

CHAPTER 4

CHAPTER 4

In this chapter, the concepts of $\tau^*g^*\alpha$ -closed sets and $\tau^*g^*\alpha$ -continuous maps in topological spaces are studied. Properties, characterizations and applications are analyzed.

SECTION 4.1

ON $\tau^*g^*\alpha$ -CLOSED SETS IN TOPOLOGICAL SPACES

In this section, the concepts of $\tau^*g^*\alpha$ -closed sets and $\tau^*g^*\alpha$ -open sets in topological spaces are studied. Properties, characterizations and applications are analyzed.

4.1.1 Preliminaries

In this section the preliminary definitions are discussed.

Definition 4.1.1.1

A subset A of a topological space (X, τ) is called τ^* -**generalized pre-regular closed** (briefly τ^* gpr-closed) set if $\text{pcl}^*(A) \subseteq U$ whenever $A \subseteq U$ and U is τ^* -regular open in (X, τ) .

Definition 4.1.1.2

A subset A of a topological space (X, τ) is called τ^* -**semi-generalized closed** (briefly τ^* sg-closed) set if $\text{scl}^*(A) \subseteq U$ whenever $A \subseteq U$ and U is τ^* semi-open in (X, τ) .

Definition 4.1.1.3

A subset A of a topological space (X, τ) is called τ^* -**generalized semi-closed** (briefly τ^* gs-closed) set if $\text{scl}^*(A) \subseteq U$ whenever $A \subseteq U$ and U is τ^* -open in (X, τ) .

Definition 4.1.1.4

A subset A of a topological space (X, τ) is called τ^* **g α^* -closed** set if $\text{cl}_\alpha^*(A) \subseteq \text{int}^*(U)$ whenever $A \subseteq U$ and U is τ^* α -open in (X, τ) .

Definition 4.1.1.5

A subset A of a topological space (X, τ) is called τ^* **generalized α -closed** (briefly τ^* g α -closed) set if $\text{cl}_\alpha^*(A) \subseteq U$ whenever $A \subseteq U$ and U is τ^* α -open in (X, τ) .

Definition 4.1.1.6

A subset A of a topological space (X, τ) is called τ^* α -**generalized closed** (briefly τ^* α g-closed) set if $\text{cl}_\alpha^*(A) \subseteq U$ whenever $A \subseteq U$ and U is τ^* -open in (X, τ) .

Definition 4.1.1.7

A subset A of a topological space (X, τ) is called τ^* α -**generalized regular closed** (briefly τ^* α gr-closed) set if $\text{cl}_\alpha^*(A) \subseteq U$ whenever $A \subseteq U$ and U is τ^* α g-open in (X, τ) .

Definition 4.1.1.8

A subset A of a topological space (X, τ) is called τ^* **g $\#$ -closed** set if $\text{cl}^*(A) \subseteq U$ whenever $A \subseteq U$ and U is τ^* α g-open in (X, τ) .

Definition 4.1.1.9

A subset A of a topological space (X, τ) is called $\tau^* \mathbf{wg}\alpha$ -closed set if $\text{cl}_\alpha^*(\text{int}^*(A)) \subseteq U$ whenever $A \subseteq U$ and U is $\tau^* \alpha$ -open in (X, τ) .

Definition 4.1.1.10

A subset A of a topological space (X, τ) is called $\tau^* \mathbf{w}\alpha \mathbf{g}$ -closed set if $\text{cl}_\alpha^*(\text{int}^*(A)) \subseteq U$ whenever $A \subseteq U$ and U is τ^* -open in (X, τ) .

Definition 4.1.1.11

A topological space (X, τ) is said to be $\tau^* \mathbf{wg}\alpha \mathbf{T}_{\tau^* \mathbf{g}^* \alpha}$ -space if every $\tau^* \mathbf{wg}\alpha$ -closed subset of (X, τ) is $\tau^* \mathbf{g}^* \alpha$ -closed in (X, τ) .

Definition 4.1.1.12

A topological space (X, τ) is said to be $\tau^* \mathbf{g}^* \alpha \mathbf{T}_{\tau^* \mathbf{c}}$ -space if every $\tau^* \mathbf{g}^* \alpha$ -closed subset of (X, τ) is τ^* -closed in (X, τ) .

4.1.2 $\tau^* \mathbf{g}^* \alpha$ -closed sets in topological spaces**Definition 4.1.2.1**

A subset A of a topological space (X, τ) is said to be $\tau^* \mathbf{g}^* \alpha$ -closed set in (X, τ) if $\text{cl}_\alpha^*(A) \subseteq G$ whenever $A \subseteq G$ and G is $\tau^* \mathbf{g}\alpha$ -open in (X, τ) .

Notation 4.1.2.2

- (i) The collection of all $\tau^*g^*\alpha$ -closed sets of (X,τ) is denoted by $\tau^*G^*\alpha C(X,\tau)$.
- (ii) The collection of all $\tau^*g^*\alpha$ -open sets of (X,τ) is denoted by $\tau^*G^*\alpha O(X,\tau)$.

Theorem 4.1.2.3

If a subset A of a topological space (X,τ) is τ^* closed, then it is $\tau^*g^*\alpha$ -closed.

Proof

Let G be a $\tau^*g^*\alpha$ -open set containing A . Then $G \supseteq A = \text{cl}(A)$, as A is τ^* closed. Since $\text{cl}(A) \supseteq \text{cl}^*(A) \supseteq \text{cl}_\alpha^*(A)$, $G \supseteq \text{cl}_\alpha^*(A)$. Hence A is a $\tau^*g^*\alpha$ -closed set in (X,τ) .

The converse of Theorem (4.1.2.3) need not be true as seen from the following example

Example 4.1.2.4

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Then the subset $\{c\}$ is $\tau^*g^*\alpha$ -closed but not closed.

Corollary 4.1.2.5

If a subset A of a topological space (X,τ) is τ^* regular closed, then it is $\tau^*g^*\alpha$ -closed but not conversely.

Proof

Since every τ^* regular closed set is τ^* closed and every τ^* closed set is $\tau^*g^*\alpha$ -closed. Hence every τ^* regular closed set is $\tau^*g^*\alpha$ -closed.

Theorem 4.1.2.6

If a subset A of a topological space (X, τ) is $\tau^*g^*\alpha$ -closed, then it is $\tau^*g\alpha$ -closed.

Proof

Let G be a $\tau^*\alpha$ -open set containing A . Then G is a $\tau^*g\alpha$ -open set, as every $\tau^*\alpha$ -open set is $\tau^*g\alpha$ -open. Thus $G \supseteq A$ and $G \supseteq cl_\alpha^*(A)$, as A is $\tau^*g^*\alpha$ -closed. Thus $G \supseteq cl_\alpha^*(A)$. Hence A is $\tau^*g\alpha$ -closed in (X, τ) .

The converse of Theorem (4.1.2.6) need not be true as seen from the following example.

Example 4.1.2.7

Let $X = \{a, b, c, \}$ and $\tau = \{X, \phi, \{a, b\}\}$. Then the subset $\{c\}$ is $\tau^*g\alpha$ -closed but not $\tau^*g^*\alpha$ -closed in (X, τ) .

Theorem 4.1.2.8

If a subset A of a topological space (X, τ) is $\tau^*\alpha$ -closed, then it is $\tau^*g^*\alpha$ -closed

Proof

Let G be an $\tau^*g\alpha$ -open set containing A . Then $G \supseteq A = cl_\alpha^*(A)$, as every $\tau^*\alpha$ -open set is $\tau^*g\alpha$ -open. Thus $G \supseteq cl_\alpha^*(A)$. Hence A is $\tau^*g^*\alpha$ -closed in (X, τ) .

The converse of Theorem (4.1.2.8) need not be true as seen from the following example:

Example 4.1.2.9

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then the subset $\{c\}$ is $g\alpha$ -closed but not $\tau^*g^*\alpha$ -closed in (X, τ) but not $\tau^*\alpha$ -closed

Theorem 4.1.2.10

If a subset A of a topological space (X, τ) is $\tau^*g^*\alpha$ -closed, then it is $\tau^*\alpha g$ -closed

Proof

Let G be an open set containing A . Then G is a $\tau^*g\alpha$ -open set containing A . Thus $G \supseteq cl_{\alpha}^*(A)$, as A is a $\tau^*g^*\alpha$ -closed set. Therefore A is $\tau^*\alpha g$ -closed in (X, τ) .

The converse of Theorem (4.1.2.10) need not be true as seen from the following example:

Example 4.1.2.11

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Then the subset $\{a, c\}$ is $\tau^*\alpha g$ -closed but not $\tau^*g^*\alpha$ -closed in (X, τ) .

Proposition 4.1.2.12

If a subset A of a topological space (X, τ) is $\tau^*g^*\alpha$ -closed, then it is $\tau^*wg\alpha$ -closed.

Proof

Since every $\tau^*g^*\alpha$ -closed set is $\tau^*g\alpha$ -closed and every $\tau^*g\alpha$ -closed set is $\tau^*wg\alpha$ -closed. Hence every $\tau^*g^*\alpha$ -closed set is $\tau^*wg\alpha$ -closed.

Proposition 4.1.2.13

If a subset A of a topological space (X, τ) is $\tau^*g^*\alpha$ -closed, then it is $\tau^*w\alpha g$ -closed.

Proof

Since every $\tau^*g^*\alpha$ -closed set is $\tau^*wg\alpha$ -closed and every $\tau^*wg\alpha$ -closed set is $\tau^*w\alpha g$ -closed. Hence every $\tau^*g^*\alpha$ -closed set is $\tau^*w\alpha g$ -closed.

Theorem 4.1.2.14

If a subset A of a topological space (X, τ) is $\tau^*g^\#$ -closed, then it is $\tau^*g^*\alpha$ -closed.

Proof

Let G be a $g\alpha$ -open set containing A . Then G is an $\tau^*\alpha g$ -open set containing A . Since A is $\tau^*g^\#$ -closed, $G \supseteq cl^*(A) \supseteq cl_\alpha^*(A)$. Hence $G \supseteq cl_\alpha^*(A)$. Therefore A is $\tau^*g^*\alpha$ -closed.

The converse of Theorem (4.1.2.14) need not be true as seen from the following example.

Example 4.1.2.15

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. Then the subset $\{b\}$ is both $\tau^*g^*\alpha$ -closed and $\tau^*wg\alpha$ -closed but not $\tau^*g^\#$ -closed in (X, τ) .

Theorem 4.1.2.16

If a subset A of a topological space (X, τ) is $\tau^*g^*\alpha$ -closed, then it is τ^*gs -closed.

Proof

Let G be an open set containing A . Then G is a $\tau^*g\alpha$ -open set containing A . Since A is $\tau^*g^*\alpha$ -closed, $G \supseteq cl_\alpha^*(A)$. Thus $G \supseteq cl_\alpha^*(A) \supseteq scl^*(A)$. Hence A is τ^*gs -closed in (X, τ) .

The converse of Theorem (4.1.2.16) need not be true as seen from the following example.

Example 4.1.2.17

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Then the subset $\{a, b\}$ is $\tau^*g^*\alpha$ -closed but not $\tau^*g^*\alpha$ -closed in (X, τ) .

Theorem 4.1.2.18

If a subset A of a topological space (X, τ) is $\tau^*g^*\alpha$ -closed, then it is $\tau^*\alpha$ gr-closed.

Proof

Let A be a $\tau^*g^*\alpha$ -closed set in (X, τ) . Let G be a τ^* regular open set of (X, τ) such that $A \subseteq G$. Then G is a $\tau^*g\alpha$ -open set containing A , since every regular open set is $\tau^*g\alpha$ -open. Thus $G \supseteq cl_\alpha^*(A)$, as A is $\tau^*g^*\alpha$ -closed. Therefore A is $\tau^*\alpha$ gr-closed.

The converse of Theorem (4.1.2.18) need not be true as seen from the following example.

Example 4.1.2.19

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then the subset $\{b\}$ is $\tau^*\alpha$ gr-closed but not $\tau^*g^*\alpha$ -closed in (X, τ) .

Theorem 4.1.2.20

If a subset A of a topological space (X, τ) is $\tau^*g^*\alpha$ -closed, then it is τ^*g pr-closed.

Proof

Since every $\tau^*g^*\alpha$ -closed set is $\tau^*\alpha$ gr-closed and every $\tau^*\alpha$ gr-closed set is τ^*g pr-closed. Hence every $\tau^*g^*\alpha$ -closed set is τ^*g pr-closed.

4.1.3 Properties of $\tau^*g^*\alpha$ -closed sets in topological spaces

Theorem 4.1.3.1

A set A is $\tau^*g^*\alpha$ -closed if and only if $\text{cl}_\alpha^*(A) - A$ contains no non-empty $\tau^*g\alpha$ -closed set.

Proof

Assume that A is $\tau^*g^*\alpha$ -closed. Let G be a $\tau^*g\alpha$ -closed set such that $G \subseteq \text{cl}_\alpha^*(A) - A$. Then G^c is a $\tau^*g\alpha$ -open set containing A . From the definition of $\tau^*g^*\alpha$ -closed, $\text{cl}_\alpha^*(A) \subseteq G^c$. That is $G \subseteq (\text{cl}_\alpha^*(A))^c$. This implies that $G \subseteq \text{cl}_\alpha^*(A) \cap (\text{cl}_\alpha^*(A))^c = \phi$. That is $\text{cl}_\alpha^*(A) - A$ contains no non-empty $\tau^*g\alpha$ -closed set.

To prove the sufficiency, let us assume that $\text{cl}_\alpha^*(A) - A$ contains no non-empty $\tau^*g\alpha$ -closed set. Let $A \subseteq G$ where G is a $\tau^*g\alpha$ -open subset of (X, τ) . If $\text{cl}_\alpha^*(A)$ is not contained in G , then $\text{cl}_\alpha^*(A) \cap G^c$ is a non-empty $\tau^*g\alpha$ -closed set of $\text{cl}_\alpha^*(A) - A$. Thus obtain a contradiction. Therefore $\text{cl}_\alpha^*(A) \subseteq G$ and hence A is $\tau^*g^*\alpha$ -closed.

Proposition 4.1.3.2

A $\tau^*g^*\alpha$ -closed set A is $\tau^*\alpha$ -closed if and only if $\text{cl}_\alpha^*(A) - A$ is $\tau^*g\alpha$ -closed.

Proof

Since A is $\tau^*\alpha$ -closed, $\text{cl}_\alpha^*(A) = A$. Then $\text{cl}_\alpha^*(A) - A = \phi$ is $\tau^*\alpha$ -closed and hence $\tau^*g\alpha$ -closed.

To prove the sufficiency, by Theorem (4.1.3.1) $\text{cl}_\alpha^*(A) - A$ contains no non-empty $\tau^*g\alpha$ -closed set. That is $\text{cl}_\alpha^*(A) - A = \phi$. Therefore $\text{cl}_\alpha^*(A) = A$. Hence A is $\tau^*\alpha$ -closed

Theorem 4.1.3.3

Let (X, τ) be a topological space and let $B \subseteq A \subseteq X$. If B is a $\tau^*g^*\alpha$ -closed set relative to A and A is a $\tau^*g^*\alpha$ -closed subset of (X, τ) . Then B is a $\tau^*g^*\alpha$ -closed set relative to (X, τ) .

Proof

Let $B \subseteq G$ and G be a $\tau^*g\alpha$ -open set in (X, τ) . Then $B \subseteq A \cap G$. Since B is $\tau^*g^*\alpha$ -closed relative to A , $(cl_\alpha^*)_A(B) \subseteq A \cap G$. That is $A \cap cl_\alpha^*(B) \subseteq A \cap G$. Thus $A \cap cl_\alpha^*(B) \subseteq G$ and $A \cap cl_\alpha^*(B) \cup (cl_\alpha^*(B))^c \subseteq G \cup (cl_\alpha^*(B))^c$. Since A is $\tau^*g^*\alpha$ -closed in (X, τ) , $cl_\alpha^*(A) \subseteq G \cup (cl_\alpha^*(B))^c$. Also $B \subseteq A$ implies $cl_\alpha^*(B) \subseteq cl_\alpha^*(A)$. Thus $cl_\alpha^*(B) \subseteq cl_\alpha^*(A) \subseteq G \cup (cl_\alpha^*(B))^c$. Therefore $cl_\alpha^*(B) \subseteq G$, since $cl_\alpha^*(B)$ is not contained in $(cl_\alpha^*(B))^c$. Thus B is a $\tau^*g^*\alpha$ -closed set relative to X .

Proposition 4.1.3.4

If A is $\tau^*g^*\alpha$ -closed and F is closed in a topological space (X, τ) , then $A \cap F$ is $\tau^*g^*\alpha$ -closed in (X, τ) .

Proof

Since $A \cap F$ is closed in A , $cl_A^*(A \cap F) = A \cap F$ in A . Let $A \cap F \subseteq G$, where G is $\tau^*g\alpha$ -open in A . Then $(cl_\alpha^*)_A(A \cap F) \subseteq cl_A^*(A \cap F) = A \cap F \subseteq G$. Thus $(cl_\alpha^*)_A(A \cap F)$ and hence $A \cap F$ is $\tau^*g^*\alpha$ -closed in A . Hence $A \cap F$ is $\tau^*g^*\alpha$ -closed in (X, τ) .

Theorem 4.1.3.5

If A is a $\tau^*g^*\alpha$ -closed set in (X,τ) and $A \subseteq B \subseteq cl_\alpha^*(A)$, then B is a $\tau^*g^*\alpha$ -closed set in (X,τ) .

Proof

Let U be a $\tau^*g\alpha$ -open set in (X,τ) such that $B \subseteq U$ and hence $A \subseteq U$. Since A is $\tau^*g^*\alpha$ -closed, $cl_\alpha^*(A) \subseteq U$. Since $B \subseteq cl_\alpha^*(A)$, $cl_\alpha^*(B) \subseteq (cl_\alpha^*(cl_\alpha^*(A))) = cl_\alpha^*(A) \subseteq U$. Hence $cl_\alpha^*(B) \subseteq U$ which implies that B is $\tau^*g^*\alpha$ -closed in (X,τ) .

Theorem 4.1.3.6

Let (X,τ) be a topological space and $A \subseteq Y \subseteq X$. If A is a $\tau^*g^*\alpha$ -closed set in X , then A is $\tau^*g^*\alpha$ -closed relative to Y .

Proof

Let A be a $\tau^*g^*\alpha$ -closed set in X and let $A \subseteq Y \cap G$, where G is $\tau^*g\alpha$ -open in (X,τ) . Since A is $\tau^*g^*\alpha$ -closed in (X,τ) and $A \subseteq G$ implies $cl_\alpha^*(A) \subseteq G$. That is $Y \cap cl_\alpha^*(A) \subseteq Y \cap G$ where $Y \cap cl_\alpha^*(A)$ is $\tau^*\alpha$ -closure of A in (Y,σ) . Thus, A is $\tau^*g^*\alpha$ -closed relative to Y .

Theorem 4.1.3.7

If A is $\tau^*g\alpha$ -open and $\tau^*g^*\alpha$ -closed in (X,τ) , then A is $\tau^*\alpha$ -closed in (X,τ) .

Proof

Let A be $\tau^*g\alpha$ -open and $\tau^*g^*\alpha$ -closed in (X,τ) . Then $cl_\alpha^*(A) \subseteq A$. Since $A \subseteq cl_\alpha^*(A)$ for every subset A of X . Hence $cl_\alpha^*(A) = A$. This implies that A is $\tau^*\alpha$ -closed in (X,τ) .

Theorem 4.1.3.8

For each $x \in X$, either $\{x\} \in \tau^*G\alpha C(X,\tau)$ or its complement $X - \{x\} \in \tau^*G^*\alpha C(X,\tau)$.

Proof

Assume that $\{x\} \notin \tau^*G\alpha C(X,\tau)$, for each $x \in X$. Therefore $X - \{x\} \notin \tau^*G\alpha O(X,\tau)$. Then X is the only $\tau^*g\alpha$ -open set containing $X - \{x\}$. Hence $\text{cl}_\alpha^*(X - \{x\})$ which implies that $X - \{x\}$ is $\tau^*g^*\alpha$ -closed in (X,τ) . Hence $X - \{x\} \in \tau^*G^*\alpha C(X,\tau)$.

Definition 4.1.3.9

The τ^* generalized $^*\alpha$ -closure of a subset A (briefly $\tau^*g^*\text{cl}_\alpha^*(A)$) in (X,τ) is defined as follows:

$$\tau^*g^*\text{cl}_\alpha^*(A) = \bigcap \{F \subseteq X : A \subseteq F \text{ and } F \text{ is } g^*\alpha\text{-closed in } (X,\tau)\}.$$

4.1.4 $\tau^*g^*\alpha$ -open sets in topological spaces**Definition 4.1.4.1**

A subset A of a topological space (X,τ) is called τ^* generalized $^*\alpha$ -open (briefly $\tau^*g^*\alpha$ -open) if its complement A^c is τ^* generalized $^*\alpha$ -closed in (X,τ) .

Notation 4.1.4.2

The collection of all $\tau^*g^*\alpha$ -open sets of (X,τ) is denoted by $\tau^*G^*\alpha O(X,\tau)$.

Proposition 4.1.4.3

- (i) Every τ^* -open set is $\tau^*g^*\alpha$ -open.
- (ii) Every τ^* regular open set is $\tau^*g^*\alpha$ -open.
- (iii) Every $\tau^*g^*\alpha$ -open set is $\tau^*wg\alpha$ -open
- (iv) Every $\tau^*g^*\alpha$ -open set is $\tau^*w\alpha g$ -open.
- (v) Every $\tau^*g^*\alpha$ -open set is $\tau^*g\alpha$ -open.
- (vi) Every $\tau^*g^*\alpha$ -open set is $\tau^*\alpha g$ -open.
- (vii) Every $\tau^*g^*\alpha$ -open set is $\tau^*wg^*\alpha$ -open.
- (viii) Every $\tau^*g^\#$ -open set is $\tau^*g^*\alpha$ -open.
- (ix) Every $\tau^*\alpha$ -open set is $\tau^*g^*\alpha$ -open.

The converses of (i), (ii), (iii), (iv), (v), (vi), (vii), (viii) and (ix) need not be true.

Theorem 4.1.4.4

A subset A is $\tau^*g^*\alpha$ -open in (X, τ) if and only if $G \subseteq \text{int}_\alpha^*(A)$ whenever $A \supseteq G$ and G is $\tau^*g\alpha$ -closed in (X, τ) .

Proof

Let A be a $\tau^*g^*\alpha$ -open set and let G be a $\tau^*g\alpha$ -closed set in (X, τ) contained in A . Then G^c is a $\tau^*g\alpha$ -open set in (X, τ) containing A^c . Since A^c is $\tau^*g^*\alpha$ -closed, $\text{cl}_\alpha^*(A^c) \subseteq G^c$. This implies $G \subseteq \text{int}_\alpha^*(\text{cl}(A))$.

To prove the sufficiency, assume that $G \subseteq \text{int}_\alpha^*(A)$ whenever $G \subseteq A$ and G is $\tau^*g\alpha$ -closed in (X, τ) . Then G^c is a $\tau^*g\alpha$ -open set containing A^c . Now $\text{int}_\alpha^*(A) \subseteq G$ is equivalent to $G^c \subseteq \text{cl}_\alpha^*(A^c)$. Therefore A^c is $\tau^*g^*\alpha$ -closed. Hence A is $\tau^*g^*\alpha$ -open.

Proposition 4.1.4.5

If A and B are $\tau^*g^*\alpha$ -open sets in (X, τ) , then $A \cap B$ is $\tau^*g^*\alpha$ -open in (X, τ) .

Proof

Similar to Proposition 4.1.4.3

Proposition 4.1.4.6

If A is $\tau^*g^*\alpha$ -closed in (X, τ) , then $\text{cl}_\alpha^*(A) - A$ is $\tau^*g^*\alpha$ -open.

Proof

Let $A \subseteq X$ be $\tau^*g^*\alpha$ -closed and F be a $\tau^*g\alpha$ -closed set such that $F \subseteq \text{cl}_\alpha^*(A) - A$. By Theorem (4.1.3.1), $F = \emptyset$, which implies that $F \subseteq \text{int}_\alpha^*(\text{cl}_\alpha^*(A) - A)$. This shows that $\text{cl}_\alpha^*(A) - A$ is $\tau^*g^*\alpha$ -open.

SECTION 4.2

ON $\tau^*g^*\alpha$ -CONTINUOUS MAPS IN TOPOLOGICAL SPACES

In this section, the concept of $\tau^*g^*\alpha$ -continuous maps $\tau^*g^*\alpha$ -open maps $\tau^*g^*\alpha$ -closed maps are studied. The relations between $\tau^*g^*\alpha$ -continuous maps with various existing continuous maps are discussed. Properties, characterizations of such maps are studied.

4.2.1 Preliminaries

Definition 4.2.1.1

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be τ^*g -continuous if $f^{-1}(V)$ is τ^*g -open in (X, τ) for every σ^* -open set V in (Y, σ) .

Definition 4.2.1.2

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\tau^* \alpha$ -**continuous** if $f^{-1}(V)$ is $\tau^* \alpha$ -open in (X, τ) for every σ^* -open set V in (Y, σ) .

Definition 4.2.1.3

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\tau^* g\alpha$ -**continuous** if $f^{-1}(V)$ is $\tau^* g\alpha$ -open in (X, τ) for every σ^* -open set V in (Y, σ) .

Definition 4.2.1.4

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\tau^* \alpha$ **g-continuous** if $f^{-1}(V)$ is $\tau^* \alpha$ g-open in (X, τ) for every σ^* -open set V in (Y, σ) .

Definition 4.2.1.5

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be τ^* **semi-continuous** if $f^{-1}(V)$ is τ^* semi-open in (X, τ) for every σ^* -open set V in (Y, σ) .

Definition 4.2.1.6

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be τ^* **sg-continuous** if $f^{-1}(V)$ is sg-open in (X, τ) for every σ^* -open set V in (Y, σ) .

Definition 4.2.1.7

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be τ^* **gs-continuous** if $f^{-1}(V)$ is τ^* gs-open in (X, τ) for every σ^* -open set V in (Y, σ) .

Definition 4.2.1.8

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be τ^* **completely-continuous** if $f^{-1}(V)$ is τ^* -regular open in (X, τ) for every σ^* -open set V in (Y, σ) .

Definition 4.2.1.9

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\tau^* \alpha$ -**irresolute** if $f^{-1}(V)$ is $\tau^* \alpha$ -open in (X, τ) for every $\sigma^* \alpha$ -open set V in (Y, σ) .

Definition 4.2.1.10

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\tau^* \text{wg} \alpha$ -**continuous** if $f^{-1}(V)$ is $\tau^* \text{wg} \alpha$ -open in (X, τ) for every open set V in (Y, σ) .

Definition 4.2.1.11

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\tau^* \text{w} \alpha$ -**g-continuous** if $f^{-1}(V)$ is $\tau^* \text{w} \alpha$ -g-open in (X, τ) for every open set V in (Y, σ) .

Definition 4.2.1.12

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\tau^* \text{g}^\#$ -**continuous** if $f^{-1}(V)$ is $\tau^* \text{g}^\#$ -open in (X, τ) for every open set V in (Y, σ) .

Definition 4.2.1.13

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\tau^* \text{g}$ -**pre-continuous** if $f^{-1}(V)$ is $\tau^* \text{gpr}$ -open in (X, τ) for every open set V in (Y, σ) .

Definition 4.2.1.14

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\tau^* \alpha$ -**pre-continuous** if $f^{-1}(V)$ is $\tau^* \alpha$ -pr-open in (X, τ) for every open set V in (Y, σ) .

4.2.2 $\tau^*g^*\alpha$ -continuous maps

Definition 4.2.2.1

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be τ^* **generalized** α -**continuous** (briefly $\tau^*g^*\alpha$ -continuous) if the inverse image of every σ^* -closed set in (Y, σ) is $\tau^*g^*\alpha$ -closed in (X, τ) .

Proposition 4.2.2.2

Every τ^* -continuous map is $\tau^*g^*\alpha$ -continuous.

Proof

Assume that $f : (X, \tau) \rightarrow (Y, \sigma)$ is a τ^* -continuous map. Let V be any σ^* -closed set in (Y, σ) . Then $f^{-1}(V)$ is τ^* -closed in (X, τ) . By Proposition (4.1.2.3), every τ^* -closed set is $\tau^*g^*\alpha$ -closed and hence $f^{-1}(V)$ is $\tau^*g^*\alpha$ -closed in (X, τ) . Therefore f is $\tau^*g^*\alpha$ -continuous.

The converse of the Proposition (4.2.2.2) need not be true as seen from the following example:

Example 4.2.2.3

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is $\tau^*g^*\alpha$ -continuous but not continuous, since for the σ^* -closed set $\{c\}$ in (Y, σ) , $f^{-1}(\{c\}) = \{c\}$ is not σ^* -closed in (X, τ) .

Proposition 4.2.2.4

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\tau^*g^*\alpha$ -continuous map where (X, τ) is a $\tau^*g^*\alpha T \tau^*$ -c-space. Then f is τ^* -continuous.

Proof

Let V be any σ^* -closed set in (Y, σ) . Since $f: (X, \tau) \rightarrow (Y, \sigma)$ is $\tau^*g^*\alpha$ -continuous, $f^{-1}(V)$ is $\tau^*g^*\alpha$ -closed in (X, τ) . Since (X, τ) is a $\tau^*g^*\alpha T_{\tau^*C}$ space, $f^{-1}(V)$ is τ^* -closed in (X, τ) . Hence f is τ^* -continuous.

Proposition 4.2.2.5

Every completely τ^* -continuous map is $\tau^*g^*\alpha$ -continuous.

Proof

Assume that $f: (X, \tau) \rightarrow (Y, \sigma)$ is τ^* -completely continuous. Let V be any σ^* -closed set in (Y, σ) . Then $f^{-1}(V)$ is τ^* -regular-closed in (X, τ) . By Corollary (4.1.2.3), every τ^* -regular-closed set is $\tau^*g^*\alpha$ -closed and hence $f^{-1}(V)$ is $\tau^*g^*\alpha$ -closed in (X, τ) . Therefore f is $\tau^*g^*\alpha$ -continuous.

The converse of the Proposition (4.2.2.5) need not be true as seen from the following example:

Example 4.2.2.6

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b, c\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = c, f(c) = a$. Then f is $\tau^*g^*\alpha$ -continuous but not τ^* -completely continuous, since $\{a\}$ is σ^* -closed in (Y, σ) but $f^{-1}(\{a\}) = \{c\}$ is not τ^* -regular closed in (X, τ) .

Proposition 4.2.2.7

Every $\tau^*g^*\alpha$ -continuous map is $\tau^*g\alpha$ -continuous.

Proof

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be $\tau^*g^*\alpha$ -continuous and let V be any σ^* -closed set in (Y, σ) . Then $f^{-1}(V)$ is $\tau^*g^*\alpha$ -closed in (X, τ) . Since every $\tau^*g^*\alpha$ -closed set is $\tau^*g\alpha$ -closed and hence $f^{-1}(V)$ is $\tau^*g\alpha$ -closed in (X, τ) . Hence f is $\tau^*g\alpha$ -continuous.

The converse of the Proposition (4.2.2.7) need not be true as seen from the following example:

Example 4.2.2.8

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = a, f(c) = c$. Then f is $\tau^*g\alpha$ -continuous but not $\tau^*g^*\alpha$ -continuous, since for the α^* -closed set $\{b, c\}$ in (Y, σ) , $f^{-1}(\{b, c\}) = \{a, c\}$ is not $\tau^*g^*\alpha$ -closed in (X, τ) .

Proposition 4.2.2.9

Every $\tau^*g^*\alpha$ -continuous map is $\tau^*\alpha g$ -continuous.

Proof

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be $\tau^*g^*\alpha$ -continuous and let V be any σ^* -closed set in (Y, σ) . Then $f^{-1}(V)$ is $\tau^*g^*\alpha$ -closed in (X, τ) . By Theorem (4.1.2.10), every $\tau^*g^*\alpha$ -closed set is $\tau^*\alpha g$ -closed and hence $f^{-1}(V)$ is $\tau^*\alpha g$ -closed in (X, τ) . Therefore f is $\tau^*\alpha g$ -continuous.

The converse of the Proposition (4.2.2.9) need not be true as seen from the following example:

Example 4.2.2.10

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = a, f(c) = c$. Then f is $\tau^* \alpha g$ -continuous but not $\tau^* g^* \alpha$ -continuous, since $\{b, c\}$ is α^* -closed in (Y, σ) but $f^{-1}(\{b, c\}) = \{a, c\}$ is not $\tau^* g^* \alpha$ -closed in (X, τ) .

Proposition 4.2.2.11

Every $\tau^* g^* \alpha$ -continuous map is $\tau^* wg \alpha$ -continuous.

Proof

Assume that f is $\tau^* g^* \alpha$ -continuous and let V be any σ^* -closed set in (Y, σ) . Then $f^{-1}(V)$ is $\tau^* g^* \alpha$ -closed in (X, τ) . By Theorem (4.1.2.12), every $\tau^* g^* \alpha$ -closed set is $\tau^* wg \alpha$ -closed, hence $f^{-1}(V)$ is $\tau^* wg \alpha$ -closed in (X, τ) . Therefore f is $\tau^* wg \alpha$ -continuous.

The converse of the Proposition (4.2.2.11) need not be true as seen from the following example:

Example 4.2.2.12

Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, d\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = a, f(c) = d, f(d) = c$. Then f is $\tau^* wg \alpha$ -continuous but not $\tau^* g^* \alpha$ -continuous, since for the closed set $\{b, c\}$ in (Y, σ) , $f^{-1}(\{b, c\}) = \{a, d\}$ is not $\tau^* g^* \alpha$ -closed in (X, τ) .

Proposition 4.2.2.13

Let (X, τ) be a $\tau^*wg\alpha$ $T_{\tau^*g^*\alpha}$ -space and let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\tau^*wg\alpha$ -continuous map. Then f is $\tau^*g^*\alpha$ -continuous.

Proof

Let V be any σ^* -closed set in (Y, σ) . Since f is $\tau^*wg\alpha$ -continuous, $f^{-1}(V)$ is $\tau^*wg\alpha$ -closed in (X, τ) . Since (X, τ) is a $\tau^*wg\alpha$ $T_{\tau^*g^*\alpha}$ -space, $f^{-1}(V)$ is $\tau^*g^*\alpha$ -closed in (X, τ) . Hence f is $\tau^*g^*\alpha$ -continuous.

Proposition 4.2.2.14

Every $\tau^*g^*\alpha$ -continuous map is $\tau^*w\alpha g$ -continuous.

Proof

Assume that $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tau^*g^*\alpha$ -continuous. Since f is $\tau^*wg\alpha$ -continuous and every $\tau^*wg\alpha$ -continuous map is $\tau^*w\alpha g$ -continuous. Hence f is $\tau^*w\alpha g$ -continuous.

The converse of the above Corollary need not be true as seen from the following example:

Example 4.2.2.15

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = c$, $f(c) = b$, then f is $\tau^*w\alpha g$ -continuous but not $\tau^*g^*\alpha$ -continuous, since for the α^* -closed set $\{a, c\}$ in (Y, σ) , $f^{-1}(\{a, c\}) = \{a, b\}$ is not $\tau^*g^*\alpha$ -closed in (X, τ) .

Theorem 4.2.2.16

If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a map, then every $\tau^* \alpha$ -continuous map is $\tau^* g^* \alpha$ -continuous,

Proof

Let V be any σ^* -closed set in (Y, σ) . Since f is $\tau^* \alpha$ -continuous, $f^{-1}(V)$ is $\tau^* \alpha$ -closed in (X, τ) . Since every $\tau^* \alpha$ -closed set is $\tau^* g^* \alpha$ -closed, $f^{-1}(V)$ is $\tau^* g^* \alpha$ -closed in (X, τ) . Therefore f is $\tau^* g^* \alpha$ -continuous.

The converse of the Theorem (4.2.2.16) need not be true as seen from the following examples

Example 4.2.2.17

Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a, b, c\}\}$ and $\sigma = \{Y, \phi, \{d\}, \{a, d\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = d, f(c) = a, f(d) = c$. Then f is $\tau^* g^* \alpha$ -continuous but not $\tau^* \alpha$ -continuous, since $\{b, c\}$ is σ^* -closed in (Y, σ) but $f^{-1}(\{b, c\}) = \{a, d\}$ is not $\tau^* \alpha$ -closed in (X, τ) .

Theorem 4.2.2.18

If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a map, then every $\tau^* g^\#$ -continuous map is $\tau^* g^* \alpha$ continuous,

Proof

Let f be a $\tau^* g^\#$ -continuous map and let V be any σ^* -closed set in (Y, σ) . Then $f^{-1}(V)$ is $\tau^* g^\#$ -closed in (X, τ) . By Proposition (4.1.2.14),

every $\tau * g^\#$ -closed set is $\tau * g^* \alpha$ -closed, hence $f^{-1}(V)$ is $\tau * g^* \alpha$ -closed in (X, τ) . Therefore f is $\tau * g^* \alpha$ -continuous.

The converse of the above Theorem need not be true as seen from the following examples

Example 4.2.2.19

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = a, f(c) = c$. Then f is $\tau * g^* \alpha$ -continuous but not $\tau * g^\#$ -continuous, since for the σ^* -closed set $\{a\}$ in (Y, σ) , $f^{-1}(\{a\}) = \{b\}$ is not $\tau * g^\#$ -closed in (X, τ) .

Theorem 4.2.2.20

If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a map, then every $\tau * g^* \alpha$ -continuous map is $\tau * g_s$ -continuous

Proof

Let V be any σ^* -closed set in (Y, σ) . Then $f^{-1}(V)$ is $\tau * g^* \alpha$ -closed in (X, τ) , as f is $\tau * g^* \alpha$ -continuous. By Proposition (4.1.2.16), every $\tau * g^* \alpha$ -closed set is $\tau * g_s$ -closed, hence $f^{-1}(V)$ is $\tau * g_s$ -closed in (X, τ) . Therefore f is $\tau * g_s$ -continuous.

The converse of the Theorem (4.2.2.20) need not be true as seen from the following examples.

Example 4.2.2.21

Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{b, c\}, \{a, b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = a, f(b) = d,$

$f(c) = b, f(d) = c$. Then f is $\tau^*g^*\alpha$ -continuous but not $\tau^*g^*\alpha$ -continuous, since for the σ^* -closed set $\{a, c, d\}$ in (Y, σ) , $f^{-1}(\{a, c, d\}) = \{a, b, d\}$ is not $\tau^*g^*\alpha$ -closed in (X, τ) .

Theorem 4.2.2.22

If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a map, then every $\tau^*g^*\alpha$ -continuous map is $\tau^*\alpha_{gr}$ continuous.

Proof

Let V be any σ^* -closed set in (Y, σ) . Then $f^{-1}(V)$ is $\tau^*g^*\alpha$ -closed in (X, τ) , as f is $\tau^*g^*\alpha$ -continuous. Since every $\tau^*g^*\alpha$ -closed set is $\tau^*\alpha_{gr}$ -closed, $f^{-1}(V)$ is $\tau^*\alpha_{gr}$ -closed in (X, τ) . Therefore f is $\tau^*\alpha_{gr}$ -continuous.

Example 4.2.2.23

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{c\}, \{b, c\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = c, f(c) = a$. Then f is $\tau^*\alpha_{gr}$ -continuous but not $\tau^*g^*\alpha$ -continuous, since $\{a, b\}$ is σ^* -closed in (Y, σ) but $f^{-1}(\{a, b\}) = \{a, c\}$ is not $\tau^*g^*\alpha$ -open in (X, τ) .

Theorem 4.2.2.24

Every $\tau^*g^*\alpha$ -continuous map is τ^*g_{pr} -continuous.

Proof

Assume that $f: (X, \tau) \rightarrow (Y, \sigma)$ is $\tau^*g^*\alpha$ -continuous. Since every $\tau^*g^*\alpha$ -continuous map is $\tau^*\alpha_{gr}$ -continuous and every $\tau^*\alpha_{gr}$ -

continuous map is $\tau^*g^*\alpha$ -continuous. Hence every $\tau^*g^*\alpha$ -continuous map is τ^* gpr-continuous.

Remark 4.2.2.25

The following examples show that τ^* semi-continuous and $\tau^*g^*\alpha$ -continuous maps are independent:

Example 4.2.2.26

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = c$, $f(c) = b$. Then f is semi-continuous but not $\tau^*g^*\alpha$ -continuous, since $\{a\}$ is open as well as σ^* -closed in (Y, σ) , $f^{-1}(\{a\}) = \{a\}$ is not $\tau^*g^*\alpha$ -closed in (X, τ) .

Example 4.2.2.27

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b$, $f(b) = c$, $f(c) = a$. Then f is $\tau^*g^*\alpha$ -continuous but not τ^* semi-continuous, since $\{a, c\}$ is σ^* -closed in (Y, σ) but $f^{-1}(\{a, c\}) = \{b, c\}$ is not τ^* semi-closed in (X, τ) .

Theorem 4.2.2.28

If a map $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tau^*\alpha$ -irresolute, then it is $\tau^*g^*\alpha$ -continuous.

Proof

Let V be any σ^* -closed set in (Y, σ) . Then V is $\tau^*\alpha$ -closed in (Y, σ) . Since f is $\tau^*\alpha$ -irresolute, $f^{-1}(V)$ is $\tau^*\alpha$ -closed in (X, τ) . By

Proposition (4.1.2.8) every $\tau^* \alpha$ -closed set is $\tau^* g^* \alpha$ -closed and hence $f^{-1}(V)$ is $\tau^* g^* \alpha$ -closed in (X, τ) . Therefore f is $\tau^* g^* \alpha$ -continuous.

The converse of the Proposition (4.2.2.28) need not be true as seen from the following example:

Example 4.2.2.29

Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{d\}, \{b, d\}, \{b, c, d\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = d, f(b) = b, f(c) = a, f(d) = c$. Then f is $\tau^* g^* \alpha$ -continuous but not $\tau^* \alpha$ -irresolute, since for the $\tau^* \alpha$ -closed set $\{a, b, c\}$ in (Y, σ) , $f^{-1}(\{a, b, c\}) = \{b, c, d\}$ is not $\tau^* \alpha$ -closed in (X, τ) .

Remark 4.2.2.30

The following examples show that $\tau^* g$ -continuous and $\tau^* g^* \alpha$ -continuous maps are independent:

Example 4.2.2.31

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = c, f(b) = a, f(c) = b$. Then f is $\tau^* g$ -continuous but not $\tau^* g^* \alpha$ -continuous, since $\{b, c\}$ is σ^* -closed in (Y, σ) but $f^{-1}(\{b, c\}) = \{a, c\}$ is not $\tau^* g^* \alpha$ -closed in (X, τ) .

Example 4.2.2.32

Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b, f(b) = c, f(c) = a$. Then f is $\tau^* g^* \alpha$ -continuous but not $\tau^* g$ -continuous, since for the σ^* -closed set $\{c\}$ in (Y, σ) , $f^{-1}(\{c\}) = \{b\}$ is not $\tau^* g$ -closed in (X, τ) .

Theorem 4.2.2.33

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tau * g^* \alpha$ -continuous if and only if the inverse image of every σ^* -open set in (Y, σ) is $\tau * g^* \alpha$ -open in (X, τ) .

Proof

Let V be an σ^* -open set in (Y, σ) . Then V^c is σ^* -closed in (Y, σ) . Since f is $\tau * g^* \alpha$ -continuous, $f^{-1}(V^c)$ is $\tau * g^* \alpha$ -closed in (X, τ) . But $f^{-1}(V^c) = X - f^{-1}(V)$. Hence $f^{-1}(V)$ is $\tau * g^* \alpha$ -open in (X, τ) .

To prove the sufficiency, assume that the inverse image of every τ^* -open set in (Y, σ) is $\tau * g^* \alpha$ -open in (X, τ) . Let V be a σ^* -closed set in (Y, σ) . Then V^c is σ^* -open in (Y, σ) . By assumption, $f^{-1}(V^c) = X - f^{-1}(V)$ is $\tau * g^* \alpha$ -open in (X, τ) which implies that $f^{-1}(V)$ is $\tau * g^* \alpha$ -closed in (X, τ) . Hence f is $\tau * g^* \alpha$ -continuous.

Theorem 4.2.2.34

Every $\tau * g^* \alpha$ -continuous map is $\tau * wg \alpha$ -continuous.

Proof

Assume that f is a $\tau * g^* \alpha$ -continuous map. Let V be any σ^* -closed set in (Y, σ) . Then $f^{-1}(V)$ is $\tau * g^* \alpha$ -closed in (X, τ) . By Theorem (4.1.2.12), every $\tau * g^* \alpha$ -closed set is $\tau * wg \alpha$ -closed and hence $f^{-1}(V)$ is $\tau * wg \alpha$ -closed in (X, τ) . Thus f is $\tau * wg \alpha$ -continuous.

The converse of the Proposition (4.2.2.24) need not be true as seen from the following example:

Example 4.2.2.35

Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{Y, \phi, \{c\}, \{b, c\}, \{b, c, d\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = d, f(b) = c, f(c) = a, f(d) = b$. Then f is $\tau^*wg\alpha$ -continuous but not $\tau^*g^*\alpha$ -continuous, since for the σ^* -closed set $\{a, d\}$ in (Y, σ) , $f^{-1}(\{a, d\}) = \{a, c\}$ is not $\tau^*g\alpha$ -closed in (X, τ) .

Theorem 4.2.2.36

Let (X, τ) and (Z, η) be any two topological spaces and let (Y, σ) be a $\tau^*g^*\alpha T_{\tau^*c}$ -space. If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are $\tau^*g^*\alpha$ -continuous maps, then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is also a $\tau^*g^*\alpha$ -continuous map.

Proof

Let F be a closed set in (Z, η) . Since g is $\tau^*g^*\alpha$ -continuous, $g^{-1}(F)$ is $\tau^*g^*\alpha$ -closed in (Y, σ) . Since (Y, σ) is a $\tau^*g^*\alpha T_{\tau^*c}$ -space, $g^{-1}(F)$ is σ^* -closed in (Y, σ) . Since $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tau^*wg^*\alpha$ -continuous, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\tau^*g^*\alpha$ -closed in (X, τ) . Hence $g \circ f$ is a $\tau^*g^*\alpha$ -continuous map.

Theorem 4.2.2.37

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an $\tau^*\alpha$ -irresolute map and let $g : (Y, \sigma) \rightarrow (Z, \eta)$ be a $\tau^*g^*\alpha$ -continuous map. If (Y, σ) is a $\tau^*g^*\alpha T_{\alpha}$ -space, then the composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is $\tau^*\alpha$ -continuous.

Proof

Let F be a σ^* -closed set in (Z, η) . Since g is $\tau^*g^*\alpha$ -continuous, $g^{-1}(F)$ is $\tau^*g^*\alpha$ -closed in (Y, σ) . Since (Y, σ) is a $\tau^*g^*\alpha T_\alpha$ -space, $g^{-1}(F)$ is $\sigma^*\alpha$ -closed in (Y, σ) . Since f is $\tau^*\alpha$ -irresolute, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\tau^*\alpha$ -closed in (X, τ) . Hence $g \circ f$ is $\tau^*\alpha$ -continuous.

Theorem 4.2.2.38

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\tau^*g\alpha$ -continuous, τ^* pre- α -closed map. Then $f(A)$ is $\sigma^*\alpha g$ -closed in (Y, σ) for every $\tau^*g^*\alpha$ -closed set A in (X, τ) .

Proof

Let A be a $\tau^*g^*\alpha$ -closed set in (X, τ) and let V be an σ^* -open set in (Y, σ) such that $f(A) \subseteq V$ which implies that $A \subseteq f^{-1}(V)$. Since V is σ^* -open in (Y, σ) and f is $\tau^*g\alpha$ -continuous, $f^{-1}(V)$ is $\tau^*g\alpha$ -open in (X, τ) . Since A is $\tau^*g^*\alpha$ -closed and $A \subseteq f^{-1}(V)$, $cl_\alpha^*(A) \subseteq f^{-1}(V)$ which implies that $f(cl_\alpha^*(A)) \subseteq f(f^{-1}(V)) \subseteq V$. Hence $f(cl_\alpha^*(A)) \subseteq V$. Since f is τ^* pre- α -closed, $f(cl_\alpha^*(A))$ is $\sigma^*\alpha$ -closed in (Y, σ) . Therefore $cl_\alpha^*(f(cl_\alpha^*(A))) \subseteq f(cl_\alpha^*(A))$. Since $A \subseteq cl_\alpha^*(A)$, $f(A) \subseteq f(cl_\alpha^*(A))$ which implies that $cl_\alpha^*(f(A)) \subseteq cl_\alpha^*(f(cl_\alpha^*(A))) = f(cl_\alpha^*(A)) \subseteq V$. Hence $cl_\alpha^*(f(A)) \subseteq V$. Therefore $f(A)$ is $\sigma^*\alpha g$ -closed in (Y, σ) .

Theorem 4.2.2.39

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a $\tau^*g\alpha$ -continuous, τ^* pre- α -closed map where (Y, σ) is a ${}_{\alpha}T_b$ -space, then $f(A)$ is $\sigma^*g^*\alpha$ -closed in (Y, σ) for every $\tau^*g^*\alpha$ -closed set A in (X, τ) .

Proof

Let A be a $\tau^*g^*\alpha$ -closed set in (X, τ) and V be a $\sigma^*g\alpha$ -open set in (Y, σ) such that $f(A) \subseteq V$. Since every $\sigma^*g\alpha$ -open set is $\sigma^*\alpha g$ -open, V is $\sigma^*\alpha g$ -open in (Y, σ) . Since (Y, σ) is a ${}_{\alpha}T_b$ -space, V is σ^* -open in (Y, σ) . Since f is $\tau^*g\alpha$ -continuous, $f^{-1}(V)$ is $\tau^*g\alpha$ -open in (X, τ) . Since A is $\tau^*g^*\alpha$ -closed in (X, τ) and $A \subseteq f^{-1}(V)$, $\text{cl}_{\alpha}^*(A) \subseteq f^{-1}(V)$ which implies that $f(\text{cl}_{\alpha}^*(A)) \subseteq f(f^{-1}(V))$. Hence $\text{cl}_{\alpha}^*(A) \subseteq V$. Since f is σ^* -pre- α -closed, $f(\text{cl}_{\alpha}^*(A))$ is $\sigma^*\alpha$ -closed in (Y, σ) . Therefore $\text{cl}_{\alpha}^*(A) \subseteq (f(\text{cl}_{\alpha}^*(A))) \subseteq (f(\text{cl}_{\alpha}^*(A))) \subseteq V$. Since $A \subseteq \text{cl}_{\alpha}^*(A)$, $f(A) \subseteq f(\text{cl}_{\alpha}^*(A))$ which implies that $\text{cl}_{\alpha}^*(f(A)) \subseteq f^{-1}(V)$. $\text{cl}_{\alpha}^*(A) \subseteq (f(\text{cl}_{\alpha}^*(A))) \subseteq (f(\text{cl}_{\alpha}^*(A))) \subseteq V$. Hence $f(A)$ is $\sigma^*g^*\alpha$ -closed in (Y, σ) .