Characterisation of Fuzzy Subgroups of a Finite Cyclic Group

By

Jayalakshmi. S.

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE-641 043 IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

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Introduction

INTRODUCTION

The concept of fuzzy set was first introduced by Zadeh in 1965. Since then the theory of fuzzy sets has developed enormously leading to a number of interesting articles on fuzzy topologies and fuzzy algebraic systems. The concept of fuzzy groups was introduced by Rosenfeld in 1971. In 1981 Das has developed this concept of fuzzy group and has obtained a characterisation of fuzzy subgroups of a prime cyclic group. The concept of fuzzy groups and fuzzy neighbourhood rings were studied by Ahsanullah and Ganguly in 1988 and 1990 and fuzzy vector spaces and fuzzy topological spaces by KatSaras and Liu in 1977.

The aim of this dissertation is to study in detail the following two articles.

(i) Fuzzy groups by Azriel Rosenfeld [11]

(ii) Fuzzy groups and level subgroups by Sivarama krishna Das [12]

In the first Chapter fuzzy subgroupoid, fuzzy ideal, lattice of fuzzy groupoid and fyzzy ideal, fuzzy homomorphism and fuzzy subgroups are studied.

In the first section of this chapter fundamental ideas of fuzzy sets that are needed for the study are collected.

In section two, fuzzy subgroupoid and fuzzy ideal are defined as follows:

A fuzzy subset μ is said to be a fuzzy subgroupoid if, for all $x, y \in S$, $\mu(xy) \geq \min(\mu(x), \mu(y))$

A fuzzy subset μ is said to be a fuzzy ideal if $\mu(xy) \geq \max(\mu(x), \mu(y))$

Some of interesting results proved here are as follows:

(i) If A is a groupoid, then the characteristic function ψ_A is a fuzzy subgroupoid.

(ii) If the characteristic function ψ_A of A is a fuzzy subgroupoid, then A is a subgroupoid.

It is proved here that similar results hold for ideals and fuzzy ideals.

Section three is devoted to the study of lattices of fuzzy subgroupoids and fuzzy ideals.

If T is a subset of G, (T) is the subgroupoid generated by T, (ψ_T) is the fuzzy subgroupoid generated by ψ_T, then the author has proved $(\psi_T) = \psi_{(T)}$

In section four we study fuzzy homomorphism and prove the following results.

(i) A homomorphic pre-image of a fuzzy subgroupoid or fuzzy ideal is a fuzzy subgroupoid or fuzzy ideal respectively.

(ii) A homomorphic image of a fuzzy subgroupoid (ideal) which has the sup property is a fuzzy subgroupoid (ideal).

The study of fuzzy subgroup is dealt with in section five. A fuzzy subgroup is defined as follows:

If S is a group a fuzzy subgroupoid μ of S is called a fuzzy subgroup of S if $\mu(x^{-1}) \geq \mu(x)$ for all $x \in S$.

A few interesting results are as follows:

(i) ψ_T is a fuzzy subgroup <=> T is a subgroup.

(ii) The fuzzy subgroup generated by the characteristic function of a set is just the characteristic function of the subgroup generated by the set.

(iii) Under suitable condition a homomorphic image (or) pre-image of a fuzzy subgroup is a fuzzy subgroup.

(iv) Fuzzy ideals in a group are just the constant functions.

In chapter two fuzzy subgroups and level subgroups are analysed. Here a characterisation of fuzzy subgroups of a prime cyclic group is obtained.

In the first section of this chapter we start with the definition of level subgroups.

Let G be a group and μ be a fuzzy subgroup of G. The subgroups $\mu_t = \{x \varepsilon G | \mu(x) \geq t\}$ for $t \varepsilon [0,1]$ and $t \leq \mu(e)$ are called level subgroups of μ.

The two important theorems discussed here are as follows:

(i) Any subgroup H of a group G can be realised as a level subgroup of some fuzzy subgroup of G.

(ii) If $\bar{A}$ is the collection of all fuzzy subgroups of a group G and $\bar{B}$ is the collection of all level subgroups of members of $\bar{A}$, then there is a one-one correspondence between the subgroups of G and the equivalence classes of level subgroups.

In 1971 Azriel Rosenfeld [11] has obtained a characterisation of a fuzzy subgroup of a prime cyclic group in terms of the membership function. Section two of this chapter is devoted to the study of a similar characterisation of fuzzy subgroups of a finite cyclic group G obtained by Sivaramakrishna Das [12] in 1981. The characterisation of Sivaramakrishna Das is stated as follows:

Let G be a finite cyclic group. Any fuzzy subset μ of G is a fuzzy subgroup if there exists a maximal chain of subgroups $C_1 \subsetneq C_2 \subsetneq \ldots \subsetneq C_r = G$ such that for the numbers $t_o, t_1, \ldots t_r \in Im(\mu)$ with $t_o > t_1 > \ldots t_r$
We have $\mu(e) = t_o$, $\mu(\hat{C}_1) = t_1 \ldots \mu(\hat{C}_r) = t_r$
Where $\hat{C}_i = C_i - C_{i-1}$, $i=1,2,\ldots r$.
Conversely, any given fuzzy subgroup μ satisfy such a condition.

Review of Literature

REVIEW OF LITERATURE

In 1965 L.A. Zadeh introduced the concept of fuzzy set which is a generalisation of an ordinary set. Since its inception the theory of fuzzy set has developed in many directions and is finding applications in a wide variety of fields. In 1968 chang introduced the concept of fuzzy topological spaces. In 1976 Lowen modified the definition of chang and developed the theory of fuzzy topological spaces. In 1971 Azries Rosenfeld studied fuzzy set theory with regard to algebraic structure like groupoids, group, subgroups, ideals, homomorphisms, etc.

In 1975 Negoita and Ralescu and in 1979 Anthony and Sherwood redefined fuzzy group and they developed the theory of fuzzy group.

In 1977 Katsaras and Liu applied the concept of fuzzy set theory to elementary theory of vector spaces and topological vector spaces. In 1979 Foster introduced the concept of product fuzzy subgroups of the groups with respect to minimum function and in 1984 Abuosman introduced the concept product fuzzy subgroups of the groups with respect to t-norm and generalised the results of Foster.

In 1981 Sivaramakrishna Das generalised same basic properties of group theory in terms of fuzzy group and he obtained a characterisation of all fuzzy subgroups of finite cyclic group.

In 1988 Phullendu Das redefined fuzzy subgroups using the notion of t-norm and obtained several results which are generalisations of the results of Ketsaras and Liu.

Chapter I

CHAPTER – I

FUZZY SUBGROUPS

Chapter one is devoted to the study of the article "Fuzzy groups" by **Azriel Rosenfeld [11]** in 1971. In this the author has defined fuzzy subgroupoid, fuzzy ideal, fuzzy subgroups etc and has obtained some interesting results connecting subgroupoids and fuzzy subgroupoids, ideals and fuzzy ideals and soubgroups and fuzzy subgroups and a characterisation of fuzzy subgroup of a cylcic group of prime order P.

In section one of this chapter a few preliminary results on fuzzy sets are collected.

Fuzzy subgroupoid and fuzzy ideal are studied in section two. Here the author has shown that A is subgroupoid (ideal) iff the characteristic function ψ_A is a fuzzy subgroupoid (fuzzy ideal).

As an arbitrary intersection of fuzzy subgroupoid (fuzzy ideal) is a fuzzy subgroupoid (fuzzy ideal).

In section three the author has proved that the characteristic function of the subgroupoid generated by a subset T is same as the fuzzy subgroupoid generated by the characteristic function ψ_T (ie) $(\psi_T)=\psi_{(T)}$

In Section four homomorphic image and pre-image of fuzzy subgroupoid and fuzzy ideal are studied.

Section five devoted to the study of fuzzy subgroup and its properties. Here the author has obtained a characterisation of fuzzy subgroup of a cyclic group of prime order P.

SECTION 1.1 : PRELIMINARIES ON FUZZY SETS:-

In this section some fundamental definitions and results on fuzzy subsets that are needed for our study are collected.

Definition 1.1.1

Let X be a set and I = [0,1]. By a fuzzy subset on X. We mean a function $f:X \dashrightarrow I$.

Remark: 1.1.2

Every subset $A \subseteq X$ can be identified with the fuzzy subset ψ_A (characteristic function on A) defined by

$$\psi_A(x)=1 \text{ if } x \in A$$
$$=0 \quad \text{ if } x \notin A$$

Notation:-

The collection of all fuzzy subsets on X is denoted by I^X

Definition:1.1.3.

For any two fuzzy subsets f,g on X, the fuzzy subsets $f \cup g$ (f join g) and $f \cap g$ (f meet g) on X are defined as follows:

$$(f \cup g)(x) = \sup(f(x), g(x)) \text{ for } x \in X$$

$$(f \cap g)(x) = \inf(f(x), g(x)) \text{ for } x \in X$$

Definition:- 1.1.4

The complement of a fuzzy subset on X is defined as the fuzzy subset 1-f on X and is denoted by f'

(i.e.) $f'(x) = (1-f)(x) = 1-f(x)$ for $x \in X$

Properties of fuzzy subsets:- 1.1.5

Let $f_1, f_2, \ldots\ldots\ldots\ldots f_n$ be fuzzy subsets on X
Then,

1) $\left(\bigcup_{i=1}^{n} f_i\right)(x) = (f_1 \cup f_2 \cup \ldots\ldots\ldots\ldots \cup f_n)(x)$ for $x \in X$

$= \max\{f_i(x) / i=1,2,\ldots\ldots n\}$

2) $\left(\bigcap_{i=1}^{n} f_i\right)(x) = (f_1 \cap f_2 \cap \ldots\ldots\ldots\ldots \cap f_n)(x)$ for $x \in X$

$= \min\{f_i(x) / i=1,2,\ldots\ldots n\}$

Let $\{f_\lambda | \lambda \in \Lambda\}$ be an arbitrary collection of fuzzy subsets on X. Then,

3) $\left(\bigcup_{\lambda \in \Lambda} f_\lambda\right)(x) = \sup_{\lambda \in \Lambda} f_\lambda(x)$, for $x \in X$

4) $\left(\bigcap_{\lambda \in \Lambda} f_\lambda\right)(x) = \inf_{\lambda \in \Lambda} f_\lambda(x)$, for $x \in X$

5) $\left(1-\bigcup_{\lambda \in \Lambda} f_\lambda\right) = \bigcap_{\lambda \in \Lambda} (1-f_\lambda)$

Definition 1.1.6

The inclusion of fuzzy sets on X is defined as follows:-

$\mu \subseteq \vartheta$ means $\mu(x) \subseteq \vartheta(x)$ for all $x \in X$.

Definition 1.1.7

A subset which contains intersection and union is called a lattice.

(i.e.) A subset which contains inf and sup is called a lattice.

Definition 1.1.8

For any subset S of L, if sup S and inf $S \in L$ then, L is a complete lattice.

Notation:-

The least and greatest elements in I^X are the constant functions 0 and 1. These functions are denoted as ϕ_0 and ϕ_X respectively.

Definition 1.1.9.

A mapping ϕ from a group G into a group $\bar{G}$ is said to be a homomorphism if for all $a, b \in G$,

$$\phi(ab) = \phi(a)\ \phi(b)$$

Definition 1.1.10.

A set X which is closed under a binary operation is known as a groupoid.

Definition 1.1.11.

A set S which is closed under binary composition and in which the binary composition is asscoative is known as Semi-group.

Definition 1.1.12.

Let μ be a fuzzy subset on X and f be a function defined on X. The fuzzy subset ϑ in f(x) defined by

$$\vartheta(y) = \sup_{x \in f^{-1}(y)} \mu(x)$$ for all $y \in f(X)$ is called the

image of μ under f.

Definition 1.1.13.

Let ϑ be a fuzzy subset on f(X). The fuzzy subset $\mu = f.\vartheta$ on X defined by

$$\mu(x) = \vartheta(f(x))$$ for all $x \in X$ is called

the preimage of ϑ under f.

Definition 1.1.14.

A fuzzy subset μ on X is f- invarient if

$f(x) = f(y) \Rightarrow \mu(x) = \mu(y)$

Definition 1.1.15. CHAIN

A set with a relation which linearely ordered is called a chain.

Definition 1.1.16. LINEARELY ORDERING

i) If $x<y$ and $y<x$ then $x=y$ and

ii) $x < y$ or $y < x$ whenever x and y are distinct
Number of union of the domain and the range of $<$

Definition 1.1.17. MAXIMAL

A chain C is maximal if there exists no chain which contains C.

SECTION-2

FUZZY SUBGROUPOIDS AND FUZZY IDEAL

In this section fuzzy subgroupoids and fuzzy ideals which are generalations of groupoids and ideals are defined and some of the properties are studied.

Definition.1.2.1.

A fuzzy subset μ on S will be called a fuzzy sub groupoid if for all x, y in S

$\mu(xy) \geq \min(\mu(x), \mu(y))$

Definition.1.2.2.

A fuzzy subset μ on S will be called a fuzzy left ideal if $\mu(xy) \geq \mu(y)$ for all $x, y \in S$.

Definition 1.2.3.

A fuzzy subset μ on S will be called a fuzzy right ideal if $\mu(xy) \geq \mu(x)$ for all $x, y \in S$.

Definition.1.2.4.

A fuzzy subset μ on S will be called a fuzzy ideal if it is a fuzzy left and right ideal.

REMARK: 1.2.5.

Equivalently a fuzzy subset μ on S is a fuzzy ideal if $\mu(xy) \geq \max(\mu(x), \mu(y)$ for all $x, y \in S$

PROPOSITION: 1.2.6.

Every fuzzy (left, right) ideal is a fuzzy subgroupoid.

PROOF:-

Let μ be a fuzzy ideal on S.

Let $x, y \in S$. Then we have

$$\mu(xy) \geq \max(\mu(x), \mu(y)) \geq \min(\mu(x), \mu(y))$$

(i.e) $\mu(xy) \geq \min(\mu(x), \mu(y))$ for all $x, y \in S$

Hence μ is a fuzzy subgroupoid on S.

PROPOSITION: 1.2.7.

For any fuzzy subgroupoid μ on S, we have $\mu(x^n) \geq \mu(x)$ for all $x \in S$ where x^n is any composite of x's.

PROOF:-

Let μ be a fuzzy subgroupoid on S.

For any $x, y \in S$, we have

$\mu(xy) \geq \min(\mu(x), \mu(y))$ ----------> (1)

Putting $y=x$ in (1) we get

$\mu(x^2) \geq \min(\mu(x), \mu(x))$

(i.e.) $\mu(x^2) \geq \mu(x)$

Therefore, the result is true for $n=1$ and 2.

We shall prove the result by the method of induction.

Assume the result is true for all positive integers $\leq n$ and we shall prove it for n+1.

$\mu(x^{n+1}) \geq \min(\mu(x), \mu(x^n))$ [From (1)]

But $\mu(x^n) \geq \mu(x)$ for all $x \in X$ [By induction hyphothesis]

Hence $\mu(x^{n+1}) \geq \mu(x)$

Hence the Result.

THEOREM:- 1.2.8.

For any $t \in [0,1]$, $\{Z \,/\, Z \in S, \mu(z) \geq t\}$ is a subgroupoid or (left, right) ideal if μ is a fuzzy subgroupoid or fuzzy (left, right) ideal.

PROOF:-

Let μ be a fuzzy subgroupoid on S.

For any $t \in [0,1]$,

let $A = \{Z \,/\, Z \in S, \mu(z) \geq t\}$

Claim:-

A is a subgroupoid of S.

Let $x, y \in A$.

$\Rightarrow \mu(x) \geq t$ and $\mu(y) \geq t$

$\Rightarrow \min(\mu(x), \mu(y)) \geq t$

Since μ is a fuzzy subgroupoid of S, We have $\mu(xy) \geq \min(\mu(x), \mu(y))$

Hence $\mu(xy) \geq t$

$\Rightarrow xy \in A$

$\Rightarrow$ A is a subgroupoid.

Hence the claim

Let μ be a fuzzy (left, right) ideal on S.

Claim:-

A is a left (right) ideal of S.

Let x, y $\in$ A and s $\in$ S.

$\Rightarrow \mu(x) \geq t$

Since μ is a fuzzy (left, right) ideal,

$\mu(xy) \geq \mu(y)$ and

$\mu(xy) \geq \mu(x)$

$\mu(sx) \geq \mu(x) \geq t$

(i.e.) $\mu(sx) \geq t$

$\Rightarrow sx \in A$.

Similarly we can prove

$ys \in A$

Hence A is a (left, right) ideal of S.

PROPOSITION 1.2.9.

If A is a groupoid of S, then the characteristic function ψ_A is a fuzzy subgroupoid on S

PROOF:-

Let A be a groupoid and ψ_A be the characteristic function of A.

Claim:-

ψ_A is a fuzzy subgroupoid.

Let x, y ε A.

Then $\psi_A(x) = 1$ and $\psi_A(y) = 1$.

Since A is a groupoid, xy ε A

Therefore $\psi_A(xy) = 1$

Therefore $\psi_A(xy) \geq \min(\psi_A(x), \psi_A(y))$

Therefore ψ_A is a fuzzy subgroupoid on S.

Hence the claim.

PROPOSITION 1.2.10.

If the characteristic function ψ_A of A is a fuzzy subgroupoid on S then A is a subgroupoid of S.

PROOF:

Let ψ_A be a subgroupoid on S.

Claim:-

A is a subgroupoid of S.

Let x, y ε A. Then $\psi_A(x) = \psi_A(y) = 1$

Since ψ_A is a fuzzy subgroupoid,

$\psi_A(xy) \geq \min(\psi_A(x), \psi_A(y))$

$\Rightarrow \psi_A(xy) = 1$

$\Rightarrow xy \varepsilon A$

Therefore A is a subgroupoid of S

Hence the claim

PROPOSITION: 1.2.11.

If A is (left, right) ideal, then the characteristic ψ_A is a fuzzy (left, right) ideal.

PROOF:-

Let A be a (left, right) ideal.

Claim:-

ψ_A is a fuzzy (left, right) ideal.

Let x, y ε A,

Then $\psi_A(x) = 1$ and $\psi_A(y) = 1$

Also $\psi_A(xy) = 1$

Therefore $\psi_A(xy) \geq \psi_A(y)$ and

$\psi_A(xy) \geq \psi_A(x)$

Hence ψ_A is a fuzzy ideal.

Hence the claim.

PROPOSITION:1.2.12

If the characteristic function ψ_A of A is a fuzzy (left, right) ideal, then A is a (left, right) ideal.

PROOF:- Let ψ_A be a fuzzy ideal on S

Claim:- A is an ideal.

Let $x \in A$ and $s \in S$

$\Rightarrow \psi_A(x) = 1$

Since ψ_A is a fuzzy (left, right) ideal

$\psi_A(xy) \geq \psi_A(y)$ &

$\psi_A(xy) \geq \psi_A(x)$

Therefore $\psi_A(xs) \geq \psi_A(x) = 1$

$\Rightarrow \psi_A(xs) = 1$

$\Rightarrow xs \in A$

Similarly we can prove $sx \in A$

Therefore A is an ideal

Hence the claim

SECTION - 3

The lattices of fuzzy subgroupoids and fuzzy ideals.

Here it is shown that the set of all fuzzy subgroupoids as well as fuzzy ideals form a complete lattice and the subgroupoid lattice of S is identified with a sublattice of the fuzzy subgroupoid lattice of S.

PROPOSITION: 1.3.1

The intersection of any collection of fuzzy subgroupoids is a fuzzy subgroupoid.

PROOF:-

Let $\{\mu_i \,/\, i \in \wedge\}$ be any collection of fuzzy subgroupoids on S.

Then $[\cap\mu_i]\,(xy) = \inf[\mu_i(xy)] \geq \inf[\min \mu_i(x), \mu_i(y)]$

$= \min[\inf \mu_i(x), \inf\mu_i(y)]$

$[\cap\mu_i]\,(xy) = \min\,[[\cap\mu_i]\,(x), [\cap(\mu_i)]\,(y)]$

Hence the intersection of any set of fuzzy subgroupoids is a fuzzy subgroupoid.

REMARK: 1.3.2.

The set of all fuzzy subgroupoids on S form a complete lattice. In this lattice the inf of a set of fuzzy subgroupoids μ_i is $\cap\mu_i$ and sup is the least μ (i.e. the $\cap$of all μ's) which contains $\cup\mu_i$.

Definition 1.3.3.

The fuzzy subgroupoid (σ) generated by the fuzzy set σ is defined as the least fuzzy sub groupoid which contains σ

THEOREM 1.3.4.

$(\psi_T) = \psi_{(T)}$, where

i) (ψ_T) - fuzzy subgroupoid generated by ψ_T

ii) (T) - subgroupoid generated by T

iii) $\psi_{(T)}$ - characteristic function of (T)

PROOF:-

Let μ be a fuzzy subgroupoid, that contains ψ_T

Then $\mu(x) = 1$ for all $x \in T$.

Since μ is a fuzzy subgroupoid, $\mu = 1$ for any composite of elements of T and

Therefore μ contains $\psi_{(T)}$

Thus $\psi_{(T)}$ is contained in the intersection of all such μ's.

(i.e.) $\psi_{(T)}$ is contained in the intersection of all such μ's which contains ψ_T.

(i.e.) $(\psi_T) \subseteq \psi_{(T)}$

Conversely $\psi_{(T)}$ itself is one such μ which contains ψ_T

[By proposition 1.2.8]

(i.e.) $\psi_{(T)} \subseteq (\psi_T)$

Hence $\psi_{(T)} = (\psi_T)$

Hence the theorem

REMARK:- 1.3.5.

The subgroupoid lattice of S can be regarded as a sublattice of the fuzzy subgroupoid lattice of S.

PROPOSITION : 1.3.6

The $\cap$ or $\cup$ of any set of fuzzy (left, right) ideal is a fuzzy (left, right) ideal.

PROOF :

Let $\{\mu_i / \ i \ \varepsilon \Delta\}$ be any arbitrary collection of fuzzy (left,right) ideal.

Then, $[\cap\mu_i]\ (xy) = \inf\ [\mu_i\ (xy)]$

$= \inf\ [\mu_i\ (y)]$

$= [\cap\ \mu_i]\ (y).$

Similarly,

$[\ \cup\ \mu_i]\ (xy) = (\ \cup\ \mu_i)(y)$

Hence, the intersection or union of any set of fuzzy (left,right) ideal is a fuzzy (left,right) ideal.

PROPOSITION : 1.3.7

The set of all fuzzy ideals of S forms a complete sublattice.

PROPOSITION : 1.3.8.

Let S be a semi-group and has a left - identity. Then the left ideal generated by $T \subseteq S$ is ST and the fuzzy left ideal generated by ψ_T is ψ_{ST}.

REMARK : 1.3.9

Let T be a subset of S and f be defined on S. If $\mu = \psi_{(T)}$, then the image ϑ of μ under f is $\psi_{f(T)}$.

PROOF :

For, $x\ \varepsilon\ S$,

$\mu(x) = \psi_W(f(x))$

$$= \psi_{f^{-1}(W)}(x).$$

REMARK : 1.3.10

Let f be defined on S and let $W \subseteq f(s)$. Then the preimage μ of ψ_W under f is $\psi_{f^{-1}(W)}$.

PROOF :

For, $y \in f(S)$

$$\vartheta(y) = \sup_{x \in f^{-1}(y)} \psi_T(x)$$

$$= \psi_{f(T)}(y).$$

SECTION 4. HOMOMORPHISM

In this section the homomorphic image and preimage of a fuzzy groupoid (ideal) is discussed. It is proved that the homomorphic preimage of a fuzzy groupoid (ideal) is a fuzzy groupoid (ideal) and the image of fuzzy groupoid (ideal) is a fuzzy groupoid provided it has the sup property.

PROPOSITION : 1.4.1

The homomorphic preimage of a fuzzy subgroupoid or (left, right) ideal is a fuzzy subgroupoid or (left, right) ideal respectively.

PROOF :

Let f be a homomorphism defined on S and ϑ be a fuzzy subgroupoid on f(s)

CLAIM:-

The homomorphic preimage μ of a fuzzy subgroupoid ϑ under f is a fuzzy subgroupoid.

Let $x, y \in S$.

Then $\mu(xy) = \vartheta(f(xy)) = \vartheta\ (f(x)f(y)) \geq \min\ (\vartheta\ f(x), \vartheta\ f(y))$

$\mu(xy) \geq \min\ (\mu(x), \mu(y))$

Hence the claim

Let ϑ be a fuzzy (left, right) ideal on f(s)

CLAIM:- The homomorphic preimage μ of a fuzzy ideal ϑ under f is a fuzzy ideal.

Let x, y ε S

Then $\mu(xy) = \vartheta(f(xy)) = \vartheta(f(x)f(y)) \geq \max(\vartheta f(x) \vartheta f(y))$

$\mu(xy) \geq \max(\mu(x), \mu(y))$

Hence the claim.

Definition .1.4.2.

A fuzzy subset μ on S has the sup property if for any subset $T \subseteq S$ there exists a $t_o \varepsilon T$ such that

$$\mu(t_o) = \sup_{t \varepsilon T} \mu(t)$$

EXAMPLE:- 1.4.3.

Let μ be a fuzzy set on S which can take only finitely many values. Then μ has the sup property. In particular if it is a characteristic function then μ has the sup property.

PROPOSITION:- 1.4.4.

The homomorphic image of a fuzzy subgroupoid which has the sup property is a fuzzy subgroupoid and similarly for (left, right) ideals.

PROOF:- Let f be a homomorphism defined on S.

Let μ be a fuzzy subgroupoid on S which has the sup property.

CLAIM:- The image of a fuzzy subgroupoid μ which has the sup property under the homomorphism f is a fuzzy subgroupoid.

Let $x, y \in S$

Then $f(x), f(y) \in f(S)$ Let $x_o \in f^{-1}(f(x))$ and $y_o \in f^{-1}(f(y))$ be such that

$$\mu(x_o) = \sup_{t \in f^{-1}(f(x))} \mu(t)$$

and $\mu(y_o) = \sup_{t \in f^{-1}(f(x))} \mu(t)$ respectively

consider $\vartheta(f(x)f(y)) = \sup_{z \in f^{-1}(f(x)f(y))} \mu(z)$

$$\geq \min(\mu(x_o), \mu(y_o))$$

$$= \min(\vartheta(f(x)), \vartheta(f(y)))$$

Therefore $\vartheta(f(x)f(y)) \geq \min(\vartheta f(x), \vartheta f(y))$

Hence the claim.

CLAIM:-

Let μ be a fuzzy (left, right) ideal in S which has the sup property.

Let $x, y \in S$.

Then $f(x), f(y) \in f(s)$.

Let $x_o \in f^{-1}(f(x))$, $y_o \in f^{-1}(f(y))$ be such that

$$\mu(x_o) = \sup_{t \in f^{-1}(f(x))} \mu(t)$$

$$\mu(y_o) = \sup_{t \in f^{-1}(f(y))} \mu(t) \qquad \text{respectively}$$

Consider

$$\vartheta\,(f(x)f(y)) = \sup_{z \in f^{-1}(f(x)f(y))} \mu(z)$$

$$\geq \max\,(\mu(x_o),\ \mu(y_o))$$

$$= \max\,(\vartheta f(x),\ \vartheta f(y))$$

$$\vartheta(f(x)f(y) \geq \max\,(\vartheta f(x),\ \vartheta f(y))$$

Hence the claim.

REMARK: 1.4.5

Let f be any function defined on S and ϑ be a fuzzy set on f(s), Then the image of the preimage of ϑ under f is just ϑ itself as

$$\sup_{x \in f^{-1}(y)} \vartheta(f(x)) = \vartheta(y) \text{ for all } y \in S$$

REMARK: 1.4.6

Let f be any function defined on S and μ is any fuzzy set on S. Then the preimage of the image μ under f always contains μ since

$\sup_{z \in f^{-1}(f(x))} \mu(z) \geq \mu(x)$ for all $x \in S$.

NOTE:-1.4.7.

If μ is f - invarient then, $\sup_{z \in f^{-1}(f(x))} \mu(z) = \mu(x)$, $\forall x \in S$.

(i.e.) The preimage of the image of μ under f is μ itself.

PROPOSITION. 1.4.8.

There is a one - to - one correspondence between the f-invarient fuzzy sets on S and the fuzzy sets on f(s).

RESULT 1.4.9.

There is a one - to - one correspondence between the f-invarient fuzzy subgroupoids on s and the fuzzy subgroupoids on f(s) provided that the former have the sup property.

SECTION-5 FUZZY SUBGROUPS

In this section fuzzy subgroup is defined and a relation between a subgroup and a fuzzy subgroup is obtained. It is shown that the homomorphic image or preimage of a fuzzy subgroup is a fuzzy subgroup provided the sup property holds in the former case and the fuzzy ideals on a group are just the constant functions on S.

Definition 1.5.1.

If S is a group, a fuzzy subgroupoid μ on S will be called a fuzzy subgroup on S if

$\mu(x^{-1}) \geq \mu(x)$ for all $x \in S$

PROPOSITION 1.5.2

ψ_T is a fuzzy subgroup if and only if T is a subgroup.

PROOF:

Let ψ_T be a fuzzy subgroup.

CLAIM:

T is a subgroup.

Since ψ_T is a fuzzy subgroup.

$\psi_T(x^{-1}) \geq \psi_T(x)$ for all $x \in S$.

and $\psi_T(xy) \geq \min(\psi_T(x), \psi_T(y))$

Let $x, y \in T$.

=> $\psi_T(x) = 1$ and $\psi_T(y) = 1$

But $\psi_T(xy) \geq \min(\psi_T(x), \psi_T(y))$

$\psi_T xy) \geq \min(1,1)$

$\psi_T(xy) = 1$

=> $xy \in T$

We know that $\psi_T(x^{-1}) \geq \psi_T(x)$

(i.e.) $\psi_T(x^{-1}) \geq \psi_T(x) = 1$

$\psi_T(x^{-1}) \geq 1$

=> $x^{-1} \in T$

Therefore T is a subgroup

Hence the claim.

Conversely assume that T is a subgroup.

CLAIM:- ψ_T is a fuzzy subgroup.

Let $x, y \in T$ => $\psi_T(x) = 1$ and $\psi_T(y) = 1$.

Since T is a subgroup $xy \in T$ and $x^{-1} \in T$.

(i.e.) $\psi_T(xy) = 1$ and $\psi_T(x^{-1}) = 1$

=> $\psi_T(xy) \geq \min(\psi_T(x), \psi_T(y))$ and

$\psi_T(x^{-1}) \geq \psi_T(x)$

Therefore ψ_T is a fuzzy subgroup.

Hence the claim.

PROPOSITION 1.5.3.

The intersection of any collection of fuzzy subgroup is a fuzzy subgroup.

PROOF: Let $\{\mu_i\}$ be a collection of fuzzy subgroups.

CLAIM: $\cap\mu_i$ is a fuzzy subgroup.

We know the intersection of any collection of fuzzy subgroupoids is a fuzzy subgroupoid (by proposition 1.3.1)

Consider $\cap\mu_i(x^{-1}) = \inf [\mu_i(x^{-1})]$

$\geq \inf [\mu_i(x)]$

$\cap\mu_i(x^{-1}) \geq \cap \mu_i(x)$

$\Rightarrow \cap\mu_i$ is a fuzzy subgroup.

(ie.) The $\cap$ of a collection of fuzzy subgroups is a fuzzy subgroup.

Hence the claim.

PROPOSITION 1.5.4.

The fuzzy subgroup generated by the characteristic function of a set is just the characteristic function of the subgroup generated by the set.

PROOF:-

The proof is obvious from Theorem 1.3.4. and Proposition 1.5.2.

PROPOSITION 1.5.5.

Let μ be a fuzzy subgroup on S, Then $\mu(x^{-1}) = \mu(x)$ and $\mu(x) \leq \mu(e)$ for all $x \in S$, where e is the identity element of S.

PROOF:- Let μ be a fuzzy subgroup on S and $x \varepsilon S$

CLAIM:- $\mu(x^{-1}) = \mu(x)$ and $\mu(x) \leq \mu(e)$

Since μ is a fuzzy subgroup

$\mu(xx^{-1}) \geq \min(\mu(x), \mu(x^{-1}))$

and $\mu(x^{-1}) \geq \mu(x)$

As $\mu(x) = \mu[(x^{-1})^{-1}] \geq \mu(x^{-1})$

=> $\mu(x^{-1}) = \mu(x)$ for all $x \varepsilon S$.

As $\mu(e) = \mu(xx^{-1}) \geq \min(\mu(x), \mu(x^{-1}))$

$= \mu(x)$

(i.e.) $\mu(x) \leq \mu(e)$ for all $x \varepsilon S$

Hence the proof.

COROLLARY 1.5.6.

$\{x / \mu(x) = \mu(e)\}$ is a subgroup.

PROOF:- Let $G_u = \{x / \mu(x) = \mu(e)\}$

Let $x, y \varepsilon G_u$

Then $\mu(x) = \mu(e)$ and $\mu(y) = \mu(e)$

Since μ is a fuzzy subgroup

$\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ and

$\mu(x^{-1}) \geq \mu(x)$

Therefore $\mu(xy) \geq \min(\mu(e), \mu(e))$

Therefore $\mu(xy) \geq \mu(e)$

But $\mu(xy) \leq \mu(e)$

Therefore $\mu(xy) = \mu(e)$

Therefore $xy \in G_u$

$\mu(x^{-1}) \geq \mu(x) = \mu(e)$

But $\mu(x^{-1}) \leq \mu(e)$

Therefore $\mu(x^{-1}) = \mu(e)$

Therefore. $x^{-1} \in G_u$

Therefore G_u is a subgroup.

PROPOSITION: 1.5.7.

$\mu(xy^{-1}) = \mu(e)$ implies $\mu(x) = \mu(e)$

PROOF:

Let $\mu(xy^{-1}) = \mu(e)$

As $\mu(x) = \mu(x(y^{-1}y))$

$= \mu((xy^{-1})y) \geq \min(\mu(xy^{-1}), \mu(y))$

$= \min(\mu(e), \mu(y))$

$\mu(x) = \mu(y)$

PROPOSITION 1.5.8.

μ is constant on each coset of G_u

PROOF:-

Given μ is a fuzzy subgroup on S.

A coset of G_u is of the form

$aG_u = \{ab / b \in G_u\}$, where $a \in S$

CLAIM:-

μ is constant on each coset of G_u

Let $b \in G_u$. This => $\mu(b) = \mu(e)$

For $a \in S$,

$\mu(a) = \mu((ab^{-1})b) \geq \min(\mu(ab^{-1}), \mu(b))$

$= \min(\mu(ab^{-1}), \mu(e))$

But $\mu(ab^{-1}) \leq \mu(e)$ [by Proposition 1.5.5]

Hence $\mu(a) \geq \mu(ab^{-1})$

But $\mu(ab^{-1}) \geq \min(\mu(a), \mu(b^{-1}))$

Since G_u is a subgroup, $b \in G_u$ => $b^{-1} \in G_u$

=> $u(b^{-1}) = e$.

Hence $\mu(ab^{-1}) \geq \mu(a)$

(i.e) $\mu(a) \geq \mu(ab^{-1}) \geq \mu(a)$. This is true for all $b^{-1} \in G_u$

and hence true for all $b \in G_u$

Hence $\mu(a) \geq \mu(ab) \geq \mu(a)$ $\forall b \in G_u$.

(i.e) $\mu(ab) = \mu(a)$ for all $b \in G_u$

Hence μ is constant on each coset of G_u.

PROPOSITION 1.5.10

If G_u has finite index, μ has the sup property.

PROOF:-

Let G_u have a finite index

(i.e) The number of distinct cosets of G_u is finite.

Since μ is constant on each coset of G_u [By Proposition 1.5.8] μ takes on only finitely many values.

$\Rightarrow$ μ has the sup property.

Hence the proof.

PROPOSITION 1.5.11

μ is a fuzzy subgroup on S $\Leftrightarrow$ $\mu(xy^{-1}) \geq \min(\mu(x),\mu(y))$ for all x,y in s.

PROOF:-

Let μ be a fuzzy subgroup on S and $x,y \in S$.

CLAIM:-

$\mu(xy^{-1}) \geq \min(\mu(x), \mu(y^{-1}))$

Since μ is a fuzzy subgroup

$\mu(y^{-1}) = \mu(y)$ [By Proposition 1.5.5]

Hence $\mu(xy^{-1}) \geq \min(\mu(x), \mu(y))$ for all $x,y \in S$.

Conversely assume that

$\mu(xy^{-1}) \geq \min(\mu(x), \mu(y))$ for all $x,y \in S$.

CLAIM:-

μ is a fuzzy subgroup on S.

When y=x, we have

$\mu(xx^{-1}) \geq \min(\mu(x), \mu(x))$.

(i.e) $\mu(e) \geq \mu(x)$, $\forall x \in S$.

Now $\mu(y^{-1}) = \mu(ey^{-1}) \geq \min(\mu(e), \mu(y))$.

$\Longrightarrow \mu(y^{-1}) \geq \mu(y)$.

Consider

$\mu(xy) = \mu(x(y^{-1})^{-1}) \geq \min (\mu(x), \mu(y^{-1}))$

$\mu(xy) \geq \min (\mu(x), \mu(y))$

Hence the claim

PROPOSITION 1.5.12.

A group cannot be the U of two proper fuzzy subgroups.

PROOF:

Let μ and ϑ be proper fuzzy subgroups of S such that

$\mu(x) = 1$ or $\vartheta(x) = 1 \ \forall x \in S$

Let u, V in S be such that

$\mu(u) = 1$ and $\mu(v) < 1$

$\vartheta(u) < 1$ and $\vartheta(V) = 1$

We know that $\mu(u^{-1}) = \mu(u)$ [by proposition 1.5.5.]

Hence $\mu(u^{-1}) = 1$

Consider $\mu(V) = \mu(u^{-1}(uV))$

$\geq \min (\mu(u^{-1}), \mu(uV))$

If $\mu(uV) = 1$

Then $\mu(V) \geq \min (\mu(u^{-1}), \mu(uV)) = 1$

Which is a contradiction

Consider $\vartheta(u) = \vartheta(uVV^{-1})$

$\geq \min (\vartheta (uV), \vartheta(V^{-1}))$

But $\vartheta(V^{-1}) = \vartheta(V) = 1$

If $\vartheta(uV) = 1$

Then $\vartheta(V) \geq \min(\vartheta(uV), \vartheta(V^{-1})) = 1$

which is a contradiction

Hence a group cannot be the union of two proper fuzzy subgroups

PROPOSITION:1.5.13

A homomorphic image or preimage of a fuzzy subgroup is a fuzzy subgroup (in the former case, provided the sup property holds).

PROOF:

Let f be a homomorphism defined on S.

Let ϑ be a fuzzy subgroup on f(S). Then the pre-image μ of ϑ under f is given by $\mu(x) = \vartheta(f(x))$ for all $x \in S$

CLAIM:-

μ is a fuzzy subgroup on S

We know μ is a fuzzy subgroupoid [By proposition 1.4.1]

Consider $\mu(x^{-1}) = \vartheta(f(x^{-1})) = \vartheta(f(x)^{-1})$[Since f is a homomorphism)

$\geq \vartheta(f(x))$ [Since ϑ is a fuzzy subgroup]

$= \mu(x)$

$\geq \min(\vartheta(uV), \vartheta(V^{-1}))$

But $\vartheta(V^{-1}) = \vartheta(V) = 1$

If $\vartheta(uV) = 1$

Then $\vartheta(V) \geq \min(\vartheta(uV), \vartheta(V^{-1})) = 1$

which is a contradiction

Hence a group cannot be the union of two proper fuzzy subgroups

PROPOSITION:1.5.13

A homomorphic image or preimage of a fuzzy subgroup is a fuzzy subgroup (in the former case, provided the sup property holds).

PROOF:

Let f be a homomorphism defined on S.

Let ϑ be a fuzzy subgroup on f(S). Then the pre-image μ of ϑ under f is given by $\mu(x) = \vartheta(f(x))$ for all $x \in S$

CLAIM:-

μ is a fuzzy subgroup on S

We know μ is a fuzzy subgroupoid [By proposition 1.4.1]

Consider $\mu(x^{-1}) = \vartheta(f(x^{-1})) = \vartheta(f(x)^{-1})$[Since f is a homomorphism)

$\geq \vartheta(f(x))$ [Since ϑ is a fuzzy subgroup]

$= \mu(x)$

(i.e.) $\mu(x^{-1}) \geq \mu(x) \; \forall \; x \varepsilon S$

$\Rightarrow$ μ is a fuzzy subgroup

Hence the claim.

Let μ be a fuzzy subgroup on S. Then the image ϑ of μ under f is given by.

$$\vartheta(y) = \sup_{x \varepsilon f^{-1}(y)} \mu(x) \quad \text{for all } y \; \varepsilon \; f(S)$$

Given $f(x) \varepsilon f(S)$, Let $x_o \varepsilon f^{-1}(f(x))$ be such that

$$\mu(x_o) = \sup_{t \varepsilon f^{-1}(f(x)} \mu(t)$$

CLAIM:- ϑ is a fuzzy subgroup

We know ϑ is a fuzzy subgroupoid [By proposition 1.4.1]

Consider

$$\vartheta(f(x)^{-1}) = \sup_{z \varepsilon f^{-1}(f(x)^{-1})} \mu(z) \geq \mu(x_o^{-1}) \qquad \text{[By Remark 1.4.6]}$$

(i.e.) $\vartheta \; (f(x)^{-1}) \geq \mu(x_o)$ (Since μ is a fuzzy subgroup)

$= \vartheta(f(x))$

(i.e.) $\vartheta \; (f(x^{-1}) \geq \vartheta(f(x))$ for all $x \; \varepsilon \; S$

$\Rightarrow$ ϑ is a fuzzy subgroup

PROPOSITION 1.5.14

The fuzzy (left, right) ideals on a group S are just the constant functions on S.

PROOF:

Let μ be a constant function on S.

$\Rightarrow \mu(xy) = \mu(x) = \mu(y)$ for all x, y ε S

$\Rightarrow \mu(xy) \geq \max(\mu(x), \mu(y))$ -------> (1)

$\Rightarrow \mu$ is a fuzzy ideal

Conversely

Let μ be a fuzzy left ideal on the group S

Since μ is a fuzzy left ideal,

$\mu(xy) \geq \mu(y)$ for all x,y ε S ---> (2)

Putting y = e, we have

$\mu(xe) \geq \mu(e)$

(i.e.) $\mu(x) \geq \mu(e)$ for all x

Putting $x = y^{-1}$ in (2), we have

$\mu(y^{-1}y) \geq \mu(y)$ for all y

In Particular

$\mu(e) \geq \mu(x)$

(i.e.) $\mu(x) = \mu(e)$ for all x

(i.e.) $\mu = \mu(e)$ is a constant function

Hence the proof

PROPOSITION: 1.5.15

Let G_p be the cyclic group of prime order P and let μ

be any fuzzy subgroup of G_p. Then $\mu(x) = \mu(1) \le \mu(0)$ for all $x \neq 0$ in G_p and conversely such μ is a fuzzy subgroup.

PROOF:-

Let μ be a fuzzy subgroup of G_p such that

$\mu(x) = \mu(1) \le \mu(o)$ for all $x \neq 0$ in G_p

CLAIM:-

μ is a fuzzy subgroup.

Let $x, y \neq 0$ in G_p.

$\Rightarrow \mu(x) = \mu(1) \le \mu(o)$ and

$\mu(y) = \mu(1) \le \mu(o)$.

If $xy = o$ in G_p then,

$\mu(xy) = \mu(o) \ge \min(\mu(x), \mu(y))$.

Suppose if $xy \neq o$ in G_p, then,

$\mu(xy) = \mu(1) \le \mu(o)$

Therefore $\mu(xy) = \min(\mu(x), \mu(y))$.

Hence $\mu(xy) \ge \min(\mu(x), \mu(y)$ for all $x, y \neq 0$ in G_p.

Let $x \neq 0$ in G_p

$x^{-1} \neq 0$

$\Rightarrow \mu(x) = \mu(x^{-1}) = \mu(1)$

Hence μ is a fuzzy subgroup.

Hence the claim.

Conversely, assume that μ is a fuzzy subgroup on G_p.

CLAIM:-

$\mu(x) = \mu(1) \leq \mu(o)$ for all $x \neq 0$ in G_p

Let $x \neq 0$ and $y \neq 0$ in G_p

x can be written in terms of y and

y can be written in terms of x.

(i.e.) $x = y^l$ (say), and

$y = x^q$ (say)

Therefore $\mu(x) = \mu(y^l) \geq \mu(y)$ and

$\mu(y) = \mu(x^q) \geq \mu(x)$.

(i.e.) $\mu(x) \geq \mu(y) \geq \mu(x)$

$\Rightarrow \mu(x) = \mu(1) \leq \mu(0)$ for all $x \neq 0$ in G_p

Hence the claim.

Chapter II

CHAPTER 2

CHARACTERISATION OF FUZZY SUB GROUPS OF A FINITE CYCLIC GROUP

This chapter is devoted to a study of characterisation of fuzzy subgroups of finite cyclic group. In the first section of this chapter fuzzy level sub group of a fuzzy subgroup is studied. In this section every subgroup H of a group G is identified with same level subgroup of some fuzzy sub group on G. Based on this it is shown that there is a one - one correspondence between the subgroups of G and the equivalence classes of level subgroups of a fuzzy subgroups on G.

The characterisation of all fuzzy groups of a prime cyclic group in terms of membership function is obtained in 1971 by Rosenfeld [11] . In section two of this chapter we study a similar characterisation of all fuzzy subgroups of finite cyclic group G in terms of maximal chain of subgroups of G obtained by Sivaramakrishna Das in 1981 [12].

SECTION 1

FUZZY GROUPS AND LEVEL SUBGROUPS

In the first section of this chapter level subgroup of a fuzzy subgroup is discussed. It is shown that

i) Any subgroup H of G can be identified with same level subgroup of some fuzzy subgroup on G.

ii) There is a one - one correspondence between the subgroups of G and the equivalence classes of level subgroups of fuzzy subgroups of G.

Definition:2.1.1

Let μ be a fuzzy subset on S. For $t \in [0,1]$, $\mu_t = \{x \in S / \mu(x) \geq t\}$ is called a level subset of the fuzzy set μ.

THEOREM 2.1.2

Let G be a group and μ be a fuzzy subgroup of G. Then the level subset μ_t, for $t \in [0,1]$, $t \leq \mu(e)$ is a subgroup of G, where e is the identity of G.

PROOF:

Let μ be a fuzzy subgroup of G. Then for $t \in [0,1]$, the level subset

$\mu_t = \{x \in G / \mu(x) \geq t\}$ and $e \in \mu_t$

Claim: μ_t is a subgroup of G

Let $x, y \varepsilon \mu_t$

$\Rightarrow \mu(x) \geq t$ and $\mu(y) \geq t$.

Since μ is a fuzzy subgroup,

$\mu(x y) \geq \min(\mu(x), \mu(y))$

Therefore, $\mu(xy) \geq t$

$\Rightarrow xy \varepsilon \mu_t$

Therefore μ_t satisfies closure property

Let $x \varepsilon \mu_t$

$\Rightarrow \mu(x) \geq t$

But $\mu(x^{-1}) \geq \mu(x)$

Therefore $\mu(x^{-1}) \geq t$

$\Rightarrow x^{-1} \varepsilon \mu_t$

Hence every element in μ_t has an inverse in μ_t

Therefore μ_t is a subgroup of G.

Hence the theorem.

THEOREM: 2.1.3

Let G be a Group and μ be a fuzzy subset of G such that μ_t is a subgroup of G for all $t \varepsilon [0,1]$ and $t \leq \mu(e)$. Then μ is a fuzzy subgroup of G.

PROOF:

Let μ be a fuzzy subset of G and for $t \varepsilon [0,1]$, μ_t is a subgroup of G and $t \leq \mu(e)$.

Therefore $e \varepsilon \mu_t$ for every $t \varepsilon [0,1]$.

Claim::

μ is a fuzzy subgroup of G.

Let $x, y \varepsilon G$.

$\mu(x)= t_1$ and $\mu(y) = t_2$

Then $x \varepsilon\ \mu_{t_1}$ and $y\ \varepsilon\ \mu_{t_2}$

without loss of generality we can assume

that $t_1 < t_2$

As

$\mu_{t_1} = \{x \varepsilon G / \mu(x) \geq t_1\}$ and

$\mu_{t_2} = \{x \varepsilon G / \mu(y) \geq t_2\}$,

Hence we get $\mu_{t_2} \subset \mu_{t_1}$

As μ_{t_1} is a subgroup,

$x, y \varepsilon\ \mu_{t_1} \Rightarrow xy \varepsilon\ \mu_{t_1}$

Therefore $\mu(xy) \geq t_1 = \min \{\mu(x), \mu(y)\}$

Therefore $\mu(xy) \geq \min \{\mu(x), \mu(y)\}$

Let $x \varepsilon G$ and $\mu(x)=t$

Then $x \varepsilon \mu_t$

Since μ_t is a subgroup. $x^{-1}\ \varepsilon\ \mu_t$

$\mu(x^{-1}) \geq \mu(x)$ [Since $\mu(x)=t$]

Therefore μ is a fuzzy subgroup of G.

DEFINITION 2.1.4 LEVEL SUBGROUPS

Let G be a group and μ be a fuzzy subgroup of G. Then the Subgroups μ_t, $t \varepsilon [0,1]$ and $t \leq \mu(e)$ are called level Subgroups of μ.

REMARK 2.1.5

If G is finite, then the number of level subgroups of a fuzzy subgroup μ is finite.

REMARK 2.1.6

Every level subgroup is a subgroup of G [By Theorem 2.1.2]

NOTE It is to be noted that not all the level subgroups are distinct.

THEOREM 2.1.7

Let G be a group and μ be a fuzzy subgroup of G. Two subgroups μ_{t_1} and μ_{t_2} with $(t_1<t_2)$ of μ are equal <=> There is no x of G such that $t_1 < \mu(x)<t_2$.

PROOF

Let μ be a fuzzy subgroup of G and μ_{t_1} and μ_{t_2} are equal.

Claim:

There is no x of G such that $t_1 < \mu(x) < t_2$.

Suppose there exists $x \in G$ such that

$t_1 < \mu(x) < t_2$.

=> $\mu(x) > t_1$ and $\mu(x) < t_2$ and

$\mu_{t_2} \subseteq \mu_{t_1}$

=> $x \in \mu_{t_1}$ and $x \notin \mu_{t_2}$.

=> a contradiction to the fact that two subgroups are equal.

Therefore there exists no such $x \in G$ such that $t_1<\mu(x)<t_2$.

Conversely assume that there exists no x such that $t_1<\mu(x)<t_2$.

Claim:

$\mu_{t_1} = \mu_{t_2}$

μ_{t_1} and μ_{t_2} are two subgroups with $t_1<t_2$

$\mu_{t_1} = \{x \in G / \mu(x) \geq t_1\}$

$\mu_{t_2} = \{x \in G / \mu(x) \geq t_2\}$

As $t_2>t_1$, $\mu_{t_2} \subseteq \mu_{t_1}$ ——————> (1)

Let $x \in \mu_{t_1}$

Then $\mu(x) \geq t_1$ and there exist no $x \varepsilon G$ such that

$t_1 < \mu(x) < t_2$

Therefore $\mu(x) > t_2$

$\Rightarrow x \varepsilon \mu_{t_2}$

$\Rightarrow \mu_{t_1} \subseteq \mu_{t_2} \longrightarrow$ (2)

From (1) and (2) $\mu_{t_1} = \mu_{t_2}$

Hence the claim.

RESULT 2.1.8.

Let G be a finite group of order n and μ be a fuzzy subgroup of G. Let $Im(\mu) = \{t_i | \mu(x) = t_i$ for some $x \varepsilon G\}$. Then $\{\mu_{t_i}\}$ are the only level subgroups of μ.

PROOF

Let G be a finite group of order n

and $G = \{x_1, x_2, \ldots\ldots, x_n\}$

Let μ be a fuzzy subgroup of G. Then for $t \varepsilon [0,1]$,

the level subgroup μ_t is a subgroup of G.

Let $Im(\mu) = \{t_i | \mu(x) = t_i$ for some $x \varepsilon G\}$

Claim::-

$\{\mu_{t_i} | \mu(x) = t_i$ for $x \varepsilon G\}$ are the only level subgroups of μ.

Let $y \neq e$ be in μ_t

$\Rightarrow \mu(y) \geq t$ [since μ_t is a level subgroup]

Therefore $\mu(y) \geq t$

Given that $t = \mu(e)$

$\Rightarrow \mu(y) \geq \mu(e)$

Contradication to proposition 1.5.5.

Therefore $\mu_t = \{e\}$

Hence the claim

Let $t \varepsilon [0,1]$, $t \notin Im(\mu)$ and

$t_i < t < t_j$, where $t_i, t_j \varepsilon Im(\mu)$

Therefore $\mu_{t_i} = \mu_t = \mu_{t_j}$ [By Theorem 2.1.7]

If $t < t_r$, where t_r is the least element in $Im(\mu)$

Then for $t_r = \inf\{t_i | \mu(x) = t_i$ for some $x \varepsilon G\}$

Claim::-

$G \subset \mu_{t_r} = \mu_t$.

PROOF:-

Given $x \varepsilon G$,

$\mu(x) \geq t_i > t_r$

$\Rightarrow \mu(x) \geq t_r$

$\Rightarrow x \varepsilon \mu_{t_r}$

Therefore $G \subset \mu_{t_r} = \mu_t$ [since $t \notin Im\mu$]

Hence the claim.

Since, μ_{t_r} are the subgroups of G,

$\mu_{t_r} \subset G$.

Therefore $\mu_{t_r} = G$

$\mu_t = \mu_{t_r} = G$

Therefore, for any $t \varepsilon [0,1]$, the level subgroup is one of $\{\mu_{t_i}\}$, where $t_i \varepsilon Im(\mu)$

Hence the proof.

THEOREM: 2.1.9.

Any subgroup H of a group G can be realised as a level subgroup of some fuzzy subgroup of G.

PROOF:-

Let μ be a fuzzy subset of G defined by

$$\mu(x) = t \text{ if } x \varepsilon H$$
$$= 0 \text{ if } x \notin H, 0<t<1$$

Claim::

μ is a fuzzy subgroup of G.

CASE (i)

Let x, y ε G

If x,yεH,

Then xyεH Since H is a subgroup of G.

=> $\mu(xy) = t$ and

$\mu(xy) \geq \min(\mu(x), \mu(y))$

Let xεH, Then $\mu(x) = t$

Since H is a subgroup of G, $x^{-1}\varepsilon H$

=> $\mu(x^{-1}) = t$

=> $\mu(x^{-1}) \geq \mu(x)$

Therefore μ is a fuzzy subgroup of G.

CASE (ii)

Let x, y ε G

Suppose x ε H and y $\notin$ H

=> $\mu(x) = t$ and $\mu(y) = 0$ and xy $\notin$H

=> $\mu(xy) = 0$

$\mu(xy) \geq \min(\mu(x), \mu(y))$

Since xεH, $x^{-1}\varepsilon H$

=> $\mu(x^{-1}) = t$

Therefore $\mu(x^{-1}) \geq \mu(x)$ [since $\mu(x) = t$]

Therefore μ is a fuzzy subgroup of G.

CASE (iii)

Let $x, y \in G$

$x \notin H$ and $y \notin H$,

$\Rightarrow \mu(x) = 0$ and $\mu(y) = 0$

Let us assume $xy \notin H$

Then $\mu(xy) = 0$

$$\mu(xy) = \min\{\mu(x), \mu(y)\}$$

Let $x \notin H$. Then $x^{-1}H$.

$\Rightarrow \mu(x^{-1}) = 0 = \mu(x)$

Therefore μ is a fuzzy subgroup of G.

CASE (iv)

Let $x, y \in G$ be such that $x \notin H$ and $y \notin H$ and $xy \in H$

$\Rightarrow \mu(x) = \mu(y) = 0$ and $\mu(xy) = t$

$\Rightarrow \mu(xy) \geq \min\{\mu(x), \mu(y)\}$

Let $x \notin H$. Then $x^{-1} \notin H$

$\Rightarrow \mu(x^{-1}) = 0 = \mu(x)$

Therefore μ is a fuzzy subgroup of G and the subgroup H can be realised as the level subgroup μ_t of the fuzzy subgroup G.

Hence the proof.

THEOREM.2.1.10.

Let $\bar{X}$ be the collection of fuzzy subgroups of group G and $\bar{B}$ be the collection of all level subgroups of members of $\bar{X}$. Then there is a one-one correspondence between the subgroups of G and the equivalence classes of level subgroups (under a suitable equivalence relation on B).

PROOF:-

Let $\bar{A}$ be the collection of all fuzzy subgroups of a group G and $\bar{B}$ be the collection of all level subgroups of members of $\bar{A}$. Then

$\bar{B} = \bar{A} \times I$, where $I = [0,1]$

Let the relation '~' in $\bar{B}$ be defined by $(A,t) \sim (B,t')$

$\Leftrightarrow A_t = B_{t'}$

We shall prove '~' is an equivalence relation.

REFLEXIVITY:-

For any fuzzy subgroup A of the group G and $t \in [0,1]$,

$A_t = A_t$

Therefore $(A,t) \sim (A, t)$.

Therefore '~' is reflexivity.

SYMMETRY

Let A and B be two fuzzy subgroup of the group G and $t, t' \in [0,1]$.

Let $(A,t) \sim (B,t')$

Then $A_t = B_{t'}$

Therefore $B_{t'} = A_t$

Hence $(B,t') \sim (A,t)$

Therefore '~' is symmetry.

TRANSITIVITY

Let A,B, and C be fuzzy subgroups of the group G and $t, t', t'' \in [0,1]$

Let $(A,t) \sim (B,t')$

$(B,t') \sim (C,t'')$

$(A,t) \sim (B,t') \Rightarrow A_t = B_{t'} \longrightarrow (1)$

$(B,t') \sim (C,t'') \Rightarrow B_{t'} = C_{t''} \longrightarrow (2)$

From (1) and (2)

$A_t = B_{t'} = C_{t''}$

$\Rightarrow A_t = C_{t''}$

(i.e.) $(A,t) \sim (C,t'')$

Therefore $\sim$ is a equivalence relation on $\overline{B}$

So '$\sim$' partitions $\overline{B}$ into equivalence classes

Let $[A_t]$, $[B_{t'}]$, $[C_{t''}]$ .... be the equivalence class of level subgroups in $\overline{B}$.

Let $\overline{B}$ be the set of all equivalence classes determined by '$\sim$'.

Let H be a subgroup of G. By Theorem 2.1.9 H can be realised as a level subgroup of some fuzzy subgroup A.

(i.e.) $H = A_t$ for some $t \in [0,1]$

Claim::-

There is a one-one correspondence between the subgroups of G and the equivalence classes of level subgroups of fuzzy subgroups of the group G.

Let $[A_t]$ be the equivalence class of the level subgroup determined by A_t.

Let $\overline{G}$ be the set of all subgroups of G.

Let $\psi: \overline{G} \longrightarrow \overline{B}$ such that

$\psi(H) = [A_t]$

Let A be a fuzzy subgroup of G. For every equivalence class $[A_t]$ we get a element

$A_t = \{x \in G \mid A(x) \geq t, \forall x \in G\}$ and A_t is a subgroup of G

Hence the claim.

Conversely for any fuzzy subgroup A of the group G and $t \varepsilon [0,1]$, $A_t = \{x \varepsilon G | A(x) \geq t\}$ is a subgroup of G.

REMARK 2.1.11

In the chain $C(\mu)$ of level subgroups of the fuzzy subgroup μ of the group G, $\{e\}$ is the smallest level subgroup and G is the largest level subgroup.

REMARK 2.1.12.

All the subgroups of G need not form a chain.

We shall find a fuzzy subgroup μ of G which accommodates as many subgroups of G as possible in $C(\mu)$.

THEOREM. 2.1.13.

Let G be a finite group such that

$G = C_{P_1} \times C_{P_2} \times \ldots\ldots \times C_{P_i}$

Where C_{P_i} are Prime cyclic groups of orders P_i. Then there exists a fuzzy subgroup μ such that $C(\mu)$ is a maximal chain of length 2.

PROOF:

We prove by induction on r.

If $r=1$, then $G = C_{P_i}$ is a prime cyclic group of order P_1

Define μ: $G \rightarrow I$ as $\mu(e) = t_o$ and $\mu(x) = t_1$ for $x \neq e$ and $t_o \geq t_1$. Then μ is a fuzzy subgroup and $\mu_{t_o} = \{e\}$,

$\mu_{t_1} = \{ x \varepsilon G | \mu(x) \geq t, \forall x \varepsilon G\}$ and

$\mu_{t_1} = G = C_{P_1}$

and $\mu_{t_o} \subset \mu_{t_1}$

Then $\{\mu_{t_0}, \mu_{t_1}\}$ is the maximal chain of level subgroups of length 2.

Hence the theorem is true for r=1.

Let $r>1$ and we assume that the theorem is true for the integers $\leq r-1$ and we shall prove it for r.

Let $H = C_{p_1} \times C_{p_2} \times \ldots\ldots\ldots \times C_{p_{r-1}}$

Then $G = H \times C_{p_r}$

Define a fuzzy set $\mu : G \longrightarrow [0,1]$ by $\mu(e) = t_0$ and

$\mu\,[\hat{C_{p_1}}] = t_1$,

$\mu\,[C_{p_1} \hat{\times} C_{p_2}] = t_2, \ldots\ldots$

$\mu\,[H \hat{\times} C_{p_r}] = t_r$ where $t_0 > t_1 \ldots\ldots > t_r$ and

$\hat{C_{p_1}} = C_{p_1} - (e)$

$C_{p_1} \hat{\times} C_{p_2} = C_{p_1} \times C_{p_2} - C_{p_1}$ and so on.

Claim:: μ is a fuzzy subgroup of G.

PROOF: Let $x, y \in G$ and $x, y \in H$

Since H is a subgroup, $xy \in H$

By induction there exists a fuzzy subgroup μ such that

$\mu(xy) \geq \min(\mu(x), \mu(y))$

$\mu(x^{-1}) \geq \mu(x)$

Let $x, y \in G$ and $x \in H$, $y \in G-H$

$\Rightarrow xy \notin H$

Then $\mu(xy) = t_r$.

$\mu(x) \geq t_{r-1}$ [Since $x \in H$]

$\mu(y) = t_r$ [Since $y \in C_{p_r}$]

$\Rightarrow \mu(xy) \geq \min[\mu(x), \mu(y)]$ [Since $t_{r-1} > t_r$]

and $\mu(x^{-1}) \geq \mu(x)$ for every $x \in H$

Let $x, y \in G$ and $x, y \notin H$

Then xy may or may not be in H

Let $xy \in H$

Then $\mu(xy) \geq t_{r-1}$ and

$\mu(x) = t_r$, $\mu(y) = t_r$

$\implies \mu(xy) \geq \min [\mu(x), \mu(y)]$

Let $xy \notin H$.

Then $\mu(xy) = t_r$,

$\mu(x) = t_r$ and $\mu(y) = t_r$

$\implies \mu(xy) \geq \min [\mu(x), \mu(y)]$

$\mu(x^{-1}) \geq \mu(x)$

Therefore μ is a fuzzy subgroup of G.

Let $\mu_{t_o} = (e)$, $\mu_{t_1} = C_{p_1}$, $\mu_{t_2} = C_{p_1} \times C_{p_2}, \ldots.$

$\mu_{t_r} = H \times C_{p_r}$

Then $\mu_{t_o} \subset \mu_{t_1} \subset \ldots\ldots \subset \mu_{t_r}$ and

$C(\mu) = \{ \mu_{t_o}, \mu_{t_1}, \ldots\ldots \mu_{t_r} \}$ is the maximal chain of

lenght r+1

REMARK: 2.1.14

In the same way we can find fuzzy subgroups with maximal $C(\mu)$ in the following cases.

i) G is a cyclic P-group

ii) G is a product of cyclic P-groups

iii) G is a Finte abelian group.

Section - 2.2

CHARACTERISATION OF FUZZY SUBGROUPS OF A FINITE CYCLIC GROUP.

In 1971 Rosenfeld has obtained a characterisation of all fuzzy groups of a prime cyclic group in terms of membership function. In this section we study a similar characterisation of all fuzzy subgroups of finite cyclic group G is obtained by Sivaramakrishna Das in 1981. Here the characterisation is obtained in terms of maximal chain of subgroups of G.

THEOREM 2.2.1

Let G be a cyclic P- group of order P^n, where P is a prime. Let μ be a fuzzy subgroup of G then for $x, y \in G$

i) If $O(x) > O(y)$, then $\mu(y) \geq \mu(x)$

ii) If $O(x) = O(y)$, then $\mu(x) = \mu(y)$

Proof:-

Let $O(G) = P^n$

We prove by induction on n

If n=1, then $O(G) = P$, a prime Number.

In this case we have $\mu(y) \geq \mu(x)$ if $O(x) > O(y)$ and $\mu(y) = \mu(x)$ if $O(x) = O(y)$ [by Proposition 1.5.15]

Assume that the result is true for all integers $\leq n-1$.

Let G be a cyclic group of order p^n and H be a subgroup of G of order p^{n-1}.

Let μ be a fuzzy subgroup of G.

Claim:- For $x, y \in G$

i) If $O(x) > O(y)$ then $\mu(y) \geq \mu(x)$

ii) If $O(x) = O(y)$ then $\mu(x) = \mu(y)$.

proof:-

Let x,y be any two elements in G.

Case (i)

Let $x, y \in G$, and $x, y \in H$

By induction the result follows.

Case (ii)

Let $x, y \in G$, $x \notin H$ and $y \in H$

$x \notin H \Rightarrow O(x) = p^n$

$y \in H \Rightarrow O(y) = p^r$ where $r \leq n-1$.

Therefore x is a generator of G.

$\Rightarrow y = x^l$ for some integer l.

$\Rightarrow \mu(y) = \mu(x^l) \geq \mu(x)$

$\mu(y) \geq \mu(x)$.

Case (iii)

Let $x, y \in G$ and $x, y \notin H$

$\Rightarrow O(x) = p^n$ and $O(y) = p^n$

Therefore x and y are the generator of G

Therefore $x^m = y$ and $y^l = x$ for some integers m and l

$\mu(y) = \mu(x^m) > \mu(x)$ --> (1)

$\mu(x) = \mu(y^l) > \mu(y)$ --> (2)

From (1) and (2).

$\mu(y) = \mu(x)$

Hence the theorem.

Note:-

The following example shows that the above theorem is not true in general.

Example :- 2.2.2.

Consider Kliens 4 - group.

Let $G = \{a,b/a^2=b^2=(ab)^2 = e\}$

Let the fuzzy subset $\mu: G \to [0,1]$ be defined as

$\mu(e) = t_o$, $\mu(a) = t_1$, $\mu(b) = t_2 = \mu(ab)$

where $t_o > t_1 > t_2$

Claim::-

μ is a fuzzy subgroup of G

As $\mu(a) = t_1$, $\mu(b) = t_2$ and $\mu(ab) = t_2$,

$\mu(ab) \geq \min \{\mu(a), \mu(b)\}$ and

$\mu(a^{-1}) \geq \mu(a)$ [Since $a^{-1} \in G$, G is a Group]

Therefore μ is a fuzzy subgroup of G.

But $\mu(a) \neq \mu(b)$ even though $O(a) = O(b)$.

THEOREM 2.2.3

Let G be a finite cyclic group. Any fuzzy subset μ of G is a fuzzy subgroup if there exists a maximal chain of subgroup.

(i.e) $C_o \subsetneq C_1 \subsetneq \ldots \subsetneq C_r = G$

Such that for the numbers $t_o, t_1 \ldots t_r \in Im(\mu)$

With $t_o > t_1 > \ldots > t_r$ we have

$\mu(e) = t_o$, $\mu(\hat{C}_1) = t_1 \; . \; . \; . \; . \; . \; . \; \mu(\hat{C}_r) = t_r$

Where $\hat{C}_i = C_i - C_{i-1}$, $i = 1,2 \; . \; . \; . \; r$ conversely any given fuzzy subgroup μ satisfy such a condition.

Proof:-

Let $\mu : G \rightarrow [0,1]$ be fuzzy subset and

$C_o \subsetneq C_1 \subsetneq C_2 \; . \; . \; . \; . \; \subsetneq C_r$ be a maximal chain of sub- groups such that for the numbers $t_o > t_1 \; . \; . \; . \; . \; . > t_r$ We have

$\mu(e) = t_o$, $\mu(\hat{C}_i) = t_i$ for $i = 1,2....r$ and

$\hat{C}_i = C_i - C_{i-1}$.

Claim:-

μ is a Fuzzy subgroup of G

Let $x,y \in G$

Let $x,y \in C_i$ but not in C_{i-1}

Then $\mu(x) = t_i$ and $\mu(y) = t_i$

$\Rightarrow xy \in C_i$ or C_{i-1} [Since $C_o, C_1 . \; . \; . \; C_i$ are the subgroups of G]

$xy \in C_i \Longrightarrow \mu(xy) = t_i$

$xy \in C_{i-1} \Longrightarrow \mu(xy) = t_{i-1}$

But $t_{i-1} > t_i$

$\Longrightarrow \mu(xy) \geq t_i$

$\Longrightarrow \mu(xy) \geq \min(\mu(x), \mu(y))$

$x \in C_i \Longrightarrow \mu(x) = t_i$

$x^{-1} \in C_i$ [Since $C_o, C_1 .... C_i$ are the subgroups of G]

$\mu(x^{-1}) \geq t_i = \mu(x)$

$\mu(x^{-1}) \geq \mu(x)$

Therefore μ is a fuzzy subgroup of G

Hence the claim

Let G be a fin ite cyclic group and

Let μ be a fuzzy subgroup of G

Claim:-

There exists a maximal chain of subgroups.

$(e) = C_o \subsetneq C_1 \subsetneq C_2 \; .... \; C_r = G$ such that

for the Numbers $t_o, t_1, t_2 . . . t_r \in Im(\mu)$ with

$t_o > t_1 . . . . > t_r$

We have $\mu(e) = t_o$, $\mu(\hat{C_1}) = t_1 . . . \mu(\hat{C_r}) = t_r$.

Where $\hat{C_i} = C_i - C_{i-1}$, i-1,2,3...r.

By theorem 2.1.2.

μ_{t_o} μ_{t_1} μ_{t_2}..... μ_{t_r} are the only level subgroups of μ.

The level Subgroups of the fuzzy group μ form a chain and it is the maximal chain.

Therefore $C(\mu) : \mu_{t_o} \subset \mu_{t_1} \subset ...... \mu_{t_r}$

Where $\mu_{t_o} = \{e\}$ and $\mu_{t_r} = G$ where $t_o > t_1 . . . > t_r$

Case (i) $C(\mu)$ is maximal chain

Let $c_i = \mu_{t_i}$

(i.e) $e = C_o = \mu_{t_o}$ and $c_1 = \mu_{t_1}$

$\Rightarrow e = C_o \subsetneq C_1 . . . . \subsetneq C_r = G$.

$C(\mu) = C_o \subsetneq C_1 . . . . \subsetneq C_r$ is a maximal chain of length r+1.

Hence $C(\mu)$ is a maximal chain.

Case (ii) $C(\mu)$ is not a maximal chain.

We define $C(\mu)$ by introducing subgroups of G.

(i.e) $C(\mu) = C_o \subsetneq C_1 . . . . \subsetneq C_s$

where $C_o = \mu_{t_o} = (e)$ and $C_s = \mu_{t_r} = G$

For all C_i between μ_{t_o} and μ_{t_1}

$\mu(\hat{C}_i) = t_i$

where $\mu_{t_o} = (C_o)$ and μ_{t_1} (= C_j for some j)Similarly C_k between μ_{t_i} and $\mu_{t_{i+1}}$

$\mu(\hat{C}_k) = t_{i+1}$

C_s between $\mu_{t_{r-1}}$ and μ_{t_r}

$\mu(\hat{C}_s) = t_r$ thus

$\mu(C_o) = t_o$, $\mu(\hat{C}_1) = \ldots \quad \mu(\hat{C}_j) = t_1$

(i.e) $\mu(\hat{C}_1) = \mu(\hat{C}_2) \ldots \mu(\hat{C}_j) = t_1$

and $\mu(\hat{C}_{j+1}) \ldots \mu(\hat{C}_k) = t_2$

where $\hat{C}_1 = C_1 - C_o$ and

$\hat{C}_2 = C_2 - C_1 \ldots, \hat{C}_s = C_s - C_{s-1}$ and

$t_o > t_1 \ldots > t_r$

Therefore $C(\mu)$ is a chain of subgroups with $\mu(e) = t_o$.

$\mu(\hat{C}_1) = t_1$

Hence the result

Corollary 2.2.4

If G is a cyclic P-Group of order p^i, then the necessary and sufficient condition for a fuzzy subset μ of G to be fuzzy subgroup is that for all elements x such that $O(x) = p^i$ we have $\mu(x) = t_i$, $i=0,1\ldots r$ with $t_o > t_1 > \ldots > t_r$.

Summary and Conclusion

SUMMARY AND CONCLUSION

In this thesis we have attempted to give a detailed survey of the fundamental results on fuzzy groupoids and fuzzy groups. For this purpose we have discussed the following two papers

(i) "FUZZY GROUPS" BY AZRIEL ROSENFELD [1971]

(ii)"FUZZY GROUPS AND LEVEL SUBGROUPS" BY SIVA-RAMAKRISHNA DAS [1981]

It is very interesting to study the inter relationship between subgroups and fuzzy subgroups and ideals and fuzzy ideals. An important characterisation of fuzzy subgroups of a finite cyclic group is obtained by Das.

Based on the definition of fuzzy subgroups and fuzzy groups stranger algebraic structures like fuzzy rings, fuzzy vector spaces, fuzzy module etc. are developed and studied by various authors in the subsequent years [1967, 1977, 1982, 1983, 1984, 1988]

We hope that a deep study of these concepts will lead to many interesting open problems yielding a very good scope for further research.

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