end{cases} \quad \text{and} \quad \nu_{A_1 \cup A_2}(x) = \begin{cases} 0.4 & \text{if } x = 0, 2 \\ 0.5 & \text{if } x = 1 \\ 0.7 & \text{if } x = 3 \end{cases}$$

Clearly  $A_1 \cap A_2$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$ .

We can generalize the above theorem as follows

**Corollary 5.1.5:** Let  $\{A_i \mid i \in \Omega\}$  be a family of intuitionistic fuzzy Z-Subalgebras of a Z-algebra  $X$ . Then  $\bigcap_{i \in \Omega} A_i$  is an intuitionistic fuzzy Z-Subalgebra of  $X$ .

**Theorem 5.1.6:** An intuitionistic fuzzy set  $A = (\mu_A, \nu_A)$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$  if and only if the fuzzy sets  $\mu_A$  and  $(\nu_A)^c$  are fuzzy Z-Subalgebras of  $X$ .

**Proof:** Let  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$  then

$$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} \quad \text{and} \quad (1)$$

$$\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\} \quad \forall x, y \in X \quad (2)$$

For every  $x, y \in X$ , we have,

$$\begin{aligned} (\nu_A)^c(x * y) &= 1 - \nu_A(x * y) \geq 1 - \max\{\nu_A(x), \nu_A(y)\} = \min\{1 - \nu_A(x), 1 - \nu_A(y)\} \\ &= \min\{(\nu_A)^c(x), (\nu_A)^c(y)\} \end{aligned}$$

Hence  $\mu_A$  and  $(\nu_A)^c$  is a fuzzy Z-Subalgebras of  $X$ .

Conversely, for every  $x, y \in X$ , we have,

$$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} \quad \text{and} \quad (3)$$

$$\begin{aligned} 1 - v_A(x * y) &= (v_A)^c(x * y) \geq \min\{(v_A)^c(x), (v_A)^c(y)\} = \min\{1 - v_A(x), 1 - v_A(y)\} \\ &= 1 - \max\{v_A(x), v_A(y)\} \end{aligned}$$

$$\text{i.e., } v_A(x * y) \leq \max\{v_A(x), v_A(y)\} \quad (4)$$

By (3) and (4) we get,  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-Subalgebra of  $X$ .

**Theorem 5.1.7:**  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$  if and only if  $\oplus A = (\mu_A, (\mu_A)^c)$  and  $\otimes A = ((v_A)^c, v_A)$ , both are intuitionistic fuzzy Z-Subalgebras of  $X$ .

**Proof:** Let  $A = (\mu_A, v_A)$  be an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$ .

Let  $x, y \in X$ . Then,

$$(i) \quad \mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

$$(ii) \quad v_A(x * y) \leq \max\{v_A(x), v_A(y)\}$$

$$\begin{aligned} (iii) \quad (\mu_A)^c(x * y) &= 1 - \mu_A(x * y) \leq 1 - \min\{\mu_A(x), \mu_A(y)\} = \max\{1 - \mu_A(x), 1 - \mu_A(y)\} \\ &= \max\{(\mu_A)^c(x), (\mu_A)^c(y)\} \end{aligned}$$

$$\begin{aligned} (iv) \quad (v_A)^c(x * y) &= 1 - v_A(x * y) \geq 1 - \max\{v_A(x), v_A(y)\} = \min\{1 - v_A(x), 1 - v_A(y)\} \\ &= \min\{(v_A)^c(x), (v_A)^c(y)\} \end{aligned}$$

From (i) and (iii), we get  $\oplus A$  is an intuitionistic fuzzy Z-Subalgebra of  $X$ .

And, from (ii) and (iv), we get  $\otimes A$  is an intuitionistic fuzzy Z-Subalgebra of  $X$ .

**Conversely,** for any  $x, y \in A$ ,  $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$  and

$v_A(x * y) \leq \max\{v_A(x), v_A(y)\}$  implies  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-Subalgebra of  $X$ .

**Theorem 5.1.8:** If  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$ , then

$U(\mu_A; s)$  and  $L(v_A; t)$  are Z-Subalgebras of  $X$  for all  $s, t \in [0, 1]$ .

**Proof:** (i) Let  $x, y \in U(\mu_A; s)$ . Then  $\mu_A(x) \geq s$  and  $\mu_A(y) \geq s$ .

It follows that,  $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min\{s, s\} \geq s$  so that  $x * y \in U(\mu_A; s)$ .

Hence,  $U(\mu_A; s)$  is a Z-Subalgebra of  $X$ .

(ii) Let  $x, y \in L(v_A; t)$ . Then,

$$v_A(x) \leq t \text{ and } v_A(y) \leq t$$

It follows that,  $v_A(x * y) \leq \max\{v_A(x), v_A(y)\} \leq \max\{t, t\} = t$  so that  $x * y \in L(v_A; t)$ .

Hence,  $L(v_A; t)$  is a Z-Subalgebra of  $X$ .

From (i) and (ii), we get the upper  $s$ -level subset  $U(\mu_A; s)$  and lower  $t$ -level subset  $L(v_A; t)$  of  $A$  are Z-Subalgebras of  $X$ .

**Theorem 5.1.9:** Let  $A = (\mu_A, v_A)$  be an intuitionistic fuzzy set in a Z-algebra  $X$  such that the sets  $U(\mu_A; s)$  and  $L(v_A; t)$  are Z-Subalgebras of  $X$  for every  $s, t \in [0, 1]$ . Then  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-Subalgebra of  $X$ .

**Proof :** Suppose there exist  $x_0, y_0 \in X$  such that  $\mu_A(x_0 * y_0) < \min\{\mu_A(x_0), \mu_A(y_0)\}$  (1)

Let  $\mu_A(x_0) = \theta_1$ ,  $\mu_A(y_0) = \theta_2$  and  $\mu_A(x_0 * y_0) = s$ . Then,  $s < \min\{\theta_1, \theta_2\}$ .

$$\text{Let us consider, } s_1 = \frac{1}{2}[\mu_A(x_0 * y_0) + \min\{\mu_A(x_0), \mu_A(y_0)\}] = \frac{1}{2}[s + \min\{\theta_1, \theta_2\}]$$

$$\text{Therefore, } \theta_1 > s_1 = \frac{1}{2}[s + \min\{\theta_1, \theta_2\}] > s \text{ and } \theta_2 > s_1 = \frac{1}{2}[s + \min\{\theta_1, \theta_2\}] > s$$

Hence,  $\min\{\theta_1, \theta_2\} > s_1 > s = \mu_A(x_0 * y_0)$ , so that  $x_0 * y_0 \notin U(\mu_A; s_1)$ .

But,  $\mu_A(x_0) = \theta_1 \geq \min\{\theta_1, \theta_2\} > s_1$  and  $\mu_A(y_0) = \theta_2 \geq \min\{\theta_1, \theta_2\} > s_1$  implies

$x_0, y_0 \in U(\mu_A; s_1)$ , which is a contradiction.

Hence  $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ , for all  $x, y \in X$ .

Suppose there exists  $x_0, y_0 \in X$  such that  $v_A(x_0 * y_0) > \max\{v_A(x_0), v_A(y_0)\}$  (2)

Let  $v_A(x_0) = \eta_1$ ,  $v_A(y_0) = \eta_2$  and  $v_A(x_0 * y_0) = t$ . Then,  $t > \max\{\eta_1, \eta_2\}$ .

$$\text{By taking } t_1 = \frac{1}{2}[v_A(x_0 * y_0) + \max\{v_A(x_0), v_A(y_0)\}] = \frac{1}{2}[t + \max\{\eta_1, \eta_2\}]$$

$$\text{Therefore, } \eta_1 < t_1 = \frac{1}{2}[t + \max\{\eta_1, \eta_2\}] < t \text{ and } \eta_2 < t_1 = \frac{1}{2}[t + \max\{\eta_1, \eta_2\}] < t$$

Hence,  $\max\{\eta_1, \eta_2\} < t_1 < t = v_A(x_0 * y_0)$  so that  $x_0 * y_0 \notin L(v_A; t_1)$ .

But  $v_A(x_0) = \eta_1 \leq \max\{\eta_1, \eta_2\} < t_1$  and  $v_A(y_0) = \eta_2 \leq \max\{\eta_1, \eta_2\} < t_1$  implies  $x_0, y_0 \in L(v_A; t_1)$  which is a contradiction.

Thus,  $v_A(x * y) \leq \max\{v_A(x), v_A(y)\}$ , for all  $x, y \in X$ .

Therefore,  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-Subalgebra of  $X$ .

**Theorem 5.1.10:** Any Z-Subalgebra of a Z-algebra  $X$  can be realized as both the upper s-level and lower t-level Z-Subalgebras of some intuitionistic fuzzy Z-Subalgebras of  $X$ .

**Proof :** Let  $Q$  be a Z-Subalgebra of a Z-algebra  $X$  and let  $A$  be an intuitionistic fuzzy set in  $X$  defined by

$$\mu_A(x) = \begin{cases} s, & \text{if } x \in Q \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad v_A(x) = \begin{cases} t, & \text{if } x \in Q \\ 1, & \text{otherwise} \end{cases} \quad \text{for all } x \in X \text{ and } s, t \in (0,1) \text{ and } s+t < 1.$$

**To prove:**  $A$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$ . Let  $x, y \in X$ .

**Case 1:** If  $x, y \in Q$  then  $x * y \in Q$ . Also,  $\mu_A(x) = s, v_A(x) = t, \mu_A(y) = s, v_A(y) = t$ ,  $\mu_A(x * y) = s$  and  $v_A(x * y) = t$ .

Hence,  $\mu_A(x * y) = s = \min\{s, s\} = \min\{\mu_A(x), \mu_A(y)\}$  and  $v_A(x * y) = t = \max\{t, t\} = \max\{v_A(x), v_A(y)\}$

**Case 2:** If  $x \in Q$  and  $y \notin Q$ , then  $\mu_A(x) = s, v_A(x) = t$  and  $\mu_A(y) = 0, v_A(y) = 1$ . Thus,  $\mu_A(x * y) \geq 0 = \min\{s, 0\} = \min\{\mu_A(x), \mu_A(y)\}$  and  $v_A(x * y) \leq 1 = \max\{t, 1\} = \max\{v_A(x), v_A(y)\}$

**Case 3:** If  $x \notin Q$  and  $y \in Q$ , then  $\mu_A(x) = 0, v_A(x) = 1$  and  $\mu_A(y) = s, v_A(y) = t$ . Thus,  $\mu_A(x * y) \geq 0 = \min\{0, s\} = \min\{\mu_A(x), \mu_A(y)\}$  and  $v_A(x * y) \leq 1 = \max\{1, t\} = \max\{v_A(x), v_A(y)\}$

**Case 4:** If  $x \notin Q$  and  $y \notin Q$ , then  $\mu_A(x) = 0, v_A(x) = 1$  and  $\mu_A(y) = 0, v_A(y) = 1$ . Thus,  $\mu_A(x * y) \geq 0 = \min\{0, 0\} = \min\{\mu_A(x), \mu_A(y)\}$  and  $v_A(x * y) \leq 1 = \max\{1, 1\} = \max\{v_A(x), v_A(y)\}$

Therefore,  $A$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$ .

Obviously,  $U(\mu_A; s) = Q = L(v_A; t)$ .

**Theorem 5.1.11:** Let  $Q$  be a subset of a  $Z$ -algebra  $X$  and  $A$  be an intuitionistic fuzzy set on  $X$  which is given in the proof of Theorem 5.1.10. If  $Q$  be realized as upper  $s$ -level  $Z$ -Subalgebra, lower  $t$ -level  $Z$ -Subalgebra of intuitionistic fuzzy  $Z$ -Subalgebra  $A$  of  $X$ , then  $Q$  is a  $Z$ -Subalgebra of  $X$ .

**Proof:** Let  $Q$  be a subset of a  $Z$ -algebra  $X$  and  $A$  be an intuitionistic fuzzy  $Z$ -Subalgebra of  $X$  such that  $U(\mu_A; s) = Q = L(v_A; t)$ .

**To prove:**  $Q$  is an intuitionistic fuzzy  $Z$ -Subalgebra of a  $Z$ -algebra  $X$ .

Let  $x, y \in Q$ . Then  $\mu_A(x) \geq s$ ,  $\mu_A(y) \geq s$ ,  $v_A(x) \leq t$  and  $v_A(y) \leq t$ .

Thus,  $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min\{s, s\} = s$

$v_A(x * y) \leq \max\{v_A(x), v_A(y)\} \leq \max\{t, t\} = t$

Hence,  $x * y \in Q$

Therefore,  $Q$  is a  $Z$ -Subalgebra of a  $Z$ -algebra  $X$ .

As a generalization of Theorem 5.1.10, we prove the following theorem.

**Theorem 5.1.12:** Let  $(X, *, 0)$  be a  $Z$ -algebra. Then any given chain of  $Z$ -Subalgebras  $Q_0 \subset Q_1 \subset \dots \subset Q_r = X$ , there exists an intuitionistic fuzzy  $Z$ -Subalgebra  $A$  of  $X$  whose level (both upper level and lower level)  $Z$ -Subalgebras are exactly the  $Z$ -Subalgebras of this chain.

**Proof:** Consider set of numbers  $s_0 > s_1 > \dots > s_r$  and  $t_0 < t_1 < \dots < t_r$ , where each set  $s_i, t_i \in [0, 1]$  and  $s_i + t_i \leq 1$ .

Let  $A = (\mu_A, v_A)$  be an intuitionistic fuzzy set of a  $Z$ -algebra  $X$  is defined by

$$\mu_A(x) = \begin{cases} s_0, & \text{if } x \in Q_0 \\ s_i, & \text{if } x \in Q_i - Q_{i-1}, i = 1, 2, \dots, r \end{cases} \quad (1)$$

$$\text{and } v_A(x) = \begin{cases} t_0, & \text{if } x \in Q_0 \\ t_i, & \text{if } x \in Q_i - Q_{i-1}, i = 1, 2, \dots, r \end{cases} \quad (2)$$

**Case 1:** Let  $x, y \in Q_i - Q_{i-1}$ . Therefore, by (1) and (2),  $\mu_A(x) = \mu_A(y) = s_i$  and  $v_A(x) = v_A(y) = t_i$ .

Since  $Q_i$  is a Z-Subalgebra,  $x * y \in Q_i \Rightarrow x * y \in Q_i - Q_{i-1}$  or  $x * y \in Q_{i-1}$ . In any case, we conclude that

$$\mu_A(x * y) \geq s_i = \min\{s_i, s_i\} = \min\{\mu_A(x), \mu_A(y)\}$$

$$\text{and } v_A(x * y) \leq t_i = \max\{t_i, t_i\} = \max\{v_A(x), v_A(y)\}$$

**Case 2:** Let  $x \in Q_i - Q_{i-1}$  and  $y \in Q_j - Q_{j-1}$  for  $i > j$ . Therefore, by (1) and (2),  $\mu_A(x) = s_i$ ,  $\mu_A(y) = s_j$ ,  $v_A(x) = t_i$  and  $v_A(y) = t_j$ . Since  $Q_i$  is a Z-Subalgebra of  $X$  and  $Q_j \subset Q_i$ ,  $x * y \in Q_i$ .

$$\text{Hence, } \mu_A(x * y) \geq s_i = \min\{s_i, s_j\} = \min\{\mu_A(x), \mu_A(y)\}$$

$$\text{and } v_A(x * y) \leq t_i = \max\{t_i, t_j\} = \max\{v_A(x), v_A(y)\}$$

Thus  $A$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$ .

From (1) and (2), it follows that  $\text{Im}(\mu_A) = \{s_0, s_1, \dots, s_r\}$  and  $\text{Im}(v_A) = \{t_0, t_1, \dots, t_r\}$ . Hence, the level Z-Subalgebras of  $A$  are given by the chain of Z-Subalgebras

$$U(\mu_A; s_0) \subset U(\mu_A; s_1) \subset \dots \subset U(\mu_A; s_r) = X \tag{3}$$

$$\text{and } L(v_A; t_0) \subset L(v_A; t_1) \subset \dots \subset L(v_A; t_r) = X \tag{4}$$

Now,  $U(\mu_A; s_0) = \{x \in X \mid \mu_A(x) \geq s_0\} = Q_0 = \{x \in X \mid v_A(x) \leq t_0\} = L(v_A; t_0)$ .

Clearly,  $Q_i \subseteq U(\mu_A; s_i)$  and  $Q_i \subseteq L(v_A; t_i)$ . If  $x \in U(\mu_A; s_i)$  and  $L(v_A; t_i)$ , then  $\mu_A(x) \geq s_i$  and  $v_A(x) \leq t_i$  which implies that  $x \notin Q_j$  for  $j > i$ .

Hence,  $\mu_A(x) \in \{s_0, s_1, \dots, s_i\}$  and  $v_A(x) \in \{t_0, t_1, \dots, t_i\}$  and so  $x \in Q_k$  for some  $k \leq i$ . As  $Q_k \subseteq Q_i$ , it follows that  $x \in Q_i$ . Therefore,  $U(\mu_A; s_i) = Q_i = L(v_A; t_i)$  for  $i = 0, 1, \dots, r$ . This completes the proof.

**Note:** If  $X$  is a finite Z-algebra, then the number of Z-Subalgebra of  $X$  is finite, but on the other hand the number of level Z-Subalgebras of an intuitionistic fuzzy Z-Subalgebra  $A$  appears to be infinite. But since every level Z-Subalgebras is indeed a Z-Subalgebra of  $X$ , not all these Z-Subalgebras are distinct. The next theorem characterizes this aspect.

**Theorem 5.1.13:** Let  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

Then,

- (i) two upper s-level Z-Subalgebras  $U(\mu_A; s_1)$  and  $U(\mu_A; s_2)$  (with  $s_1 < s_2$ ) of A are equal if and only if there is no  $x \in X$  such that  $s_1 \leq \mu_A(x) < s_2$ .
- (ii) two lower t-level Z-Subalgebras  $L(\nu_A; t_1)$  and  $L(\nu_A; t_2)$  (with  $t_1 > t_2$ ) of A are equal if and only if there is no  $x \in X$  such that  $t_1 \geq \nu_A(x) > t_2$ .

**Proof:**

(i) Assume that  $U(\mu_A; s_1) = U(\mu_A; s_2)$  for  $s_1 < s_2$  and assume that there exists  $x \in X$  such that  $s_1 \leq \mu_A(x) < s_2$ . Then  $U(\mu_A; s_2)$  is a proper subset of  $U(\mu_A; s_1)$ , which is a contradiction.

Conversely, suppose that there is no  $x \in X$  such that  $s_1 \leq \mu_A(x) < s_2$ .

Since  $s_1 < s_2$ , we have  $U(\mu_A; s_2) \subseteq U(\mu_A; s_1)$  (1)

If  $x \in U(\mu_A; s_1)$ , then  $\mu_A(x) \geq s_1$  and so  $\mu_A(x) \geq s_2$ , because  $\mu_A(x)$  does not lie between  $s_1$  and  $s_2$ . Hence,  $x \in U(\mu_A; s_2)$ , which implies that  $U(\mu_A; s_1) \subseteq U(\mu_A; s_2)$ . (2)

From (1) and (2) we get,  $U(\mu_A; s_1) = U(\mu_A; s_2)$ .

(ii) Assume that  $L(\nu_A; t_1) = L(\nu_A; t_2)$  for  $t_1 > t_2$  and assume that there exists  $x \in X$  such that  $t_1 \geq \nu_A(x) > t_2$ . Then  $L(\nu_A; t_2)$  is a proper subset of  $L(\nu_A; t_1)$ , which is a contradiction.

Conversely, suppose that there is no  $x \in X$  such that  $t_1 \geq \nu_A(x) > t_2$ .

Since  $t_1 > t_2$ , we have  $L(\nu_A; t_2) \subseteq L(\nu_A; t_1)$  (3)

If  $x \in L(\nu_A; t_1)$ , then  $\nu_A(x) \leq t_1$  and so  $\nu_A(x) \leq t_2$ , because  $\nu_A(x)$  does not lie between  $t_1$  and  $t_2$ . Hence,  $x \in L(\nu_A; t_2)$ , which implies that  $L(\nu_A; t_1) \subseteq L(\nu_A; t_2)$ . (4)

From (3) and (4) we get,  $L(\nu_A; t_1) = L(\nu_A; t_2)$ .

**Theorem 5.1.14:** Let X be a finite Z-algebra and  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy Z-Subalgebra of X.

- (i) If  $\text{Im}(\mu_A) = \{s_1, \dots, s_n\}$ , then the family of Z-Subalgebras  $U(\mu_A; s_i), i = 1, 2, \dots, n$  constitutes all the upper s-level Z-Subalgebras of A.

(ii) If  $\text{Im}(v_A) = \{t_1, \dots, t_n\}$ , then the family of Z-Subalgebras  $L(v_A; t_i)$ ,  $i = 1, 2, \dots, n$  constitutes all the lower t-level Z-Subalgebras of A.

**Proof:**

(i) Let  $s \in [0, 1]$  and  $s \notin \text{Im}(\mu_A)$ . Suppose  $s_1 < s_2 < \dots < s_n$  without loss of generality. If  $s \leq s_1$ , then  $U(\mu_A; s_1) = X = U(\mu_A; s)$ . If  $s > s_n$ , then obviously  $U(\mu_A; s) = \phi$ . If  $s_{i-1} < s < s_i$ , then by Theorem 5.1.13, we get  $U(\mu_A; s) = X = U(\mu_A; s_i)$ . Then for any  $s \in [0, 1]$ , the upper s-level Z-Subalgebra is one of  $\{U(\mu_A; s_i) \mid i = 1, 2, \dots, n\}$ .

(ii) Let  $t \in [0, 1]$  and  $t \notin \text{Im}(v_A)$ . Suppose  $t_1 > t_2 > \dots > t_n$  without loss of generality. If  $t \geq t_1$ , then  $L(v_A; t_1) = X = L(v_A; t)$ . If  $t < t_n$ , then obviously  $L(v_A; t) = \phi$ . If  $t_{i-1} > t > t_i$ , then by Theorem 5.1.13, we get  $L(v_A; t) = X = L(v_A; t_i)$ . Then for any  $t \in [0, 1]$ , the lower t-level Z-Subalgebra is one of  $\{L(v_A; t_i) \mid i = 1, 2, \dots, n\}$ .

**Theorem 5.1.15:** Let  $A = (\mu_A, v_A)$  be an intuitionistic fuzzy Z-Subalgebra of a Z-algebra X. Then

(i) If  $\text{Im}(\mu_A)$  is finite, say  $\{s_1, \dots, s_n\}$ , then for any  $s_i, s_j \in \text{Im}(\mu_A)$ ,  $U(\mu_A; s_i) = U(\mu_A; s_j)$  implies  $s_i = s_j$ .

(ii) If  $\text{Im}(v_A)$  is finite, say  $\{t_1, \dots, t_n\}$ , then for any  $t_i, t_j \in \text{Im}(v_A)$ ,  $L(v_A; t_i) = L(v_A; t_j)$  implies  $t_i = t_j$ .

**Proof:** (i) Assume that  $s_i \neq s_j$ , say  $s_i < s_j$ . If  $x \in U(\mu_A; s_j)$ , then  $\mu_A(x) \geq s_j > s_i$ , which implies that  $x \in U(\mu_A; s_i)$ . Let  $x \in X$  be such that  $s_i < \mu_A(x) < s_j$ . Then  $x \in U(\mu_A; s_i)$ , but  $x \notin U(\mu_A; s_j)$ . Hence,  $U(\mu_A; s_j) \subset U(\mu_A; s_i)$  and  $U(\mu_A; s_j) \neq U(\mu_A; s_i)$ , a contradiction. Therefore  $s_i = s_j$ .

(ii) Assume that  $t_i \neq t_j$ , say  $t_i > t_j$ . If  $x \in L(v_A; t_j)$ , then  $v_A(x) \leq t_j < t_i$ , which implies that  $x \in L(v_A; t_i)$ . Let  $x \in X$  be such that  $t_i > v_A(x) > t_j$ . Then  $x \in L(v_A; t_i)$ , but  $x \notin L(v_A; t_j)$ . Hence,  $L(v_A; t_j) \subset L(v_A; t_i)$  and  $L(v_A; t_j) \neq L(v_A; t_i)$ , a contradiction. Therefore  $t_i = t_j$ .

**Theorem 5.1.16:** Let h be a Z-homomorphism from a Z-algebra  $(X, *, 0)$  onto a Z-algebra  $(Y, *, 0')$  and  $A = (\mu_A, v_A)$  be an intuitionistic fuzzy Z-Subalgebra of X with sup-inf property.

Then the image  $h(A) = \left\langle \left\langle y, \mu_{h(A)}(y), \nu_{h(A)}(y) \right\rangle \mid y \in Y \right\rangle$  of  $A$  under  $h$  is an intuitionistic fuzzy Z-Subalgebra of  $Y$ .

**Proof:** Let  $a, b \in Y$  with  $x_0 \in h^{-1}(a)$  and  $y_0 \in h^{-1}(b)$  such that

$$\mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t); \quad \mu_A(y_0) = \sup_{t \in h^{-1}(b)} \mu_A(t) \quad \text{and}$$

$$\nu_A(x_0) = \inf_{t \in h^{-1}(a)} \nu_A(t); \quad \nu_A(y_0) = \inf_{t \in h^{-1}(b)} \nu_A(t)$$

$$\begin{aligned} \mu_{h(A)}(a * b) &= \sup_{t \in h^{-1}(a * b)} \mu_A(t) \geq \mu_A(x_0 * y_0) \geq \min \{ \mu_A(x_0), \mu_A(y_0) \} \\ &= \min \left\{ \sup_{t \in h^{-1}(a)} \mu_A(t), \sup_{t \in h^{-1}(b)} \mu_A(t) \right\} \\ &= \min \{ \mu_{h(A)}(a), \mu_{h(A)}(b) \} \end{aligned}$$

$$\begin{aligned} \text{Also } \nu_{h(A)}(a * b) &= \inf_{t \in h^{-1}(a * b)} \nu_A(t) \leq \nu_A(x_0 * y_0) \leq \max \{ \nu_A(x_0), \nu_A(y_0) \} \\ &= \max \left\{ \inf_{t \in h^{-1}(a)} \nu_A(t), \inf_{t \in h^{-1}(b)} \nu_A(t) \right\} \\ &= \max \{ \nu_{h(A)}(a), \nu_{h(A)}(b) \} \end{aligned}$$

Hence  $h(A)$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $Y$ .

**Theorem 5.1.17** Let  $h : X \rightarrow Y$  be a Z-homomorphism of Z-algebras and  $B$  be an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $Y$ . Then the inverse image of  $B$ ,  $h^{-1}(B) = \left\langle \left\langle x, \mu_{h^{-1}(B)}(x), \nu_{h^{-1}(B)}(x) \right\rangle \mid x \in X \right\rangle$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $X$ .

Converse is true if  $h$  is an Z-epimorphism.

**Proof:** Let  $B$  be an intuitionistic fuzzy Z-Subalgebra of a Z-algebra  $Y$ .

Let  $x, y \in X$ . Then,

$$\begin{aligned} \mu_{h^{-1}(B)}(x * y) &= \mu_B(h(x * y)) = \mu_B(h(x) * h(y)) \geq \min \{ \mu_B(h(x)), \mu_B(h(y)) \} \\ &= \min \{ \mu_{h^{-1}(B)}(x), \mu_{h^{-1}(B)}(y) \} \end{aligned}$$

$$\begin{aligned} \text{Also, } \nu_{h^{-1}(B)}(x * y) &= \nu_B(h(x * y)) = \nu_B(h(x) * h(y)) \leq \max \{ \nu_B(h(x)), \nu_B(h(y)) \} \\ &= \max \{ \nu_{h^{-1}(B)}(x), \nu_{h^{-1}(B)}(y) \} \end{aligned}$$

Thus  $h^{-1}(B)$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

On the other hand, assume that h is an Z-epimorphism and  $h^{-1}(B)$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

Let  $y_1, y_2 \in Y$ . Since h is an Z-epimorphism there exists  $x_1, x_2 \in X$  such that  $h(x_1) = y_1$  and  $h(x_2) = y_2$ .

This implies  $x_1 = h^{-1}(y_1)$  and  $x_2 = h^{-1}(y_2)$ .

$$\begin{aligned} \text{Now, } \mu_B(y_1 *' y_2) &= \mu_B(h(x_1) *' h(x_2)) = \mu_B(h(x_1 * x_2)) = \mu_{h^{-1}(B)}(x_1 * x_2) \\ &\geq \min\{\mu_{h^{-1}(B)}(x_1), \mu_{h^{-1}(B)}(x_2)\} \\ &= \min\{\mu_B(h(x_1)), \mu_B(h(x_2))\} \\ &= \min\{\mu_B(y_1), \mu_B(y_2)\} \end{aligned}$$

$$\begin{aligned} \text{and } \nu_B(y_1 *' y_2) &= \nu_B(h(x_1) *' h(x_2)) = \nu_B(h(x_1 * x_2)) = \nu_{h^{-1}(B)}(x_1 * x_2) \\ &\leq \max\{\nu_{h^{-1}(B)}(x_1), \nu_{h^{-1}(B)}(x_2)\} \\ &= \max\{\nu_B(h(x_1)), \nu_B(h(x_2))\} \\ &= \max\{\nu_B(y_1), \nu_B(y_2)\} \end{aligned}$$

Hence B is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra Y.

**Definition 5.1.18:** Let h be an Z-endomorphism of Z-algebras and A be an intuitionistic fuzzy set in a Z-algebra X. We define a **new intuitionistic fuzzy set**  $A^h = (\mu_{A^h}, \nu_{A^h})$  in X as  $\mu_{A^h}(x) = \mu_A(h(x))$  and  $\nu_{A^h}(x) = \nu_A(h(x))$  for all  $x \in X$ .

**Theorem 5.1.19:** Let h be an Z-endomorphism of a Z-algebra  $(X, *, 0)$ . If A be an intuitionistic fuzzy Z-Subalgebra of X. Then  $A^h$  is also an intuitionistic fuzzy Z-Subalgebra of X.

**Proof:** Let  $x, y \in X$ . Then

$$\begin{aligned} \text{(i) } \mu_{A^h}(x * y) &= \mu_A(h(x * y)) = \mu_A(h(x) * h(y)) \geq \min\{\mu_A(h(x)), \mu_A(h(y))\} \\ &= \min\{\mu_{A^h}(x), \mu_{A^h}(y)\} \\ \text{(ii) } \nu_{A^h}(x * y) &= \nu_A(h(x * y)) = \nu_A(h(x) * h(y)) \leq \max\{\nu_A(h(x)), \nu_A(h(y))\} \\ &= \max\{\nu_{A^h}(x), \nu_{A^h}(y)\} \end{aligned}$$

From (i) and (ii) we get,  $A^h$  is an intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

**Theorems 5.1.20:** Let A and B be any two intuitionistic fuzzy Z-Subalgebras of a Z-algebra X. Then  $A \times B$  is an intuitionistic fuzzy Z-Subalgebra of  $X \times X$ .

**Proof:** Take  $(x_1, y_1)$  and  $(x_2, y_2) \in X \times X$ . Then

$$\begin{aligned} \mu_{A \times B}[(x_1, y_1) * (x_2, y_2)] &= \mu_{A \times B}[(x_1 * x_2), (y_1 * y_2)] \\ &= \min\{\mu_A(x_1 * x_2), \mu_B(y_1 * y_2)\} \\ &\geq \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\} \\ &= \min\{\min\{\mu_A(x_1), \mu_B(y_1)\}, \min\{\mu_A(x_2), \mu_B(y_2)\}\} \\ &= \min\{\mu_{A \times B}(x_1, y_1), \mu_{A \times B}(x_2, y_2)\} \end{aligned}$$

$$\begin{aligned} \nu_{A \times B}[(x_1, y_1) * (x_2, y_2)] &= \nu_{A \times B}[(x_1 * x_2), (y_1 * y_2)] \\ &= \max\{\nu_A(x_1 * x_2), \nu_B(y_1 * y_2)\} \\ &\leq \max\{\max\{\nu_A(x_1), \nu_A(x_2)\}, \max\{\nu_B(y_1), \nu_B(y_2)\}\} \\ &= \max\{\max\{\nu_A(x_1), \nu_B(y_1)\}, \max\{\nu_A(x_2), \nu_B(y_2)\}\} \\ &= \max\{\nu_{A \times B}(x_1, y_1), \nu_{A \times B}(x_2, y_2)\} \end{aligned}$$

This proves that the Cartesian product of any two intuitionistic fuzzy Z-Subalgebras of a Z-algebra X is again an intuitionistic fuzzy Z-Subalgebra of X.

### 5.2 Intuitionistic Fuzzy Z-Ideals in Z-Algebras

In this section we introduce the notion of Intuitionistic Fuzzy Z-ideal of a Z-algebra. Also we have proved some simple but elegant results.

**Definition 5.2.1:** An intuitionistic fuzzy set  $A = (\mu_A, \nu_A)$  in a Z-algebra  $(X, *, 0)$  is called an **intuitionistic fuzzy Z-ideal** of X if it satisfies the following conditions:

- (i)  $\mu_A(0) \geq \mu_A(x)$  and  $\nu_A(0) \leq \nu_A(x)$
- (ii)  $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$
- (iii)  $\nu_A(x) \leq \max\{\nu_A(x * y), \nu_A(y)\}$ , for all  $x, y \in X$ .

**Example 5.2.2:** Let  $X = \{0, 1, 2, 3\}$  be a set with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	0	1	1	3
2	0	1	2	2
3	0	3	2	3

Then  $(X, *, 0)$  is a Z-algebra.

Define an intuitionistic fuzzy set  $A = (\mu_A, \nu_A)$  in X as follows:

$\mu_A(x) = 0.8$  for all  $x = 0, 1, 2, 3$  and  $\nu_A(x) = 0.1$  for all  $x = 0, 1, 2, 3$ . Then,  $A = (\mu_A, \nu_A)$  is an intuitionistic fuzzy Z-ideal of a Z-algebra X.

**Theorem 5.2.3:** Intersection of any two intuitionistic fuzzy Z-ideals of a Z-algebra X is again an intuitionistic fuzzy Z-ideal of X.

**Proof :** For every  $x, y \in X$

$$\begin{aligned} \mu_{A \cap B}(0) &= \min\{\mu_A(0), \mu_B(0)\} \geq \min\{\mu_A(x), \mu_B(x)\} = \mu_{A \cap B}(x) \\ \nu_{A \cup B}(0) &= \max\{\nu_A(0), \nu_B(0)\} \leq \max\{\nu_A(x), \nu_B(x)\} = \nu_{A \cup B}(x) \\ \mu_{A \cap B}(x) &= \min\{\mu_A(x), \mu_B(x)\} \geq \min\{\min\{\mu_A(x * y), \mu_A(y)\}, \min\{\mu_B(x * y), \mu_B(y)\}\} \\ &= \min\{\min\{\mu_A(x * y), \mu_B(x * y)\}, \min\{\mu_A(y), \mu_B(y)\}\} \\ &= \min\{\mu_{A \cap B}(x * y), \mu_{A \cap B}(y)\} \end{aligned}$$

$$\begin{aligned} v_{A \cup B}(x) &= \max\{v_A(x), v_B(x)\} \leq \max\{\max\{v_A(x * y), v_A(y)\}, \max\{v_B(x * y), v_B(y)\}\} \\ &= \max\{\max\{v_A(x * y), v_B(x * y)\}, \max\{v_A(y), v_B(y)\}\} \\ &= \max\{v_{A \cup B}(x * y), v_{A \cup B}(y)\} \end{aligned}$$

Hence  $A \cap B$  is an intuitionistic fuzzy Z-ideal of a Z-algebra X.

We generalize the above theorem as follows.

**Theorem 5.2.4:** Let  $\{A_i \mid i \in \Omega\}$  be a family of intuitionistic fuzzy Z-ideals of a Z-algebra X.

Then  $\bigcap_{i \in \Omega} A_i$  is an intuitionistic fuzzy Z-ideal of X.

**Lemma 5.2.5:** An intuitionistic fuzzy set  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-ideal of a Z-algebra X if and only if the fuzzy sets  $\mu_A$  and  $(v_A)^c$  are fuzzy Z-ideals of X.

**Proof :** Let  $A = (\mu_A, v_A)$  be an intuitionistic fuzzy Z-ideal of a Z-algebra X.

It is clear that  $\mu_A$  is a fuzzy Z-ideal of X.

Now, for any  $x, y \in X$ , we have,

$$(v_A)^c(0) = 1 - v_A(0) \geq 1 - v_A(x) = (v_A)^c(x)$$

$$\begin{aligned} \text{and } (v_A)^c(x) &= 1 - v_A(x) \geq 1 - \max\{v_A(x * y), v_A(y)\} = \min\{1 - v_A(x * y), 1 - v_A(y)\} \\ &= \min\{(v_A)^c(x * y), (v_A)^c(y)\} \end{aligned}$$

Hence  $(v_A)^c$  is a fuzzy Z-ideal of X.

**Conversely,** for every  $x, y \in X$ , we have,

$$(i) \mu_A(0) \geq \mu_A(x) \text{ and } \mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$$

$$(ii) 1 - v_A(0) = (v_A)^c(0) \geq (v_A)^c(x) = 1 - v_A(x) \quad \Rightarrow \quad v_A(0) \leq v_A(x) \text{ and}$$

$$\begin{aligned} 1 - v_A(x) = (v_A)^c(x) &\geq \min\{(v_A)^c(x * y), (v_A)^c(y)\} = \min\{1 - v_A(x * y), 1 - v_A(y)\} \\ &= 1 - \max\{v_A(x * y), v_A(y)\} \end{aligned}$$

$$\text{That is, } v_A(x) \leq \max\{v_A(x * y), v_A(y)\}$$

From (i) and (ii) we get  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-ideal of a Z-algebra X.

**Theorem 5.2.6:** Let  $A = (\mu_A, v_A)$  be an intuitionistic fuzzy set in a Z-algebra X. Then

$A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-ideal of X if and only if  $\oplus A = (\mu_A, (\mu_A)^c)$  and

$\otimes A = ((v_A)^c, v_A)$  are intuitionistic fuzzy Z-ideals of X.

**Proof:** Let  $A = (\mu_A, v_A)$  be an intuitionistic fuzzy Z-ideal of a Z-algebra X.

Let  $x, y \in X$ . Then,

- (i)  $\mu_A(0) \geq \mu_A(x)$  and  $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$
- (ii)  $v_A(0) \leq v_A(x)$  and  $v_A(x) \leq \max\{v_A(x * y), v_A(y)\}$
- (iii)  $(\mu_A)^c(0) = 1 - \mu_A(0) \leq 1 - \mu_A(x) = (\mu_A)^c(x)$
- (iv)  $(\mu_A)^c(x) = 1 - \mu_A(x) \leq 1 - \min\{\mu_A(x * y), \mu_A(y)\} = \max\{1 - \mu_A(x * y), 1 - \mu_A(y)\}$   
 $= \max\{(\mu_A)^c(x * y), (\mu_A)^c(y)\}$
- (v)  $(v_A)^c(0) = 1 - (v_A)^c(0) \geq 1 - v_A(x) = (v_A)^c(x)$
- (vi)  $(v_A)^c(x) = 1 - v_A(x) \geq 1 - \max\{v_A(x * y), v_A(y)\} = \min\{1 - v_A(x * y), 1 - v_A(y)\}$   
 $= \min\{(v_A)^c(x * y), (v_A)^c(y)\}$

From (i), (iii) and (iv) we get  $\oplus A = (\mu_A, (\mu_A)^c)$  is an intuitionistic fuzzy Z-ideal of X.

And, from (ii), (v) and (vi) we get  $\otimes A = ((v_A)^c, v_A)$  is an intuitionistic fuzzy Z-ideal of X.

**Conversely**, assume that  $\oplus A = (\mu_A, (\mu_A)^c)$  and  $\otimes A = ((v_A)^c, v_A)$  are intuitionistic fuzzy Z-ideals of a Z-algebra X. For any  $x, y \in A$ ,

- (i)  $\mu_A(0) \geq \mu_A(x)$  and  $v_A(0) \leq v_A(x)$
- (ii)  $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$  and
- (iii)  $v_A(x) \leq \max\{v_A(x * y), v_A(y)\}$

Hence  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-ideal of a Z-algebra X.

**Theorem 5.2.7:** An intuitionistic fuzzy set  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-ideal of a Z-algebra X if and only if for all  $s, t \in [0,1]$ , the sets  $U(\mu_A; s)$  and  $L(v_A; t)$  are either empty or Z-ideals of X.

**Proof:** Assume that  $U(\mu_A; s) \neq \emptyset \neq L(v_A; t)$  for any  $s, t \in [0,1]$ .

Since  $\mu_A(0) \geq s$  and  $v_A(0) \leq t$ , it is clear that  $0 \in U(\mu_A; s) \cap L(v_A; t)$

Let  $x * y \in U(\mu_A; s)$  and  $y \in U(\mu_A; s)$  for all  $x, y \in X$ . Then,  $\mu_A(x * y) \geq s$  and  $\mu_A(y) \geq s$ .

Now,  $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\} \geq s \Rightarrow x \in U(\mu_A; s)$

Hence  $U(\mu_A; s)$  is an Z-ideal of a Z-algebra X.

Let  $x, y \in X$  such that  $x * y \in L(v_A; t)$  and  $y \in L(v_A; t)$ . Then  $v_A(x * y) \leq t$  and  $v_A(y) \leq t$ .

Now,  $v_A(x) \leq \max\{v_A(x * y), v_A(y)\} \leq t \Rightarrow x \in L(v_A; t)$ .

Hence  $L(v_A; t)$  is an Z-ideal of a Z-algebra X.

**Conversely**, assume that for each  $s, t \in [0, 1]$ , the sets  $U(\mu_A; s)$  and  $L(v_A; t)$  are either empty or Z-ideals of a Z-algebra X.

For any  $x \in X$ , let  $\mu_A(x) = s$  and  $v_A(x) = t$ .

Then  $x \in U(\mu_A; s) \cap L(v_A; t) \Rightarrow U(\mu_A; s) \neq \emptyset \neq L(v_A; t)$

Since  $U(\mu_A; s)$  and  $L(v_A; t)$  are Z-ideals of X,  $0 \in U(\mu_A; s) \cap L(v_A; t)$

Hence,  $\mu_A(0) \geq s = \mu_A(x)$  and  $v_A(0) \leq t = v_A(x) \quad \forall x \in X$ . (1)

Suppose there exist  $x', y' \in X$  such that  $\mu_A(x') < \min\{\mu_A(x' * y'), \mu_A(y')\}$ .

By taking  $s_0 = \frac{1}{2}[\mu_A(x') + \min\{\mu_A(x' * y'), \mu_A(y')\}]$

We have,  $\mu_A(x') < s_0 < \min\{\mu_A(x' * y'), \mu_A(y')\}$

Hence  $x' \notin U(\mu_A; s_0)$  but  $x' * y' \in U(\mu_A; s_0)$  and  $y' \in U(\mu_A; s_0)$ .

That is,  $U(\mu_A; s_0)$  is not an Z-ideal of X, which is a contradiction.

Therefore,  $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$ . (2)

If there exist  $a, b \in X$  such that,  $v_A(a) > \max\{v_A(a * b), v_A(b)\}$ .

Then by taking  $t_0 = \frac{1}{2}[v_A(a) + \max\{v_A(a * b), v_A(b)\}]$

We have,  $\max\{v_A(a * b), v_A(b)\} < t_0 < v_A(a)$

Therefore,  $a * b \in L(v_A; t_0)$  and  $b \in L(v_A; t_0)$

But  $a \notin L(v_A; t_0)$ . That is,  $L(v_A; t_0)$  is not an Z-ideal of X, which is a contradiction.

Hence,  $v_A(x) \leq \max\{v_A(x * y), v_A(y)\}$ . (3)

By (1), (2) and (3), we get  $A = (\mu_A, v_A)$  is an intuitionistic fuzzy Z-ideal of a Z-algebra X.

**Theorem 5.2.8:** Let  $h$  be a homomorphism from a Z-algebra  $(X, *, 0)$  onto a Z-algebra  $(Y, *, 0')$  and  $A$  be an intuitionistic fuzzy Z-ideal of  $X$  with sup-inf property. Then the image of  $A$ ,  $h(A) = \left\langle \left\langle y, \mu_{h(A)}(y), \nu_{h(A)}(y) \right\rangle \middle| y \in Y \right\rangle$  is an intuitionistic fuzzy Z-ideal of  $Y$ .

**Proof:** Let  $a, b \in Y$  with  $x_0 \in h^{-1}(a)$  and  $y_0 \in h^{-1}(b)$  such that

$$\mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t) ; \mu_A(y_0) = \sup_{t \in h^{-1}(b)} \mu_A(t)$$

$$\nu_A(x_0) = \inf_{t \in h^{-1}(a)} \nu_A(t) ; \nu_A(y_0) = \inf_{t \in h^{-1}(b)} \nu_A(t)$$

Now,

$$(i) \quad \mu_{h(A)}(0') = \sup_{t \in h^{-1}(0')} \mu_A(t) \geq \mu_A(0) \geq \mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t) = \mu_{h(A)}(a)$$

$$(ii) \quad \nu_{h(A)}(0') = \inf_{t \in h^{-1}(0')} \nu_A(t) \leq \nu_A(0) \leq \nu_A(x_0) = \inf_{t \in h^{-1}(a)} \nu_A(t) = \nu_{h(A)}(a)$$

$$(iii) \quad \min\{\mu_{h(A)}(a *' b), \mu_{h(A)}(b)\} = \min\left\{ \sup_{t \in h^{-1}(a *' b)} \mu_A(t), \sup_{t \in h^{-1}(b)} \mu_A(t) \right\}$$

$$\leq \min\{\mu_A(x_0 *' y_0), \mu_A(y_0)\}$$

$$\leq \mu_A(x_0)$$

$$= \sup_{t \in h^{-1}(a)} \mu_A(t)$$

$$= \mu_{h(A)}(a)$$

$$(iv) \quad \max\{\nu_{h(A)}(a *' b), \nu_{h(A)}(b)\} = \max\left\{ \inf_{t \in h^{-1}(a *' b)} \nu_A(t), \inf_{t \in h^{-1}(b)} \nu_A(t) \right\}$$

$$\geq \max\{\nu_A(x_0 *' y_0), \nu_A(y_0)\}$$

$$\geq \nu_A(x_0)$$

$$= \inf_{t \in h^{-1}(a)} \nu_A(t)$$

$$= \nu_{h(A)}(a)$$

Hence  $h(A)$  is an intuitionistic fuzzy Z-ideal of a Z-algebra  $Y$ .

**Theorem 5.2.9:** Let  $h : (X, *, 0) \rightarrow (Y, *, 0')$  be a Z-homomorphism of Z-algebras and  $B$  be an intuitionistic fuzzy Z-ideal of a Z-algebra  $Y$ . Then the inverse image of  $B$ ,  $h^{-1}(B) = \left\langle \left\langle x, \mu_{h^{-1}(B)}(x), \nu_{h^{-1}(B)}(x) \right\rangle \middle| x \in X \right\rangle$  is an intuitionistic fuzzy Z-ideal of a Z-algebra  $X$ .

**Proof:** Let  $x, y \in X$ . Now it is clear that

$$\mu_{h^{-1}(B)}(0) = \mu_B(h(0)) = \mu_B(0') \geq \mu_B(h(x)) = \mu_{h^{-1}(B)}(x)$$

$$v_{h^{-1}(B)}(0) = v_B(h(0)) = v_B(0') \leq v_B(h(x)) = v_{h^{-1}(B)}(x)$$

$$\begin{aligned} \text{Then, } \mu_{h^{-1}(B)}(x) &= \mu_B(h(x)) \geq \min\{\mu_B(h(x) *' h(y)), \mu_B(h(y))\} \\ &= \min\{\mu_B(h(x *' y)), \mu_B(h(y))\} \\ &= \min\{\mu_{h^{-1}(B)}(x *' y), \mu_{h^{-1}(B)}(y)\} \end{aligned}$$

$$\begin{aligned} \text{Also, } v_{h^{-1}(B)}(x) &= v_B(h(x)) \leq \max\{v_B(h(x) *' h(y)), v_B(h(y))\} \\ &= \max\{v_B(h(x *' y)), v_B(h(y))\} \\ &= \max\{v_{h^{-1}(B)}(x *' y), v_{h^{-1}(B)}(y)\} \end{aligned}$$

Hence  $h^{-1}(B)$  is an intuitionistic fuzzy Z-ideal of a Z-algebra X.

**Theorem 5.2.10:** Let  $h: (X, *, 0) \rightarrow (Y, *', 0')$  be an Z-epimorphism of Z-algebras. Let B be an intuitionistic fuzzy set of a Z-algebra Y. If  $h^{-1}(B)$  is an intuitionistic fuzzy Z-ideal of a Z-algebra X then B is an intuitionistic fuzzy Z-ideal of a Z-algebra Y.

**Proof:** Let  $y \in Y$ , there exists  $x \in X$  such that  $h(x) = y$ . Then

$$(i) \quad \mu_B(y) = \mu_B(h(x)) = \mu_{h^{-1}(B)}(x) \leq \mu_{h^{-1}(B)}(0) = \mu_B(h(0)) = \mu_B(0')$$

$$(ii) \quad v_B(y) = v_B(h(x)) = v_{h^{-1}(B)}(x) \geq v_{h^{-1}(B)}(0) = v_B(h(0)) = v_B(0')$$

Let  $x, y \in Y$ . Then there exists  $a, b \in X$  such that  $h(a) = x$  and  $h(b) = y$ . It follows that

$$\begin{aligned} (iii) \quad \mu_B(x) &= \mu_B(h(a)) = \mu_{h^{-1}(B)}(a) \geq \min\{\mu_{h^{-1}(B)}(a *' b), \mu_{h^{-1}(B)}(b)\} \\ &= \min\{\mu_B(h(a *' b)), \mu_B(h(b))\} \\ &= \min\{\mu_B(h(a) *' h(b)), \mu_B(h(b))\} \\ &= \min\{\mu_B(x *' y), \mu_B(y)\} \end{aligned}$$

$$\begin{aligned} (iv) \quad v_B(x) &= v_B(h(a)) = v_{h^{-1}(B)}(a) \leq \max\{v_{h^{-1}(B)}(a *' b), v_{h^{-1}(B)}(b)\} \\ &= \max\{v_B(h(a *' b)), v_B(h(b))\} \\ &= \max\{v_B(h(a) *' h(b)), v_B(h(b))\} \\ &= \max\{v_B(x *' y), v_B(y)\} \end{aligned}$$

Hence  $B$  is an intuitionistic fuzzy  $Z$ -ideal of a  $Z$ -algebra  $Y$ .

**Theorem 5.2.11:** Let  $h$  be an  $Z$ -endomorphism of  $Z$ -algebra  $(X, *, 0)$ . If  $A$  be an intuitionistic fuzzy  $Z$ -ideal of  $X$ . Then intuitionistic fuzzy set  $A^h = (\mu_{A^h}, \nu_{A^h})$  is also an intuitionistic fuzzy  $Z$ -ideal of  $X$ .

**Proof:** Let  $x, y \in X$ ,

$$\mu_{A^h}(0) = \mu_A(h(0)) \geq \mu_A(h(x)) = \mu_{A^h}(x)$$

$$\nu_{A^h}(0) = \nu_A(h(0)) \leq \nu_A(h(x)) = \nu_{A^h}(x)$$

$$\begin{aligned} \mu_{A^h}(x) &= \mu_A(h(x)) \geq \min\{\mu_A(h(x) * h(y)), \mu_A(h(y))\} = \min\{\mu_A(h(x * y)), \mu_A(h(y))\} \\ &= \min\{\mu_{A^h}(x * y), \mu_{A^h}(y)\} \end{aligned}$$

$$\begin{aligned} \nu_{A^h}(x) &= \nu_A(h(x)) \leq \max\{\nu_A(h(x) * h(y)), \nu_A(h(y))\} = \max\{\nu_A(h(x * y)), \nu_A(h(y))\} \\ &= \max\{\nu_{A^h}(x * y), \nu_{A^h}(y)\} \end{aligned}$$

Hence  $A^h$  is an intuitionistic fuzzy  $Z$ -ideal of a  $Z$ -algebra  $X$ .

**Theorem 5.2.12:** Let  $A$  and  $B$  be two intuitionistic fuzzy  $Z$ -ideals in a  $Z$ -algebra  $X$ . Then  $A \times B$  is an intuitionistic fuzzy  $Z$ -ideal of  $X \times X$ .

**Proof:** Take  $(x_1, x_2) \in X \times X$ .

$$\text{Then } \mu_{A \times B}(0, 0) = \min\{\mu_A(0), \mu_B(0)\} \geq \min\{\mu_A(x_1), \mu_B(x_2)\} = \mu_{A \times B}(x_1, x_2)$$

$$\text{and } \nu_{A \times B}(0, 0) = \max\{\nu_A(0), \nu_B(0)\} \leq \max\{\nu_A(x_1), \nu_B(x_2)\} = \nu_{A \times B}(x_1, x_2)$$

Now take  $(x_1, x_2), (y_1, y_2) \in X \times X$ . Then

$$\begin{aligned} \mu_{A \times B}(x_1, x_2) &= \min\{\mu_A(x_1), \mu_B(x_2)\} \\ &\geq \min\{\min\{\mu_A(x_1 * y_1), \mu_A(y_1)\}, \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\} \\ &= \min\{\min\{\mu_A(x_1 * y_1), \mu_B(x_2 * y_2)\}, \min\{\mu_A(y_1), \mu_B(y_2)\}\} \\ &= \min\{\mu_{A \times B}((x_1 * y_1), (x_2 * y_2)), \mu_{A \times B}(y_1, y_2)\} \\ &= \min\{\mu_{A \times B}((x_1, x_2) * (y_1, y_2)), \mu_{A \times B}(y_1, y_2)\} \end{aligned}$$

$$\begin{aligned} \nu_{A \times B}(x_1, x_2) &= \max\{\nu_A(x_1), \nu_B(x_2)\} \\ &\leq \max\{\max\{\nu_A(x_1 * y_1), \nu_A(y_1)\}, \max\{\nu_B(x_2 * y_2), \nu_B(y_2)\}\} \\ &= \max\{\max\{\nu_A(x_1 * y_1), \nu_B(x_2 * y_2)\}, \max\{\nu_A(y_1), \nu_B(y_2)\}\} \end{aligned}$$

$$\begin{aligned} &= \max \{v_{A \times B}((x_1 * y_1), (x_2 * y_2)), v_{A \times B}(y_1, y_2)\} \\ &= \max \{v_{A \times B}((x_1, x_2) * (y_1, y_2)), v_{A \times B}(y_1, y_2)\} \end{aligned}$$

Hence  $A \times B$  is an intuitionistic fuzzy Z-ideal of  $X \times X$ .