A STUDY ON πgb-CLOSED SETS IN TOPOLOGICAL SPACES

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PAVITHRA, D
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A DISSERTATION SUBMITTED TO THE
AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER
EDUCATION FOR WOMEN COIMBATORE – 641 043

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

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CERTIFIED AS A BONAFIDE RESEARCH WORK

Parvathi 7/5/13
Signature of the Head of the Department

Parvathi 7/5/13
Signature of the Guide

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CONTENTS

INTRODUCTION

INTRODUCTION

Generalized open sets play a very important role in general topology and they are now the research topics of many researchers worldwide. Indeed a significant topic in general topology and real analysis concerns the variously modified forms of continuity, separation axioms etc., by utilizing generalized open sets. As a generalisation of open sets, the notion of b-open sets was introduced by Andrijervic [1996]. This type of sets was discussed by Ekici and Caldas [2004] under the name of γ-open sets. The class of b-open sets is contained in the class of semi-preopen sets and contains the class of semi-open sets and the class of preopen sets. Moreover, it generates the same topology as the class of preopen sets. Since the advent of these notions, several research papers with interesting results in different aspects came to existence. Al-Omari and Noorani [2009], Ekici [2005], Dontchev [1996] and Janaki[2009].

Levine [1970] introduced the concept of generalized closed sets in topological spaces. Extensive research on generalizing closedness was done in recent years , as the notions of generalized semi-closed , α-generalized closed , generalized semi-preclosed sets were investigated by Dontchev [1995], Maki, Devi and Balachandran [1994] and Maki, Umehara and Noiri [1996] respectively. In continuation of the study of generalized closed sets, Al-Omari and Noorani [2009] have introduced the notion of generalized b-closed sets. Since then many authors have contributed to the study of the various concepts using the notion of generalized b-closed sets.

The following articles are chosen for our discussion:

1) Andrijievic [1987], On b-open sets
2) Al-Omari and Noorani[2009], On generalized b-closed sets.
3) Sreeja and Janaki [2011], On πgb closed-sets in Topological spaces
4) Sinem Caglar Akgun and Gulhan Aslim [2012] On πgb-Closed Sets

and Related Topics

In chapter I we discuss the contributions of Andrijievic [1996] towards the study of b-open sets. Andrijevic [1996] introduced the notion of b-open sets as a generalization of open sets. The class of b-open sets is finer than that of preopen sets and semi-open sets but coarser than that of semi-preopen sets. The union of any family of b-open sets is b-open. Regarding intersection, it is shown that the intersection of an open set and a b-open set is b-open and also the intersection of an α-set and a b-open set is b-open. Moreover, some interesting results on the topology generated by b-open sets are discussed. It is proved that the class of b-open sets generate the same topology as the class of preopen sets.

In chapter II we discuss the concept of generalized b-closed (briefly, gb-closed) sets introduced by Al-Omari and Noorani [2009]. This class of gb-closed sets is finer than that of closed, α-closed, preclosed, b-closed, g-closed, αg-closed and gp-closed sets. Some interesting characterizations of these sets are discussed. A necessary and sufficient condition for a gb-closed set to be b-closed is studied. Since every open set is b-open, it is always true that b-d(A) $\subset$ d(A) for any subset A of a topological space X (where d(A) denotes the derived set of A and b-d(A) denotes the b-derived set of A). Moreover, if d(A) = b-d(A), then cl(A) = bcl(A). Also, union of gb-closed sets A and B is gb-closed provided d(A) $\subset$ b-d(A) and d(B) $\subset$ b-d(B) and intersection of a open gb-closed set and a b-closed set is gb-closed. Also, b-closure of a gb-closed set is a gb-closed set. An interesting characterization of extremally disconnected spaces in terms of gb-closed sets is studied. Al-Omari and Noorani [2009] have introduced the notion of ap-b-continuous maps and ap-b-closed maps using gb-closed sets and studied their relationship of ap-b-continuous map with continuous, contra-b-continuous and perfectly continuous maps and the relationship of ap-b-closed map with pre-closed, contra-b-closed, perfectly closed.

Thefollowing implications hold:

perfectly continuous $\Rightarrow$ contra-b-continuous

$\Downarrow$ $\Downarrow$

continuous $\Rightarrow$ ap-b-continuous

perfectly closed $\Rightarrow$ contra-b-closed

$\Downarrow$ $\Downarrow$

pre-closed $\Rightarrow$ ap-b-closed

The converse of the above implications need not hold. Moreover, the following interesting results discussed are as follows:

(1) gb-closedness is preserved under

(a) an ap-b-continuous and b-closed map

and (b) a continuous and b-closed map

(2) Composition of

(a) b-continuous and contra-b-irresolute maps is contra-b-continuous.

(b) b-irresolute and contra-b-irresolute maps is contra-b-irresolute.

(c) ap-b-closed and pre-closed maps is ap-b-closed.

(d) continuous and ap-b-continuous maps is ap-b-continuous.

(3) Every $T_{1/2}$-space is a T_{gs}-space and every T_{gs}-space is a b-$T_{1/2}$ space. This chapter is concluded by discussing the characterizations of T_{gs}-spaces using the concepts of ap-b-continuous maps and ap-b-closed maps.

In chapter III we discuss the concepts of πgb-closed sets and almost πgb-continuous maps introduced by Sreeja and Janaki [2011] and Sinem Caglar Akgun and Gulhan Aslim [2012] respectively. The class of πgb-closed sets is finer than that of closed, α-closed, preclosed, gb-closed, g-closed, πg-closed, πgα-closed, πgs-closed and πgp-closed sets but coarser than that of πgsp-closed set. Some interesting properties of these sets are discussed. Finite union (intersection) of πgb-closed sets need not be πgb-closed. The union of two

πgb-closed sets is πgb-closed provided d(A) $\subset$ b-d(A) and d(B) $\subset$ b-d(B) and intersection of πgb-closed set and a b-closed set is πgb-closed. Some of the characterizations of πgb-$T_{1/2}$-spaces and πgb-spaces are discussed. The characterization and the relationships among πgb-continuous functions, other continuous functions and πgb-irresolute functions are analysed. The composition of two πgb-continuous functions need not be πgb-continuous. The properties of characterizations of almost πgb-continuous functions are analysed.

REVIEW OF LITERATURE

REVIEW OF LITERATURE

Andrijevic [1996] introduced the notion of b-open sets. This type of sets was discussed by Ekici and Caldas [2004] under the name of γ-open sets. Levine [1970] introduced the notion of generalized closed sets. As a continuation of this study, Al-Omari and Noorani [2009] introduced the notion of generalized b-closed sets. Sreeja and Janaki [2011] introduced the notion of πgb-closed and πgb-continuous sets. As a continuation of this study, Sinem Caglar Akgun and Gulhan Aslim [2012] introduced the notion of On πgb- Closed Sets and Related Topics. Since the advent of these notions, several authors have contributed to the study of these concepts and several worthwhile research papers have been published.

In this chapter we have collected some of the articles published on these concepts from the available literature; it is not exhaustive.

Andrijevic [1996] has introduced and studied a new class of generalized open sets in a topological space, called b-open sets. This class is contained in the class of semi-preopen sets and contains the class of semi-open sets and the class of preopen sets. It is proved that the class of b-open sets generates the same topology as the class of preopen sets.

Akdag [2007] has studied some properties of b-I-open sets and obtained several characterizations of b-I-continuous functions and investigated their relationship with other types of functions.

Rajesh [2007] has introduced and characterized almost b-continuous functions using b-open sets.

Vinayagamoorthi and Nagaveni [2007] introduced a new class of generalized α b spaces and analyzed some of their properties.

Al-Omari and Noorani [2008] have introduced the notions of locally b-closed, b-t-set, b-B-set, locally b-closed continuous, b-t-continuous, b-B-continuous functions and obtained decomposition of continuity and complete continuity.

Al-Omari and Noorani [2009] have introduced the class of generalized b-closed sets. This notion is used to consider weak and stronger forms of continuities associated with these sets. These notions are used to give a new characterization of extremally disconnected spaces and also T_{gs}-spaces.

Caldas, Jafari and Rajesh [2009] have defined totally b-continuous functions using b-closed sets and b-open sets and obtained relationships between this new class and other classes of existing known functions.

Lellis Thivagar and MeeraDevi [2010] have introduced and studied new forms of nearly open sets namely, (1, 2)t-set and (1,2)B-set. Using these notions some of the concepts and results of Classical Topological Spaces has been generalized to Bitopological spaces. Further the notions of (1, 2)b-open and (1,2) locally closed sets are introduced and some of their properties are discussed.

Sarsak, Gowrisankar and Rajesh [2010] introduced and studied pre-bg-closed functions using b-open sets induced by Andrijevic [1996].

Ganster and Steiner [2007] have introduced many relationships between some known types of generalized closed sets and b-generalized closed sets. [Hussein [2011]] has investigated many relationships between some known types of generalized closed sets and b-generalized closed sets and obtained some new characterizations of extremally disconnected spaces, T_{gs}-spaces and sg-submaximal spaces.

Benchalli and Karnel [2011] have introduced a new class of fuzzy sets called fuzzy b-generalized closed sets and studied its properties. Further fuzzy b-generalized continuous, fuzzy b-irresolute maps and fuzzy $bT_{1/2}$-spaces are introduced and fuzzy separation axioms are investigated with the help of fuzzy b-open sets.

Benchalli and Karnel [2011] have introduced a new form of fuzzy compact spaces namely fuzzy b-compact spaces, b-closed spaces and fbg-compact spaces with the concept of fuzzy b-open sets. Some characterizations, hereditary property and the invariance of fuzzy b-compact spaces, b-closed spaces and fbg-compact spaces under fuzzy mappings are investigated.

Bharathi, Bhuvaneswari and Chandramathi [2011] have introduced the notions of generalized locally b-closed sets(glbc sets) and generalized locally b-continuous maps (glbc maps) which are weaker forms of locally closed sets and LC continuous maps, respectively in topological spaces and obtained certain results relating to them.

Mustafa [2011] have introduced a new class of functions called weakly ωb-continuous functions and investigated several properties, characterizations and relationships with other existing concepts, such as ωb-continuous and weakly b-continuous functions.

Rajesh and Salleh [2011] have introduced a new class of topological spaces called b-$T_{1/2}$-space in terms of b-open sets and b-kernal and have investigated some of their fundamental properties.

A new class of functions called generalized αb-continuous mappings (denoted by gαb-continuous) has been introduced by Vinayagamoorthi and Nagaveni [2012] and some of their properties. Further a study on generalized αb-open maps and generalized αb-closed maps are analysed.

Muthuvel and Parimelazhagan [2012] have introduced and studied the concept of a new class of closed sets called b^*-closed sets and investigated some of their properties.

Vidhya and Parimelazhagan [2012] have introduced generalized* b-closed sets in topological spaces and studied some of its basic properties and investigated the relations between the associated topologies.

Poongothai and Parimelazhagan [2012] have introduced the concept of strongly b^* - closed set (breifly sb^*-closed set) in a topological space (X, τ) and investigated the relation between the associated topologies.

Rajesh [2012] introduced the notion of b-closed spaces and investigated its fundamental properties.

Bharathi, Bhuvaneswari and Chandramathi [2012] have introduced and investigated a new class of sets and maps between topological spaces called strongly generalized b-closed sets and g*b continuous maps, respectively. The concept of strongly generalized b-closed maps and their properties are investigated.

The contributions of Andrijevic [1996], Al-Omari and Noorani [2009], Sreeja and Janaki [2011] and Sinem Caglar Akgun and Gulhan Aslim[2012] have discussed in detail.

CHAPTER 1

CHAPTER - I

b - OPEN SETS

In this chapter we discuss the contributions of Andrijevic [1996] towards the study of b-open sets. Andrijevic [1996] introduced the notion of b-open sets as a generalization of open sets. The class of b-open sets is finer than that of preopen sets and semiopen sets but coaser than that of semi-preopen sets. The union of any family of b-open sets is b-open. Regarding intersection it is shown that the intersection of an open set and a b-open set is b-open and also the intersection of an α-set and a b-open set is b-open. Moreover, some intersecting results on the topology generated by b-open sets are discussed. It is proved that the class of b-open sets generates the same topology as the class of preopen sets. Some interesting properties of these concepts are discussed.

Section 1.1

Preliminary definitions and results

In this section we give the preliminary definition and results needed for our discussion.

Definition : 1.1.1

Let (X,τ) be a topological space. A subset S of a space X is called

(1) an **α-openset** if $S \subset int(cl(int(S)))$

(2) **semi-open** if $S \subset cl(int(S))$

(3) **preopen** if $S \subset int(cl(S))$

(4) **β-open or semi-preopen** if $S \subset cl(int(cl(S)))$

Notation : 1.1.2

The complement of α-openset, semi-open, preopen and β-open are called α-openset, semi-open, preopen and β-open respectively. We denote the classes of these sets in a space (X, τ) by, SO(X), PO(X) and βO(X) or SPO(X) respectively.

Definition : 1.1.3

A subaet S of a topological space (X, τ) is semi-closed if and only if int(cl(S)) $\subset$ S.

Remark : 1.1.4

In general, SO(X) need not be a topology on X, but the intersection of a semi open set and an open set is semi open. The same holds for PO(X) and SPO(X).

Definition : 1.1.5

Let (X,τ) be a topological space. For a subset S of a space X, the **semi-closure** of S, denoted by **scl S** is the intersection of all semi-closed subsets of X containing S. The semi- interior of S, denoted by sint S, is the union of all semi-open subsets of X contained in S.

Definition : 1.1.6

Let (X,τ) be a topological space. For a subset S of a space X, the **pre-closure** of S, denoted by **pcl S** is the intersection of all pre-closed subsets of X containing S. The pre- interior of S, denoted by pint S, is the union of all pre-open subsets of X contained in S.

Definition : 1.1.7

Let (X,τ) be a topological space. For a subset S of a space X, the **semi-preclosure** of S, denoted by **spcl S** is the intersection of all semi-preclosed subsets of X containing S. The semi-pre interior of S, denoted by spint S, is the union of all semi-preopen subsets of X contained in S. By cl_α and int_α we denote the closure and the interior operator in (X,τ_α). The collection of all α-open sets τ_α on a space X is a topology on x.

Definition : 1.1.8

Let (X,τ) be a topological space. For a subset S of a space X, the α-closure of S, denoted by cl_α is the intersection of all α-closed subsets of X containing S. The α-interior of S, denoted by int_α, is the union of all α-open subsets of X contained in S. By cl_α and int_α we denote the closure and the interior operator in (X,τ_α). The collection of all α-open sets τ_α on a space X is a topology on x.

Definition : 1.1.9

Let (X,τ) be a topological space. Let S be a subset of a space X. Then

(1) $cl_\alpha(S) = S \cup cl(int(cl(S)))$, $int_\alpha S = S \cap int(cl(int(S)))$

(2) $scl(S) = S \cup int(cl(S))$, $sint\, S = S \cap cl(int(S))$

(3) $pcl(S) = S \cup cl(int(S))$, $pint(S) = S \cap int(cl(S))$

(4) $spcl(S) = S \cup int(cl(int(S)))$, $spint(S) = S \cap cl(int(cl(S)))$

Definition : 1.1.10

Let (X,τ) be a topological space.Let S be a subset of a space X. Then

(1) $scl(sint(S)) = sint(S) \cup int(cl(int(S)))$

(2) $pcl(pint(S)) = pint(S) \cup cl(int(S))$

(3) $spcl(spint(S)) = spint(spcl(S))$

(4) $int(scl(S)) = pint(cl(S)) = pint(scl(S)) = scl(pint(S)) = int(cl(S))$

(5) $int(pcl(S)) = scl(int(S)) = spcl(int(S)) = int(spcl(S)) = int(cl(int(S)))$

Section 1.2

Some properties of b-open sets

In this section we consider a new class of generalized open sets, namely, b-open sets and study some of its properties.

Definition : 1.2.1

A subset S of a space X is called **b-open** if $S \subset cl(int(S)) \cup int(cl(S))$.

The class of all b-open sets in X will be denoted by **BO(X)**.

If S is a subset of a space X, the b-interior of S denoted by bint S, is the largest b-open set contained in S.

If S is a subset of a space X, the b-closure of S denoted by bcl S, is the smallest b-closed set contained in S.

Theorem : 1.2.2

(1) Every open set is a b-open set.

(2) Every preopen set is a b-open set.

(3) Every semi-open set is a b-open set.

(4) Every b-open set is a semi-preopen set.

Proof :

(1) Let S be an open set.

Then S = int (S)

$\Rightarrow$ $S \subset cl(int(S))$ (i)

Always $S \subset cl(S)$

$\Rightarrow$ $int\ S \subset int(cl(S))$

$\Rightarrow$ $S \subset int(cl(S))$ (ii)

From (i) and (ii) we get,

$S \subset cl(int(S)) \cup int(cl(S))$

Therefore S is b-open.

Hence every open set is a b-open.

(2) Let S be a preopen set.

By definition, $S \subset int(cl(S))$

$\subset cl(int(S)) \cup int(cl(S))$

Therefore S is b-open.

Hence every preopen set is a b-open set.

(3) Let S be a semi-open set.

By definition, $S \subset cl(int(S))$

$\subset cl(int(S)) \cup int(cl(S))$

Therefore S is b-open.

Hence every semi-open set is a b-open set.

(4) Let S be a b-open set.

By definition, $S \subset cl(int(S)) \cup int(cl(S))$

Take $x \in S$

Then $x \in cl(int(S))$ or $x \in int(cl(S))$.

Case (1)

Suppose $x \in cl(int (S))$

Since $S \subset cl (S)$, $int S \subset int(cl (S))$

Therefore $cl(int (S)) \subset cl(int(cl (S)))$

Hence $x \in cl(int(cl(S)))$

Case (2)

Suppose $x \in int(cl (S))$

Since $int(cl S) \subset cl(int(cl(S)))$, we get $x \in cl(int(cl(S)))$.

From case(1) and (2), we get $S \subset cl(int(cl(S)))$

Hence every b-open set is semi-preopen.

Remark : 1.2.3

From the above theorem we get, $PO(X) \cup SO(X) \subset BO(X) \subset SPO(X)$

The following examples show that the inclusions cannot be replaced with equalities:

Example : 1.2.4

A b-open set which is neither semi-open nor pre-open.

Consider the set R of real numbers with usual topology and let

$S = [0, 1] \cup ((1, 2) \cap Q)$ where Q stands for the set of rational numbers.

$int(S) = (0, 1)$

$cl(int(S)) = [0, 1]$ (1)

$cl(S) = (0, 1) \cup [1, 2]$

$int(cl(S)) = (0, 1) \cup (1, 2)$ (2)

From (1) and (2) we get, $S \subset cl(int(S)) \cup int(cl(S))$

Hence S is b-open.Since $cl(int(S)) = [0, 1]$, $S \not\subset cl(int(S))$

Hence S is not semi-open. Since $int(cl(S)) = (0, 1) \cup (1, 2)$, $S \not\subset int(cl(S))$.

Hence S is not pre open.Therefore S is a b-open set which is neither semi-open nor pre-open.

Example : 1.2.5

A semi-pre open set which is not a b-open set.

Let $T = [0, 1] \cap Q$ then T is semi-pre open but not b-open. $cl(T) = [0,1]$

$int(cl(T)) = (0, 1)$

$cl(int(cl(T))) = [0, 1]$

Therefore $T \subset cl(int(cl(T)))$

Hence T is semi-pre open.

$int(cl(T)) = (0, 1)$

$int(T) = \phi$

$cl(int(T)) = \phi$

Therefore $cl(int(T)) \cup int(cl(T)) = (0, 1)$.

$\therefore T \not\subset cl(int(T)) \cup int(cl(T))$.

Hence T is not b-open set.

Theorem : 1.2.6

For a subset S of a space X the following are equivalent:

(a) S is b-open

(b) $S = \text{pint } S \cup \text{sint } S$

(c) $S \subset \text{pcl}(\text{pint } S)$

Proof :

(a)⇒(b)

Let S be b-open, that is $S \subset cl(int(S)) \cup int(cl(S))$.

Then by Definition:1.1.9 we have

$$\text{pint } S \cup \text{sint } S = (S \cap int(cl(S))) \cup (S \cap cl(int(S)))$$

$$= (S \cap int(cl(S))) \cup (S \cap cl(int(S))) = S$$

(b) ⇒(c)

Definitions 1.1.9 and 1.1.10 imply

$$S = \text{pint } S \cup \text{sint } S = \text{pint } S \cup (S \cap cl(int(S)))$$

$$\subset \text{pint } S \cup cl(int(S))$$

$$= \text{pcl}(\text{pint}S)$$

(c) ⇒(a)

By Definitions 1.1.9 and 1.1.10, we have

$$S \subset \text{pcl}(\text{pint } S) = \text{pint } S \cup cl(int(S))$$

$\subset int(cl(S)) \cup cl(int(S))$, and so S is a b-open.

Remark : 1.2.7

From 1.2.6 (b) every b-open set can be represented as a union of a preopen set and a semi open set.

Proposition : 1.2.8

S is semi-preopen if and only if $S \subset \text{sint } (\text{scl } S)$

Proof:

Assume that S is semi open.

That is $S = \text{sint } S$

$S \subset \text{scl } (\text{sint } S)$

$S \subset scl\ S$

Conversely,

Assume that $S \subset scl\ (sint\ S)$

To prove S is semi-open.

That is to prove $sint\ S \subset S$

As $S \subset scl\ (sint\ S) \subset scl\ S$

scl S is semi-open.

Theroem : 1.2.9

Let S be a b-open set such that int S = ϕ. Then S is preopen.

Proof :

Let S be a b-open set.

Then $S \subset cl(int(S) \cup int(cl(S))$

Given int S = ϕ

$\Rightarrow$ cl(int S) = cl (ϕ) = ϕ

Therefore $S \subset int(cl(S))$

Hence S is a preopen.

Theorem : 1.2.10

(a) The union of any family of b-open sets is a b-open set.

(b) The intersection of an open and b-open set is a b-open set.

Proof :

(a) Let $\{ A_\alpha \}$ be a family of b-open sets.

Let $A = \cup A_\alpha$ Since A_α is a b-open we get, $A_\alpha \subset cl(int\ A_\alpha) \cup int(cl\ A_\alpha)$

Consider $A = \cup A_\alpha \subset \cup (cl(int(A_\alpha)) \cup int(cl(A_\alpha)))$

$= (\cup (cl(int(A_\alpha)))) \cup (\cup (int(cl(A_\alpha))))$

$\subseteq (cl(\cup int(A_\alpha))) \cup (int(\cup cl(A_\alpha)))$

$\subseteq \quad (cl(int(\bigcup A_\alpha))) \cup (int(cl(\bigcup A_\alpha)))$

$= \quad cl(int(A)) \cup int(cl(A))$

Hence the union of any family of b-open sets is a b-open set.

(b) Let A be open then A is b-open and $A = int(A) \subseteq bint(A)$

Let B be b-open then $B = bint(B)$

Now $A \cap B \subseteq bint(A) \cap bint(B)$

$= bint(A \cap B)$

Always $bint(A \cap B) \subset A \cap B$.

Hence $A \cap B = bint(A \cap B)$

Therefore $A \cap B$ is b-open.

Theorem : 1.2.11 [Andrijevic [1984]]

For any $S \subset X$, $cl_\alpha(int_\alpha S) = cl(int\ S)$ and $int_\alpha(cl_\alpha S) = int(cl\ S)$.

We deduce from the above theorem the following result, analogous to those established for the other three classes of generalized open sets.

Theorem : 1.2.12

Let (X, τ) be a space. Then

(1) τ and τ_α have the same class of b-open sets.

(2) The intersection of an α-set and a b-open set is a b-open set.

Proof :

(1) Let A be a b-open set in τ.

Then $A \subset cl(int(A)) \cup int(cl(A))$

$= cl_\alpha(int_\alpha(A)) \cup int_\alpha(cl_\alpha(A))$

Therefore A is a b-open in τ_α.

Similarly, if A is b-open in τ_α then as

$A \subset cl_\alpha(int_\alpha(A)) \cup int_\alpha(cl_\alpha(A))$

$= cl(int(A)) \cup int(cl(A))$

We get A is b-open in τ.

Therefore τ and τ_α have the same class of b-open sets.

(2) Proof follows from the definition of α-open set and b-open set.

Definition : 1.2.13

A subset S of a space X is called **b-closed** if X - S is b-open. The class of all b-closed is denoted by BC(X). S is b-closed if and only if $int(cl\ S) \cap cl(int\ S) \subset S$.

Definition : 1.2.14

If S is a subset of a space X, S is b-closed if and only if $int(cl\ S) \cap cl(int\ S) \subset S$.

Theorem : 1.2.15

Let S be a subset of a space X. Then

(1) $bcl(S) = scl(S) \cap pcl(S)$

(2) $bint(S) = sint(S) \cup pint(S)$

Proof :

(1) Since bcl(S) is a b-closed set, we have

$bcl(S) \supset int(cl(bcl(S))) \cap cl(int(bcl(S))) \supset int(cl(S)) \cap cl(int(S))$ and so.

$bcl(S) \supset S \cup (int(cl(S)) \cap cl(int(S)))$

$= (S \cup int(cl(S))) \cap (S \cup cl(int(S))$

$= scl(S) \cap pcl(S)$

(2) bint(S) $\supset$ cl(int(bint(S))) $\cup$ int(cl(bint(S))) $\supset$ cl(int(S)) $\cup$ int(cl(S))

and so. bint(S) $\supset$ S ∩ (cl(int(S)) $\cup$ (S ∩ int(cl(S)))

= sint(S) ∩ pint(S)

Theorem : 1.2.16

Let S be a subset of a space X. Then

1) bcl(int S) = int(bcl S) = int(cl(int S))
2) bint(cl S) = cl(bint S) = cl(int(cl S))
3) bcl(sint (S)) = scl(sint (S))
4) bint(scl (S)) = sint(scl (S))
5) sint(bcl (S)) = scl (S) ∩ cl(int (S))
6) scl(bint (S)) = sint (S) $\cup$ int(cl (S))
7) pint(bcl (S)) = bcl(pint (S)) = pint(pcl (S))
8) pcl(bint (S)) = bint(pcl (S)) = pcl(pint (S))
9) spint(bcl (S)) = bcl(spint (S)) = sint(scl (S)) ∩ pcl (S)
10) spcl(bint(S)) = bint(spcl (S)) = scl(sint (S)) $\cup$ pint (S)

Proposition : 1.2.1

Let S be a subset of a space X. Then bint(bcl(S)) = bcl(bint(S)).

Section 1.3

On the topology generated by b-open sets

Although none of SO(X), PO(X), SPO(X) and BO(X) is a topology on X, each of these classes generates a topology in a natural way.

Let $\tau(\mathbb{S})$ = {V$\subset$ X / V∩S∈ $\mathbb{S}$ whenever S∈ $\mathbb{S}$}, where $\mathbb{S}$ stands for SO(X), PO(X), SPO(X) and BO(X) respectively. Clearly $\tau(\mathbb{S})$ is a topology on X larger than τ.

[Njastad [1965]] showed that $\tau(\mathbb{S}) = \tau_\alpha$ for $\mathbb{S}$ = SO(X).

The topology generated by PO(X) denoted by τ_γ **.** The closure and interior of a set S in (X, τ_γ) are denoted by cl_γ(S) and int_γ(S). [Ganster and Andrijevic-[1988]] obtained that $\tau(\mathbb{S}) = \tau$ for $\mathbb{S}$ = SPO(X).

The topology generated by b-open sets will be denoted by τ_b and we shall prove that $\tau_b = \tau_\gamma$**.**

Proposition : 1.3.1 [Ganster and Andrijevic [1988]]

The intersection of a semi-open set and a preopen set is a semi-pre open set.

Theorem : 1.3.2 [Ganster and Andrijevic [1988]]

For a space (X, τ) and x$\in$ X the following are equivalent:

(a) {x} $\in$ SPO(X)

(b) {x} $\in$ PO(X)

(c) {x} $\in \tau_\gamma$

Proof :

(a) $\Rightarrow$ (b)

Since $\tau(\mathbb{S}) = \tau_\gamma$ where $\mathbb{S}$ = SPO(X), we get

{x} $\in$ SPO(X)

$\Rightarrow$ {x}$\in \tau(\mathbb{S})$

$\Rightarrow$ {x}$\in \tau_\gamma$

$\Rightarrow$ {x}$\in$ PO(X)

Hence (a) $\Rightarrow$ (b)

(b) $\Rightarrow$(c)

Since τ_γ is a topology generated by PO(X), we get

$\{x\} \in PO(X) \Rightarrow \{x\} \in \tau_\gamma$

Hence (b) $\Rightarrow$(c)

(c) $\Rightarrow$(a)

Since $\tau(\mathbb{S}) = \tau_\gamma$ where $\mathbb{S}$ = SPO(X),

$\{x\} \in \tau_\gamma$

$\Rightarrow \{x\} \in \tau(\mathbb{S})$

$\Rightarrow \{x\} \in SPO(X)$

Hence (c) $\Rightarrow$(a)

Proposition :1.3.3 [Andrijevic [1987]

Let S be a subset of a space X. Then $cl_\gamma(int(S)) = cl(int(S))$.

Theorem : 1.3.4 [Andrijevic [1992]

Let S be a subset of a space X. Then $S \in \tau_\gamma$ if and only if $S = G \cup H$ with $G \in \tau_\alpha$ and $\{h\} \in PO(X)$ for every $h \in H$.

Corollary : 1.3.5

Let $V \in \tau_\gamma$ and $G \in SO(X)$. Then $V \cap G \in BO(X)$.

Theorem : 1.3.6

Let (X, τ) be a space. Then $\tau_\gamma \subset \tau_{b.}$

Proof :

Let $V \in \tau_\gamma$ and $S \in BO(X)$. Then $S = sint(S) \cup pint(S)$ by Theorem : 1.2.6.

Hence $V \cap S = (V \cap pint(S)) \cup (V \cap sint(S))$ is b-open and so $V \in \tau_{b.}$

Lemma : 1.3.7

Let (X, τ) be a space and $V \in \tau$. Then $S = V - int(cl(int(V)))$ is a pre open set.

Proof :

Since $int(cl(int(V)))$ is a semi-closed, S is b-open. On the other hand, $int(S) = \phi$ and so S is preopen .

Lemma : 1.3.8

Let (X, τ) be a space and $V \in \tau_b$. Then $sint(V) = int_\alpha(V)$.

Proof :

Let $x \in sint(V) = V \cap cl(int(V))$ and suppose $x \notin int(cl(int$

$(V)))$.

Then $x \notin (int(V))$ and $\{x\} \cup int\ V$ is semi-open. Also $V - int(cl(int(V)))$ is pre open by Lemma 1.3.7. Hence $\{x\} = (\{x\} \cup int(V)) \cap (V - int(cl(int(V))))$ is semi-preopen by Proposition 1.3.1, and so $\{x\} \in \tau\gamma$ by Theorem 1.3.2. Since $x \in cl(int(V))$ and as $cl(int(V)) = cl_\gamma(int(V))$ by Theorem 1.3.3,

we get $x \in cl_\gamma(int(V))$ (1)

Since $\{x\}$ is preopen and $x \notin (int(V))$, we get $\{x\} \cap int\ V = \phi$

$\therefore x \notin cl\gamma\ int\ V$, a contradiction to (1)

Therefore $x \in int(cl(int(V))$ that is $x \in int_\alpha(V)$.

Theorem : 1.3.9

Let (X, τ) be a space. Then $\tau_b = \tau_\gamma$.

Proof :

Let $V \in \tau_b$ and $S \in PO(X)$. Put $G = V \cap cl(int(V))$ and $H = V - cl(int(V))$

$sint(V) = V \cap cl(int(V))$

$V = sint\ V \cup H$

$= int_{\alpha} V \cup H$

Since $H \in \tau_b$ and int $H = \phi$, we have that $H \cap S \in PO(X)$

Therefore $V \cap S = (int_{\alpha} V \cap S) \cup (H \cap S)$ is preopen and so $V \in \tau_{\gamma}$.

Therefore $\tau_b \subseteq \tau_{\gamma}$

By Theorem : 1.3.6, $\tau_b = \tau_{\gamma}$

Hence proved.

CHAPTER 2

CHAPTER – II

ON GENERALIZED b-CLOSED SETS

In this chapter we discuss the concept of generalized b-closed sets (briefly, gb-closed sets) introduced by Al-Omari and Noorani [2009]. The class of gb-closed sets is finer than that of closed, α-closed, pre closed, b-closed, g-closed, αg-closed, gp-closed and b-closed sets. Some interesting characterization of these sets are discussed. A necessary and sufficient condition for a gb-closed set to be b-closed is obtained. Since every open set is b-open, it is always true that b-d(A) $\subset$ d(A) for any subset A of a topological space X (where d(A) denotes the derived set of A and b-d(A) denotes the b-derived set of A). Moreover, if d(A) = b-d(A), then cl(A) = bcl(A). Also union of gb-closed sets A and B is gb-closed provided b-d(A) $\subset$ d(A) and b-d(B) $\subset$ d(B) and intersection of an open gb-closed set and a b-closed set is gb-closed. Also b-closure of a gb-closed set is a gb-closed set. A characterization of extremally disconnected spaces in terms of gb-closed sets is obtained. Al-Omari, and Noorani [2009] have introduced some weak and strong forms of continuity, namely, ap-b-continuity, contra-b-continuity and studied their relationship with other forms of continuity. Al-Omari, and Noorani [2009] have also contributed to the study of the behavior of gb-closed sets under these mappings. The chapter is concluded by discussing characterizations of T_{gs}-spaces using the concepts of ap-b-continuous maps and ap-b-closed maps.

Section 2.1

Preliminaries

Definition : 2.1.1 [Levine [1970]]

A subset A of a space (X, τ) is called a **generalized closed set** (briefly,**g-closed**) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open.

Definition : 2.1.2 [Maki, Devi and Balachandran [1994]]

A subset A of a space (X, τ) is called an **α-generalized closed set** (briefly,**αg-closed**) if αcl (A) $\subseteq$ U Whenever A $\subseteq$ U and U is open.

Definition : 2.1.3 [Maki, Umehara and Noiri [1996]]

A subset A of a space (X, τ) is called a **generalized preclosed set** (briefly,**gp-closed**) if pcl (A) $\subseteq$ U Whenever A $\subseteq$ U and U is open.

Definition : 2.1.4 [Dontchev [1995]]

A subset A of a space (X, τ) is called a **generalized semi-preclosed set** (briefly,**gsp-closed**) if spcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open.

Definition : 2.1.5 [Arya and Nour [1990]]

A subset A of a space (X, τ) is called a **generalized semiclosed set** (briefly,**gs-closed**) Arya and Nour [1990] if scl(A) $\subseteq$ U Whenever A $\subseteq$ U and U is open.

Definition : 2.1.6 [Bhattacharyya and Lahiri [1987]]

A subset A of a space (X, τ) is called a **semi generalized closed set** (briefly,**sg-closed**) if scl (A) $\subseteq$ U Whenever A $\subseteq$ U and U is semi-open.

Complement of g-closed sets are called **g-open sets.**

Complement of gp-closed, αg-closed, gsp-closed, gs-closed and sg-closed sets are called **gp-open, αg-open, gsp-open, gs-open and sg-open sets** respectively.

Theorem : 2.1.7 [Dontchev, Ganster and Noiri [2000]]

Let A be a subset of a topological space (X, τ). If A$\in$SO(X), then pcl(A) = cl(A).

Definition : 2.1.8

A space (X,τ) is **extremally disconnected** if the closure of every open subset of X is open.

Theorem : 2.1.9

For a space X, the following statements are equivalent :

(1) (X, τ) is extremally disconnected

(2) $scl(A \cup B) = scl(A) \cup scl(B)$ for all $A, B \subseteq X$

(3) The union of two semi-closed subsets of X is semi-closed

(4) The union of two sg-closed subsets of X is sg-closed

(5) Every semi-preclosed subset of X is preclosed

(6) Every sg-closed subset of X is preclosed

(7) Every semi-closed subset of X is preclosed

(8) Every semi-closed subset of X is α-closed

(9) Every semi-closed subset of X is gα-closed

Definition : 2.1.10 [Ekici [2005]]

A function $f : X \longrightarrow Y$ is said to be **b-continuous** if for each $x \in X$ and each open set V of Y containing f(x), there exists $U \in bO(X, x)$ such that $f(U) \subseteq V$.

Definition : 2.1.11 [Dontchev,1996]]

A function $f : X \longrightarrow Y$ is said to be **contra continuous** if $f^{-1}(V)$ is closed in X for each open set V of Y.

Definition : 2.1.12 [Nasef,2005]]

A function $f : X \longrightarrow Y$ is said to be **contra b-continuous** if $f^{-1}(V)$ is b-closed in X for each open set V of Y.

Definition : 2.1.13 [Ekici [2005]]

A function $f : X \longrightarrow Y$ is said to be **b-irresolute** if for each b-open set V in Y, $f^{-1}(V)$ is b-open in X.

Definition : 2.1.14 [Ekici [2005]]

A function $f : X \longrightarrow Y$ is said to be **b-closed** if for every b-closed subset A of X, f(A) is b-closed in Y .

Definition : 2.1.15 [Ekici [2005]]

A function $f : X \longrightarrow Y$ is said to be **b-open** if for every b-open subset A of X, f(A) is b-open in Y .

Definition : 2.1.16

A map $f : X \longrightarrow Y$ is said to be **contra b-closed** if f(U) is b-open in Y for each closed set U of X.

Definition : 2.1.17

A map $f : X \longrightarrow Y$ is said to be **contra b-open** if f(U) is b-closed in Y for each open set U of X.

Section 2.2

Generalized b-closed sets

In this section, we investigate the class of generalized b-closed sets and study some of its fundamental properties. Several characterizations of generalized b-closed sets are given.

Definition : 2.2.1 [Ganster and Steiner [2007]]

Let X be a topological space. A subset A of X is called a **generalized b-closed Set** (briefly, **gb-closed set**) if $bcl(A) \subseteq U$, whenever $A \subseteq U$ and U is open.

The complement of a generalized b-closed set is called **generalized b-open** (briefly, **gb-open**).

Notation :2.2.2

The collection of all gb-closed subsets of X is denoted by gBC(X) and the collection of all gb-open subsets of X is denoted by gbO(X).

Implication digram of union closed sets

Closed $\Rightarrow$ α-closed $\Rightarrow$ pre-closed $\Rightarrow$ b-closed

$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$

g-closed $\Rightarrow$ αg-closed $\Rightarrow$ gp-closed $\Rightarrow$ gb-closed

Theorem : 2.2.3

Every b-closed set is gb-closed.

Proof:

Let A be a b- closed set .Then bcl A = A. Hence bcl(A) $\subset$ U whenever A $\subseteq$ U and U is open. Hence A is gb-closed.

The converse of Theorem : 2.2.3 need not be true

Example : 2.2.4

Let X = {a, b, c} and let τ = { ϕ, X, {a} }. Then the family of all b-closed sets of X is { ϕ, X, {b}, {c}, {b, c} } but the family of all gb-closed sets of X is { ϕ, X, {b}, {c}, {a, b}, {b, c}, {a, c} }. It is clear that {a, c} is gb-closed but not b-closed in X

Theorem : 2.2.5 Necessary and sufficient condition for a gb-closed set to be b-closed.

Let A be a gb-closed subset of (X, τ). Then bcl(A) – A does not contain any non-empty closed sets.

Proof :

Let F be a closed set such that F $\subseteq$ bcl(A) – A.

Then X-F is open and A $\subseteq$ X – F. Since A is gb-closed, bcl(A) $\subseteq$ X – F and thus F $\subseteq$ X – bcl(A).

This implies that F $\subseteq$ (X – bcl(A)) $\cap$ (bcl(A)-A) = ϕ.

Hence F = ϕ.

Theorem : 2.2.6

Let A be a gb-closed set. Then A is b-closed if and only if bcl(A) – A is closed.

Proof :

Let A be a gb-closed set. If A is b-closed, then bcl(A) – A = ϕ which is a closed set.

Conversely, let bcl(A) – A be closed. Then, by Theorem 2.2.5, bcl(A) – A does not contain any non-empty closed subset and since bcl(A) – A is a closed subset of itself, bcl(A) – A = ϕ. Hence A = bcl(A) and so A is a b-closed set.

Definition : 2.2.7

Let A be a subset of a space X. A point x $\in$ X is said to be a b-limit point of A if for each b-open set U containing x, U $\cap$ (A - {x}) $\neq$ ϕ.

The set of all b-limit points of A is called the b-derived set of A and is denoted by b-d(A).

Remark : 2.2.8

Since every open set is b-open, b-d(A) $\subseteq$ d(A) for any subset A $\subseteq$ X Where d(A) is the derived set of A.

Lemma : 2.2.9

If d(A) = b - d(A) then cl(A) = bcl(A).

Proposition : 2.2.10

If $d(A) \subseteq b\text{-}d(A)$ for every subset A of X . Then for subsets F and B of X, $bcl(F \cup B) = bcl(F) \cup bcl(B)$.

Theorem : 2.2.11

If A and B are gb-closed sets such that $d(A) \subseteq b\text{-}d(A)$ and $d(B) \subseteq b\text{-}d(B)$. Then $A \cup B$ is gb-closed.

Proof :

Let U be an open set such that $A \cup B \subseteq U$.

Since A and B are gb-closed sets $bcl(A) \subseteq U$ and $bcl(B) \subseteq U$.

Since $d(A) \subseteq b\text{-}d(A)$, we get $d(A) = b\text{-}d(A)$ and by Lemma 2.2.9,

$cl(A) = bcl(A)$.

Similarly, $cl(B) = bcl(B)$.

Thus $bcl(A \cup B) \subseteq cl(A \cup B) = cl(A) \cup cl(B)$

$= bcl(A) \cup bcl(B) \subseteq U$

Which implies that $A \cup B$ is gb-closed.

Definition : 2.2.12

Let $B \subseteq A \subseteq X$. Then we say that **B is gb-closed relative to A** if $bcl_A(B) \subseteq U$ where $B \subseteq U$ and U is open in A.

Theorem : 2.2.13

Let $B \subseteq A \subseteq X$ where A is a gb-closed and an open set. Then B is gb-closed relative to A if and only if B is gb-closed in X.

Proof :

Let A be both gb-closed and open. Then $bcl(A) \subseteq A$. Since $B \subset A$. $bcl(B) \subseteq bcl(A) \subseteq A$. Therefore $A \cap bcl(B) = bcl(B)$.

Now from the fact that $A \cap bcl(B) = bcl_A(B)$

$$\Rightarrow bcl(B) = bcl_A(B) \subseteq A.$$

Let B is gb-closed relative to A and U is an open subset of X such that $B \subseteq U$.

Then $B = B \cap A \subseteq U \cap A$ and $U \cap A$ is open in A.

Since B is gb-closed relative to A, $bcl(B) = bcl_A(B) \subseteq U \cap A \subseteq U$.

Therefore B is gb-closed in X.

Conversely, if B is gb-closed in X and U is an open subset of A such that $B \subseteq U$.

Then $U = V \cap A$ for some open subset V of X.

As $B \subseteq V$ and B is gb-closed in X, $bcl(B) \subseteq V$.

Then $bcl_A(B) = bcl(B) \cap A \subseteq V \cap A = U$.

Therefore B is gb-closed relative to A.

Theorem : 2.2.14

Let A be an open and gb-closed set. Then $A \cap F$ is gb-closed whenever $F \in BC(X)$.

Proof :

Since A is gb-closed and open, $bcl(A) \subseteq A$ and hence A is b-closed. Hence $A \cap F$ is b-closed in X which implies that $A \cap F$ is gb-closed in X.

Theorem : 2.2.15

If A is a gb-closed set and B is any set such that $A \subseteq B \subseteq bcl(A)$, then B is a gb-closed set.

Proof :

Let $B \subseteq U$ where U is open set.

Since A is gb-closed and $A \subseteq U$,

$bcl(A) \subseteq U$ and also $bcl(B) \subseteq bcl(A)$.

Therefore bcl(B) $\subseteq$ U and hence B is a gb-closed set.

Theorem : 2.2.16

A subset A $\subseteq$ X is gb-open if and only if F $\subseteq$ bint(A) whenever F is a closed set and F $\subseteq$ A.

Proof :

Let A be a gb-open set and suppose F $\subseteq$ A, where F is closed.

Then X – A is a gb-closed set contained in the open set X – F.

Hence bcl(X – A) $\subseteq$ X – F and X - bint(A) $\subseteq$ X – F.

Hence F $\subseteq$ bint(A).

Conversely, if F is a closed set with F $\subseteq$ bint(A) and F $\subseteq$ A, then

X – bint(A) $\subseteq$ X – F.

Then bcl(X – A) $\subseteq$ X – F.

Hence X – A is a gb-closed set and A is a gb-open set.

Theorem : 2.2.17

A space X is extremally disconnected if and only if every gb-closed subset of X is gp-closed.

Proof :

Suppose that X is extremally disconnected. Let A be gb-closed and let U be an open set containing A. Then bcl(A) $\subseteq$ U.

Since bcl(A) = A $\cup$ [int(cl(A)) $\cap$ cl(int(A))],

[int(cl(A)) $\cap$ cl(int(A))] $\subseteq$ U (1)

Since int(A) $\subset$ int(cl(A)), int(A) = cl(int(A)) $\cap$ int(A)

$\therefore$ cl(int (A)) = cl [int(cl(A)) $\cap$ (int(A))]

$\subseteq$ [cl(int(cl(A))) $\cap$ cl(int(A))]

$= \text{int(cl(A))} \cap \text{cl(int(A))}$ (since int(cl A) is closed)

$\subseteq$ U (by 1)

pcl(A) = $A \cup$ cl(int A) $\subseteq$ U

Hence A is gp-closed.

Conversely, let every gb-closed subset of X be gp-closed.

Let $A \subseteq X$ be regular open.

Then bcl(A) = $A \cup [\text{int(cl(A))} \cap \text{cl(int(A))}]$

$= A \cup [A \cap \text{cl(int(A))}] \subseteq A$

Then A is gb-closed and so gp-closed.

Since every regular open set is semi open and A is gp-closed, cl(A) = pcl A. Therefore A is closed and X is extremally disconnected.

Section 2.3

ap-b-continuous, ap-b-closed and contra b-continuous maps

In this section we discuss the notions of ap-b-continuous and ap-bclosed maps which are defined using the notion of gb-closed sets and study their relationships with various types of mappings.

Definition : 2.3.1

A map $f : X \to Y$ is said to be **approximately b-continuous** (**ap-b-continuous**) if bcl(F) $\subseteq f^{-1}(U)$ whenever U is an open subset of Y and F is a gb-closed subset of X such that $F \subseteq f^{-1}(U)$.

Definition : 2.3.2

A map $f : X \to Y$ is said to be **approximately b-closed** (briefly, **ap-b-closed**) if $f(F) \subseteq$ bint(V) whenever V is a gb-open subset of Y, F is an closed subset of X and $f(F) \subseteq V$.

Definition : 2.3.3

A map $f : X \longrightarrow Y$ is said to be **approximately b-open** (**ap-b-open**) if $bcl(F) \subseteq f(U)$ whenever U is an open subset of X, F is a gb-closed subset of Y and $F \subseteq f(U)$.

Theorem : 2.3.4

Let $f : X \longrightarrow Y$ be a function. Then

(1) If f is contra b-continuous, then f is ap-b-continuous

(2) If f is contra b-closed, then f is ap-b-closed

(3) If f is contra b-open, then f is ap-b-open

Proof :

(1) Let U be an open set in Y and F be a gb-closed subset of X.

Such that $F \subseteq f^{-1}(U)$.

Since f is contra b-continuous, $bcl(F) \subseteq bcl(f^{-1}(U)) = f^{-1}(U)$.

Hence f is ap-b-continuous.

(2) Let F be an closed subset of X and V be a gb-open subset of Y.

Such that $f(F) \subseteq V$.Therefore $f(F) = bint(f(F) \subseteq bint(V)$.

Hence f is ap-b-closed.

(3) Let U be an open subset in X and F be a gb-closed subset of Y.

Such that $F \subseteq f(U)$.

Then $F = bcl(F) \subseteq f(U)$.

Hence f is ap-b-open.

Example : 2.3.5

An ap-b-continuous map which is not contra b-continuous

Let X = {a, b, c} and τ = { ϕ, X, {b}, {c}, {b, c} } then BC(X) = { ϕ, X, {a}, {b}, {c}, {a, c}, {a, b} } = gbC(X). Let $f : (X, \tau) \longrightarrow (X, \tau)$ be the identity map. Then f is ap-b-continuous. Since every gb-closed is b-closed, f is ap-b-continuous but not contra b-continuous.

Example : 2.3.6

An ap-b-closed map which is not contra b-closed

Let X = {a, b, c} with τ = { ϕ, X, {a} } and Y = {a, b, c} with σ = { ϕ, Y, {a}, {a b} } then BO(X) = { ϕ, X, {a}, {a, b}, {a, c} }, gbC(X) = { ϕ, X, {b}, {c}, {a, b}, {a, c},{b, c} }. And BO(Y) = { ϕ, Y, {a}, {a, b}, {a, c} }, gbC(Y) = { ϕ, Y, {b}, {c}, {b, c}, {a, c} }. Let f : (X, τ) $\rightarrow$ (Y, σ) be the identity map. Since the only gb-open subset of (Y, σ) containing the image of closed set V in X is Y, f is ap-b-closed but not contra b-closed.

Definition : 2.3.7 [Noiri [1979/80]]

A function f : X $\rightarrow$ Y is said to be **perfectly continuous** if the inverse image of every open set in Y is clopen in X.

Definition : 2.3.8

A function f : X $\rightarrow$ Y is said to be **perfectly closed** if the image of every closed set in X is clopen in Y .

Definition : 2.3.9

A function f : X $\rightarrow$ Y is said to be **pre-closed** if for every pre-closed subset A of X, f(A) is pre-closed in Y.

Definition : 2.3.10

A function f : X $\rightarrow$ Y is said to be **pre-open** if for every pre-open subset A of X, f(A) is pre-open in Y.

Proposition : 2.3.11

(i) Every continuous map is ap-b-continuous

(ii) pre-closed maps are ap-b-closed

(iii) pre-open maps are ap-b-open.

The converse of the statements (i), (ii) and (iii) need not be true

Example : 2.3.11

An ap-b-continuous map which is not continuous

Let X = {a, b, c} with τ = { ϕ, X, {b}, {c}, {b, c} } and Y = {a, b, c}, with σ = { ϕ, Y, {a, b} }, then BO(X) = { ϕ, X, {b}, {c}, {a, b}, {a, c}, {b,c} }. And BO(Y) = { ϕ, Y, {a}, {b}, {a, b}, {a, c}, {b,c} }. Let f : (X, τ) $\longrightarrow$ (Y, σ) be the identity map. Then f is ap-b-continuous (Since f is contra b-continuous) but not continuous.

Example : 2.3.12

An ap-b-closed map which is not pre-closed

Let X = {a, b, c} with τ = { ϕ, X, {b}, {c}, {b, c} } and Y = {a, b, c} with σ = { ϕ, Y, {a} } then BO(X) = { ϕ, X, {b}, {c}, {a, b}, {a, c},{b,c} }. And BO(Y) = { ϕ, Y, {a}, {a, b}, {a, c} }. Let f : (X, τ) $\longrightarrow$ (Y, σ) be the identity map. Since f is contra b-closed. Then f is ap-b-closed but not pre-closed map.

Clearly, the following two diagrams (i) and (ii) hold and none of its implications is reversible:

(i) perfectly continuous $\Rightarrow$ contra-b-continuous

$\Downarrow$ $\Downarrow$

Continuous $\Rightarrow$ ap-b-continuous

(ii) perfectly closed $\Rightarrow$ contra-b-closed

$\Downarrow$ $\Downarrow$

pre-closed $\Rightarrow$ ap-b-closed

Theorem : 2.3.13

Let f : (X,τ) $\longrightarrow$ (Y, σ) be a map.

(1) If the open and b-closed sets of (X,τ) coincide then f is ap-b-continuous if mand only if f is contra b-continuous.

(2) If the open and b-closed sets of (Y, σ) coincide then f is ap-b-closed if and only if f is contra-b-closed.

(3) If the open and b-closed sets of (Y, σ) coincide then f is ap-b-open if and only if f is contra-b-open.

Proof :

(1) Assume f is ap-b-continuous. Let A be an arbitrary subset of (X,τ) such that A $\subseteq$ U, where U is open in X. Then bcl(A) $\subseteq$ bcl(U) = U. Therefore all subsets of (X,τ) are gb - closed . For any open set V of (Y, σ). Then f^{-1}(V) is gb-closed in (X,τ). Since f is ap-b-continuous bcl(f^{-1} (V)) $\subseteq$ f^{-1}(V).

Therefore bcl(f^{-1}(V)) = f^{-1}(V). Thus f^{-1}(V) is b-closed in (X,τ) and f is contra b-continuous.

Conversely , assume f is contra b-continuous. Let F $\subseteq$ f^{-1}(U), where U is open in Y and F is a gb-closed subset of X. Therefore bcl(F) $\subseteq$ bcl(f^{-1}(U)). Hence f is ap-b- continuous.

(2) Let f be ap-b-closed map. All the subsets of (Y, σ) are gb-open.

Therefore for any closed subset F of (X, τ), f (F) is gb-open in Y. Since f is ap-b-closed, f (F) $\subseteq$ bint(f(F)). Therefore f (F) = bint(f(F)). Hence f (F) is b-open inY. Therefore f is contra b-closed.

Conversely, assume f is contra b-closed. Let f (F) $\subseteq$ V, where V is a gb-open subset of Y and F is a closed subset of X. Therefore f(F) $\subseteq$ bint(f (F)) $\subseteq$ bint(V). Then f is ap-b- closed.

(3) Assume f is a ap-b-open map. All the subsets of (Y, σ) are gb-closed. Let U be open subset of (X, τ) and F be gb-closed in Y such that F $\subseteq$ f (U).

Since f is ap-b-open, bcl(F) $\subseteq$ f(U) and f is contra b-open.

Conversely, assume f is a contra-b-open. All the subsets of (Y, σ) are gb-closed.

Let U be open subset of (X, τ) and F be b-closed in Y such that A $\subseteq$ U.

bcl F $\subseteq$ bcl f(U) =f(U). Therefore f(U) is b-closed. F is ap-b-open.

Theorem : 2.3.14 [Andrijevic [1996]]

If a map f : X $\longrightarrow$ Y is surjective, b-irresolute and ap-b-closed, then the inverse image of each gb-closed set in Y is gb-closed in X.

Proof :

Let A be a gb-closed subset of Y. Suppose that f^{-1}(A) $\subseteq$ U where U is an open subset of X. Then X – U $\subseteq$ f^{-1}(Y – A)

$\Rightarrow$ f (X – U) $\subseteq$ (Y – A).

Since f is ap-b-closed. f(X – U) $\subseteq$ bint(Y – A)

= Y – bcl(A)

X – U $\subseteq$ X – (f^{-1}(bcl(A)).

Hence f^{-1}(bcl(A) $\subseteq$ U.

Since f is b-irresolute, f^{-1}(bcl(A)) is b-closed.

Hence bcl(f^{-1}(A)) $\subseteq$ bcl(f^{-1}(bcl(A)))

= (f^{-1}(bcl(A)) $\subseteq$ U.

This implies that f^{-1}(A) is gb-closed in X.

Theorem : 2.3.15

If a map $f : X \longrightarrow Y$ is surjective, b-irresolute and ap-b-open then the inverse image of each gb-open set in Y is gb-open in X.

Proof :

Let A be a gb-open set of Y. To prove $f^{-1}(A)$ is gb-open in Y.

Let $F \subseteq f^{-1}(A)$ where F is an closed in X. To prove $F \subseteq bint(f^{-1}(A))$

$F \subseteq f^{-1}(A) \Rightarrow f^{-1}(Y\text{-}A) \subset X\text{-}F$ (or)

$(Y\text{-}A) \subset f(X\text{-}F)$

Since f is ap-b-open, $bcl(Y\text{-}A) \subset f(X\text{-}F)$

$Y\text{-}bintA \subset f(X\text{-}F)$

$f^{-1}(Y\text{-}bintA) \subset f^{-1}(f(X\text{-}F))$

$\subset X\text{-}F$

$X\text{-} f^{-1}(bintA) \subset X\text{-}F$

$f^{-1}(bintA)$ is b-open.

$F \subset f^{-1}(bintA)$

$\subset intb(f^{-1}(bintA))$

$\subset bint(f^{-1}(A))$

Hence the proof.

Theorem 2.3.16

If a map $f : X \longrightarrow Y$ is ap-b-continuous and b-closed, then the image of each gb-closed set in X is gb-closed in Y.

Proof :

Let F be a gb-closed subset of X. Let $f(F) \subseteq V$, where V is an open set of Y.Then $F \subseteq f^{-1}(V)$.Since f is ap-b-continuous, $bcl(F) \subseteq f^{-1}(V)$.Then $f(bcl(F)) \subseteq V$

. Therefore $bcl(f(F)) \subseteq bcl(f(bcl(F)))$

$= f(bcl(F))$

$\subseteq V$

Hence f(F) is gb-closed in Y.

Theorem : 2.3.17

If f : X $\longrightarrow$ Y is a continuous and b-closed function, then f(A) is gbclosed in Y for every gb-closed set A of X.

Proof :

Let A be gb-closed in X. Let f(A) $\subseteq$ V, where V be any open set in Y.

Since f is continuous, $f^{-1}(V)$ is open in X and A $\subseteq$ $f^{-1}(V)$.

Then bcl(A) $\subseteq$ $f^{-1}(V)$ and f(bcl(A)) $\subseteq$ V. Since f is b-closed, f(bcl(A)) is b-closed in Y. Hence bcl(f(A) $\subseteq$ bcl(f(bCl(A)))

$= f(bcl(A))$

$\subseteq V.$

Hence f(A) is gb-closed in Y.

Definition : 2.3.18

A map f : X $\longrightarrow$ Y is said to be **contra-b-irresolute** if $f^{-1}(U)$ is b-closed in X for each U$\in$BO(Y).

Theorem 2.3.19

Let f : X $\longrightarrow$ Y and g : Y $\longrightarrow$ Z be two maps such that ($g\circ f$) : X $\longrightarrow$ Z,

(1) If f is pre-closed and g is ap-b-closed then ($g\circ f$) is ap-b-closed.

(2) If f is ap-b-closed and g is b-open and g^{-1} preserves gb-open sets then ($g\circ f$) is ap-b-closed.

(3) If f is ap-b-continuous and g is continuous then ($g\circ f$) is ap-b-continuous.

Proof :

(1) Suppose B is an arbitrary closed subset in X and A is a gb-open subset of Z for which $(g\circ f)(B) \subseteq A$. Then f(B) is closed in Y because f is pre-closed. Since g is ap-b-closed, $g(f(B)) \subseteq bint(A)$. Hence $(g\circ f)$ is ap-b-closed.

(2) Suppose B is an arbitrary closed subset of X and A is a gb-open subset of Z for which $(g\circ f)(B) \subseteq A$. Hence $f(B) \subseteq g^{-1}(A)$.

Then $f(B)\ ) \subseteq bint(g^{-1}(A))$ because $g^{-1}(A)$ is gb-open and f is ap-b-closed.

Hence $(g\circ f)(B) = g(f(B)) \subseteq g(bint(g^{-1}(A))) \subseteq bint(g(g^{-1}(A))) \subseteq bint(A)$.

Hence $(g\circ f)$ is ap-b-closed.

(3) Suppose F is an arbitrary gb-closed subset of X and $U\in O(Z)$ for which $F\subseteq (g\circ f)^{-1}(U)$. Then $g^{-1}(U)\in O(Y)$ because g is continuous. Since f is ap-b-continuous, then $bcl(F)\subseteq f^{-1}(g^{-1}(U))=(g\circ f)^{-1}(U)$. Hence $(g\circ f)$ is ap-b-continuous.

Section 2.4

Characterization of T_{gs}-spaces

In this section we discuss the concept of T_{gs}-spaces and obtain characterizations of these spaces by using the concept of ap-b-continuous maps and ap-b-closed maps.

Definition : 2.4.1 [Levine [1970]]

A space X is a **$T_{1/2}$-space** if every g-closed set is closed or equivalently if every singleton is open or closed.

Definition : 2.4.2 [Ganster and Steiner [2007]]

A topological space X is said to be a **T_{gs}-space** if every gs-closed subset of (X, τ) is sg-closed.

Theorem : 2.4.3 [Ganster and Steiner [2007]]

A topological space X is a T_{gs}-space if and only if every gb-closed set is b-closed.

Theorem : 2.4.4

Every $T_{1/2}$-space is a T_{gs}-space.

Proof :

Let (X,τ) be $T_{1/2}$ and suppose $A \subseteq X$ is not a b-closed set.Let $x \in bcl(A)-A$, then $\{x\} \subseteq bcl(A) - A$. Since X is $T_{1/2}$-space, {x} is a closed set and hence A is not a gb-closed set.

Definition : 2.4.5 [Abd EL-Monsef, EL-Atik and El-Sharkasy [2005]]

A space (X,τ) is said to be a **b-$T_{1/2}$** space if every each singleton is either b-open or b-closed set.

Lemma : 2.4.6 [Abd EL-Monsef, EL-Atik and El-Sharkasy [2005]]

Let (X,τ) be a topological space.Then the space $bO(X,\tau)$ is a b-$T_{1/2}$ space.

Example : 2.4.7

A T_{gs} - space which is not a $T_{1/2}$ – space

Let X = {a, b, c} with τ = { ϕ, X, {a, b} } then bC(X) = gbC(X) = { ϕ, X, {a}, {b}, {c}, {a, c}, {b, c} }. Then X is a T_{gs}-space but not a $T_{1/2}$-space.

Example : 2.4.8

A $T_{1/2}$ - space which is not a T_{gs} – space.

Let X = {a, b, c, d} with τ= { ϕ,X, {a, b}, {b, c, d} } then

bC(X) = { ϕ, X, {a}, {c}, {d}, {a, c}, {a, d}, {c, d}, {a, c, d} }. And U = {a, b, c} is gb-closed since the only open set containing U is X. Therefore X is not a T_{gs}-space but a b-$T_{1/2}$-space.

The following diagram holds and none of its implications is reversible:

$T_{1/2}$-space $\Rightarrow$ T_{gs}-space $\Rightarrow$ b-$T_{1/2}$-space

Theorem : 2.4.9

For a topological space X, if every gb-closed subset of X is closed, then X is a $T_{1/2}$-space.

Proof :

Let $x \in X$. If {x} is not closed, then A = X – {x} is not open and hence A is gb-closed, since the only open set containing A is X. Hence A is closed and {x} is open in X and so X is a $T_{1/2}$-space.

Definition : 2.4.10

A function $f : X \longrightarrow Y$ is called **gb-continuous** if $f^{-1}(V)$ is gb-closed in X for every closed set V of Y.

Definition : 2.4.11

A function $f : X \longrightarrow Y$ is called **gb-irresolute** if $f^{-1}(V)$ is gb-closed in X for every gb-closed set V of Y.

Theorem : 2.4.12

Let $f : X \longrightarrow Y$ be a function.

(1) If f is gb-irresolute and X is a T_{gs}-space, then f is b-irresolute.
(2) If f is gb-continuous and X is a T_{gs}-space, then f is b-continuous.

Proof :

(1) Let V be b-closed in Y. Then V is gb-closed in Y and since f is gb-irresolute, then $f^{-1}(V)$ is gb-closed in X. Since X is a T_{gs}-space, $f^{-1}(V)$ is b-closed in X. Hence f is b-irresolute.
(2) Let V be closed in Y. Since f is gb-continuous, , $f^{-1}(V)$ is b-closed in X. Since X is T_{gs}-space, $f^{-1}(V)$ is b-closed in X. Hence f is b-continuous.

Theorem : 2.4.13

If the bijective function $f : X \longrightarrow Y$ is b-irresolute and open then f is gb-irresolute.

Proof :

Let V be gb-closed and let $f^{-1}(V) \subseteq U$, where U is open in X. Then $V \subseteq f(U)$, since f(U) is open and V is gb-closed in Y. Then $bcl(V) \subseteq f(U)$ and hence $f^{-1}(bcl(V)) \subseteq U$. Since f is b-irresolute and since bcl(V) is a b-closed set, $f^{-1}(bcl(V))$ is a b-closed set in X.

Hence $bcl(f^{-1}(V)) \subseteq bcl(f^{-1}(bcl(V))) = f^{-1}(bcl(V)) \subseteq U$. So $f^{-1}(V)$ is gb-closed and f is gb-irresolute.

Example : 2.4.14

A map which is b-irresolute but not gb-irresolute

Let X = {a, b, c} with τ = { ϕ, X, {a}, {a, b} } and Y = {a, b, c} with σ = { ϕ, Y, {a} } then bC(X) = { ϕ, X, {b}, {c}, {b, c} } and bC(Y) = { ϕ, Y, {b}, {c}, {b, c} }. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be the identity map. Then f is b-irresolute but not gb-irresolute, since V = {a, b} belongs to gb-closed of (Y, σ) but not gb-closed of (X,τ).

Example : 2.4.15

A map which is gb-irresolute but not b-irresolute

Let X = {a, b, c}, with τ = { ϕ,X, {a} }, then bC(X) = { ϕ, X, {b}, {c}, {b, c} }, and gbC(X)={ ϕ, X, {b}, {c}, {b, c}, {a, c}, {a, b}}. Let $f : (X, \tau) \longrightarrow (X, \tau)$ be defined by f(a) = f(c) = b, f(b) = c. Then f is gb-irresolute but not b-irresolute, since V = {b} is b-closed and $f^{-1}\{b\}$ = {a, c} is not b-closed of (X, τ).

Theorem : 2.4.16

Let $f : X \longrightarrow Y$ be an open, b-irresolute and b-closed surjective function. If X is a T_{gs}-space, then Y is T_{gs}-space.

Proof :

Let F be a gb-closed set of Y. Let G be open subset of X such that $f^{-1}(F) \subseteq G$. Then $F \subseteq f(G)$ and f(G) is open.

Since F is gb-closed, then $bcl(F) \subseteq f(G)$ and $f^{-1}(bcl(F)) \subseteq G$.

Since f is b-irresolute, $f^{-1}(bcl(F))$ is b-closed and

$bcl(f^{-1}(bcl(F))) = f^{-1}(bcl(F)) \subseteq G$ also $bcl(f^{-1}(F)) \subseteq bcl(f^{-1}(bcl(F))) \subseteq G$.

Hence $f^{-1}(F)$ is gb-closed in X. Since X is a T_{gs}-space, $f^{-1}(F)$ is b-closed in X and so F is b-closed in Y. Hence Y is a T_{gs}-space.

Theorem : 2.4.17

Let X be a topological space. Then the following statements are equivalent:

(1) X is a T_{gs}-space.

(2) For every space Y and every map $f : X \longrightarrow Y$, f is ap-b-continuous.

Proof :

(1) $\Rightarrow$ (2)

Let F be a gb-closed subset of X and suppose that $F \subseteq f^{-1}(U)$, where U is open. Since (X,τ) is $T_{gs,}$-space, F is b-closed and $bcl(F) = F \subseteq f^{-1}(U)$.

Then f is ap-b-continuous.

(2) $\Rightarrow$(1)

Let B be a gb-closed subset of X and let Y be the set X with topology $\sigma = \{ \phi, B, Y\}$. Let $f : X \longrightarrow Y$ be the identity map. By assumption f is ap-b-continuous. Since B is gb-closed in X and open in Y and $B \subseteq f^{-1}(B)$,

$bcl(B) \subseteq f^{-1}(B) = B$.

Hence B is b-closed in X and hence (X, τ) is a T_{gs}-space.

Theorem : 2.4.18

Let Y be a topological space. Then the following statements are equivalent:

(1) Y is a T_{gs}-space.

(2) For every space (X, τ) and every map f : X $\longrightarrow$ Y, f is ap-b-closed (or ap-b-open).

Proof :

Let Y be T_{gs}-space. Every gb-closed set is b-closed.

Let F be an closed subset of X.

f(F) $\subseteq$ V where V is a gb-open subset of Y.

To prove : f(F) $\subseteq$ bint(V)

Every gb-open set is b-open.

Therefore V is b-open.

V = bint(V)

Hence f(F) $\subseteq$ V.

Conversely, let B be a gb-closed subset of X and B^c be a gb-open.

V is a gb-open subset of Y, B^c is an closed subset of X.

$f(B^c) \subseteq B^c$.

$B^c = f(B^c) \subseteq bint(B^c) \subseteq B^c$.

$B^c \subseteq bint(B^c)$.

B^c is b-open.

Therefore B is b-closed. Hence the proof.

CHAPTER 3

CHAPTER - III

On πgb-closed sets in topological spaces

Chapter III deals with the concepts of πgb-closed sets. The class of πgb-closed sets is finer than that of closed, α-closed, preclosed, gb-closed, g-closed, πg-closed, πgα-closed πgs-closed and πgp-closed sets but coarser than that of πgsp-closed set. Some interesting properties of these sets are discussed. Finite union (intersection) of πgb-closed sets need not be πgb-closed. Since every closed set is b-closed, d(A) ⊂ b-d(A) for any subset A of a topological space X. Also, the union of πgb-closed sets A and B is πgb-closed provided d(A) ⊂ b-d(A) and d(B) ⊂ b-d(B) and intersection of πgb-closed set and a b-closed set is πgb-closed. The concept of πgb-$T_{1/2}$ space and πgb-spaces are introduced and discussed some of their characterizations.

The notion of πgb-continuous functions and πgb-irresolute functions are introduced and discussed the characterization and the relation among πgb-continuous and πgb-irresolute functions.

The implication relationship of πgb- continuous and other continuous functions are discussed. Also, proved that the composition of two πgb-continuous functions need not be πgb-continuous. The properties and the characterizations of almost πgb continuous functions are analysed. The concept of pre b-closed function are discussed.

The chapter is concluded by discussing the π-open map by using the concepts of πgb-continuous functions, πgb-irresolutes and pre b-closed functions.

Section : 3.1

Preliminaries

Definition : 3.1.1 [Dontchev and Noiri, [2000]]

A subset A of a space (X,τ) is called πg-closed if cl(A) ⊂ U whenever A ⊂ U and π is open.

Definition : 3.1.2

A subset A of a space (X,τ) is called πgp-closed if pcl(A) $\subset$ U whenever $A \subset U$ and π is open.

Definition : 3.1.3 [Janaki [2009]]

A subset A of a space (X,τ) is called $\pi g\alpha$-closed if αcl(A) $\subset$ U whenever $A \subset U$ and π is open.

Definition : 3.1.4 [Sarsak and Rajesh [2010]]

A subset A of a space (X,τ) is called πgsp-closed if spcl(A) $\subset$ U whenever $A \subset U$ and π is open.

Definition : 3.1.5 [Aslim [2006]]

A subset A of a space (X,τ) is called πgs-closed if scl(A) $\subset$ U whenever $A \subset U$ and π is open.

Definition : 3.1.6 [Aslim [2006]]

A function $f : (X,\tau) \rightarrow (Y, \sigma)$ is called π- irresolute if $f^{-1}(V)$ is π -closed in (X,τ) for every π -closed v of (Y, σ).

Section : 3.2

πgb-closed sets

Definition : 3.2.1

A subset A of (X, τ) is called πgb-closed if bcl(A) $\subset$ U whenever $A \subset U$ and U is π open in (X, τ).

Notation : 3.2.2

The family of all πgb-closed subsets of the space (X, τ) is denoted by πGBC(τ).

Theorem : 3.2.3

(1) Every closed set is πgb-closed.

(2) Every g-closed set is πgb-closed.

(3) Every α-closed set is πgb-closed.

(4) Every pre-closed set is πgb-closed.

(5) Every b-closed set is πgb-closed.

(6) Every gb-closed set is πgb-closed.

(7) Every πg-closed set is πgb-closed.

(8) Every πgp-closed set is πgb-closed.

(9) Every πgα-closed set is πgb-closed.

(10) Every πgs-closed set is πgb-closed.

(11) Every πgb-closed set is πgsp-closed.

Example : 3.2.4

The converse of theorem:3.2.3 (1) need not be true.

Let X = {a,b,c,d} and τ = { ϕ , {a} , {d} , {a,d} , {c,d} , {a,c,d} , X }. Let A = {c}. Then A is πgb-closed but not closed.

Example : 3.2.5

The converse of theorem:3.2.3 (2) need not be true.

Let X = {a,b,c,d} and τ = { ϕ , {a} , {d} , {a,d} , {c,d} , {a,c,d} , X }. Let A = {c}. Then A is πgb-closed but not g-closed.

Example : 3.2.6

The converse of theorem:3.2.3 (3) need not be true.

Let $X = \{a,b,c,d\}$ and $\tau = \{ \phi , \{a\} , \{d\} , \{a,d\} , \{c,d\} , \{a,c,d\} , X \}$. Let $A = \{c\}$. Then A is πgb-closed but not α-closed.

Example : 3.2.7

The converse of theorem:3.2.3 (4) need not be true.

Let $X = \{a,b,c,d\}$ and $\tau = \{ \phi , \{a\} , \{d\} , \{a,d\} , \{c,d\} , \{a,c,d\} , X \}$. Let $A = \{c\}$. Then A is πgb-closed but not pre-closed.

Example : 3.2.8

The converse of theorem:3.2.3 (5) need not be true.

Let $X = \{a,b,c,d\}$ and $\tau = \{ \phi , \{a\} , \{d\} , \{a,d\} , \{c,d\} , \{a,c,d\} , X \}$. Let $A = \{c\}$. Then A is πgb-closed but not gb-closed.

Example : 3.2.9

The converse of theorem:3.2.3 (6) need not be true.

Let $X = \{a,b,c,d\}$ and $\tau = \{ \phi , \{a\} , \{d\} , \{a,d\} , \{c,d\} , \{a,c,d\} , X \}$. Let $A = \{c\}$. Then A is πgb-closed but not πg -closed.

Example : 3.2.10

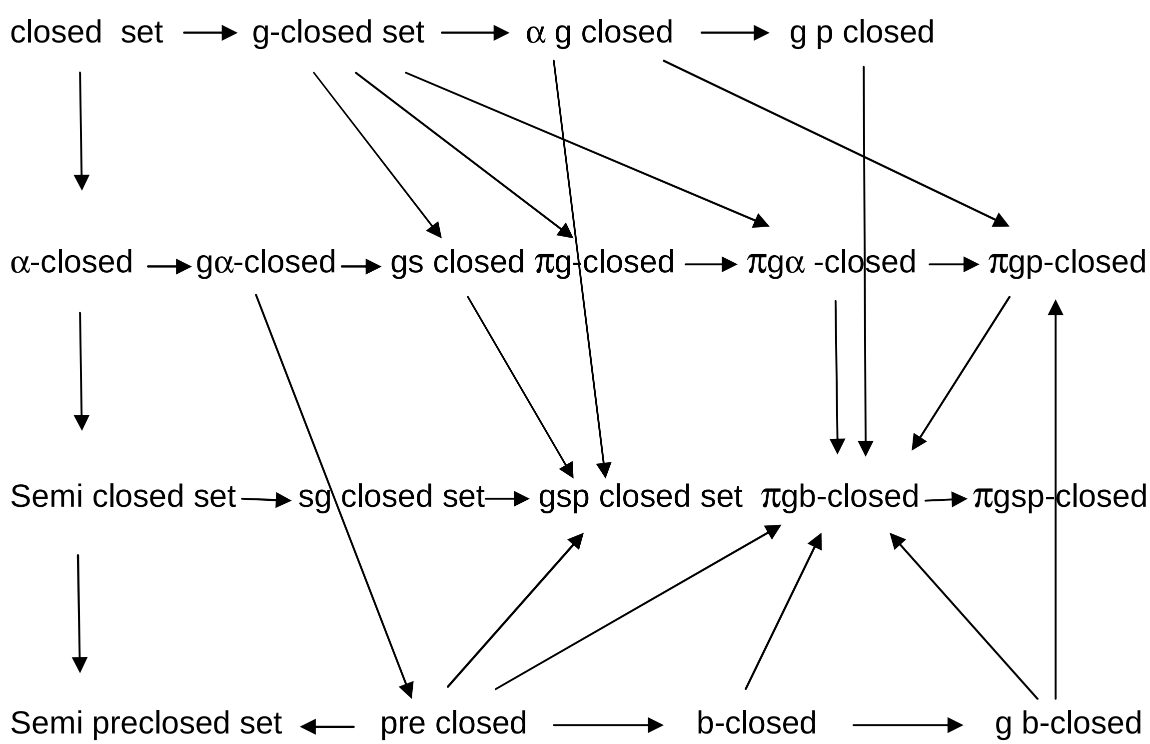
The converse of theorem:3.2.3 (7) need not be true.

Let $X = \{a,b,c,d\}$ and $\tau = \{ \phi , \{a,b\} , \{c,d\} , \{a,c,d\} , X \}$. Let $A = \{a,b\}$. Then A is πgb-closed but not πgp -closed.

Example : 3.2.11

Let $X = \{a,b,c,d,e\}$ and $\tau = \{ \phi , \{a,b\} , \{c,d\} , \{a,b,c,d\} , X \}$. Let $A = \{a\}$. Then A is πgb-closed but not πgα -closed.

Remark : 3.2.12



Theorem : 3.2.13

If A is π-open and πgb-closed then A is b-closed.

Proof :

Let A be π -open and πgb-closed. Since A is π -open and πgb-closed, bcl(A) ⊂ Aand so A = bcl(A). Hence A is b-closed.

Theorem : 3.2.14

Let A be πgb-closed in (X, τ). Then bcl(A)-A does not contain any non empty π-closed set.

Proof :

Let F be a non empty π-closed set such that F ⊂ bcl(A) -A. Then

A⊂ X - F and X - F is π -open Since A is πgb-closed bcl(A) ⊂ X -F

Hence $F \subset bcl(A) \cap (X - bcl(A))$

Implies that $F = \phi$. Which is a contradiction. Therefore bcl(A) – A does not contain any non empty π-closed set.

Theorem : 3.2.15

Let A be πgb-closed in (X, τ). Then A is b-closed if and only if bcl(A) - A is π-closed.

Proof :

Let A be b-closed. Then bcl(A) = A. This implies bcl(A) - A = ϕ which is π-closed. Assume bcl(A) – A is π-closed. Then bcl(A) – A = ϕ. Hence bcl(A) = A. Hence A is b-closed.

Remark : 3.2.16

Finite union of πgb-closed sets need not be πgb-closed.

Examble : 3.2.17

Consider X = {a,b,c} , τ = { ϕ , {a} , {b} ,{a,b} , X }. Let A = {a}, B = {b}. A and B are πgb-closed sets but $A \cup B$ = {a,b} is not πgb-closed .

Remark : 3.2.18

Finite intersection of πgb-closed sets need not be πgb-closed.

Examble : 3.2.19

Consider X = {a,b,c,d} , τ = { ϕ , {a} , {b} ,{a,b} , {a,b,c} ,{a,b,d} ,X }. Let A = {a,b,c}, B = {a,b,d}. A and B are πgb-closed sets but $A \cap B$ = {a,b} is not πgb-closed.

Definition : 3.2.20 [Al-Omari and Noorani [2009]]

Let (X, τ) be topological space and $A \subset X$. A point $x \in X$ is said to be a b-limit point of A if and only if every b-open set containing x contains a point of A different from x.

Theorem : 3.2.21

Let A and B be πgb-closed sets in (X, τ) such that $d(A) \subset b\text{-}d(A)$ and $d(B) \subset b\text{-}d(B)$. Then $A \cup B$ is πgb-closed.

Proof :

Let U be a π-open set such that $A \cup B \subset U$. Since A and B are πgb-closed sets, $bcl(A) \subset U$ and $bcl(B) \subset U$.

Since $d(A) \subset b\text{-}d(A)$ and $d(B) \subset b\text{-}d(B)$

$cl(A) = bcl(A)$ and $cl(B) = bcl(B)$.

Hence $bcl\,(A \cup B) \subset cl(A \cup B) = cl(A) \cup cl(B)$

$= bcl(A) \cup bcl(B) \subset U$

Hence $A \cup B$ is πgb-closed.

Theorem : 3.2.22

If A is a πgb-closed set and B is any set such that $A \subset B \subset bcl(A)$, then B is a πgb-closed set.

Proof :

Let $B \subset U$ and U be π-open. Given $A \subset B$. Then $A \subset U$.

Since A is πgb-closed, $bcl(A) \subset U$

$bcl(B) \subset bcl(A) \subset U$.

Hence B is a πgb-closed set.

Section :3.3

πgb-open sets

Definition : 3.3.1

A set $A \subset X$ is called πgb-open if its complement is πgb-closed.

The family of all πgb-open subsets of the space (X, τ) is denoted by $\pi GBO(\tau)$.

Definition : 3.3.2

A set $A \subset X$ is πgb-open if and only if $F \subset bint(A)$ whenever F is π-closed and $F \subset A$.

Proof :

Assume that $A \subset X$ is πgb-open. Let F be π-closed such that $F \subset A$. Then $X - A \subset X - F$. Since $X - A$ is πgb-closed and $X - F$ is π-open, $bcl(X - A) \subset X - F$.

That is $X - bint(A) \subset X - F$. Hence $F \subset bint(A)$.

Conversely, assume that F is π-closed and $F \subset A$ such that $F \subset bint(A)$.

Let $X - A \subset U$, where U is π-open. Then $X - U \subset A$ and since $X - U$ is π-closed, $X - U \subset bint(A)$

(i.e) $bcl(X-A) \subset U$.

Hence $X - A$ is πgb-closed and A is πgb-open.

Theorem : 3.3.3

If $bint(A) \subset B \subset A$ and A is πgb-open, then B is πgb-open.

Proof :

Let $bint(A) \subset B \subset A$. Then $X - A \subset X–B \subset bcl(X–A)$. Since $X–A$ is πgb-closed. $X - A \subset X–B \subset bcl(X–A)$

Hence $(X–B)$ is πgb-closed (by Theorem : 3.2.22)

Remark : 3.3.4

For any $A \subset X$, $bint(bcl(A)-A) = \phi$.

Theorem : 3.3.5

If $A \subset X$ is πgb-closed then $bcl(A) - A$ is πgb-open.

Proof :

Let A be πgb-closed and F be π-closed set

Such that $F \subset bcl(A) –A$ by the Remark : 3.3.4 $bint(bcl(A) - A) = \phi$

$F = \phi$. Hence $F \subset bint(bcl(A) –A)$. Hence $bcl(A) –A$ is πgb-open.

Lemma : 3.3.6

Let $A \subset X$. If A is open or dense, then $\pi O(A , \tau_A) = \{ V \cap A$ Such that $V \in \pi O(X , \tau) \}$.

Theorem : 3.3.7

Let $B \subset A \subset X$ where A is πgb-closed and π-open, then B is πgb-closed relative to A if and only if B is πgb-closed in X.

Proof :

Let $B \subset A \subset X$ where A πgb-closed and π-open set. Let B be πgb-closed in A. Let $B \subset U$ where U is π-open in X. Since $B \subset A$,

$B = B \cap A \subset U \cap A$, therefore $bcl(B) = bcl_A(B) \subset U \cap A \subset U$

Hence B is πgb-closed in X.

Conversely, let B be πgb-closed in X. Let $B \subset O$ where O is π-open in A.

Then $O = U \cap A$ where U is π-open in X. Therefore $B \subset O = U \cap A \subset U$.

Since B is πgb-closed in X, $bcl(B) \subset U$.

Hence $bcl_A(B) = A \cap bcl(B) \subset U \cap A = O$.

Hence B is πgb-closed relative to A.

Hence the proof.

Theorem : 3.3.8

Let A be a π-open and πgb-closed set.

Then $A \cap F$ is πgb-closed whenever $F \in bc(X)$.

Proof :

Since A is πgb-closed and π-open, $bcl(A) \subset A$ and hence A is b-closed. Hence $A \cap F$ is b-closed in X. Therefore $A \cap F$ is πgb-closed in X.

Definition : 3.3.9

A space (X , τ) is called a πgb-$T_{1/2}$ space if every πgb-closed set is b-closed.

Theorem : 3.3.10

(1) $BO(\tau) \subset \pi GBO(\tau)$

(2) A space (X , τ) is πgb-$T_{1/2}$ if and only if $BO(\tau) = \pi GBO(\tau)$.

Proof :

(1) Let A be b-open, then $X – A$ is b-closed so $X – A$ is πgb-closed. Hence A is πgb-open. Hence $BO(\tau) \subset \pi GBO(\tau)$.

(2) Let (X , τ) be πgb-$T_{1/2}$ space. Let $A \in \pi GBO(\tau)$.

Then $X – A$ is πgb-closed and hence $X – A$ is b-closed.

Hence $A \in BO(\tau)$. Hence $\pi GBO(\tau) = BO(\tau)$.

Conversely,

Let $BO(\tau) = \pi GBO(\tau)$. Let A be πgb-closed. Then $X - A$ is πgb-open and $X - A \in \pi GBO(\tau)$.

$X - A \in BO(\tau)$.

Hence A is b-closed.

Therefore (X, τ) is πgb-$T_{1/2}$ space.

Proposition : 3.3.11

Let A be a subset of (X, τ) and $x \in X$. Then $x \in bcl(A)$ if and only if $V \cap A \neq \phi$ for every b-open set V containing x.

Theorem : 3.3.12

For a topological space (X, τ) the following are equivalent.

(1) X is πgb-$T_{1/2}$ space.

(2) Every singleton set is either π-closed or b-open.

Proof :

(1) $\Rightarrow$ (2)

Let X be πgb-$T_{1/2}$ space. Let $x \in X$ and assume that $\{x\}$ is not π-closed. Then $X - \{x\}$ is not π-open. Hence $X - \{x\}$ is trivially a πgb-closed. Since X is πgb-$T_{1/2}$ space, $X - \{x\}$ is b-closed. Therefore $\{x\}$ is b-open. Hence (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1)

Assume every singleton of X is either π-closed or b-open.

Let A be a πgb-closed set. Let $\{x\} \in bcl(A)$.

Case : (1)

Let $\{x\}$ be π-closed and assume that $\{x\} \notin A$.

Since $\{x\} \in bcl(A)-A$. We get a contradiction to Theorem : 3.2.14

Therefore $\{x\} \in A$.

Hence $bcl(A) \subset A$.

Case : (2)

Let $\{x\}$ be b-open.

Since $\{x\} \in bcl(A)$

$\{x\} \cap A \neq \phi$ Implies $\{x\} \in A$.

Therefore $bcl(A) \subset A$.

Hence A is b-closed.

Section : 3.4

πgb-continuous and πgb-irresolute functions

Definition : 3.4.1

A function $f : (X, \tau) \to (Y, \sigma)$ is called πgb- continuous if every $f^{-1}(V)$ is πgb-closed in (X, τ) for every closed set V of (Y, σ).

Definition : 3.4.2 [Ekici and Baker [2007]]

A function $f : X \to Y$ is said to be m-π-closed if f(V) is π-closed in Y for every π-closed set V of X.

Definition : 3.4.3

A function $f : (X, \tau) \to (Y, \sigma)$ is called πgb- irresolute if $f^{-1}(V)$ is πgb-closed in (X, τ) for every πgb-closed set V in (Y, σ).

Proposition : 3.4.4

Every πgb- irresolute function is πgb- continuous.

Example : 3.4.5

The converse need not be true.

Consider X = {a, b, c}, τ = { ϕ, X, {a}, {b}, {a, b} }, σ = { ϕ, X, {a} }.

Let f : (X, τ) $\longrightarrow$ (Y, σ) be the identity map.

Then f is πgb-continuous but not πgb-irresolute.

Remark : 3.4.6

Composition of two πgb- continuous functions need not be πgb-continuous.

Example : 3.4.7

Let X = {a, b, c,d}, τ = { ϕ, X, {b},{c}, {b, c} }, σ = { ϕ, X, {a, b, d} }.

= { ϕ, X, {a, d} }.Define f : (X, τ) $\longrightarrow$ (X, σ) by f(a) = a, f(b) = c, f(c) = b, f(d)= d.

Define g : (X, σ) $\longrightarrow$ (X,) by g(a) = d, g(b) = c, g(c) = b, g(d) = a.

Then f and g are πgb-continuous but (g $\circ$ f) is not πgb-continuous.

Proposition : 3.4.8

(a) Every πgb-continuous function need not be gb-continuous or πg-continuous,

(b) Every πgb-continuous function need not be πgp-continuous,

(c) Every πgb-continuous function need not be πgs-continuous.

Example : 3.4.9

Let X = {a, b, c, d, e}, τ = { ϕ, X, {a},{c, d}, {a, c, d},{b, c, d, e} },and

Y = {x, y, z, t}, σ = { ϕ, Y, {x, y, z},{t} }. Define a function f: X $\longrightarrow$ Y as follows:

f(a) = z, f(b) = f(e) = t, f(c) = y and f(d) = x. Then f is πgb-continuous but it is not gb-continuous.

Example : 3.4.10

Let X = {a, b, c, d}, τ = { ϕ, X, {a},{b, c}, {a, b, c},{a, b}, {b} },and Y = {x, y, z}, σ = { ϕ, Y, {x, y}, {x, z},{x} }. Define a function f: X $\longrightarrow$ Y as follows: f(a) = y, f(b) = f(d) = x, f(c) = z. Then f is a πgb-continuous function which is neither πg-continuous nor πgp-continuous.

Example : 3.4.11

Let X be the real numbers with the ususal topology and Y = {0,1} with the

topology σ = { ϕ, Y, {1}}. We define the function f: X $\longrightarrow$ Y such as

$$f(x) = \begin{cases} 0, & x \in (0,2) \setminus Q \\ 1, & x \notin (0,2) \setminus Q \end{cases}$$

Then f is πgb-continuous but it is not πgs-continuous.

Theorem : 3.4.12

Let f: X $\longrightarrow$ Y be a function. Then the following statements are equivalent:

(1) f is πgb-continuous;

(2) The inverse image of every open set in Y is πgb-open in X.

Proof :

(1) $\Rightarrow$ (2)

Let U be a open subset of X. Then Y – U is closed.

Since f is πgb-continuous, f^{-1}(Y-U) = X – f^{-1}(U) is πgb-closed in X. Hence f^{-1}(U) is πgb-open in X.

(2) $\Rightarrow$ (1)

Let V be a closed subset of Y. Then Y – V is open and by hypothesis (2) f^{-1}(Y-V) = X – f^{-1}(V) is πgb-open in X.

Hence $f^{-1}(V)$ is πgb-closed. Therefore, f is πgb-continuous.

Theorem : 3.4.13

If $f: X \longrightarrow Y$ is πgb-continuous then $f(\pi gb\text{-}cl(A)) \subset cl(f(A))$ for every Subset A of X.

Proof :

Let A be a subset of X. Since f is πgb-continuous and $A \subset f^{-1}(cl(f(A)))$, we obtain $\pi gb\text{-}cl(A) \subset f^{-1}(cl(f(A))$ and then $f(\pi gb\text{-}cl(A)) \subset cl(f(A))$.

Example : 3.4.14

The converse need not be true

Let $X = \{a, b, c, d\}$, $\tau = \{ \phi, X, \{a, b\},\{d\}, \{a, b, d\} \}$ and $\sigma = \{ \phi, X, \{d\} \}$.

Define the function $f: (X, \tau) \longrightarrow (X, \sigma)$ such as $f(a) = c$, $f(b) = a$, $f(c) =d$, $f(d)=b$. Then $f(\pi gb\text{-}cl(A)) \subset cl(f(A))$ for every Subset A of X. Since $\{a, b, c\}$ is closed in (X, σ) but $f^{-1}(\{a, b, c\}) = \{a, b, d\}$ is not πgb-closed in (X, τ), f is not πgb-continuous.

Theorem : 3.4.15

Let $f: X \longrightarrow Y$ be a function. Then the following statements are equivalent:

(1) For each $x \in X$ and each open set V containing f(X) there exists a πgb-open set U containing x such that $f(U) \subset V$.

(2) $f(\pi gb\text{-}cl(A)) \subset cl(f(A))$ for every Subset A of X.

Proof :

(1) $\Rightarrow$ (2)

Let $y \in f(\pi gb\text{-}cl(A))$ and let V be any open neighborhood of y.

Then there exists a $x \in X$ and a πgb -open set U such that $f(x) = y$, $x \in U$, $x \in \pi$gb-cl(A) and $f(U) \subset V$. By lemma : 3.2.18, $U \cap A \neq \phi$ and hence $f(A) \cap V \neq \phi$.

Therefore, $y = f(x) \in$ cl(f(A)). Therefore f(πgbcl A) $\subset$ cl f(A).

(2) $\Rightarrow$ (1)

Let $x \in X$ and V be any open set containing f(x). Let $A = f^{-1}(Y\text{-}V)$. Since f(πgb-cl(A)) $\subset$ cl(f(A)) $\subset$ Y – V, πgbcl A $\subset f^{-1}(Y\text{-}U) = A$. Hence πgb-cl(A) = A.

Since $x \notin \pi$gb-cl(A) there exists a πgb -open set U containing x such that $U \cap A = \phi$ and hence $f(U) \subset f(X\text{-}A) \subset V$.

Theorem : 3.4.16

Let X be an extremely disconnected space and $f: X \to Y$ be a function. If f is πgb-continuous and m-π-closed then f is πgb-irresolute.

Proof :

Let A be a πgb-closed subset of Y. Let $f^{-1}(A) \subset U$, where U is π-open in X. Then $X – U \subset f^{-1}(Y\text{-}A)$.

Hence $f(X – U) \subset (Y\text{-}A)$.Since f is m-$\pi$-closed, f(X-U) is π-closed.

Since Y – A is πgb-open, f(X-U) $\subset$ bint(Y-A) = Y-bcl(A).

Hence f^{-1}(bcl(A)) $\subset$ U. Since f is πgb-continuous and X is extremely disconnected, f^{-1}(cl(A)) is πgb-closed.

Therefore bcl(f^{-1}(bcl(A))) $\subset$ U and hence

bcl(f^{-1}(A)) $\subset$ bcl(f^{-1}(bcl(A))) $\subset$ U.

Hence $f^{-1}(A)$ is πgb-closed. Hence f is πgb-irresolute.

Definition : 3.4.17 [Keskin and Noiri [2007]]

A function f: $X \longrightarrow Y$ is said to be almost b-continuous if $f^{-1}(V)$ is b-closed in X for every regular closed set V of Y.

Definition : 3.4.18

A function f: $X \longrightarrow Y$ is said to be almost πgb-continuous if $f^{-1}(V)$ is πgb-closed in X for every regular closed set V of Y.

Theorem : 3.4.19

For a function f: $X \longrightarrow Y$, the following statements are equivalent:

(1) f is almost πgb-continuous;

(2) $f^{-1}(V)$ is πgb-open in X for every regular open set V of Y;

(3) $f^{-1}(int(cl(V)))$ is πgb-open in X for every open set V of Y;

(4) $f^{-1}(cl(int(V)))$ is πgb-closed in X for every closed set V of Y.

Proof :

(1) $\Rightarrow$ (2)

Let V be a regular open subset of Y.

Since Y – V is regular closed and f is almost πgb-continuous

$f^{-1}(Y-V) = X - f^{-1}(V)$ is πgb-closed in X.

Hence $f^{-1}(V)$ is πgb-open in X.

(2) $\Rightarrow$ (1)

Let V be a regular closed subset of Y.

Then Y – V is regular open.

By the hypothesis, $f^{-1}(Y-V) = X - f^{-1}(V)$ is πgb-open in X.

Hence $f^{-1}(V)$ is πgb-closed. Hence f is πgb-continuous.

(2) $\Rightarrow$ (3)

Let V be an open subset of Y. Then int(cl(V)) is regular open.

By the hypothesis, $f^{-1}(int(cl(V)))$ is πgb-open in X.

(3) $\Rightarrow$ (2)

Let V be a regular open subset of Y. Since V = int(cl(V)) and every regular open set is open then $f^{-1}(V)$ is πgb-open in X.

(3) $\Rightarrow$ (4)

Let V be a closed subset of Y. Then Y – V is open. By the hypothesis $f^{-1}(int(cl(Y-V))) = f^{-1}(Y-cl(int(V))) = X - f^{-1}(cl(int(V)))$ is πgb-open in X. Therefore $f^{-1}(cl(int(V)))$ is πgb-closed in X.

(4) $\Rightarrow$ (3)

Let V be a open subset of Y. Then Y – V is closed. By the hypothesis $f^{-1}(cl(int(Y-V))) = Y - intl(cl(V)) = X - f^{-1}(int(cl(V)))$ is πgb-closed in X. Therefore $f^{-1}(int(cl(V)))$ is πgb-open in X.

Remark : 3.4.19

(a) Every πgb-continuous function is almost πgb-continuous.

(b) Every almost b-continuous function is almost πgb-continuous.

Example : 3.4.20

Let X = {a, b, c, d}, τ = { ϕ, X, {a, b},{d}, {a, b, d} } and σ = { ϕ, X, {d} }. Define the function f: (X, τ) $\longrightarrow$ (X, σ) such as f(a) = c, f(b) = a, f(c) =d, f(d)=b. f is almost πgb-continuous. For the regular closed set {a, b, c} of (X, σ), f^{-1}({a, b, c}) = {a, b, d} is not πgb-closed in (X, τ), and hence f is not πgb-continuous.

Example : 3.4.21

Let X = {a, b, c, d, e}, τ = { ϕ, X, {a},{c, d}, {a, c, d},{b, c, d, e} }, and Y = {x, y, z, t}, σ = { ϕ, Y, {x, y, z},{t} }. Define a function f: X $\longrightarrow$ Y as follows:

$f(a) = z$, $f(b) = f(e) = t$, $f(c) = y$ and $f(d) = x$. f is almost πgb-continuous. For the regular closed set $\{x, y, z\}$ of (Y, σ), we have $f^{-1}(\{x, y, z\}) = \{a, c, d\}$ is not b-closed in (X, τ) and hence f is not almost b-continuous.

Definition : 3.4.22 [Sreeja and Janaki [2011]]

A topological space X is said to be πgb-$T_{1/2}$ space if every πgb-closed set is b-closed.

Theorem : 3.4.23

Let X be a πgb-$T_{1/2}$ topological space. Then $f: X \longrightarrow Y$ is almost πgb-continuous if and only if f is almost b-continuous.

Proof :

Necessity

Let A be a regular closed subset of Y and $f: X \longrightarrow Y$ be an almost πgb-continuous function.

Then $f^{-1}(A)$ is πgb-closed in X. Since X is πgb-$T_{1/2}$ space, $f^{-1}(A)$ is b-closed in X. Hence f is almost b-continuous.

Sufficiency

Suppose that f is almost b-continuous and A be a regular closed subset of Y. Then $f^{-1}(A)$ is b-closed in X.

Since every b-closed set is πgb-closed then $f^{-1}(A)$ is πgb-closed.

Therefore, f is almost πgb-continuous.

Theorem : 3.4.24

Let X be a πgb-$T_{1/2}$ space and $f: X \longrightarrow Y$ be a function. Then

(1) f is almost πgb-continuous if and only if f is almost b-continuous,

(2) f is πgb-continuous if and only if f is b-continuous.(or gb-continuous).

Definition : 3.4.25

A function f: $X \to Y$ is said to be pre b-closed if f(U) is b-closed in Y for each b-closed set U in X.

Theorem : 3.4.26

Let f : $(X, \tau) \to (Y, \sigma)$ be π-irresolute and pre b-closed map. Then f(A) is πgb-closed in Y for every πgb-closed set A of X.

Proof :

Let A be πgb-closed in X. Let $f(A) \subset V$ where V is π-open in Y. Then $A \subset f^{-1}(V)$ and A is πgb-closed in X.

Implies $bcl(A) \subset f^{-1}(V)$.

Hence $f(bcl(A)) \subset V$.

Since f is pre b-closed, $bcl(f(A)) \subset bcl(f(bcl(A))) = f(bcl(A)) \subset V$.

Hence f(A) is πgb-closed in Y.

Definition : 3.4.27

A topological space X is a πgb-space if every πgb-closed set is closed.

Propoistion : 3.4.28

Every πgb space is πgb-$T_{1/2}$ space.

Theorem :3.4.29

(1) If f is πgb-irresolute and X is πgb-$T_{1/2}$ space, then f is b-irresolute.

(2) If is πgb-continuous and X is πgb-$T_{1/2}$ space, then f is b-continuous.

Proof :

(1) Let V be b-closed in Y. Since f is πgb-irresolute, $f^{-1}(V)$ is πgb-closed in X. Since X is πgb-$T_{1/2}$ space, $f^{-1}(V)$ is b-closed in X. Hence f is b-

irresolute.

(2) Let V be closed in Y. Since f is πgb-continuous, $f^{-1}(V)$ is πgb-closed in X. By assumption, it is b-closed. Therefore f is b-continuous.

Definition : 3.4.30 Janaki [2009]

A function $f: (X, \tau) \to (Y, \sigma)$ is π-open map if f(F) π-open in Y for every π-open set in X.

Theorem : 3.4.31

If the bijective function $f : (X, \tau) \to (Y, \sigma)$ is a b-irresolute and π-open map, then f is πgb-irresolute.

Proof :

Let V be πgb-closed in Y. Let $f^{-1}(V) \subset U$ where U is π-open in X.

Then $V \subset f(U)$ and f(U) is π-open implies $bcl(V) \subset f(U)$.

Since f is b-irresolute, $(f^{-1}(bcl(V)))$ is b-closed.

Hence $bcl(f^{-1}(V)) \subset bcl(f^{-1}(bcl(V))) = f^{-1}(bcl(V)) \subset U$.

Therefore, f is πgb-irresolute.

Theorem : 3.4.32

If $f: X \to Y$ is π-open, b-irresolute, pre b-closed surjective function.

If X is πgb-$T_{1/2}$ space, then Y is πgb-$T_{1/2}$ space.

Proof :

Let F be a πgb-closed set in Y. Let $f^{-1}(F) \subset U$ where U is π-open in X. Then $F \subset f(U)$ and F is a πgb-closed set in Y implies $bcl(F) \subset f(U)$.

Since f is b-irresolute, $bcl(f^{-1}(F)) \subset bcl(f^{-1}(bcl(F))) = f^{-1}(bcl(F)) \subset U$.

Therefore $f^{-1}(F)$ πgb-closed in X. Since X is πgb-$T_{1/2}$ space,

$f^{-1}(F)$ b-closed in X. Since f is pre b-closed, $f(f^{-1}(F)) = F$ is b-closed in Y.

Hence Y is πgb-$T_{1/2}$ space.

SUMMARY AND CONCLUSION

SUMMARY AND CONCLUSION

In this dissertation we have concentrated our study on

(i) b-open sets

(ii) generalized b-closed sets

(iii) πgb-closed sets

(iv) πgb-continuous functions

Chapter I deals with the concepts of b-open sets. The class of b-open sets is finer than that of pre open sets and semi open sets but coarser than that of semi-pre open sets.The union of any family of b-open sets is b-open. Regarding intersection, it is shown that the intersection of an open set and a b-open set is b-open and also the intersection of an α-set and a b-open set is b-open. Moreover, some interesting results on the topology generated by b-open sets are discussed.

Chapter II deals with the concepts of gb-closed sets. The class of gb-closed sets is finer than that of closed, α-closed, preclosed, b-closed, g-closed, αg-closed and gp-closed sets. Some interesting characterizations of these sets are discussed. A necessary and sufficient condition for a gb-closed set to be b-closed is obtained. Union of gb-closed sets A and B is gb-closed provided d(A) $\subset$ b-d(A) and d(B) $\subset$ b-d(B) and intersection of an open gb-closed set and a b-closed set is gb-closed. It is shown that b-closure of a gb-closed set is a gb-closed set. A characterization of extremally disconnected spaces in terms of gb-closed sets is obtained. The notion of ap-b-continuous maps and ap-b-closed maps are introduced and the relation among continuous, perfectly continuous, contra-b-continuous and ap-b-continuous maps and also among pre-closed, perfectly closed, contra-b-closed and ap-b-closed maps are discussed. It is proved that gb-closedness is preserved under an ap-b-continuous and b-closed map and also under a continuous and b-closed map. Also, it is proved that the composition of

(a) b-continuous and contra-b-irresolute maps is contra continuous

(b) b-irresolute and contra-b-irresolute maps is contra-b-irresolute

(c) ap-b-closed and pre closed maps is ap-b-closed

(d) continuous and ap-b-continuous maps is ap-b-continuous.

The chapter is concluded by discussing characterizations of Tgs-spaces by using the concepts of ap-b-continuous maps and ap-b-closed maps.

Chapter III deals with the concepts of πgb-closed sets. The class of πgb-closed sets is finer than that of closed, α-closed, preclosed, gb-closed, g-closed, πg-closed, πgα-closed πgs-closed and πgp-closed sets but coarser than that of πgsp-closed set. Some interesting properties of these sets are discussed. Finite union (intersection) of πgb-closed sets need not be πgb-closed. Since every closed set is b-closed, d(A) $\subset$ b-d(A) for any subset A of a topological space X. Also, the union of πgb-closed sets A and B is πgb-closed provided d(A) $\subset$ b-d(A) and d(B) $\subset$ b-d(B) and intersection of πgb-closed set and a b-closed set is πgb-closed. The concept of πgb-$T_{1/2}$ space and πgb-spaces are introduced and discussed some of their characterizations. The notion of πgb-continuous functions and πgb-irresolute functions and the characterization, the relation among πgb-continuous and πgb-irresolute functions are discussed.

The implication relationship of πgb-continuous and other continuous functions and also the composition of two πgb-continuous functions are discussed. The properties and the characterizations of almost πgb continuous functions and pre b-closed functions are analysed. The chapter is concluded by discussing the π-open map by using the concepts of πgb-continuous functions, πgb-irresolutes and pre b-closed functions.

These concepts can be extended to bitopological spaces and fuzzy topological spaces.

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