CHAPTER - V

CHAPTER V

ON πgb*- CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

Introduction

Continuity plays an important in the study of topological spaces. In this chapter the idea of πgb*-continuous function using the concept of πgb*-closed set was discussed. The relationships between πgb*-continuous functions and other forms of continuous functions were discussed. The characterizations and properties of πgb*-continuous functions, πgb*-space and almost πgb*-continuous function were studied.

SECTION 5.1

Preliminaries

Definition 5.1.1

A function $f : (X, \tau) \to (Y, \sigma)$ is called **continuous** if $f^{-1}(V)$ is closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.1.2

A function $f : (X, \tau) \to (Y, \sigma)$ is called **α-continuous** if $f^{-1}(V)$ is α-closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.1.3

A function $f : (X, \tau) \to (Y, \sigma)$ is called **pre-continuous** if $f^{-1}(V)$ is pre-closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.1.4

A function $f : (X, \tau) \to (Y, \sigma)$ is called **semi-continuous** if $f^{-1}(V)$ is semi-closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.1.5

A function $f : (X, \tau) \to (Y, \sigma)$ is called **gα-continuous** if $f^{-1}(V)$ is gα-closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.1.6

A function $f : (X, \tau) \to (Y, \sigma)$ is called **gp-continuous** if $f^{-1}(V)$ is gp-closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.1.7

A function $f : (X, \tau) \to (Y, \sigma)$ is called **gs-continuous** if $f^{-1}(V)$ is gs-closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.1.8

A function $f : (X, \tau) \to (Y, \sigma)$ is called **πgα-continuous** if $f^{-1}(V)$ is πgα-closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.1.9

A function $f : (X, \tau) \to (Y, \sigma)$ is called **b*-continuous** if $f^{-1}(V)$ is b*-closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.1.10

A function $f : (X, \tau) \to (Y, \sigma)$ is called **b*-irresolute** if $f^{-1}(V)$ is b*-closed in (X, τ) for every b*-closed set V in (Y, σ).

Definition 5.1.11

A function $f : X \to Y$ is said to be **almost b*-continuous** if $f^{-1}(V)$ is b*-closed in X for every regular closed set V of Y.

Section 5.2

πgb*-continuous functions

In this section πgb*-continuous functions were studied and examples were provided whereever the implications donot hold.

Definition 5.2.1

A function $f : (X, \tau) \to (Y, \sigma)$ is called **πgb*-continuous** if $f^{-1}(V)$ is πgb*-closed in (X, τ) for every closed set V of (Y, σ).

Definition 5.2.2

A function $f : (X, \tau) \to (Y, \sigma)$ is called **πgb*-irresolute** if $f^{-1}(V)$ is πgb*-closed in (X, τ) for every πgb*-closed set V in (Y, σ).

Definition 5.2.3

A function $f : (X, \tau) \to (Y, \sigma)$ is called **πgb*-closed** if f(V) is πgb*-closed in (Y, σ) for every πgb*-closed set V in (X, τ).

Theorem 5.2.4

Every continuous function is πgb*-continuous.

Proof

Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous function.
Let V be a closed set in Y.
Since f is continuous $f^{-1}(V)$ is closed in X.
As every closed set is πgb*-closed, $f^{-1}(V)$ is πgb*-closed.
Hence f is πgb*- continuous.

Remark 5.2.5

The converse of the above theorem need not be true as seen from the following example.

Example 5.2.6

Consider $X = \{a, b, c\}$, $\tau = \{ \varphi, \{a\}, \{a, b\}, \{a, c\}, X \}$ and $Y = \{ a, b, c \}$ with the topology $\sigma = \{ \varphi, Y, \{b, c\} \}$.
Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = c$, $f(c) = b$, then f is πgb*-continuous but it is not continuous.

Theorem 5.2.7

Every pre-continuous function is πgb*-continuous.

Proof

Let $f : (X, \tau) \to (Y, \sigma)$ be a pre-continuous function.
Let V be a closed subset of Y.
Since f is pre-continuous $f^{-1}(V)$ is pre-closed in X.
As every pre-closed set is πgb*-closed, $f^{-1}(V)$ is πgb*-closed.
Hence f is πgb*-continuous.

Remark 5.2.8

The converse of above theorem need not be true which can be shown by the following example.

Example 5.2.9

Let $X = \{ a, b, c, d \}$ with topology $\tau = \{ \varphi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{ a, b, d \}, X \}$ and $Y = \{ a, b, c, d \}$ with the topology $\sigma = \{ \varphi, Y, \{a\}, \{c\}, \{a, c\} \}$.
Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function.
Then f is πgb*-continuous but it is not pre-continuous.

Theorem 5.2.10

Every semi-continuous function is πgb*-continuous.

Proof

Let $f : (X, \tau) \to (Y, \sigma)$ be a semi-continuous function.
Let V be a closed subset of Y,
since f is semi-continuous $f^{-1}(V)$ is semi-closed in X.
As every semi-closed set is πgb*-closed, $f^{-1}(V)$ is πgb*-closed.
Hence f is πgb*-continuous.

Remark 5.2.11

The converse of the above theorem need not be true as seen from the following example.

Example 5.2.12

Let $X = \{ a, b, c \}$ with topology $\tau = \{ \varphi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X \}$ and $Y = \{ a, b, c \}$ with topology $\sigma = \{ \varphi, Y, \{c\} \}$.

Let $f : (X, \tau) \to (Y, \sigma)$ be defined as f(a) = b, f(b) = a and f(c) = c.

Then f is πgb*-continuous but it is not semi-continuous.

Theorem 5.2.13

Every b-continuous function is πgb*-continuous.

Proof

Let $f : (X, \tau) \to (Y, \sigma)$ be a b-continuous function.

Let V be a closed subset of Y.

Since f is b-continuous $f^{-1}(V)$ is b-closed in X.

As every b-closed set is πgb*-closed, $f^{-1}(V)$ is πgb*-closed.

Hence f is πgb*-continuous.

Remark 5.2.14

The converse of the above theorem need not be true which can be seen from the following example.

Example 5.2.15

Let $X = \{ a, b, c, d \}$ with topology $\tau = \{ \varphi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{ a, b, d \}, X \}$ and $Y = \{ a, b, c, d \}$ with topology $\sigma = \{ \varphi, Y, \{d\}\}$.

Let $f : (X, \tau) \to (Y, \sigma)$ be defined as f(a) = b, f(b) = a, f(c) = b, f(d) = d.

Then f is πgb*-continuous but it is not b-continuous.

Theorem 5.2.16

Every *g*-continuous function is πgb*-continuous.

Proof

Let $f : (X, \tau) \to (Y, \sigma)$ be a g-continuous function.

Let V be a closed subset of Y.

Since f is g-continuous $f^{-1}(V)$ is g-closed in X.

As every g-closed set is πgb*-closed, $f^{-1}(V)$ is πgb*-closed.

Hence f is πgb*-continuous.

Remark 5.2.17

The converse of the above theorem need not be true which can be seen from the following example.

Example 5.2.18

Let $X = \{ a, b, c, d \}$ with topology $\tau = \{ \varphi, \{a\}, \{d\}, \{a, d\}, \{ c, d\}, \{ a, c, d \}, X \}$ and $Y = \{ a, b, c, d \}$ with topology $\sigma = \{ \varphi, Y, \{a\}, \{a, b, d\}\}$.

Let $f : (X, \tau) \to (Y, \sigma)$ be defined as f(a) = a, f(b) = d, f(c) = c, f(d) = b.

Then f is πgb*-continuous but not g-continuous.

Theorem 5.2.19

Every gp-continuous function is πgb*-continuous.

Proof

Let $f : (X, \tau) \to (Y, \sigma)$ be a gp-continuous function.

Let V be a closed subset of Y.

Since f is gp-continuous $f^{-1}(V)$ is gp-closed in X.

As every gp-closed set is πgb*-closed, $f^{-1}(V)$ is πgb*-closed.

Hence f is πgb*-continuous.

Remark 5.2.20

The converse of the above theorem need not be true as seen from the following example.

Example 5.2.21

Let $X = \{a, b, c\}$ with topology $\tau = \{\varphi, \{a, b\}, X\}$ and let $Y = \{a, b, c$ with topology $\sigma = \{\varphi, Y, \{b\}, \{c\}, \{b, c\}\}$.

Let $f : (X, \tau) \to (Y, \sigma)$ be defined as $f(a) = a$, $f(b) = c$, $f(c) = b$.

Then f is πgb*-continuous but it is not gp-continuous.

Theorem 5.2.22

Every gs-continuous function is πgb*-continuous.

Proof

Let $f : (X, \tau) \to (Y, \sigma)$ be a gs-continuous function.

Let V be a closed subset of Y.

Since f is gs-continuous $f^{-1}(V)$ is gs-closed in X.

As every gs-closed set is πgb*-closed, $f^{-1}(V)$ is πgb*-closed.

Hence f is πgb*-continuous.

Remark 5.2.23

The converse of the above theorem need not be true can be seen from the following example.

Example 5.2.24

Let $X = \{a, b, c\}$ with topology $\tau = \{\varphi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $Y = \{a, b, c\}$ with topology$\sigma = \{\varphi, Y, \{b\}\}$.

Let $f : (X, \tau) \to (Y, \sigma)$ be defined as $f(a) = c$, $f(b) = b$, $f(c) = a$.

Then f is πgb*-continuous but is not gs-continuous

Theorem 5.2.25

Every gb-continuous function is πgb*-continuous.

Proof

Let $f : (X, \tau) \to (Y, \sigma)$ be a gb-continuous function.

Let V be a closed subset of Y.

Since f is gb-continuous $f^{-1}(V)$ is gb-closed in X.

As every gb-closed set is πgb*-closed, $f^{-1}(V)$ is πgb*-closed.

Hence f is πgb*-continuous.

Remark 5.2.26

The converse of above theorem need not be true as seen from the following example.

Example 5.2.27

Let $X = \{a, b, c, d\}$ with topology $\tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\ X\}$ and $Y = \{a, b, c, d\}$ with topology $\sigma = \{\varphi, Y, \{d\}\}$.

Let $f : (X, \tau) \to (Y, \sigma)$ be defined as $f(a) = b$, $f(b) = c$, $f(c) = a$, $f(d) = d$.

Then f is πgb*-continuous but it is not gb-continuous.

Theorem 5.2.28

Every πg-continuous function is πgb*-continuous.

Proof

Let $f : (X, \tau) \to (Y, \sigma)$ be a πg-continuous function.

Let V be a closed subset of Y.

Since f is πg-continuous $f^{-1}(V)$ is πg-closed in X.

As every πg-closed set is πgb*-closed, $f^{-1}(V)$ is πgb*-closed.

Hence f is πgb*-continuous.

Remark 5.2.29

The converse of the above theorem need not be true it can be seen from the following example.

Example 5.2.30

Let X = { a, b, c, d } with topology τ = { φ, {a}, {b}, {a, b}, { b, c}, {a, b, c}, X } and Y = { x, y, z } with topology σ = { φ, Y, { x, y}, {x, z}, {x}}.

Let f : (X, τ) → (Y, σ) be defined as follows f(a) = y, f(b) = f(d) = x, f(c) = z then f is πgb*-continuous but it is not πg-continuous.

Theorem 5.2.31

Every πgp-continuous function is πgb*-continuous.

Proof

Let f : (X, τ) → (Y, σ) be a πgp-continuous function.

Let V be a closed subset of Y, since f is πgp-continuous f^{-1} (V) is πgp-closed in X.

As every πgp-closed set is πgb*-closed, f^{-1}(V) is πgb*-closed.

Hence f is πgb*-continuous.

Remark 5.2.32

The converse of the above theorem need not be true can be seen from the following example.

Example 5.2.33

Let X = { a, b, c, d } with topology τ = { φ, {a}, {b}, {a, b}, { b, c}, {a, b, c}, X } and Y = { x, y, z } with topology σ = { φ, Y, { x, y}, {x, z}, {x}}.

Let f : (X, τ) → (Y, σ) be defined as f(a) = y, f(b) = f(d) = x, f(c) = z.

Then, f is πgb*-continuous but it is not πgp-continuous.

Theorem 5.2.34

Every πgs-continuous function is πgb*-continuous.

Proof

Let f : (X, τ) → (Y, σ) be a πgs-continuous function.

Let V be a closed subset of Y,

since f is πgs-continuous f^{-1} (V) is πgs-closed in X.

As every πgs-closed set is πgb*-closed, f^{-1}(V) is πgb*-closed.

Hence f is πgb*-continuous.

Remark 5.2.35

The converse of the above theorem need not be true as seen from the following example.

Example 5.2.36

Consider X = {a, b, c}, τ = { φ , {a} , {a, b} , {a, c} , X } and Y = { a, b, c } with the topology σ = { φ, Y, {b, c} }.

Let f : (X, τ) → (Y, σ) be defined by f(a) = a, f(b) = c, f(c) = b.

Then f is πgb*-continuous but it is not πgs-continuous.

Section 3
Characterizations of πgb*-continutinuous functions

In this section the characterizations of πgb*-continuous functions and almost πgb*-continuous functions were studied. Further the concept of πgb*-irresolute functions and πgb*-spaces were added.

Theorem 5.3.1

Let $f : X \to Y$ be a function. Then the following statements are equivalent:

1. f is πgb*-continuous;
2. The inverse image of every open set in Y is πgb*-open in X.

Proof

(1) $\Rightarrow$ (2)

Let U be open subset of X.

Then (Y – U) is closed in Y.

Since f is πgb*-continuous, $f^{-1}(Y-U) = X - f^{-1}(U)$ is πgb*-closed in X.

Hence $f^{-1}(U)$ is πgb*-open in X.

(2) $\Rightarrow$ (1)

Let V be a closed subset of Y.

Then (Y – V) is open in Y hence by hypothesis (2) $f^{-1}(Y-V) = X - f^{-1}(V)$ is πgb*-open in X.

Hence $f^{-1}(V)$ is πgb*-closed in X.

Therefore, f is πgb*-continuous.

Theorem 5.3.2

Every πgb*- irresolute function is πgb*-continuous.

Proof

Let $f : X \to Y$ be πgb*-irresolute function.

Let V be closed set in Y, then V is πgb*-closed in Y.

Since f is πgb*-irresolute $f^{-1}(V)$ is πgb*-closed in X.

Hence f is πgb*-continuous.

Remark 5.3.3

The converse of the above theorem need not be true it can be seen from the following example.

Example 5.3.4

Consider $X = Y = \{a, b, c\}$, $\tau = \{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{\varphi, X, \{a\}\}$.

Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map.

Then f is πgb*-continuous but it is not πgb*-irresolute.

Theorem 5.3.5

Let $f : X \to Y$ be a function. Then the following statements are equivalent:

(1) For each $x \in X$ and each open set V containing f(x) there exists a πgb*-open set U containing x such that $f(U) \subset V$.

(2) $f(\pi gb^*\text{-}cl(A)) \subset cl(f(A))$ for every subset A of X.

Proof

(1) $\Rightarrow$ (2)

Let $y \in f(\pi gb^*\text{-}cl(A))$ then, there exists an $x \in \pi gb^*\text{-}cl(A)$ such that $y = f(x)$.

We claim that $y \in cl(f(A))$ and let V be any open neighborhood of y.

Since $x \in \pi gb^*\text{-}cl(A)$ there exists an πgb*-open set U such that $x \in U$ and $U \cap A \neq \varphi$, $f(U) \subset V$.

Since $U \cap A \neq \varphi$, $f(A) \cap V \neq \varphi$.

Therefore, $y = f(x) \in cl\, f(A)$.

Hence $f(\pi gb^*cl\, A) \subset cl\, f(A)$.

(2) ⇒ (1)

Let $x \in X$ and V be any open set containing f(x).

Let $A = f^{-1}(Y\text{-}V)$,

since $f(\pi gb^*\text{-}cl(A)) \subset cl(f(A)) \subset (Y - V) \Rightarrow \pi gb^*cl\, A \subset f^{-1}(Y\text{-}V) = A$.

Hence $\pi gb^*\text{-}cl(A) = A$.

Since $f(x) \in V \Rightarrow x \in f^{-1}(V) \Rightarrow x \notin A \Rightarrow x \notin \pi gb^*\text{-}cl(A)$.

Thus there exists an open set U containing x such that $U \cap A = \varphi \Rightarrow f(U) \cap f(A) = \varphi$.

Therefore $f(U) \subset V$.

Definition 5.3.6

A topological space (X, τ) is a **πgb*-space** if every πgb*-closed set is closed.

Theorem 5.3.7

Every πgb*-space is πgb*-$T_{1/2}$ space.

Proof

Let (X ,τ) be a πgb*-space and let $A \subset X$ be πgb*-closed set in X.

Then A is closed

⇒ A is b*-closed ⇒ (X,τ) is a πgb*-$T_{1/2}$ space

Theorem 5.3.8

Let $f : (X, \tau) \to (Y, \sigma)$ be a function then,

(1) If f is πgb*-irresolute and X is πgb*-$T_{1/2}$ space, then f is b*-irresolute.

(2) If is πgb*-continuous and X is πgb*-$T_{1/2}$ space, then f is b*-continuous.

Proof

(1) Let V be b*-closed in Y, then V is πgb*-closed in Y.
Since f is πgb*-irresolute, $f^{-1}(V)$ is πgb*-closed in X.
And since X is πgb*-$T_{1/2}$ space, $f^{-1}(V)$ is b*-closed in X.
Hence f is b*-irresolute.

(2) Let V be closed in Y.
Since f is πgb*-continuous, $f^{-1}(V)$ is πgb*-closed in X.
And since X is πgb*-$T_{1/2}$ space, $f^{-1}(V)$ is b*-closed.
Therefore f is b*-continuous.

Definition 5.3.9

A function $f : X \to Y$ is said to be **almost πgb*-continuous** if $f^{-1}(V)$ is πgb*-closed in X for every regular closed set V of Y.

Theorem 5.3.10

For a function $f : X \to Y$, the following statements are equivalent:

1. f is almost πgb*-continuous.
2. $f^{-1}(V)$ is πgb*-open in X for every regular open set V of Y.
3. $f^{-1}(int(cl(V)))$ is πgb*-open in X for every open set V of Y.
4. $f^{-1}(cl(int(V)))$ is πgb*-closed in X for every closed set V of Y.

Proof

(1) $\Rightarrow$ (2)

Suppose f is almost πgb*-continuous.

Let V be a regular open subset of Y.

Since (Y – V) is regular closed and f is almost πgb*-continuous,

$f^{-1}(Y-V) = X - f^{-1}(V)$ is πgb*-closed in X.

Hence $f^{-1}(V)$ is πgb*-open in X.

(2) $\Rightarrow$ (1)

Let V be a regular closed subset of Y.

Then (Y – V) is regular open.

By the hypothesis, $f^{-1}(Y-V) = X - f^{-1}(V)$ is πgb*-open in X.

Hence $f^{-1}(V)$ is πgb*-closed.

Thus f is almost πgb*-continuous.

(2) $\Rightarrow$ (3)

Let V be an open subset of Y.

Then int(cl(V)) is regular open in Y.

By the hypothesis, $f^{-1}(int(cl(V)))$ is πgb*-open in X.

(3) $\Rightarrow$ (2)

Let V be a regular open subset of Y.

Since V = int(cl(V)) and every regular open set is open.

Then, $f^{-1}(V)$ is πgb*-open in X.

(3) $\Rightarrow$ (4)

Let V be a closed subset of Y.

Then (Y – V) is open in Y.

By the hypothesis,

$f^{-1}(int(cl(Y-V))) = f^{-1}(Y-cl(int(V))) = X - f^{-1}(cl(int(V)))$ is πgb*-open in X.

Therefore $f^{-1}(cl(int(V)))$ is πgb*-closed in X.

(4) $\Rightarrow$ (3)

Let V be a open subset of Y.

Then (Y – V) is closed.

By the hypothesis,

$f^{-1}(cl(int(Y-V))) = X - f^{-1}(int(cl(V)))$ is πgb*-closed in X.

Therefore, $f^{-1}(int(cl(V)))$ is πgb*-open in X.

Theorem 5.3.11

Every πgb*-continuous function is almost πgb*-continuous.

Proof

Let $f : X \rightarrow Y$ be πgb*-continuous function.

Let V be regular closed set in Y, then V is closed in Y.

Since f is πgb*-continuous function $f^{-1}(V)$ is πgb*-closed in X.

Therefore f is almost πgb*-continuous.

Theorem 5.3.12

Every almost b*-continuous function is almost πgb*-continuous.

Proof

Let $f : X \rightarrow Y$ be almost b*-continuous function and let V be regular closed set in Y.

Then, $f^{-1}(V)$ b*-closed in X,

hence $f^{-1}(V)$ is πgb*-closed in X.
Therefore f is almost πgb*-continuous.

Theorem 5.3.13

Let X be a πgb*-$T_{1/2}$ space. Then f : X → Y is almost πgb*-continuous if and only if f is almost b*-continuous.

Proof

Suppose f : X → Y is almost πgb*-continuous.
Let A be a regular closed subset of Y.
Then $f^{-1}(A)$ is πgb*-closed in X.
Since X is πgb*-$T_{1/2}$ space, $f^{-1}(A)$ is b*-closed in X.
Hence f is almost b*-continuous.
Conversely, suppose that f : X → Y is almost b*-continuous and A be a regular closed subset of Y.
Then $f^{-1}(A)$ is b*-closed in X.
Since every b*-closed set is πgb*-closed, $f^{-1}(A)$ is πgb*-closed.
Therefore, f is almost πgb*-continuous.