Chapter IV

CHAPTER IV

PROXIMITY ORDERED SPACES

Definition: 4.1

A proximity of Efremovič (called simply "proximity") on X is a binary relation δ defined on P(X), the power set of X, satisfying the axioms P1 to P5 given below:

P1: $(A, B)\in\delta$ implies $(B, A)\in\delta$.

P2: $(A, B\cup C)\in\delta$ if and only if $(A, B)\in\delta$ or $(A, C)\in\delta$.

P3: $(A, B)\in\delta$ implies $A \neq \phi \neq B$.

P4: $A\cap B \neq \phi$ implies $(A, B)\in\delta$.

P5: $(A, B)\notin\delta$ implies the existence of a subset $E\in P(X)$ such that $(A, E)\notin\delta$ and $(X-E, B)\notin\delta$.

The relation δ is said to be a **proximity of Pervin or a quasi-proximity** if it satisfies all the axioms of a proximity with the possible exception of P1 which is replaced by

P1': $(A\cup B, C)\in\delta$ if and only if $(A, C)\in\delta$ or $(B, C)\in\delta$.

A (quasi-) proximity is said to be **separated** if it satisfies

P6: $(\{x\}, \{y\})\in\delta$ implies $x = y$.

$(\{x\}, \{y\})\in\delta$ will be written as $(x, y)\in\delta$)

Definition: 4.2

If δ is a quasi-proximity on X, then there is associated another quasi-proximity δ^{-1}, called the **conjugate** of δ, which is defined by setting $(B, A) \in \delta^{-1}$ if and only if $(A, B) \in \delta$.

Definition: 4.3

If δ is a (quasi-) proximity on X, then there is defined a **topology $\tau(\delta)$** on X, via the closure operator $A \to A^{\delta} = \{x : (x, A) \in \delta\}$.

Note: 4.4

$\tau(\delta)$ is completely regular if δ is a proximity, but need not satisfy any separation axiom if δ is a quasi-proximity.

Definition: 4.5

Given two (quasi-) proximities δ_1 and δ_2 on X, δ_1 is said to be **finer than** δ_2 if $(A, B) \in \delta_1$ implies $(A, B) \in \delta_2$. It is known that finer (quasi-) proximities induce finer topologies.

Definition: 4.6

Let (X_1, δ_1) and (X_2, δ_2) be two (quasi-) proximity spaces. A function $f : X_1 \to X_2$ is said to be a **proximity mapping** if $(A, B) \in \delta_1$ implies $(f(A), f(B)) \in \delta_2$. If f is a one-to-one function such that both f and f^{-1} are proximity mappings, then f is called a **proximity isomorphism** and X_1 and X_2 are said to be **proximally isomorphic**.

Note: 4.7

It is known that proximity mappings are continuous with respect to the induced topologies.

Definition: 4.8

Let X be a non-empty set and let δ be a quasi-proximity on X. If we set $x \leq y$ if and only if $(x, y) \in \delta$, then $\leq$ is a pre-order on X. We call $\leq$ the **pre-order generated by δ**. Further, if we define δ^* by setting $(A, B) \in \delta^*$ if and only if whenever $\{A_i : i \in J_m\}$ is a finite cover of A and $\{B_j : j \in J_n\}$ is a finite cover of B, we have $(A_i, B_j) \in \delta \cap \delta^{-1}$ for some $(i, j) \in J_m \times J_n$, then δ^* is a proximity on X. We call δ^* the **proximity generated by δ**.

Definition: 4.9

A triple $(X, \delta^*, \leq)$ consisting of a non-empty set X, a pre-order $\leq$ and a proximity δ^*, both defined on X, is called a **proximity pre-ordered space** if there exists on X at least one quasi-proximity δ which generates both $\leq$ and δ^*. When this happens, δ is called a **generating quasi-proximity**.

It is easily seen that, on a proximity pre-ordered space $(X, \delta^*, \leq)$, $\leq$ is a partial order if and only if δ^* is separated. When such is the case, we call $(X, \delta^*, \leq)$ a **proximity ordered space**.

Lemma: 4.10

Let $(X, \delta^*, \leq)$ be a proximity pre-ordered space and let IO(X) and DO(X) be the upper and lower topologies of $(X, \tau(\delta^*), \leq)$. Then $\tau(\delta) \subseteq IO(X)$, $\tau(\delta^{-1}) \subseteq DO(X)$ and $\tau(\delta) + \tau(\delta^{-1}) = IO(X) + DO(x)$ for every generating quasi-proximity δ.

Proof

The set A is closed with respect to $\tau(\delta)$ if and only if it is of the form

$A = A^{\delta} = \{x : \{x, A) \in \delta\}$. Also, A is then decreasing with respect to $\leq$. Therefore open sets with respect to $\tau(\delta)$ are increasing. Similarly, sets open with respect to $\tau(\delta^{-1})$ are decreasing. Further, both $\tau(\delta)$ and $\tau(\delta^{-1})$ are coarser than $\tau(\delta^{*})$. Therefore $\tau(\delta) \subseteq IO(X)$, $\tau(\delta^{-1}) \subseteq DO(X)$. Next, if δ^{**} denotes the finest quasi-proximity compatible with $\tau(\delta) + \tau(\delta^{-1})$, then δ^{**} is finer than both δ and δ^{-1}, and hence finer than δ^{*}. This gives $\tau(\delta^{**}) = \tau(\delta) + \tau(\delta^{-1}) \supseteq \tau(\delta^{*})$. Therefore $\tau(\delta^{*}) = \tau(\delta) + \tau(\delta^{-1}) \subseteq IO(X) + DO(X) \subseteq \tau(\delta^{*})$.

Lemma: 4.11

If $(X, \delta^{*}, \leq)$ is a proximity pre-ordered space and if δ is a generating quasi-proximity, then $(A, B) \in \delta$ if and only if $(icl(A), dcl(B)) \in \delta$ where icl and dcl are defined in $(X, \tau(\delta^{*}), \leq)$.

Proof

This follows from the facts that $(A, B) \in \delta$ if and only if $(icl^{*}(\tau), dcl^{*}(B)) \in \delta$ and that $A \subseteq icl(A) \subseteq icl^{*}(A)$, $B \subseteq dcl(B) \subseteq dcl^{*}(B)$, where $icl^{*}(A)$ denotes the $\tau(\delta^{-1})$ closure of A and $dcl^{*}(B)$ denotes the $\tau(\delta)$-closure of B.

Definition: 4.12

A topological pre-ordered space $(X, \tau, \leq)$ is said to be **completely regular pre-ordered** if it satisfies the conditions

(1) $p \in P \in \tau$ implies the existence of two continuous functions $f, g: X \to [0, 1]$ such that f is increasing, g is decreasing, $f(p) = g(p) = 1$ and $\min\{f(x), g(x)\} = 0$ for $x \in X - P$ and

(2) $x \not\leq y$ implies the existence of a continuous increasing real-valued function f such that $f(x) > f(y)$.

Lemma: 4.13

Let X be a non-empty set and let δ be a quasi-proximity on X. If $(A, B) \notin \delta$, then there exists a function $f : X \rightarrow [0, 1]$ with the following properties:

1) f is continuous with respect to $\tau = \tau(\delta) + \tau(\delta^{-1})$;

2) f is increasing with respect to $\leq$, the pre-order generated by δ; and

3) $f(A) = \{1\}$, $f(B) = \{0\}$.

Proof

If $(A, B) \notin \delta$, then there exist sets P and Q such that $(A, X-P) \notin \delta$, $(P, X-Q) \notin \delta$ and $(Q, B) \notin \delta$. Now $(A, X-P) \notin \delta$ implies $(A, dcl(X-P)) \notin \delta$, that is, $A \subseteq X-dcl(X-P) = i(P)$, Similarly $(P, X-Q) \notin \delta$ implies $P^{i0} \subseteq i(Q)$ and $(Q, B) \notin \delta$ implies $icl(Q) \subseteq X-B$. Thus we have

$$A \subseteq i(P) \subseteq icl(P) \subseteq i(Q) \subseteq icl(Q) \subseteq X-B.$$

where i and icl are defined in $(X, \tau, \leq)$. This is similar to the main step in Nachbin's proof of Urysohn's lemma for ordered spaces .Proceeding as therein we obtain a function $f : X \rightarrow [0, 1]$, which is

(1) continuous with respect to τ,

(2) increasing with respect to $\leq$ and

(3) $f(A) = \{1\}$, $f(B) = \{0\}$.

Theorem: 4.14

Let $(X, \delta^*, \leq)$ be a proximity pre-ordered space. Then $(X, \tau(\delta^*), \leq)$ is a completely regular pre-ordered space.Where $\tau(\delta^*)$ is the topology associated with the δ^*

Proof

Let δ be a generating quasi-proximity on $(X, \delta^*, \leq)$. Let $p \in X$ and let P be a $\tau(\delta^*)$-neighbourhood of p. Since $\tau(\delta^*) = \tau(\delta) + \tau(\delta^{-1})$, there exist sets $U \in \tau(\delta)$ and $V \in \tau(\delta^{-1})$ such that $p \in U \cap V \subseteq P$. Clearly $(p, X-U) \notin \delta$ and therefore there exists $\tau(\delta^*)$-continuous increasing function $f: X \rightarrow [0, 1]$ such that $f(p) = 1$ and $f(X - U) = \{0\}$. Similarly, $(X-V, p) \notin \delta$ and therefore there a $\tau(\delta^*)$-continuous decreasing function $g : X \rightarrow [0, 1]$ such that $g(X-V) = \{0\}$ and $g(p) = 1$. Clearly, if $x \in X-P$ then either $f(x) = 0$ or $g(x) = 0$. Thus the first condition for $(X, \tau(\delta^*), \leq)$ to be completely regular pre-ordered is satisfied.

Next, let $x \not\leq y$. Since δ is a generating quasi-proximity, we have $(x, y) \notin \delta$ and therefore, as above, there exists a $\tau(\delta^*)$-continuous increasing function such that $f(x) > f(y)$.

Theorem: 4.15

Let $(X, \tau, \leq)$ be a completely regular pre-ordered space. Then there exists a proximity δ^* on X such that $(X, \delta^*, \leq)$ becomes a proximity pre-ordered space and that $\tau(\delta^*) = \tau$.

Proof

We define a relation δ as follows:

$(A, B) \notin \delta$ if and only if there exists a continuous increasing (decreasing) function $f: X \rightarrow [0, 1]$ such that $f(A) = \{1\}$, $f(B) = \{0\}$ ($f(A) = \{0\}$, $f(B) = \{1\}$, respectively).

We assert that δ is a quasi-proximity on X such that $x \leq y$ if and only if $(x, y) \in \delta$ and that $\tau(\delta) + \tau(\delta^{-1}) = \tau$. To prove that δ is a quasi-proximity it is sufficient to establish (P5) only. Let $(A, B) \notin \delta$ and let f be a continuous decreasing function $f: X \to [0, 1]$ such that $f(A) = \{0\}$, $f(B) = \{1\}$. We set

$$E = \{x : \frac{1}{2} \leq f(x) \leq 1\}$$

and define

$$g : [0, 1] \to [0, 1]$$

by setting

$$g(y) = 2y, \text{ for } y \in [0, \frac{1}{2}],$$

and $g(y) = 1$, for $y \in [\frac{1}{2}, 1]$.

Clearly, g is continuous increasing function. Therefore $gf : X \to [0, 1]$ is a continuous decreasing function such that $gf(A) = \{0\}$, $gf(E) = \{1\}$, Similarly, $g(1-f) : X \to [0,1]$ is a continuous increasing function such that

$$g(1-f)(X-E) = \{1\}, \qquad g(1-f)(B) = \{0\}.$$

Next, if $(x, y) \notin \delta$, then clearly $x \not\leq y$. On the other hand if $x \not\leq y$, then by the definition of a completely regular pre-ordered space there exists a continuous increasing real-valued function f such that $f(x) > f(y)$. If we set

$$k(z) = \frac{f(z) - f(y)}{f(x) - f(y)},$$

and $h(z) = \max\{\min(1, k(z)), \max(0, k(z))\}$, then $h : X \to [0, 1]$ is a continuous increasing function such that $h(x) = 1$, $h(y) = 0$, and therefore $(x, y) \notin \delta$.

Lastly we show that $\tau = \tau(\delta) + \tau(\delta^{-1})$. Let $V \in \tau(\delta)$ and let $x \in V$. Then $(x, X-V) \notin \delta$. Let $f : X \to [0, 1]$ be a continuous increasing function such that $f(x) = 1$ and $f(X-V) = \{0\}$. Clearly V is a τ-neighbourhood of x and because this holds for each $x \in V$, V is τ-open. Therefore $\tau(\delta) \subseteq \tau$. Similarly $\tau(\delta^{-1}) \subseteq \tau$.

Conversely, let $p \in X$ and P be a τ -neighbourhood of p. Since X is completely regular pre-ordered, there exists two continuous functions $f, g : x \rightarrow [0, 1]$ such that f is increasing, g is decreasing, $f(p) = g(p) = 1$ and $\min \{f(x), g(x)\} = 0$ for all $x \in X - P$. It is clear from here that $f^{-1}\{[0, 1]\}$ is a $\tau(\delta)$-neighbourhood of p, that $g^{-1}\{[0, 1]\}$ is a $\tau(\delta^{-1})$ neighbourhood of p, and that

$f^{-1}\{[0, 1]\} \cap g^{-1}\{[0, 1]\} \subseteq P$.

Therefore P is a $\tau(\delta) + \tau(\delta^{-1})$ neighbourhood of p. Since p is arbitary, it follows that every set open with respect to τ is also open with respect to $\tau(\delta) + \tau(\delta^{-1})$. Thus we have $\tau(\delta) + \tau(\delta^{-1}) = \tau$.

Clearly δ^* , the proximity generated by δ, meets all the requirements.