

**SOME INTERESTING RESULTS
ON H-CLOSED SPACES AND
GENERALISED H-CLOSED SPACES**

By

G. RAMA

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR
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INTRODUCTION

INTRODUCTION

The class of H-closed (absolutely closed) spaces was first introduced by Alexandroff and Urysohn [3]. A Hausdorff space X is said to be H-closed or absolutely closed if X is closed in every Hausdorff space in which it is imbedded. Alexandroff [1] has observed that this definition is equivalent to following definition :

“ A Hausdorff space X is H-closed or absolutely closed iff every open cover $\{U_\alpha / \alpha \in \Lambda\}$ of X has a finite subfamily $\{U_{\alpha_i} / i = 1, 2, \dots, n\}$ such that the union of their closures cover X ”.

Thus we note that H-closedness or absolute closedness is a slight weakening of compactness.

Many eminent authors, namely Bourbaki [5], Alexandroff and Hopf [2], Stone[16], Katetov [9], Velicko [19], Chen-Tung Liu [6] et al. have contributed to the study of H-closed spaces. A study of generalisations of H-closed spaces has also been carried out by many authors. Thompson [17] has generalised H-closed spaces using semi-open sets in the name of S-closed spaces. Somasundaram and Padmavathy[15] have generalised H-closed spaces using preopen sets and called such spaces as pre T_2 -closed spaces.

The main objective of this thesis is to discuss some interesting results on H-closed spaces and their generalisations. The following two articles are discussed in detail.

- (1) “Absolutely closed spaces” by Chen-Tung Liu [6]
- (2) “On generalisations of H-closed spaces” by Somasundaram and Padmavathy [15]

In Chapter I, we give the preliminary definitions and results needed for discussion.

In Chapter II, we discuss the results on absolutely closed spaces, obtained by Chen-Tung Liu [6]. He [6] has extended the definition of absolute closedness to non-Hausdorff spaces. He calls a space X to be generalised absolutely closed, if X satisfies the property that every open filter base on X has a cluster point. He [6] has established an extension analogous to Stone-Čech compactification. For this purpose he has introduced the concepts, absolute closure and largest absolute closure which are defined as follows :

If X is dense in Y and Y is absolutely closed then Y is called an absolute closure of X .

An absolute closure Y is said to be a largest absolute closure of X if for any other absolute closure T of X , and if $i : X \rightarrow T$ is the injection, then there exists $\tilde{i} : Y \rightarrow T$ (onto) such that $\tilde{i} / X = i$.

With these definitions he has obtained the following results :

“For any Hausdorff space X , there exists a Hausdorff space kX which is a largest absolute closure of X . Moreover, kX is essentially unique in the sense that if T is another largest absolute closure of X , then T is essentially the same as kX ”.

He [6] has extended the above result to non-Hausdorff spaces and has shown that if X is any topological space, there exist kX which is a largest generalised absolute closure of X and which is Hausdorff except for X . (If X is a subspace of a topological space we say X is Hausdorff space except for X if for every pair of distinct points a, b in Y which are not both in X , a, b have disjoint neighbourhoods in Y .)

The author [6] has obtained a counterexample to show that the absolute closure kX does not satisfy the characteristic property of Stone-cech compactification. However, the author is able to prove the following result :

“Let X, Y be topological spaces and suppose Y is generalised absolutely closed. Let $f : X \rightarrow Y$ be a continuous, open map, then there exists $\tilde{f} : kX \rightarrow Y$ such that $\tilde{f}/X = f$ ”.

Another important problem discussed by the author is an imbedding of a topological space as a closed subspace of an absolutely closed space.

In the concluding section of this chapter, the author discusses the category of absolutely closed spaces and continuous maps. In this category, he has obtained the following characterisations of projective and injective objects.

- (1) X is projective iff X is finite.
- (2) X is injective iff X is a singleton set.

Chapter III is devoted to the study of generalisations of H-closed spaces. Here we discuss the results obtained by Somasundaram and Padmavathy [15]. The authors [15] have generalised H-closed spaces by using preopen sets and called such spaces as pre T_2 -closed spaces.

If (X, τ) is a topological space and if $A \subset X$, then

- (i) A is said to be pre T_2 -closed relative to X iff every preopen cover \mathcal{U} of A has a finite subfamily $\mathcal{U}' \subset \mathcal{U}$ such that $A \subset \cup \{cl U / U \in \mathcal{U}'\}$.

(ii) A is said to be a pre T_2 -closed set iff $(A, \tau/A)$ is pre T_2 -closed.

Using these definitions, we have discussed the following basic properties of pre T_2 -closed spaces.

(1) If $A \subset X$ is pre T_2 -closed relative to X , then A is closed in X , if X is Hausdorff space.

(2) If A is an open set, then A is pre T_2 -closed relative to X iff A is a pre T_2 -closed set.

(3) X is pre T_2 -closed

\Leftrightarrow the closure of an open subset of X is pre T_2 -closed relative to X

\Leftrightarrow Every closed - open subset of X is a pre T_2 -closed set.

(4) $A \subset X$, is pre T_2 -closed iff every regular open cover of A has a finite subcover whose closures cover A .

(5) If $B \subset A \subset X$ and A is open in X , then B is pre T_2 -closed relative to X iff B is pre T_2 -closed relative to A .

(6) Finite union of sets which are pre T_2 -closed relative to X is pre T_2 -closed relative to X .

(7) If X is a pre T_2 -closed space and Y a compact space, then $X \times Y$ is pre T_2 -closed.

(8) Continuous image of a pre T_2 -closed space is pre T_2 -closed.

The authors [15] have also defined locally pre T_2 -closed spaces and obtained some interesting properties of such spaces. Existence of a projective maximum and a projective minimum in the class of one-point extensions of locally pre T_2 -closed Hausdorff spaces is established.

A study of p -regular closed spaces is made in section 2 of chapter III and some interesting characterisations of such spaces are discussed.

The chapter is concluded by proving that in the category of pre T_2 -closed extremally disconnected Hausdorff spaces and continuous maps, the projective objects are finite spaces.

REVIEW OF LITERATURE

REVIEW OF LITERATURE

In 1924, Alexandroff and Urysohn [3] introduced the class of H-closed spaces (absolutely closed spaces). From then onwards, many articles on H-closed spaces have been published. In 1968, Velicko [19] introduced the concept of H-sets (as a generalisation of H-closed spaces). Vermeer[20] and many others have contributed much to the study of H-sets. In 1968, Chen-Tung Liu[6] has extended the definition of H-closedness to non-Hausdorff spaces and studied some interesting properties of such spaces. In this chapter, we give a brief survey of some of the interesting articles published by various authors.

H-Closed Topological spaces.

[N.V.Velicko, 1968] [19]

In this paper the author has obtained characterisations of H-closed spaces in terms of θ -Convergence. Further an extensive study is also made on the H-closed extensions.

More characterisations of H-closed and nearly compact spaces.

[Lo Faro, Giovanni, 1983] [11]

A point p in a space X is a δ -adherent point of a filter \mathcal{F} on X if each neighbourhood U of p , $\text{int}(clU) \cap F \neq \emptyset$ for some $f \in \mathcal{F}$. A family C of subsets of X is an almost cover of X if $X \setminus \cup C$ is a singleton characterisation of H-closed, nearly compact and compact spaces are given in terms of δ -adherence of filters and in terms of almost covers.

Locally H-closed spaces .

[Haimovici - Iulian, 1985] [7]

A Hausdorff space is said to be locally H-closed if every point has an H-closed neighbourhood. In this paper, the author gives some characterisation of locally H-closed spaces and then applies a theorem from one of his earlier papers to prove, in a unified manner, results concerning H-closed extensions, locally H-closed extensions of Hausdorff spaces.

Accumulation points of nowhere dense sets in H-closed spaces.

[J.R. Porter and R.G.Woods, 1985] [13]

Kulpa and szymanski [10] showed that each point in a compact Hausdorff space without isolated points is the accumulation point of a nowhere dense set and they asked if the result is true in the setting of minimal Hausdorff spaces without isolated points. In this paper, the authors have settled this question by giving an example of a minimal Hausdorff without isolated points and a point which is not the accumulation point of any nowhere dense set. Also the authors have extended kulpa and szymanski [10]'s result by showing that a regular point in an H-closed spaces without isolated points the accumulation point of some nowhere dense subset.

Closed subspaces of H-closed spaces.

[J.Vermeer ,1985][20]

Characterisations of H-closed spaces and H-sets (a generalisation of an H-closed space) of a given space X are obtained. Moreover some internal properties of H-sets in terms of the Illiadis absoulte EX and the Hausdorff absoulte PX are derived. Some of the main results proved here are :

- 1) An H-closed space X is Urysohn iff $p^{-1}(A)$ is an H-set in PX for every H-set $A \subset X$.
- 2) If A is an H-closed subset of X then there exists a compact $B \subset EX$ such that $\pi(B)=A$ and $\pi: B \rightarrow A$ is θ - Continuous.
- 3) There exists a space X and an H-set $A \subset X$ which is not the image of a compact subset of EX.
- 4) If $\{H_i\}_i$ is a chain of H-closed subspaces in X, then $\bigcap H_i$ is the image of a compact subset of EX.

One - point extensions of locally para H-closed spaces.

[Zahid; M.I; Heath; R.W, 1985] [21]

A Hausdorff space is said to be para H-Closed if each open cover has a locally finite open refinement whose union is dense and is said to be locally para H-closed if every point has a closed neighbourhood which is para H-closed as a subspace. Similar to the setting of locally H-closed spaces the authors show there is projective maximum among the many one- point para H-closed extensions of a locally para H-closed spaces which is not para H-closed, however, there may not be projective minimum. In fact , a locally paracompact Hausdorff space which is neither locally H-closed nor para H-closed has no projective minimum among its one point para H-closed extension.

A note on H-closed spaces.

[Jankovic, Dragans, Ling paul. E, 1987] [8]

A function $f: X \rightarrow Y$ between spaces X and Y is said to be weakly continuous if for $x \in X$ and an open set U containing $f(x)$, there is an open set V containing x such that $f(V) \subseteq \text{cl } U$. A space is irreducible if every non empty open set is dense. Let I denote the class of irreducible T_1 - space. A Hausdorff space Y is H- closed iff for

each $x \in I$, an open bijection $f : X \rightarrow Y$ with closed graph is weakly continuous. Other characterisations of H -closed spaces in terms of I are given.

CHAPTER - I

CHAPTER - I

PRELIMINARY DEFINITIONS AND RESULTS

In this chapter we give the definitions and results which are needed for the discussion of this thesis.

SECTION : 1.1

PRE OPEN SETS

Definition : 1.1.1

A subset $A \subset X$ is called **preopen** if $A \subset \text{int cl}A$ and the complement of a preopen set is called **preclosed**.

Notation : 1.1.2

The family of all preopen sets is denoted by $PO(X)$ and that of preclosed sets by $PF(X)$.

Properties : 1.1.3

- 1) For a subfamily $\{B_i \mid i \in I\} \subset PO(X)$, $\cup B_i \in PO(X)$.
- 2) For $A \subset Y \subset X$ and $A \in PO(X)$, $A \in PO(Y)$ whenever Y is open in X .
- 3) For $A \subset Y \subset X$ and $A \in PF(X)$, $A \in PF(Y)$ whenever $Y \in PF(X)$.
- 4) Let $B \subset A \subset X$. If A is open in X and B is preopen in $A \Rightarrow B$ is preopen in X .

Definition : 1.1.4

The **preinterior** of a subset $A \subset X$ is the union of all preopen sets which are contained in A and it is denoted by $\mathbf{p\,int\,}A$.

Definition : 1.1.5

The intersection of all preclosed sets containing a set A is called the **preclosure** of A and it is denoted by $\mathbf{p\,cl\,}A$.

Theorem : 1.1.6

The preinterior of a set A is preopen and the preclosure is preclosed.

Theorem : 1.1.7

For $A \subset X_0 \subset X$ and X_0 is an open set in X , $\text{pcl}_{X_0}(A) = \text{pcl } A \cap X_0$ where $\text{pcl}_{X_0}A$ is the preclosure of A in X_0 .

Definition : 1.1.8

A mapping $f : X \rightarrow Y$ is **precontinuous** if $f^{-1}(V) \in \text{PO}(X)$ for every open set V of Y and f is **preopen** whenever $f(U) \in \text{PO}(Y)$ for every open set U of X .

Definition : 1.1.9

A space X is called **pre T_2** if for every pair of distinct points x and y of X , there exists disjoint preopen sets U and V containing x and y respectively.

SECTION : 1.2 **p - REGULAR SPACES****Definition : 1.2.1**

A space X is said to be **p -regular** if for each pair consisting of a point $x \in X$ and closed set F distinct from x , there exist disjoint preopen sets containing x and F .

Definition : 1.2.2

A space X is said to be an **almost p -regular** if for each pair consisting of a point $x \in X$ and a regular closed set F distinct from x , there exists disjoint preopen sets containing x and F .

Theorem : 1.2.3

A space X is p -regular if and only if for each $x \in X$ and each open set U of X containing x , there exists $V \in \text{PO}(X)$ such that $x \in V \subset \text{pcl } V \subset U$.

SECTION : 1.3**FILTERS****Definition : 1.3.1**

Let X be any set. A collection \mathcal{F} of nonempty subsets of X is said to be a **filter** on X if

- (i) $\mathcal{F} \neq \phi$
- (ii) If A and B are in \mathcal{F} , then $A \cap B$ is also in \mathcal{F} .
- (iii) If $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$.

Definition : 1.3.2

Let (X, τ) be any topological space. Then a filter \mathcal{F} on X is called an **open filter** if all the members \mathcal{F} are open sets.

Definition : 1.3.3

An **ultra filter** on a set X is a maximal filter on X . That is, a filter \mathcal{F} on a set X is an ultrafilter on X if given any filter \mathcal{F}' finer than \mathcal{F} (i.e. $\mathcal{F} \subset \mathcal{F}'$), $\mathcal{F} = \mathcal{F}'$.

Definition : 1.3.4

An **open ultrafilter** is an open filter which is maximal in the collection of open filters.

Definition : 1.3.5

Let X be any set and let \mathcal{D} be a nonempty collection of nonempty subsets of X with the property that if B and B' are in \mathcal{D} , then there exists $B'' \in \mathcal{D}$ such that

$B'' \subset B \cap B'$. Let $\mathcal{F} = \{A \mid B \subset A, B \in \mathcal{D}\}$. Then \mathcal{F} is filter on X . \mathcal{F} is said to be the filter generated by \mathcal{D} and \mathcal{D} is said to be a **basis for the filter** \mathcal{F} .

Definition : 1.3.6

An **open filter base** is a filter base consisting exclusively of open sets.

Definition : 1.3.7

If (X, τ) is a topological space and \mathcal{F} is a filter on X , then \mathcal{F} is said to **converge to x** , denoted by $\mathcal{F} \rightarrow x$, if every neighbourhood of x is a member of \mathcal{F} .

Definition : 1.3.8

A filter \mathcal{F} is said to have x as a **cluster point** (an adherent point) if every neighbourhood of x meets every member of \mathcal{F} . That is, x is a cluster point of \mathcal{F} if given any neighbourhood U of x , and any $A \in \mathcal{F}$, then $U \cap A \neq \phi$.

Theorem : 1.3.9

If \mathcal{U} is an open ultrafilter on X the following hold:

- a) If G is an open subset of X , then $G \cap U \neq \phi$ for all $U \in \mathcal{U}$ iff $G \in \mathcal{U}$.
- b) If G_1, G_2 are open subsets of X and $G_1 \cup G_2 \in \mathcal{U}$, then $G_1 \in \mathcal{U}$ or $G_2 \in \mathcal{U}$.
- c) If $G_1 \notin \mathcal{U}$, and G_1 is open, then $G_2 = \overline{(X - G_1)} \in \mathcal{U}$.
- d) If p is a cluster point of \mathcal{U} , then $\mathcal{U} \rightarrow p$, that is, \mathcal{U} converges to p .

Theorem : 1.3.10

Suppose X is an open, dense subset of a topological space Y , and \mathcal{U} is an open ultrafilter on X . Let $\mathcal{U}' = \{G : G \text{ is open in } Y, G \cap X \in \mathcal{U}\}$. Then \mathcal{U}' is an open ultrafilter on Y . Moreover, $\mathcal{U} \rightarrow p$ iff $\mathcal{U}' \rightarrow p$.

Theorem : 1.3.11

Let X be an open, dense subspace of topological space Y . If \mathcal{U}' is an open ultrafilter on Y and $\mathcal{U} = \mathcal{U}' \cap X = \{U' \cap X; U' \in \mathcal{U}'\}$, then \mathcal{U} is an open ultrafilter on X . Moreover, $\mathcal{U} \rightarrow p$ iff $\mathcal{U}' \rightarrow p$.

SECTION :1.4

In this section we give some more definitions and results which are used in this thesis.

Definition : 1.4.1

An open set U in a space X is said to be **regular open** if $U = \text{int } \text{cl } U$.

Definition : 1.4.2

A topological space X is called **extremally disconnected** if $\text{cl}(U)$ is open in X for every open set U of X , or, equivalently if every two disjoint open sets of X have disjoint closures.

Definition : 1.4.3

Let \mathcal{U} be a collection of extensions of X . An extension Y in \mathcal{U} is a **projective maximum** in \mathcal{U} if for each Z in \mathcal{U} there is continuous function from Y onto Z which leaves X pointwise fixed.

Definition : 1.4.4

Let \mathcal{U} be a collection of extensions of X . An extension Y in \mathcal{U} is a **projective minimum** in \mathcal{U} if for each Z in \mathcal{U} there is a continuous function from Z onto Y which leaves X pointwise fixed.

Theorem : (Tube Lemma) 1.4.5

Consider the product space $X \times Y$ where Y is a compact space. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$ then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighbourhood of x_0 in X .

CHAPTER - II

CHAPTER II

ABSOLUTELY CLOSED SPACES

This chapter is devoted to the study of absolutely closed spaces (ie. H-closed spaces) and generalised absolutely closed spaces. The results are due to Chen-Tung Liu [6]. In section 1, existence of a largest absolute closure of a Hausdorff space and that of a largest generalised absolute closure of an arbitrary topological space is established. Section 2 deals with the imbedding of a topological space as a closed subspace of an absolutely closed space. Section 3 is devoted to the study of projective and injective objects in the category of absolutely closed spaces (and of generalised absolutely closed spaces) and continuous maps.

SECTION : 2.1

In this section we give the definitions of absolutely closed spaces and generalised absolutely closed spaces and discuss their properties. We have also discussed here the existence of a largest absolute closure (respectively, largest generalised absolute closure), Hausdorff topological space (respectively, arbitrary topological space) which is essentially unique.

Definition : 2.1.1

A Hausdorff space X is called **absolutely closed** (ie., H-closed) if X is closed in every Hausdorff space in which it is imbedded.

From this definition it follows that a compact Hausdorff space is absolutely closed.

Alexandroff [1] has observed that this definition is equivalent to the following definition.

Definition : 2.1.2

A Hausdorff space X is **absolutely closed** or **H-closed** if and only if every open cover of X has a finite subcollection whose closures cover X .

The following theorem gives a set of equivalent conditions for a Hausdorff space to be absolutely closed, the proof of which is found in Bourbaki[5].

Theorem :2.1.3

Let X be a Hausdorff space, then the following are equivalent

- (a) X is absolutely closed.
- (b) Every open filter base on X has a cluster point.
- (c) Every open cover of X has finite dense subsystem(whose union is dense in X).
- (d) Every open ultrafilter on X converges.

Further, even if X is not Hausdorff, properties (b) - (d) remain equivalent.

The reason for restricting X to be Hausdorff in definition 2.1.1 follows from the fact that for any T_1 space X , “ X is closed in every T_1 space in which it is imbedded” is true only when X is finite. The following example illustrates this.

Example : 2.1.4

Let X be an infinite T_1 spaces. We can find an infinite sequence of distinct points, say $S = \{ a_1, a_2, \dots, a_n, \dots \} \subset X$, Let $p \notin X$ and $Y = X \cup \{p\}$, topologized as follows: For $x \in X$, neighbourhoods of x are unchanged.

We define a typical neighbourhood of p , say $U(p)$ as follows :

$U(p) = \{p\} \cup (\cup_{n \geq n_0} U_n(a_n))$ where $U_n(a_n)$ are neighbourhoods of a_n in X . Then

Y is a T_1 - topological space, and X is not closed in Y , since $p \in \text{cl}X$.

Definition : 2.1.5

Let X be a topological space, not necessarily Hausdorff; We say X is **generalised absolutely closed** if X satisfies property (b) in theorem 2.1.3 and hence properties (c) and (d) also.

Definition : 2.1.6

Let X, Y be Hausdorff spaces (respectively, arbitrary topological spaces)

Such that

- a) X is dense in Y .
- b) Y is absolutely closed (respectively, generalized absolutely closed).

We call Y an **absolute closure** (respectively, **generalised absolute closure**) of X .

Definition : 2.1.7

Let Y be an absolute closure (respectively, generalised absolute closure) of X .

We say Y is a **largest absolute closure** (respectively, **generalised absolute closure**) of X , if for any other absolute closure (respectively, generalised absolute closure) say T of X , and if $i : X \rightarrow T$ is the injection, then there exists $\tilde{i} : Y \rightarrow T$ (onto) such that

$$\tilde{i}/_X = i.$$

Properties : 2.1.8

- (a) Every continuous image of an absolutely closed space is absolutely closed, if it is Hausdorff.
- (b) Finite unions of absolutely closed spaces are absolutely closed, if they are Hausdorff spaces.
- (c) The intersection of two absolutely closed spaces may not be absolutely closed.
- (d) Product of topological spaces is absolutely closed iff each component space is absolutely closed.

For the same reasons, (a), (b), (c) hold for generalised absolutely closed spaces also.

Theorem : 2.1.9.

Let X be a Hausdorff space, there exists at least one absolute closure \tilde{X} of X . This result was proved by P.Obreanu [12]. Here we outline the construction of \tilde{X} .

Let $X^v = \{\mathcal{U} : \mathcal{U} \text{ is an open nonconvergent ultrafilter on } X\}$ Let $\tilde{X} = X \cup X^v$ be the disjoint union, under the following topology. The basic open sets of \tilde{X} are those of the form $\tilde{G} = G \cup G^v$ where $G^v = \{\mathcal{U} : \mathcal{U} \in X^v, G \in \mathcal{U}\}$. Then X is a Hausdorff space such that X is dense in \tilde{X} . Moreover, \tilde{X} is absolutely closed.

We need the following theorem and definition to get an extension analogous to stone - cech compactification.

Theorem : 2.1.10

Let X be dense in each of the Hausdorff spaces S and T . If the identity mapping on X has continuous extensions from S into T , and τ from T into S , then σ is a homeomorphism, onto and $\sigma^{-1} = \tau$.

Definition : 2.1.11

Let X be dense in each of the spaces S and T . If f is a homeomorphism from S to T , such that f is the identity on X , we say S is **essentially the same as T** relative to X .

The following theorem gives an extension analogous to stone-cech compactification.

Theorem : 2.1.12

For any Hausdorff space X , there exists a Hausdorff space κX which is a largest absolute closure of X . Moreover, κX is essentially unique in the sense that if T is another largest absolute closure of X , then T is essentially the same as κX .

Proof :

Define $\kappa X = X \cup X^\vee = \tilde{X}$ as a set as in theorem 2.1.9. We \mathcal{B} , the basis for the topology of κX as follows, $B \in \mathcal{B}$ iff either $B = G \cup \{\mathcal{P}\}$ where $\mathcal{P} \in X^\vee$, and $G \in \mathcal{P}$, or $B=G$ where G is open in X . It is easy to verify that \mathcal{B} is a basis for a topology on κX .

We will now show that κX is Hausdorff. Since X is Hausdorff and X is open in κX then any two distinct points in X have disjoint neighbourhoods in κX .

Let $x \in X$ and $\mathcal{P} \in X^\vee$. Since \mathcal{P} is a nonconvergent ultrafilter on X , there exists an open neighbourhoods G of x in X such that $G \notin \mathcal{P}$. Therefore by (a) of theorem 1.3.9 there exists $H \in \mathcal{P}$ such that $H \cap G = \emptyset$. Hence we have $H \cup \{\mathcal{P}\}$ and G as disjoint neighbourhoods of \mathcal{P} and x in κX .

Let $\mathcal{P}_1, \mathcal{P}_2$ be distinct points in X^\vee . By (a) of theorem 1.3.9, there exists $G_1 \in \mathcal{P}_1$ and $G_2 \in \mathcal{P}_2$ such that $G_1 \cap G_2 = \emptyset$. Let $W_1 = G_1 \cup \{\mathcal{P}_1\}$, and $W_2 = G_2 \cup \{\mathcal{P}_2\}$. Then W_1 and W_2 are disjoint neighbourhoods of \mathcal{P}_1 and \mathcal{P}_2 in κX .

Note that the subspace topology of X , as subspace of κX , is precisely the given topology of X and X is dense in κX .

To Prove : kX is absolutely closed.

Let \mathcal{U} be an open ultrafilter on kX , to prove \mathcal{U} converges. Put $\mathcal{P} = \{U \cap X; U \in \mathcal{U}\}$ then by theorem 1.3.11, $\mathcal{P} = \{U \in \mathcal{U}; U \subset X\}$ and \mathcal{P} is an open ultrafilter on X .

(i) If \mathcal{P} converges to a point $x \in X$, then x is a cluster point of \mathcal{U} and by (d) of theorem 1.3.9, $\mathcal{U} \rightarrow x$.

(ii) If \mathcal{P} is nonconvergent on X , then $\mathcal{P} \in X^\vee$. Let $W = G \cup \{\mathcal{P}\}$ be a neighbourhood of \mathcal{P} in kX ; Then $G \in \mathcal{P}$ and therefore $G \in \mathcal{U}$. Thus G meets every member of \mathcal{U} which in turn implies that W also meets every member of \mathcal{U} . Hence \mathcal{P} is a cluster point of \mathcal{U} in kX , thus by theorem 1.3.9, $\mathcal{U} \rightarrow \mathcal{P}$. Hence by theorem 2.1.3; kX is absolutely closed.

To Prove : kX is largest absolute closure of X

Let T be an absolute closure of X and $i: X \rightarrow T$ be the inclusion. We want to extend i to $\tilde{i}: kX \rightarrow T$ continuously. If such an \tilde{i} exists we claim that \tilde{i} must be onto. Because by property 2.1.8 (a); $\tilde{i}(kX)$ is absolutely closed and so it is closed in T .

$$\text{Also, } X \subset \tilde{i}(kX) \Rightarrow \text{cl } X \subset \text{cl } \tilde{i}(kX) = \tilde{i}(kX)$$

$$\Rightarrow T \subset \tilde{i}(kX) \Rightarrow \tilde{i}(kX) = T.$$

Construction of \tilde{i} :

Let $\mathcal{P} \in X^\vee$ Let $\mathcal{U} = \{U : U \text{ open in } T \text{ and } U \cap X \in \mathcal{P}\}$.

By theorem 1.3.10 \mathcal{U} is an open ultrafilter on T . Since T is absolutely closed by theorem 2.1.3 there exists $p \in T$. Such that $\mathcal{U} \rightarrow p$ By theorem 1.3.10 again we have

$\mathcal{P} \rightarrow p$ in T . We now define for each $\mathcal{P} \in X^\vee$, $\tilde{i}(\mathcal{P}) = p$ and for each $x \in X$, $\tilde{i}(x) = x$. We now show that \tilde{i} is continuous.

Case(i)

For each $x \in X$, let W be an open neighbourhood of x in T then $\tilde{i}^{-1}(W) \cap X = i^{-1}(W) = G$ is an open neighbourhood of x in X , hence is open in kX and $\tilde{i}(G) \subset W$.

Case(ii)

For $\mathcal{P} \in X^\vee$ we have $\tilde{i}(\mathcal{P}) = p$ where $\mathcal{P} \rightarrow p$ in T .

Let W be an open neighbourhood of p in T . There exists $G \in \mathcal{P}$ such that $G \subset W$.

Thus $G \cup \{\mathcal{P}\}$ will be an open neighbourhood of \mathcal{P} in kX such that .

$$\tilde{i}(G \cup \{\mathcal{P}\}) = \tilde{i}(G) \cup \tilde{i}\{\mathcal{P}\} = i(G) \cup \{p\} = G \cup \{p\} \subset W$$

Hence \tilde{i} is continuous.

To Prove : kX essentially unique

If Y is a largest absolute closure of X then there exists $\tilde{i} : Y \rightarrow kX$ where \tilde{i} is continuous extension of the inclusion map $i : X \rightarrow kX$. Also, there exists: $\tilde{j} : kX \rightarrow Y$ such that \tilde{j} is also a continuous extension of the inclusion map i .

Therefore by theorem 2.1.10, \tilde{j} is a homeomorphism from kX onto T such that \tilde{j} is identity on X . Hence by definition 2.1.11 kX is essentially the same as Y .

Definition : 2.1.13

If X is a subspace of a topological space Y , we say Y is **Hausdorff except for X** if for any pair of distinct points a, b in Y , which are not both in X , a, b have disjoint neighbourhoods in Y .

Analogous to theorem 2.1.12, we get the following result in the case of generalised absolute closure of X .

Theorem : 2.1.14

For any topological space X , there exists kX which is a largest generalised absolute closure of X , and which is Hausdorff except for X . Moreover, kX is essentially unique.

Proof :

kX is constructed exactly in the same way as in theorem 2.1.12. It is absolutely closed by the same proof as in that theorem. As was shown in theorem 2.1.12 it is clear that kX is Hausdorff except for X .

For the second part of the proof, we want to construct $\tilde{i} : kX \rightarrow T$, where T is any other generalised absolute closure of X , which extends the inclusion $i : X \rightarrow T$. The same argument shows that every open ultrafilter ρ on X converges to some p in T . Since now T is not necessarily Hausdorff, p may not be unique. We now define \tilde{i} as follows. For $x \in X$, we define $\tilde{i}(x) = i(x) = xi$; for $\rho \in X^\vee$, choose $p \in T$ such that $\rho \rightarrow p$, and define $i(\rho) = p$. Exactly the same proof shows \tilde{i} is continuous.

The uniqueness of kX follows easily from the following result "If X is dense in T , and T is Hausdorff except for X , and if $f : T \rightarrow T$ is continuous and $f|_X = \text{identity on } X$, then $f = \text{identity on } T$."

We have shown in theorem 2.1.12 that for a Hausdorff space X , if Y is any absolute closure of X , there exists $\tilde{i} : kX \rightarrow Y$ which extends the inclusion $i : X \rightarrow Y$ continuously. By analogy to the corresponding property of the stone-cech compactification the following question arises : "If Y is any absolutely closed space and $f : X \rightarrow Y$ is continuous, can f always be extended continuously to $\tilde{f} : kX \rightarrow Y$?"

The answer is No, which follows from the following example.

Example : 2.1.15

Let $X = \mathbb{N}$ be the set of positive integers, which is not absolutely closed. Let $Y = \mathbb{N} \times \mathbb{N}^* \cup \{\alpha\}$. Then X is mapped homeomorphically into Y by $f(x) = (x, \omega)$ for $x \in X$. Suppose there exists $\tilde{f} : kX \rightarrow Y$ such that $\tilde{f}/X = f$.

Then we have $\tilde{f}(kX - X) \subset Y - f(X) = Y - (\mathbb{N} \times \{\omega\})$. Choose $\rho \in kX - X$; then

$\tilde{f}(\rho) = p' \in Y - \mathbb{N} \times \{\omega\}$ Thus either $p' \in \mathbb{N} \times \mathbb{N}$ or $p' = \alpha$. In any case, there exists a neighbourhood W of p' which is disjoint from $\mathbb{N} \times \{\omega\}$. (In fact, from the topology of Y , every basic neighbourhood of p' is disjoint from $\mathbb{N} \times \{\omega\}$).

Let $G \cup \{\rho\}$ be an open neighbourhood of ρ in kX , where $G \in \rho$, then

$$\tilde{f}(G \cup \{\rho\}) = \tilde{f}(G) \cup f(\rho) = f(G) \cup \{p'\} = (G \times \{\omega\}) \cup \{p'\}.$$

Thus no neighbourhood of ρ is carried by \tilde{f} into W . Thus \tilde{f} is not continuous and f has no continuous extension to kX .

However the following result holds if we restrict our attention to open maps.

Theorem : 2.1.16

Let X, Y be topological spaces and suppose Y is generalised absolutely closed.

Let $f : X \rightarrow Y$ be a continuous, open map then there exists $\tilde{f} : kX \rightarrow Y$ Such that

$$\tilde{f}/X = f.$$

Proof :

Let us define $\tilde{f} : kX \rightarrow Y$ as follows : For $x \in X$, we define $\tilde{f}(x) = f(x)$; for $\rho \in kX - X$, let $f(\rho) = \{ f(G) : G \in \rho \}$. Then $f(\rho)$ is an open filter base on Y . Let \mathcal{U} be the open filter on Y generated by $f(\rho)$; we shall show that \mathcal{U} is an open ultrafilter on Y . For if not, suppose $\mathcal{U} \cup \{A\}$ where $A \notin \mathcal{U}$ and A is open in Y , is contained in some open filter; then $f(G) \cap A \neq \emptyset$ for all $G \in \rho$. Thus $f^{-1}(A) \cap G \neq \emptyset$ for all $G \in \rho$, thus $f^{-1}(A) \in \rho$. Therefore $f(f^{-1}(A)) \in f(\rho)$ and $A \in \mathcal{U}$. which is a contradiction.

Thus \mathcal{U} is an ultrafilter on Y . Therefore there exists $p' \in Y$ such that, $\mathcal{U} \rightarrow p'$. Since Y need not Hausdorff may be more than one such p' choose one arbitrarily. Let us define $\tilde{f}(p) = p'$.

To prove : \tilde{f} is continuous.

If $\rho \in kX - X$ and $f(\rho) = p'$ as described above. Let W be an open neighbourhood of p' in Y . Then $W \in \mathcal{U}$ because $\mathcal{U} \rightarrow p'$. Hence $W \supset f(G)$ for some $G \in \rho$. Thus $G \cup \{\rho\}$ will be an open neighbourhood of ρ in kX such that

$$\tilde{f}(G \cup \{\rho\}) = \tilde{f}(G) \cup \tilde{f}(\rho) = f(G) \cup \{p'\} \subset W. \text{ Thus } \tilde{f} \text{ is continuous at } \rho. \text{ If } x \in X, \text{ Let}$$

$$f(x) = p', \text{ and let } W, \text{ be an open neighbourhood of } p' \text{ in } Y \text{ Since } \tilde{f}(x) = f(x) = p',$$

then $G = f^{-1}(W)$ will be an open neighbourhood of x in X , hence an open

neighbourhood of x in kX . Such that $\tilde{f}(G) = f(G) \subset W$. Thus \tilde{f} is continuous at $x \in X$ also. Hence the result.

SECTION : 2.2

The imbedding of a topological space as a closed subspace of an absolutely closed space.

Chen -Tung Liu [6] has proved that every Hausdorff space can be imbedded as a closed subspace of an absolutely closed space. He [6] has also proved a similar result for generalised absolute closed spaces. These two results are discussed in this section.

Theorem : 2.2.1

Let X be a Hausdorff space. Then X can be imbedded as a closed subspace of an absolutely closed space K .

Proof :

Let us define K as follows, $K = (X \times N^*) \cup X^v$, where $N^* = N \cup \{\omega\}$ is the one point compactification of N ; and $X^v = kX - X$ is the set of all nonconvergent open ultrafilters on X .

A topology on K is defined as follows; $X \times N^*$ has the product topology i.e for $(x, n) \in X \times N$, a basic neighbourhood of it is of the form $G \times \{n\}$ where G is an open neighbourhood of x in X And for $(x, \omega) \in X \times \{\omega\}$, a basic neighbourhood of it is of the form $G \times [n, \omega]$ where G is an open neighbourhood of x in X and $[n, \omega]$ denotes $\{n, n+1, \dots, \omega\}$. For $\rho \in X^v$, a basic neighbourhood of ρ is defined to be $U(\rho) = (G \times N) \cup \{\rho\}$ where $G \in \rho$. Note that G is an open neighbourhood of ρ in kX .

Claim 1: This is a topology on k :

We need only to check that for $\rho \in X^v$, $\mathcal{B}(\rho) = \{U(\rho)\}$ forms an open neighbourhood system at ρ .

(a) $\mathcal{B}(\rho) \neq \emptyset$ since $X \in \rho$, also $\rho \in \bigcap \mathcal{B}(\rho)$.

(b) If $(G_1 \times N) \cup \{\rho\}$ and $(G_2 \times N) \cup \{\rho\}$ are two neighbourhoods of ρ , where $G_1, G_2 \in \rho$, then $((G_1 \times N) \cup \{\rho\}) \cap ((G_2 \times N) \cup \{\rho\}) = ((G_1 \cap G_2) \times N) \cup \{\rho\}$ is a basic neighbourhood of ρ because $G_1 \cap G_2 \in \rho$.

(c) Given $(G \times N) \cup \{\rho\} \in \mathcal{B}(\rho)$, let $y \in (G \times N) \cup \{\rho\}$; then either $y = (x, n) \in G \times N$ or $y = \rho$

If $y = (x, n)$ then $G \times \{n\}$ will be a basic neighbourhood of y contained in the given one.

If $y = \rho$, then $(G \times N) \cup \{\rho\}$ will be a basic neighbourhood of y contained in the given one.

Hence Claim 1

Claim 2: X can be imbedded as a closed subset of K .

This is clear, because we can identify X with $X \times \{\omega\}$ by mapping $f(x) = (x, \omega)$ for each $x \in X$. This f is easily seen to be a homeomorphism and $X \times \{\omega\}$ is closed in K .

Claim 3: K is Hausdorff.

Since X and N^* are Hausdorff, for distinct points in $X \times N^*$, the Hausdorff condition is satisfied. Let $(x, n) \in X \times N$ and $\rho \in X^v$. Since kX is Hausdorff, there exists an open neighbourhood G_1 of x and a basic open neighbourhood of ρ , say $G_2 \cup \{\rho\}$ where $G_2 \in \rho$. Such that $G_1 \cap (G_2 \cup \{\rho\}) = \emptyset$. Thus $G_1 \cap G_2 = \emptyset$. Now $G_1 \times \{n\}$ and $(G_2 \times N) \cup \{\rho\}$ will be two disjoint neighbourhoods of (x, n) and ρ in K .

respectively. Let $\rho_1, \rho_2 \in X^V$ where $\rho_1 \neq \rho_2$. Then there exists $G_1 \in \rho_1$ and $G_2 \in \rho_2$ such that $G_1 \cap G_2 = \emptyset$

Thus $(G_1 \times N) \cup \{\rho_1\}$ and $(G_2 \times N) \cup \{\rho_2\}$ will be two disjoint neighbourhoods of ρ_1 and ρ_2 in K .

Claim 4: K is absolutely closed.

Let \mathcal{E} be an open cover of K . We want to find a finite dense subsystem of \mathcal{E} .

Without loss of generality, we may assume that \mathcal{E} consists of basic open sets.

For each $\rho \in X^V = kX - X$, pick $W_\rho \in \mathcal{E}$ such that $\rho \in W_\rho$. We can write

$W_\rho = (G_\rho \times N) \cup \{\rho\}$ where $G_\rho \in \rho$. For each $x \in X$, consider $\{x\} \times N^*$. Let $W_0(x) \in \mathcal{E}$ be such that $(x, \omega) \in W_0(x)$. Now by assumption we can write $W_0(x) = U_0(x) \times [n_x, \omega]$ where $U_0(x)$ is an open neighbourhood of x in X and $n_x \in N$. For each $k=1, 2, \dots, n_x-1$, there exists $W_k(x) \in \mathcal{E}$ such that $(x, k) \in W_k(x)$. Now $W_k(x)$ is of the form

$W_k(x) = U_k(x) \times \{k\}$ where $U_k(x)$ is an open neighbourhood of x in X . Let $U(x) = U_0(x) \cap U_1(x) \cap \dots \cap U_{n_x-1}(x)$. Then $U(x)$ is open in kX for each $x \in X$ and

hence in turn it is open in kX for each $x \in X$. Consider

$\mathcal{C} = \{U(x) : x \in X\} \cup \{G_\rho \cup \{\rho\} : \rho \in X^V\}$. This is an open cover of kX . Since kX is

absolutely closed, there exists a finite subsystem, say $\mathcal{g} = \{U(x_1), \dots, U(x_n),$

$$G_{\rho_1} \cup \{\rho_1\}, \dots, G_{\rho_m} \cup \{\rho_m\}\}$$

satisfying ... $\text{Cl}(\cup \mathcal{g}) = kX$.

Let us define a family \mathcal{F} as follows :

$$\begin{aligned}
& (G_{p_1} \times N) \cup \{\rho_1\} \\
& (G_{p_2} \times N) \cup \{\rho_2\} \\
& \vdots \\
& (G_{p_m} \times N) \cup \{\rho_m\} \\
& U(x_1) \times [n_{x_1}, \omega], U(x_1) \times \{1\}, \dots, U(x_1) \times \{n_{x_1} - 1\} \\
& U(x_2) \times [n_{x_2}, \omega], U(x_2) \times \{1\}, \dots, U(x_2) \times \{n_{x_2} - 1\} \\
& \vdots \\
& U(x_k) \times [n_{x_k}, \omega], U(x_k) \times \{1\}, \dots, U(x_k) \times \{n_{x_k} - 1\}
\end{aligned}$$

Let now show that $\text{Cl}(\cup \mathcal{J}) = K$.

Since $X \times N$ is dense in K it is enough to show that $\text{Cl}(\cup \mathcal{J}) = X \times N$.

Let $(x, n) \in X \times N$. Let W be a basic neighbourhood of (x, n) . Then $W = G \times \{n\}$ where G is an open neighbourhood of x in X , hence open in kX . Thus G meets some member of \mathcal{G} because $\text{Cl}(\cup \mathcal{G}) = kX$. Thus either $G \cap U(x_j) \neq \phi$ for some j , $1 \leq j \leq r$

(or) $G \cap G_{p_i} \neq \phi$ for some i , $1 \leq i \leq m$.

Case (i) $G \cap U(x_j) \neq \phi$ where $1 \leq j \leq r$

a) If $n \geq n_{x_j}$, then $W = G \times \{n\}$ meets $U(x_j) \times [n_{x_j}, \omega]$ which is a member of \mathcal{J} .

$$\text{Infact } (G \times \{n\}) \cap (U(x_j) \times [n_{x_j}, \omega]) = (G \cap U(x_j)) \times \{n\} \neq \phi.$$

b) If $n < n_{x_j}$ then $1 \leq n \leq n_{x_j} - 1$ thus $W = G \times \{n\}$ meets $U(x_j) \times \{n\}$ which is a member of \mathcal{J}

Case (ii) $G \cap G_{P_i} \neq \emptyset$ for some $1 \leq i \leq m$. Then $W = G \times \{n\}$ meets $(G_{P_i} \times N) \cup \{P_i\}$ which is a member of \mathcal{J} . Hence K is absolutely closed.

Theorem : 2.2.2

Let X be any topological space. Then X can be imbedded as a closed subspace of a generalised absolutely closed space.

Proof :

We know by theorem 2.1.14, that kX which is the largest generalised absolute closure of X , still exists and is constructed in the same way. The previous construction of K in theorem 2.2.1 still applies, and exactly the same proof shows that K is generalised absolutely closed.

SECTION : 2.3

Projective and injective spaces in the category of absolutely closed spaces.

This section is devoted to the study of projective and injective spaces in the category of absolutely closed spaces (and generalised absolutely closed spaces) and continuous maps.

Definition : 2.3.1

Let \mathcal{C} be a category of topological spaces and continuous maps; we assume that all continuous maps are admissible; $X \in \mathcal{C}$ is called **projective** if, whenever $f: Y \rightarrow Z$ and $g: X \rightarrow Z$ are admissible maps where $Y, Z \in \mathcal{C}$ and f is onto, then there exists admissible map $g': X \rightarrow Y$ such that $g = f \circ g'$.

Definition : 2.3.2.

A **deleted neighbourhood** of a has the form $W - \{a\}$, where W is a usual neighbourhood of a .

Theorem : 2.3.3

In the category of Hausdorff absolutely closed spaces and continuous maps, X is projective iff X is finite .

Proof :

If X is finite, then clearly it is projective. Conversely , if X is projective, we will show that X is discrete. Suppose X is not discrete , then there exists $a \in X$ such that $\{a\}$ is not open. Let $Y = X - \{a\}$ and let α be any point not in $Y \times N^*$. We define $A = (Y \times N^*) \cup \{\alpha\}$ with the following topology.

$Y \times N^*$ has the product topology. Basic neighbourhoods of α are of the form $(G \times N) \cup \{\alpha\}$ where G is an open deleted neighbourhood of a in X . It is easily seen that this determines a topology on A .

To prove : A is Hausdorff.

For points in $Y \times N^*$, we can always separate them by disjoint neighbourhoods, since Y and N^* are both Hausdorff.

Let $(y,n) \in Y \times N$. Since $y \neq a$, there exist G_1, G_2 which are disjoint open neighbourhoods of y and a in X . Then $G_1 \times \{n\}$ and $(G_2' \times N) \cup \{\alpha\}$ where $G_2' = G_2 - \{a\}$ are disjoint neighbourhoods of (y, n) and α in A . Let $(y, \omega) \in Y \times \{\omega\}$; then again $y \neq a$. There exist G_1, G_2 which are open neighbourhood of y and a in X such that $G_1 \cap G_2 = \phi$. Then $G_1 \times [n, \omega]$ and $(G_2' \times N) \cup \{\alpha\}$, where $n \in N$ and $G_2' = G_2 - \{a\}$ will be open disjoint open neighbourhoods of (y, ω) and α in A . Hence A is Hausdorff

To prove : A is absolutely closed

The proof is exactly analogous to the proof of “ K is absolutely closed “ in theorem 2.2.1. Let us define $B = (Y \times N^*) \cup \{\alpha\}$ under the following topology.

$Y \times N^*$ has the product topology. Basic neighbourhoods of ∞ are of the form $(G \times N^*) \cup \{\infty\}$ where G is an open deleted neighbourhood of a in X . Let $i : A \rightarrow B$ be the identify map. Then i is continuous. Thus B is absolutely closed because it is a continuous image of A (by property 2.1.8(a)) and this i is onto. Let us define $j : X \rightarrow B$ by $j(a) = \infty$ and $j(y) = (y, \omega)$ for each $y \in Y$.

To Prove : j is continuous.

Let $(y, \omega) \in Y \times \{\omega\} \subset B$ and let W be a basic open neighbourhood of (y, ω) in B .

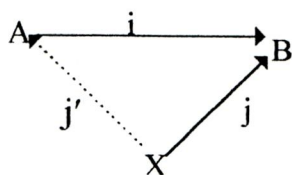
Then $W = G \times [n, \omega]$ where G is an open neighbourhood of y in Y and $n \in N$.

Thus $j(G) = G \times \{\omega\} \subset W$. Let $\infty \in B$, $a = j^{-1}(\infty)$. Take W a basic neighbourhood of ∞ in B . Then $W = G \times N^* \cup \{\infty\}$ where G is an open deleted neighbourhood of a in X .

Thus $G \cup \{a\}$ will be an open neighbourhood of a in X . Such that

$$j(G \cup \{a\}) = j(G) \cup j(a) = (G \times \{\omega\}) \cup \{\infty\} \subset (G \times N^*) \cup \{\infty\}$$

Hence $j : X \rightarrow B$ is continuous. Since X is projective, there is a continuous function $j' : X \rightarrow A$ making the following diagram commutative.



(i.e) there exists $j' : X \rightarrow A$ such that j' is continuous and $i \circ j' = j$.

Therefore $j'(Y) = Y \times \{\omega\}$ and $j'(a) = \infty$. We will show that j' is not continuous at a . For if W is any basic neighbourhood of ∞ in A , then $W = (G \times N) \cup \{\infty\}$ for some G open in Y . But there exists no open neighbourhood G' of a in X such that $j'(G') \subset W$ because $j'(G') = G' \times \{\omega\} \not\subset W$. Thus we get a contradiction. Hence X is discrete. Since X is absolutely closed, we conclude that X is finite.

The above result can be extended to the generalised absolutely closed spaces also. We give here only outline of the proof.

Theorem : 2.3.4

In the category of generalised absolutely closed spaces and continuous maps, projective spaces are precisely the finite Hausdorff spaces

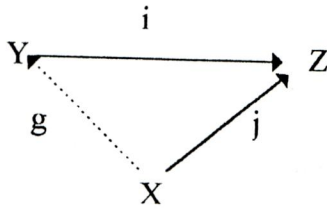
Proof :

If X is projective, generalised absolutely closed. We will show X is discrete. Thus it follows X is finite and Hausdorff.

Let $A \subset X$ be any subset of X and $B = X - A$. We will show A, B are both closed in X ; it follows then that X is discrete. Let us construct Y as follows ; $Y = K_A \cup K_B$ as a discrete union, where $K_A = (A \times N^*) \cup (kA - A)$, disjoint union, under the following topology: $A \times N^*$ has the product of topology. Let $\rho \in kA - A$; a typical neighbourhood of ρ is of the form $(G \times N) \cup \{\rho\}$ where G is open in A and $G \in \rho$. Similarly we define K_B . By theorem 2.2.2, k_A, k_B are generalised absolutely closed, and $A \times \{\omega\}$ is a closed subspace of k_A , $B \times \{\omega\}$ is a closed subspace of k_B . Thus Y is generalised absolutely closed by property 2.1.8.

Let us construct Z as follows $Z = (X \times N^*) \cup (kA - A) \cup (kB - B)$ where these sets are considered to be disjoint, under the following topology ; $X \times N^*$ has the product topology. For $\rho \in kA - A$, a typical basic neighbourhood of ρ is of the form $(G \times N) \cup \{\rho\}$, where G is open in X , and such that $G \cap A = H \in \rho$. Similarly for points in $kB - B$. It is easy to see that Z is a topological space. Let $i : Y \rightarrow Z$ be the identity map. i is continuous ; therefore Z is generalised absolutely closed.

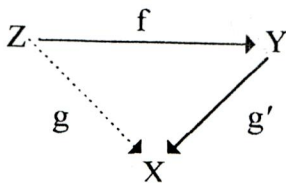
Consider $j: X \rightarrow Z$ defined by $j(x) = (x, \omega)$ for each $x \in X$, j is clearly continuous. Now look at the following diagram.



By assumption, X is projective; there exists $g: X \rightarrow Y$, continuous and $i \circ g = j$ since i is the identify map, and $j(A) = A \times \{\omega\}$, $j(B) = B \times \{\omega\}$, we conclude that $g(A) = A \times \{\omega\}$ and $g(B) = B \times \{\omega\}$. Therefore $g^{-1}(A \times \{\omega\}) = A$ and $g^{-1}(B \times \{\omega\}) = B$. We conclude that A and B are closed in X since $A \times \{\omega\}$ and $B \times \{\omega\}$ are closed in Y . Thus X is discrete, therefore it is finite and Hausdorff.

Definition : 2.3.5

Let \mathcal{C} be a category of topological spaces and continuous maps ; we assume that all continuous maps are admissible ; $X \in \mathcal{C}$ is called **injective** if, whenever $f: Z \rightarrow Y$ and $g: Z \rightarrow X$ are admissible maps where f is 1-1 and $Y, Z \in \mathcal{C}$, then there exists $g': Y \rightarrow X$ such that $g = g' \circ f$.



Theorem : 2.3.6

In the category of absolutely closed spaces and continuous maps, the injective spaces are those which consists of only a single point.

Proof :

Let $Z = \{a_{ij}, b_{ij}, c_i, \alpha, \beta, i=1,2,\dots, j=1,2,\dots\}$ where all these elements are assumed to be distinct.

Define a neighbourhood system on Z as follows:

Each a_{ij} is isolated and each b_{ij} is isolated. The basic neighbourhoods of c_i, α, β are defined as follows :

$$\mathcal{B}(c_i) = \{ V^n(c_i) = \bigcup_{j=n}^{\infty} \{a_{ij}, b_{ij}, c_i\}, n=1,2,\dots \}$$

$$\mathcal{B}(\alpha) = \{ V^n(\alpha) = \bigcup_{j=1}^{\infty} \bigcup_{i=n}^{\infty} \{a_{ij}, \alpha\}, n=1,2,\dots \}$$

$$\mathcal{B}(\beta) = \{ V^n(\beta) = \bigcup_{j=1}^{\infty} \bigcup_{i=n}^{\infty} \{b_{ij}, \beta\}, n=1,2,\dots \}$$

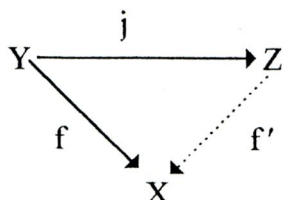
Then Z is minimal Hausdorff and hence it is absolutely closed (This example is also due to Urysohn [18]).

Let $Y = \{a_{ij}, c_i, \alpha, \beta, i=1,2,3,\dots, j=1,2,\dots\}$. Then Y is absolutely closed because $Y = A \cup B$ where $A = \{a_{ij}, c_i, \alpha, i=1,2,3,\dots, j=1,2,\dots\}$ and $B = \{\beta\}$ as subspaces of Z which are both absolutely closed.

Assume X is absolutely closed and injective.

To Prove : X is singleton.

Suppose there exists $a, b \in X$ such that $a \neq b$. Since X is Hausdorff there exists G_1, G_2 such that G_1 is an open neighbourhood of a in X and G_2 is an open neighbourhood of b in X and $G_1 \cap G_2 = \emptyset$. Let us define $f: Y \rightarrow X$ as follows $f(A) = a$ and $f(B) = b$. Thus f is continuous, because A, B are open in Y . Let $j: Y \rightarrow Z$ be the injection. Then j is one-one and continuous. Consider the following diagram.



Now, since X is injective, there exists $f' : Z \rightarrow X$ such that f' is continuous and $f = f' \circ j$. Thus $f'(A) = a$ and $f'(B) = f(B) = b$.

$$\text{Hence } (f')^{-1}(G_1) \supset (f')^{-1}(a) \supset A$$

$$(f')^{-1}(G_2) \supset (f')^{-1}(b) \supset B$$

$$\text{and } (f')^{-1}(G_1) \cap (f')^{-1}(G_2) = \phi$$

Thus $(f')^{-1}(G_1)$ and $(f')^{-1}(G_2)$ are disjoint open sets containing A and B respectively.

But in Z , the sets A, B do not have disjoint open neighbourhoods. This is a contradiction. Hence X cannot have two distinct points. Thus $X = \{\text{a single point}\}$.

Obviously. If $X = \{\text{a single point}\}$ then X is injective.

CHAPTER - III

CHAPTER - III

ON GENERALISATIONS OF H-CLOSED SPACES

By generalising the concept of H-closed spaces, Somasundaram and Padmavathy [15] have defined pre T_2 -closed spaces using preopen sets. In section 1, we discuss the properties of pre T_2 -closed spaces and locally pre T_2 -closed spaces. Moreover, the existence of a projective maximum and projective minimum in the class of one-point extensions for a locally pre T_2 -closed extremally disconnected Hausdorff space is established. Section 2 deals with p-regular closed spaces, their characterisation and their properties. In section 3 characterisation of projective objects in the category of pre T_2 -closed extremally disconnected Hausdorff spaces and continuous maps is obtained.

SECTION : 3.1***Pre T_2 -closed spaces and locally pre T_2 -closed spaces.***

In this section we give the definitions of pre T_2 -closed spaces and locally pre T_2 -closed spaces and discuss their characterisations and their one-point extensions. Moreover, construction of projective minimum and projective maximum in the set of all one-point Hausdorff pre T_2 -closed extensions of a locally pre T_2 -closed Hausdorff space is discussed.

Definition : 3.1.1

Let (X, τ) be a topological space. Let $A \subset X$. Then (i) A is said to be **pre T_2 -closed relative to X** if and only if every preopen cover \mathcal{U} of A has a finite subfamily $\mathcal{U}' \subset \mathcal{U}$ such that $A \subset \cup \{ \text{cl } U / U \in \mathcal{U}' \}$. (ii) A is said to be a **pre T_2 -closed** set if and only if $(A, \tau/A)$ is pre T_2 -closed.

Definition : 3.1.2

A space (X, τ) is said to be **locally pre T_2 -closed** if for each $x \in X$ there is an open set U containing x such that the closure of U is pre T_2 -closed .

Example : 3.1.3

Let (X, τ) be a topological space with the indiscrete topology. Every subset of X is preopen and dense in X . So for every preopen cover \mathcal{U} and $U \in \mathcal{U}$, $\text{cl } U = X$ and (X, τ) is pre T_2 -closed.

Example : 3.1.4

Consider (X, τ) with the cofinite topology. Every infinite set is preopen and dense in X . Hence (X, τ) is pre T_2 -closed .

Theorem : 3.1.5

Let (X, τ) be a topological space. Then (i) if $A \subset X$ is pre T_2 -closed relative to X , then A is closed in X , if X is Hausdorff. (ii) if $A \subset X$ is an open set, A is pre T_2 -closed relative to X iff A is a pre T_2 -closed set.

Proof :

(i) Assume X is Hausdorff and A is pre T_2 -closed relative to X . Take $x \in X - A$. Since X is Hausdorff, for each $y \in A$ there exist open neighbourhoods U_y and V_y of distinct points x and y respectively such that $U_y \cap V_y = \emptyset$. Now the collection $\{V_y / y \in A\}$ is an open cover of A and hence a preopen cover of A . Since A is pre T_2 -closed relative to X there exists a finite subset $B \subset A$ such that $A \subset \bigcup \{\text{cl } V_y / y \in B\}$. Let $U = \bigcap \{U_y / y \in B\}$. Since finite intersection of open sets is open we have U is open. Also $x \in U$ and $A \cap U = \emptyset$. i.e., $U \subset X - A$. Hence $X - A$ is open. $\therefore A$ is closed.

(ii) Assume A to be a pre T_2 -closed set. By definition, $(A, \tau/A)$ is pre T_2 -closed .

Let $\{U_y/y \in A\}$ be a preopen cover of A with $U_y \in \text{PO}(X)$, for each $y \in A$.

Consider $V_y = A \cap U_y$. Since A is open, V_y is preopen in $A \Rightarrow \{V_y / y \in A\}$ is a preopen cover of A in A . Given $(A, \tau/A)$ is pre T_2 -closed . Therefore there exists a

finite subset $B \subset A$ such that $A \subset \cup \{cl_A V_y / y \in B\}$ and $cl_A V_y = cl V_y \cap A \subset cl U_y$

Therefore $A \subset \cup \{cl U_y / y \in B\}$. Thus A is pre T_2 -closed relative to X .

Conversely assume A is pre T_2 -closed relative to X .

Let $\{U_\alpha / \alpha \in \lambda\}$ be a preopen cover of A in A i.e., each U_α is a preopen subset of A .

$$\therefore U_\alpha \subset \text{int}_A cl_A U_\alpha \subset \text{int}_A cl U_\alpha \subset \text{int} cl U_\alpha$$

$$\therefore U_\alpha \subset \text{int} cl U_\alpha$$

$\Rightarrow U_\alpha$ is preopen in X .

$\therefore \{U_\alpha\}$ will be a preopen cover of A in X . Then there is a finite subfamily $B \subset A$

such that $A \subset \cup \{cl U_\alpha / \alpha \in B\}$.

$\Rightarrow A$ is a pre T_2 -closed set.

The following two theorems give characterisations of pre T_2 -closed spaces.

Theorem : 3.1.6

Let (X, τ) be a topological space. Then the following conditions are equivalent.

- (i) X is pre T_2 -closed .
- (ii) The closure of an open subset of X is pre T_2 -closed relative to X .
- (iii) Every closed open subset of X is a pre T_2 -closed set.

Proof: (i) \Rightarrow (ii)

Assume X is pre T_2 -closed. Let U be an open subset of X and let $A = cl U$

Let \mathcal{U} be a preopen cover of A . Consider $D = \mathcal{U} \cup (X - A)$. Then D is a preopen cover of X . Since X is pre T_2 -closed there exists a finite subfamily $\{V_i, i=1,2,\dots,n\}$ in D such that $X = \bigcup \{cl V_i / i=1,2,\dots,n, V_i \in D\}$. If no V_i is $X-A$, then $A \subset \bigcup \{cl V_i / i=1,2,\dots,n, V_i \in \mathcal{U}\}$. If $V_k = X - A$, then as $U \subset \text{int } cl U$, we get $U \subset \text{int } A = X - cl(X - A) = X - cl V_k \subset \bigcup \{cl V_i / i = 1,2,\dots, k-1, k+1,\dots,n, V_i \in \mathcal{U}\}$
 $\therefore A = cl U \subset \bigcup \{cl V_i / i = 1,2,\dots, k-1, k+1,\dots,n, V_i \in \mathcal{U}\}$.

Hence (ii)

(ii) \Rightarrow (iii)

Let A be both open and closed in X . Then $A = cl A$. By (ii) A is pre T_2 -closed relative to X . Since A is an open set, by applying part (ii) of theorem 3.1.5 we get A is a pre T_2 -closed.

(iii) \Rightarrow (i)

Follows easily from part (ii) of theorem 3.1.5, since X is both open and closed.

Remark : 3.1.7

If X is extremally disconnected in the above theorem, then the closure of an open set is a pre T_2 -closed set whenever it is pre T_2 -closed relative to X .

Theorem : 3.1.8

Let (X, τ) be a topological space and $A \subset X$. Then A is a pre T_2 -closed if and only if every regular open cover has a finite subcover whose closures cover A .

Proof :

Assume A is pre T_2 -closed. Let $\{V_\alpha / \alpha \in I\}$ be a regular open cover of A . Since $V_\alpha = \text{int}(cl V_\alpha)$, V_α is preopen. As A is pre T_2 -closed, there is a finite subcollection $\{V_i / i = 1,2,\dots,n\}$ such that $A \subset \bigcup \{cl V_i / i = 1,2,\dots,n\}$. Conversely,

let $\{U_\alpha / \alpha \in I\}$ be a preopen cover of A , with $U_\alpha \in PO(X)$. Let $W_\alpha = \text{int}(\text{cl } U_\alpha)$ for every $\alpha \in I$. $\therefore \{W_\alpha / \alpha \in I\}$ is a regular open cover of A .

By assumption, there exists a finite subcollection W_1, W_2, \dots, W_k such that

$$A \subset \cup \{\text{cl } W_i / i = 1, 2, \dots, k\}$$

$$= \cup \{\text{cl int}(\text{cl } U_i) / i = 1, 2, \dots, k\}.$$

Since $U_i \subset \text{int cl } U_i$, $\text{cl } U_i \subset \text{cl}(\text{int cl } U_i)$ ----- ①

Always $\text{cl}(\text{int cl } U_i) \subset \text{cl } U_i$ ----- ②

Hence $\text{cl}(\text{int cl } U_i) = \text{cl } U_i$. \therefore We have $A \subset \cup \{\text{cl } U_i / i = 1, 2, \dots, n\}$.

Hence the theorem.

Corollary : 3.1.9

A space (X, τ) is pre T_2 -closed if and only if every regular open cover has a finite subcover, the closures of whose members cover X .

Theorem : 3.1.10

If B is an open pre closed subset of A which is open pre T_2 -closed relative to X , then B is pre T_2 -closed relative to X .

Proof : Let $\mathcal{U} = \{U_y / y \in B\}$ be a preopen cover of B relative to X . Since A is open and $A - B$ is preopen in A , $A - B$ is preopen in X . Let W be a preopen set in X such that $W \cap A = A - B$. Then $W \cap A \in PO(X)$. $\therefore A = \mathcal{U} \cup W$ is a preopen cover of A .

Since A is pre T_2 -closed, there exists a finite subfamily $\mathcal{A}' \subset \mathcal{A}$ such that

$A \subset \cup \{\text{cl } U_y / y \in \mathcal{A}'\}$. Suppose if no U_y is W , we get that the subset

$B \subset \cup \{\text{cl } U_y / U_y \in \mathcal{A}' \subset \mathcal{U}\}$. If some, $U_y = W$, then we have to prove that $B \cap \text{cl } W = \phi$.

Suppose $B \cap \text{cl } W \neq \phi$, then there exists $x \in B \cap \text{cl } W$. $\Rightarrow x \in B$ and $x \in \text{cl } W$. Since

B is open, $x \in \text{cl}W$ must imply that $B \cap W \neq \phi$, i.e., there exists $x \in B \cap W \subset A \cap W = A - B$, which is impossible. Hence $B \cap \text{cl}W = \phi$ and therefore $B \subset \cup \{ \text{cl}U_y / U_y \in \mathcal{A}' - W \}$.

Thus B is pre T_2 -closed relative to X .

Theorem : 3.1.11

Let $B \subset A \subset X$ and A be open in X . Then B is pre T_2 -closed relative to X iff B is pre T_2 -closed relative to A .

Proof : First assume that, B is pre T_2 -closed relative to A . Let $\mathcal{U} = \{U_y / y \in B\}$ be a preopen cover of B , such that $y \in U_y$. Since A is open, we have $\{U_y \cap A / U_y \in \mathcal{U}\}$ is a preopen cover of B in the relative topology of A . By assumption, there is a finite subset $C \subset B$. Such that $B \subset \cup \{ \text{cl}_A(U_y \cap A) / y \in C \}$. But $\text{cl}_A(U_y \cap A) = \text{cl}(U_y \cap A) \cap A \subset \text{cl}U_y$. Hence $B \subset \cup \{ \text{cl}U_y / y \in C \}$. Hence $B \subset \cup \{ \text{cl}U_y / y \in C \}$. Then B is pre T_2 -closed relative to X . Conversely, assume that B is pre T_2 -closed relative to X . Let $\{V_y / y \in B\}$ be a pre open cover of B with $V_y \in \text{PO}(A)$. For every y since A is open, by property 1.1.3(4) we have $V_y \in \text{PO}(X)$. $\therefore \{V_y / y \in B\}$ is a pre open cover of B in X . By assumption, there is a finite subset D of B , $B \subset \cup \{ \text{cl} V_y / y \in D \subset B \}$.

$$\begin{aligned} \text{Now } B &= B \cap A \subset \cup \{ \text{cl} V_y / y \in D \} \cap A \\ &= \cup \{ \text{cl} V_y \cap A / y \in D \} \\ &= \cup \{ \text{cl}_A V_y / y \in D \}. \end{aligned}$$

$\therefore B$ is pre T_2 -closed relative to A .

Theorem : 3.1.12

Let $\{A_i / A_i \subset X, i=1,2,\dots,n\}$ be pre T_2 -closed relative to X . Then $\cup \{A_i / i=1,2,\dots,n\}$ is pre T_2 -closed relative to X .

Proof:

Let $A = \bigcup_{i=1}^n A_i$ and let $\{U_\alpha \mid \alpha \in I\}$ be a preopen cover of A relative to X .

Then this will be a preopen cover of A_i for each i . Hence there is a finite subset I_i of I

Such that $A_i \subset \bigcup \{cl U_\alpha \mid \alpha \in I_i\}$. $\therefore A = \bigcup_{i=1}^n A_i \subset \bigcup \{cl U_\alpha \mid \alpha \in \bigcup_{i=1}^n I_i\}$. Hence we get,

from the above theorem that finite union of sets which are pre T_2 -closed relative to X are pre T_2 -closed. Regarding the products, we get that the product of pre T_2 -closed space and a compact space is pre T_2 -closed, as is seen from the following theorem.

Theorem : 3.1.13

Let X be a pre T_2 -closed space and Y a compact space. Then the product $X \times Y$ is pre T_2 -closed.

Proof :

Let \mathcal{U} be a regular open cover of $X \times Y$. Therefore for each $x \in X$, since Y is compact $\{x\} \times Y$ is compact. Since \mathcal{U} is an open cover of $\{x\} \times Y$ also, there exists a finite subcover, say, $\{U_j(x), j=1,2, \dots, n(x)\}$ of \mathcal{U} which covers $\{x\} \times Y$. By tubelemma, theorem 1.4.5 there is an open set V_x containing x such that

$$\{x\} \times Y \subset V_x \times Y \subset \{U_j(x) \mid j = 1, 2, \dots, n(x)\}$$

$$\therefore cl(V_x \times Y) = cl V_x \times Y \subset \bigcup \{cl U_j(x) \mid j = 1, 2, \dots, n(x)\}$$

$$\Rightarrow \text{int } cl(V_x \times Y) = \text{int } cl V_x \times Y \subset \text{int } \bigcup \{cl U_j(x) \mid j = 1, 2, \dots, n(x)\}$$

$$\subset \bigcup \{cl U_j(x) \mid j = 1, 2, \dots, n(x)\}.$$

Let $W_x = \text{int cl } V_x$. Then $\{W_x / x \in X\}$ is a regular open cover for X and $W_x \times Y \subset \cup \{\text{cl } U_j(x) / j=1,2,\dots,n(x)\}$. Since X is pre T_2 -closed, there exists a finite subfamily $\{W_{x_1}, W_{x_2}, \dots, W_{x_m}\}$ such that $X \subset \cup \{\text{cl } W_{x_i} / i=1,2,\dots,m\}$.

$$\begin{aligned} \text{Hence } X \times Y &\subset \cup \{\text{cl } W_{x_i} \times Y / i=1,2,\dots,m\} \\ &\subset \cup \{\cup \text{cl } U_j(x_i) / i=1,2,\dots,m, j=1,2,\dots,n(x)\}. \end{aligned}$$

$\therefore X \times Y$ is pre T_2 -closed.

Theorem : 3.1.14

Let $f : X \rightarrow Y$ be a continuous surjection. If X is pre T_2 -closed so is Y .

Proof :

Let $\{U_\alpha / \alpha \in I\}$ be a preopen cover of Y . Since U_α is preopen, $U_\alpha \subset \text{int cl } U_\alpha$ for every $\alpha \in I$. Since f is continuous, $\{f^{-1}(\text{int cl } U_\alpha) / \alpha \in I\}$ is an open cover of X and hence it is a preopen cover of X . As X is a pre T_2 -closed, there exists a finite subset $B \subset I$, such that $X \subset \cup \{\text{cl } (f^{-1}(\text{int cl } U_\alpha)) / \alpha \in B\}$. Since f is a surjection, $Y = f(X)$.

$$\begin{aligned} \text{Hence } Y &\subset f\{\cup(\text{cl } f^{-1}(\text{int cl } U_\alpha)) / \alpha \in B\} \\ &= \cup\{f(\text{cl } f^{-1}(\text{int cl } U_\alpha)) / \alpha \in B\} \\ &\subset \cup\{\text{cl}(f(\text{cl } f^{-1}(\text{int cl } U_\alpha))) / \alpha \in B\} \\ &\subset \cup\{\text{cl } (\text{int cl } U_\alpha) / \alpha \in B\} \\ &= \cup\{\text{cl } U_\alpha / \alpha \in B\}. \end{aligned}$$

$\therefore Y$ is pre T_2 -closed. Thus continuous image of a pre T_2 -closed space is pre T_2 -closed.

The following theorem gives a characterisation of a locally pre T_2 -closed extremally disconnected space.

Theorem : 3.1.15

Let (X, τ) be an extremally disconnected space. Then the following are equivalent.

- (i) X is locally pre T_2 -closed.
- (ii) Each point of X has a neighbourhood which is pre T_2 -closed .
- (iii) Each point of X has a neighbourhood which is pre T_2 -closed set.

Proof: Follows by the fact that in an extremally disconnected space, closure of an open set is open.

Theorem : 3.1.16

Let X be a locally pre T_2 -closed extremally disconnected (Hausdorff) space and Y a compact (Hausdorff) space. The product $X \times Y$ is locally pre T_2 -closed.

Proof: Take $(x, y) \in X \times Y$. Since X is locally pre T_2 -closed the point x has a neighbourhood U_x whose closure is pre T_2 -closed. As X is also extremally disconnected, $\text{cl } U_x$ is open. Hence by part (ii) of theorem 3.1.5, we get that $\text{cl } U_x$ is a pre T_2 -closed set. Let $Z = \text{cl } U_x$. Then $(Z, \tau/Z)$ is a pre T_2 -closed space. Since Y is compact, by theorem 3.1.13, we get that $Z \times Y$ is pre T_2 -closed. Now $U_x \times Y$ is an open neighbourhood of (x, y) whose closure, $\text{cl } (U_x \times Y) = \text{cl } U_x \times Y = Z \times Y$ is pre T_2 -closed. Thus $X \times Y$ is locally pre T_2 -closed.

Now let us discuss the **one-point extensions** of a pre T_2 -closed space.

Definition : 3.1.17 A topological space (Y, \mathcal{U}) is said to be an **extension** of (X, τ) if

$x \subset y$, $\mathcal{U} / X = \tau$, $\text{cl}_Y (X) = Y$. (Y, \mathcal{U}) is a **one-point pre T_2 -closed extension**.

of (X, τ) if (Y, \mathcal{U}) is an extension of (X, τ) , (Y, \mathcal{U}) is pre T_2 -closed and $(Y-X)$ is singleton.

Theorem : 3.1.18

Let (X, τ) be a Hausdorff space. If (Y, \mathcal{U}) is a one-point Hausdorff pre T_2 -closed extension of (X, τ) then

- (1) X is open in Y .
- (2) X is pre T_2 -closed relative to Y if and only if $(Y-X)$ is \mathcal{U} -open.
- (3) X is locally pre T_2 -closed.

Proof :

- (1) Let $Y - X = \{\pi\} \Rightarrow Y - \pi = X$

Since Y is Hausdorff, $\{\pi\}$ is closed. Therefore X is \mathcal{U} -open in Y .

- (2) First assume $Y - X$ is \mathcal{U} -open. Then X is closed relative to Y . Hence by(1),

we get X is a closed open subspace of Y . Since Y is pre T_2 -closed, by

theorem 3.1.6, we have X is pre T_2 -closed relative to Y . Conversely, assume X

is pre T_2 -closed relative to Y . Since Y is Hausdorff, by part (i) of theorem

3.1.5 we have X is closed in Y . Hence $Y-X$ is \mathcal{U} -open.

- (3) Since Y is Hausdorff, for each $x \in X$ there exist disjoint neighbourhoods U and

V such that $x \in U$ and $\pi \in V$, since $V \cap U = \emptyset$, $\pi \notin \text{cl}_Y U$. Since $\pi \notin U$ and X

is \mathcal{U} -open, U is open in X . Also $\text{cl}_X U = \text{cl}_Y U \cap X = \text{cl}_Y U$. Since Y is pre

T_2 -closed, by theorem 3.1.6, $\text{cl}_Y U$ is pre T_2 -closed relative to Y . Since

$\text{cl}_X U = \text{cl}_Y U \subset X \subset Y$ and since X is open by theorem 3.1.11, $\text{cl}_X U$ is pre T_2 -closed relative to X . Hence X is locally pre T_2 -closed.

Theorem : 3.1.19

Let (X, τ) be a locally pre T_2 -closed Hausdorff space which is not pre T_2 -closed. Let $X^* = X \cup \{\pi\}$ where $\pi \notin X$. Then

- (a) $\tau^* = \tau \cup \{\{\pi\} \cup V / V \in \mathcal{F}\}$ is a Hausdorff topology on X^* , where \mathcal{F} is a filter generated by $\mathcal{B} = \{U / U \in \tau, \text{ such that } (X - U) \text{ is pre } T_2\text{-closed}\}$.
- (b) (X^*, τ^*) is a one-point pre T_2 -closed extension of (X, τ) .
- (c) (X^*, τ^*) is the projective minimum in the set of all one-point Hausdorff space pre T_2 -closed extensions of (X, τ) .

Proof :

Claim(i) : \mathcal{B} is a filter base.

Since X is not pre T_2 -closed, $\emptyset \notin \mathcal{B}$. Let $\{U_i / i = 1, 2, \dots, n\} \in \mathcal{B}$.

To prove $\bigcap_{i=1}^n U_i \in \mathcal{B}$.

Since $U_i \in \mathcal{B}$, for $i = 1, 2, \dots, n$ $U_i \subset X$ and $(X - U_i)$ is pre T_2 -closed $\forall i = 1, 2, \dots, n$.

Hence by Theorem 3.1.12 $\bigcup_{i=1}^n (X - U_i)$ is pre T_2 -closed. ie., $X - \bigcap_{i=1}^n U_i$ is pre T_2 -closed.

Hence \mathcal{B} is a preopen filter base.

Claim (ii) : τ^* is a Hausdorff topology on X^* .

Consider $x, y \in X^*$.

Case (i)

If both $x, y \in X$, there is nothing to prove as (X, τ) itself is Hausdorff.

Case (ii)

If $x, y \in X$ where $x \neq \pi$ and $y = \pi$, Since X is locally pre T_2 -closed there is an open set $U \in \tau$ containing x such that the closure of U is pre T_2 -closed. Since $\text{cl } U$ is pre T_2 -closed, $X - \text{cl } U$ is an element of \mathcal{I} . Hence $\{\pi\} \cup (X - \text{cl } U)$ is an open neighbourhood of π in X^* . Also $U \cap \{\{\pi\} \cup (X - \text{cl } U)\} = \phi$.

\therefore We get two disjoint neighbourhoods x and π in X^* .

Hence τ^* is a Hausdorff topology on X^* . Hence (a) is proved.

Claim (iii) : X^* is pre T_2 -closed.

Let \mathcal{g} be a τ^* -pre open cover of X^* . Then there exists $G_0 \in \mathcal{g}$ such that $\pi \in G_0 \subset \text{int}_{X^*} \text{cl}_{X^*} G_0$. Since $\text{int}_{X^*} \text{cl}_{X^*} G_0$ is open in X^* and Since \mathcal{B} is a filter base for \mathcal{I} , there is an open set $U \subset X$ such that $X - U$ is pre T_2 -closed relative to X and $\{\pi\} \cup U \subset \text{int}_{X^*} \text{cl}_{X^*} G_0 \subset \text{cl}_{X^*} G_0$. Since X is open, $\{G \cap X / G \in \mathcal{g}\}$ is a pre open cover of $X - U$. Hence there exists $G_i \in \mathcal{g}$, $i = 1, 2, \dots, n$. Such that

$$\begin{aligned} (X - U) &\subset \cup \{ \text{cl}_X(G_i \cap X) / i = 1, 2, \dots, n \} \\ &\subset \cup \{ \text{cl}_{X^*} G_i / i = 1, 2, \dots, n \}. \end{aligned}$$

$$\begin{aligned} \text{Thus } X^* &= \{ \{\pi\} \cup U \} \cup (X - U) \\ &\subset (\text{cl}_{X^*} G_0 \cup (\cup \text{cl}_{X^*} G_i) / i = 1, 2, \dots, n). \end{aligned}$$

i.e., $X^* \subset \cup \{ \text{cl}_{X^*} G_i / i = 0, 1, 2, \dots, n \}$. Thus X^* is pre T_2 -closed. Hence (b).

Claim (iv) (X^*, τ^*) is the projective minimum in the set of all one-point pre T_2 -closed extensions of (X, τ) .

Let (Y, T) be any one point pre T_2 -closed extension of (X, τ) . Then $X \subset Y$, $T|_X = \tau$, $\text{cl}_Y(X) = Y$, (Y, T) is pre T_2 -closed and $(Y - X) = \text{singleton say } \{\xi\}$. Define $f: Y \rightarrow X^*$ by $f(x) = x \forall x \in X$ and $f(\xi) = \pi$.

Since $f|_X$ is identity and X is open in X^* and Y it is enough if we prove that f is continuous at ξ . Let G be a basic open neighbourhood of π in X . Then $G = \{\pi\} \cup U$, where $U \in \tau$. $X - U$ is pre T_2 -closed in X . As X is open in Y , by theorem 3.1.11, $X - U$ is pre T_2 -closed in Y . As Y is Hausdorff, by part(i) of theorem 3.1.5, $X - U$ is closed in Y . $\therefore Y - (X - U) = \{\xi\} \cup U$ is open in Y . Hence $\{\xi\} \cup U$ is an open neighbourhood of ξ in Y . Since $f^{-1}(G) = \{\xi\} \cup U$, we get f is continuous at ξ . Hence f is continuous and X^* is the projective minimum.

Theorem : 3.1.20

Let (X, τ) be a locally pre T_2 -closed extremally disconnected Hausdorff space which is not pre T_2 -closed. Let $X^* = X \cup \{\pi\}$ where $\pi \notin X$, then

- a) $\tau^\# = \tau \cup \{\{\pi\} \cup U / U \in \tau, (X - \text{int cl } U) \text{ is pre } T_2\text{-closed relative to } X\}$ is a Hausdorff topology on X^* .
- b) $(X^*, \tau^\#)$ is a one-point pre T_2 -closed extension of (X, τ) .
- c) $(X^*, \tau^\#)$ the projective maximum in the set of all one-point pre T_2 -closed extensions of (X, τ) .

Proof : Let $\beta = \{U / U \in \tau, (X - \text{int cl } U) \text{ is pre } T_2\text{-closed relative to } X\}$.

Claim : β is an open filter on X .

(i) Since X is not pre T_2 -closed, $\phi \notin \beta$.

(ii) It can be shown that $\bigcap_{i=1}^n U_i \in \beta$ for $U_i \in \beta$ ($i = 1, 2, \dots, n$).

(iii) Let $U \in \beta$ and $U \subset V$ with $V \in \tau$. Then $(X - \text{int cl } V) \subset (X - \text{int cl } U)$. Since

$X - \text{int cl } U = \text{cl}(X - \text{cl } U)$ and as X is extremally disconnected, we get $X - \text{int cl } U$

is open in X . Hence, as $(X - \text{int cl } U)$ is pre T_2 -closed relative to X , by theorem

3.1.5, we get that $(X - \text{int cl } U)$ is pre T_2 -closed. Therefore by theorem 3.1.6, we

get $(X - \text{int cl } V)$ is pre T_2 -closed. Therefore $V \in \beta$ and β is an open filter on X .

Hence the claim.

Let us now show that $\tau^\#$ is a Hausdorff topology on X^* .

Consider $x, y \in X^*$

Case (i) If both $x, y \in X$. Since (X, τ) is a Hausdorff space, we get the required

disjoint neighbourhoods

Case (ii) If $x, y \in X$, where $x \neq \pi$ and $y = \pi$. Since X is locally pre T_2 -closed,

there is an open set U containing x such that $\text{cl } U$ is pre T_2 -closed. As X is extremally

disconnected, $\text{cl } U$ is open. Therefore $\text{int cl } U = \text{cl } U$ Hence $\text{int cl } U$ is pre T_2 -

closed. Therefore $X - \text{int cl } U \in \beta$. Thus U and $\{\pi\} \cup \{X - \text{int cl } U\}$ are required

disjoint neighbourhoods of x and π respectively. Hence $\tau^\#$ is a Hausdorff topology

on X^* .

To prove : $(X^*, \tau^\#)$ is pre T_2 -closed

Let \mathcal{g} be a $\tau^\#$ preopen cover of X^* , there exists $G_0 \in \mathcal{g}$, such that $\pi \in G_0 \subset \text{int}_{X^*} \text{cl}_{X^*} G_0$. Now $\text{int}_{X^*} \text{cl}_{X^*} G_0$ is a $\tau^\#$ open set containing π and therefore we can find, $U \in \tau$ such that $\{\pi\} \cup U \subset \text{int}_{X^*} \text{cl}_{X^*} G_0 = U'$ and $(X - \text{int}_X \text{cl}_X U)$ is pre T_2 -closed relative to X . Now since X is open $\{G \cap X / G \in \mathcal{g}\}$ is a τ - preopen cover of $X - U$. Now, $U \subset \text{int}_X \text{cl}_X U \Rightarrow X - \text{int}_X \text{cl}_X U \subset X - U$. Therefore $\{G \cap X / G \in \mathcal{g}\}$ is also a preopen cover of $(X - \text{int}_X \text{cl}_X U)$ and since $X - \text{int}_X \text{cl}_X U$ is pre T_2 - closed, there exists $G_1, G_2, \dots, G_n \in \mathcal{g}$ such that $(X - \text{int}_X \text{cl}_X U) \subset \cup \{\text{cl}_X (G_i \cap X) / i=1,2,\dots,n\}$

$$\subset U \cup \{\text{cl}_{X^*} (G_i) / i=1,2,\dots,n, G_i \in \mathcal{g}\}.$$

Now $X^* = \{ \{\pi\} \cup \text{int}_X \text{cl}_X U \} \cup \{ X - \text{int}_X \text{cl}_X U \}$.

$$\{\pi\} \cup \text{int}_X \text{cl}_X U \subset \{\pi\} \cup \text{cl}_X U \subset \text{cl}_{X^*} U' = \text{cl}_{X^*} (\text{int}_{X^*} \text{cl}_{X^*} G_0) = \text{cl}_{X^*} G_0.$$

$$\therefore X^* \subset \text{cl}_{X^*} G_0 \cup \{ \cup \text{cl}_{X^*} G_i / i=1,2,\dots,n, G_i \in \mathcal{g} \}$$

$$= \{ \text{cl}_{X^*} G_i / i=0,1,2,\dots,n, G_i \in \mathcal{g} \}.$$

Thus $(X^*, \tau^\#)$ is pre T_2 - closed .

Let us now show that $(X^*, \tau^\#)$ is the projective maximum.

Let (Y, T') be any other one point pre T_2 - closed extension of (X, τ) .

Define $f: X^* \rightarrow Y$ by $f(x) = x \forall x \in X$ and $f(\pi) = \eta$ where $\eta \in Y - X$. Since X^* and Y are Hausdorff X is open in X^* and Y . Moreover f is the identity map on X Hence it is enough if we prove that f is continuous at π .

Let U be a T' - Open neighbourhood of η . $f^{-1}(U) = (U \cap X) \cup \{\pi\}$.

Now to prove $f^{-1}(U)$ is a $\tau^\#$ - open neighbourhood of π .

Since Y is pre T_2 - closed, $\text{cl}_Y(Y - \text{cl}_Y U)$ is pre T_2 - closed [by theorem 3.1.6]. And now $\text{cl}_Y(Y - \text{cl}_Y U) \subset X \subset Y$, where $X \in T'$. Therefore by theorem 3.1.11, $\text{cl}_Y(Y - \text{cl}_Y U)$ is pre T_2 - closed relative to X .

$$\text{Now } \text{int}_X \text{cl}_X (U \cap X) = \text{int}_Y \text{cl}_Y U \cap X$$

$$\Rightarrow Y - \text{int}_Y \text{cl}_Y U = X - \text{int}_X \text{cl}_X (U \cap X)$$

$$\Rightarrow \text{cl}_Y(Y - \text{cl}_Y U) = X - \text{int}_X \text{cl}_X U$$

$$\Rightarrow X - \text{int}_X \text{cl}_X (U \cap X) \text{ is pre } T_2 \text{ - closed relative to } X. \text{ Hence } \{\pi\} \cup (U \cap X) \in \beta$$

i.e $f^{-1}(U)$ is open in $\tau^\#$. Therefore f is continuous of η .

SECTION : 3.2

p - regular closed spaces

This section is devoted to the study of p - regular closed spaces, their characterisation and their behaviour under continuous maps.

Definition : 3.2.1

A p - regular spaces (X, τ) is said to be **p - regular closed** if it is closed in every p - regular space in which it can be embedded.

Definition : 3.2.2

A cover \mathcal{U} is said to be a **p - regular cover** if there exists a preopen cover \mathcal{V} such that preclosures of whose members refine \mathcal{U} .

Definition 3.2.3

A filter base is said to be a **p-regular filter base** if it is equivalent to a preclosed filter base.

Theorem 3.2.4

The following are equivalent for a p -regular space X

- (1) X is p -regular closed.
- (2) Every open p -regular cover has a finite subcover .
- (3) Every open p -regular filter base has non-empty adherence.

Proof : (1) \Rightarrow (2)

Assume X is p -regular closed. Let \mathcal{U} be an open p -regular cover of X and suppose \mathcal{U} does not have a finite subcover.

Let $Y = X \cup \{p\}$ where $p \notin X$

Let $\tau^* = \tau \cup \{ \{p\} \cup (X - \bigcup_{i=1}^n U_i) \mid U_i \in \mathcal{U} \}$. Then τ^* is a topology on Y .

Let us now show that (Y, τ^*) is p -regular.

Case(i)

Let $x \in Y$ and $x \neq p$. Let B be a τ^* -closed set containing neither x nor p . Then $Y - B$ is a τ^* -open set containing both x and p . Since \mathcal{U} is a p -regular cover, there exists a preopen cover \mathcal{V} of X . Such that $\{p\} \cup \{V \mid V \in \mathcal{V}\}$ is a refinement of \mathcal{U} . Since $x \in X$, $x \in V$ for some $V \in \mathcal{V}$. Moreover $Y - B = \{p\} \cup \{ (X - \bigcup_{i=1}^n U_i) \mid U_i \in \mathcal{U} \}$.

Therefore $x \in D = V \cap \{ \{p\} \cup (X - \bigcup_{i=1}^n V_i) \}$ which is a preopen set in τ^* .

Also, as $Y - B = Y - \{ \bigcup_{i=1}^n U_i \}$

$$B = \bigcup_{i=1}^n U_i.$$

Therefore we get D and $\bigcup_{i=1}^n U_i$ are disjoint τ^* - preopen sets containing x and B respectively.

Case (ii)

Suppose that $x=p$ and B is a τ^* - closed set not containing p .

Then $Y - B = \{p\} \cup (X - \bigcup_{i=1}^n U_i)$ is a τ^* - open and hence a preopen set containing p

and $\bigcup_{i=1}^n U_i$ is a τ^* openset containing B .

Case (iii)

Suppose $x \neq p$. Let B be a closed set containing p . i.e $Y-B$ is open, not containing p and therefore it is a τ -open set containing x . As X is p -regular, there exist a preopen set V containing x such that $x \in V \subset \text{Pcl } V \subset Y-B$ (since from theorem 1.2.3) Since X is open, V is preopen in Y . Thus (Y, τ^*) is p -regular. Since every openset containing p intersects X , X is not closed in Y which is a contradiction.

$\therefore \mathcal{U}$ has a finite subcover

(2) \Rightarrow (3)

Let \mathcal{F} be an open p -regular filter base without an adherent point in X .

Then $(X - \text{cl } F \mid F \in \mathcal{F})$ is an open cover of X . Let $x \in X$. Since x is not an adherent point of \mathcal{F} , $x \notin \text{cl } F_x$ for some $F_x \in \mathcal{F}$. Since X is p -regular, there exists a preopen set V_x containing x such that $x \in V_x \subset \text{Pcl } V_x \subset (X - \text{cl } F_x)$. Thus $\mathcal{V} = \{V_x : x \in X\}$ is a preopen cover of X such that $\{\text{Pcl } V_x : x \in X\}$ is a refinement of $\{x - \text{cl}$

$\{F_x \mid F_x \in \mathcal{F}\} \Rightarrow \{X - \text{cl } F_x \mid F_x \in \mathcal{F}\}$ is a p-regular cover of X . Therefore there exists finitely many members F_1, F_2, \dots, F_n of \mathcal{F} such that $X = \bigcup_{i=1}^n (X - \text{cl } F_i) \subset$

$$\bigcup_{i=1}^n (X - F_i) = X - \bigcap_{i=1}^n F_i.$$

$\Rightarrow \bigcup_{i=1}^n F_i = \phi$. This is a contradiction to the fact that \mathcal{F} is a filter base.

\therefore every open p-regular filter base has a non empty ashherence.

(3) \Rightarrow (1) Proof similar to the case of H-closed spaces.

Theorem : 3.2.5

Let X be a p-regular closed space. Let $Y \subset X$ be a closed open subset. Then Y is p-regular closed.

Proof :

Every semi-open subset of a p-regular space is p-regular. Since Y is closed open, it is semi-open and therefore p-regular. To prove $(Y, \tau/Y)$ possesses the property that every open p-regular cover has a finite subcover.

Let \mathcal{U} be an open p-regular cover of Y . Let $\mathcal{H} = \{H \subset X, U = H \cap Y; U \in \mathcal{U}\} \cup (X - Y)$ Next to prove \mathcal{H} is an open p-regular cover of X .

Since \mathcal{U} is a p-regular cover of Y , there exists a preopen cover \mathcal{V}' of Y such that $\{\text{P cl } V \mid V \in \mathcal{V}'\}$ refines \mathcal{U} . Y being open, each $V \in \mathcal{V}'$ is a preopen subset of X .

Now $(X - Y)$ is both open and closed in X . Hence $\mathcal{V} = \mathcal{V}' \cup (X - Y)$ is a preopen cover of X such that preclosure of members of \mathcal{V} refines \mathcal{H} . Thus \mathcal{H} is a p-regular. As

Y is open, H is an open p-regular cover of X . Hence $X = \bigcup_{i=1}^n H_i \cup (X - Y)$ for some

finite subfamily H_1, H_2, \dots, H_n of H . Then $Y = \bigcup_{i=1}^n (H_i \cap Y) = \bigcup_{i=1}^n U_i$. Therefore Y is p -regular closed.

Theorem : 3.2.6.

A continuous image of a p -regular closed space onto a regular space is regular closed.

Proof :

Let $f : X \rightarrow Y$ be a continuous mapping from a p -regular closed space X to a regular space Y . Let \mathcal{U} be a regular cover of Y . Then \mathcal{U} is an open cover of Y . Let \mathcal{V} be a closed refinement of \mathcal{U} . Since f is continuous $\{f^{-1}(U) ; U \in \mathcal{U}\}$ is an open cover of X . Also f being continuous $\{f^{-1}(V) ; V \in \mathcal{V}\}$ is a closed, hence a preclosed refinement of X . It is also a preclosed cover of X . Thus $f^{-1}(\mathcal{U})$ is an open regular cover of X . Hence, $X = \bigcup_{i=1}^n f^{-1}(U_i)$ for a subfamily U_1, U_2, \dots, U_n of \mathcal{U} and $Y=f(X)=\bigcup_{i=1}^n U_i$. So Y is regular closed.

SECTION : 3.3

Projective objects in the category of pre T_2 - closed extremally disconnected Hausdorff spaces and continuous maps.

Theorem : 3.3.1

In the category of pre T_2 - closed extremally disconnected Hausdorff Spaces and continuous maps, the projective objects are finite spaces.

Proof :

When X is finite, it is obvious that X is projective. Conversely, if X is projective, let us show that X is discrete.

Suppose X is not discrete, then there exists a point $a \in X$ such that $\{a\}$ is not open in X . Let $Y = X - \{a\}$. Then $\text{cl } Y = X$. Let N be the discrete set of natural numbers. Let $N^* = N \cup \{w\}$ be the one-point compactification of N . Let π be a point not in $Y \times N^*$. Let $A = (Y \times N^*) \cup \{\pi\}$. Let us consider a topology \mathcal{U} on A defined as follows

- (i) Let $Z = (x, n) \in Y \times N$. Basic neighbourhoods of Z in A are of the form $G \times \{n\}$ where G is an open neighbourhood of x in Y . For $(x, w) \in Y \times \{w\}$ basic neighbourhoods of it, in A are of the form $G \times [n, w]$ where G is an open neighbourhood of x in Y and $[n, w]$ is such that $N^* - [n, w]$ is finite.
- (ii) If $Z = \pi$, basic open neighbourhoods of it in A are of the form $(Q \times N) \cup \{\pi\}$ where Q is a deleted open neighbourhoods of a . (i.e $a \notin Q$ and $Q \cup \{a\}$ is open in X) By applying exactly the same argument as in theorem 2.3.3 we can show that (A, \mathcal{U}) is Hausdorff.

Claim : (A, \mathcal{U}) is pre T_2 - closed .

Since $\{a\}$ is closed in X , Y is open in X . As X is pre T_2 - closed, Y is locally pre T_2 - closed (by theorem 3.1.18) Since N^* is a compact space and Y is locally pre T_2 - closed, by theorem 3.1.16, we get that the product $Y \times N^*$ is locally pre T_2 - closed. Consider $B = Y \times N^*$. If the topology of $Y \times N^*$ is denoted by τ , then (B, τ)

is a locally pre T_2 - closed Hausdorff space. Now for an open set U and a point $Z \in U$, $U \in \mathcal{U}_Z$, where \mathcal{U}_Z denote the neighbourhood system at Z .

Also $\text{cl}\{Q \times \{n\}\} = \text{cl } Q \times \{n\}$ where $\text{cl } Q$ is open in Y (since Y is extremally disconnected being an open subspace of X). and also $\text{cl}\{Q \times [n, \omega]\} = \text{cl } Q \times [n, \omega]$ since $W \in [n, \omega]$ and $\text{cl}(Q \times N \cup \{\pi\}) = (\text{cl } Q \times N^*) \cup \{\pi\}$. We find that the closure of an open set in A is open in A . Hence A is **extremally disconnected**. Since \mathcal{U} is a Hausdorff topology $B = A - \{\pi\}$ is open in A , and hence B is also extremally disconnected. So (B, τ) satisfies the condition of theorem 3.1.20. Moreover $\text{cl}_A(B) = A$. So (A, \mathcal{U}) is an one point extension of (B, τ) .

To prove : (A, \mathcal{U}) is pre T_2 - closed.

Let us consider an open neighbourhood of a in x , say $Q \cup \{a\}$. Since (X, τ) is a one - point pre T_2 - closed extension of $(Y, \tau / Y)$, $Q \cup \{a\}$ is an open neighbourhood of a in $X = Y^* = Y \cup \{a\}$ endowed with the projective maximum topology $(\tau / Y)^\#$. Hence $Y - \text{int}_Y \text{cl}_Y Q$ is pre T_2 - closed relative to Y .

$$\begin{aligned} \text{Thus } B - \text{int}_B \text{cl}_B(Q \times N) &= \text{cl}_B(B - \text{cl}_B(Q \times N)) \\ &= \text{cl}_B(B - \text{cl}_Y Q \times N^*). \\ &= (Y - \text{int}_Y \text{cl}_Y Q) \times N^* \end{aligned}$$

which is pre T_2 - closed relative to B . Thus $(Q \times N) \cup \{\pi\}$ is open in A endowed with the projective maximum topology on A . Thus $\mathcal{U} \subset \tau^\#$ since $(A, \tau^\#)$ is pre T_2 - closed, we have (A, \mathcal{U}) is pre T_2 - closed. Hence we have proved that (A, \mathcal{U}) is a pre T_2 - closed extremally disconnected Hausdorff space. Now define a topology \mathcal{U}' on

A as follows. Let neighbourhoods of $Z \in Y \times N^*$ be as in (A, \mathcal{U}) and a basic \mathcal{U}' - open neighbourhood of π be of the form $(Q \times N^*) \cup \{\pi\}$ where Q is an open deleted neighbourhood of a . Since $i : (A, \mathcal{U}) \rightarrow (A, \mathcal{U}')$ is continuous. We get by theorem 3.1.14, that (A, \mathcal{U}') is pre T_2 - closed.

Define $f: X \rightarrow (A, \mathcal{U}')$ such that $f(a) = \pi$, $f(y) = (y, w)$ for each $y \in Y$.

Then $f^{-1}((Q \times N^*) \cup \{\pi\}) = Q \cup \{a\}$ is open. -----(1)

Again $f^{-1}(Q(y) \times (N^* - L)) = Q(y)$ where $Q(y)$ denote the open neighbourhood of y and L is a finite subset of N . Thus $f^{-1}(Q(y) \times (N^* - L))$ is open. -----(2)

Thus from (1) and (2), we get f is continuous. Since X is projective, there is a continuous function g making the following diagram commutative.

$$\begin{array}{ccc}
 & & i \\
 & & \longrightarrow \\
 A & \xrightarrow{\quad} & A' \\
 & \nearrow g & \nwarrow f \\
 & X &
 \end{array}$$

Here A stands for (A, \mathcal{U}) and A' stands for (A, \mathcal{U}') Thus we get $i \circ g = f$

Clearly $g(a) = \pi$. Similarly $g(y) = (y, w)$ for each $y \in Y$. Then $g^{-1}((Q \times N) \cup \{\pi\}) = \{a\}$

which is not open and this violates the continuity of g .

Thus x is discrete. Since X is pre T_2 - closed, we conclude that X is finite :

Hence X is projective iff X is finite.

*SUMMARY
AND
CONCLUSION*

SUMMARY AND CONCLUSION

Properties closely related to compactness, have long been of interest to topologists. It is well known that if a compact Hausdorff space X is imbedded in a Hausdorff space Y , the image of X is always a closed subspace of Y . This property was generalised by Alexandroff and Urysohn [3] to arbitrary Hausdorff spaces in 1924 and they called such spaces as H-closed spaces

In 1940, Katetov[9] studied minimal Hausdorff spaces and observed that these are precisely the space which are H-closed and semi regular. He[9] has also proved that any Hausdorff space X is densely imbedded in a H-closed Space X^* having the property that each bounded, real valued continuous function on X can be extended to X^* . Although X^* is the H-closed analogue of the Stone-cech compactification, it is compact only if X is compact.

In chapter II we have discussed the following result by Chen-Tung Liu[6] which is analogous to the extension property of Stone -cech compactification. "Let X, Y be topological spaces and suppose Y is generalised absolutely closed. Let $f: X \rightarrow Y$ be continuous, open map, then there exists $\tilde{f}: kX \rightarrow Y$ such that $\tilde{f}/X = f$ ". It has been shown that any Hausdorff space can be imbedded as a closed subspace of H-closed space (Theorem 2.2.1). Moreover, in the category of absolutely closed spaces and continuous maps, projective objects are finite spaces (Theorem 2.3.3) and injective objects are singletons (Theorem 2.3.6).

Generalising the notion of H-closed spaces using preopen sets, a class of pre T_2 - closed spaces have been introduced [15]. In the category of pre T_2 - closed

extremally disconnected Hausdorff spaces and continuous maps, it is shown that projective objects are finite spaces. (Theorem 3.3.1)

Though we have not discussed the results on para H-closed spaces, it is interesting to study such spaces. Raghavan [14] in his paper entitled “ On countably para H - closed spaces” has characterised the projective objects in the category of the first countable Hausdorff countably para H-closed spaces and continuous maps and established that the projective objects are precisely the discrete spaces.

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