CHAPTER IV

CHAPTER IV

PAIRWISE CLOSED SETS AND BICONTINUOUS MAPS IN BICLOSURE SPACES

In this chapter pairwise closed sets in biclosure spaces and bicontinuous maps in biclosure spaces are studied.

SECTION 4.1

PAIRWISE CLOSED SETS IN BICLOSURE SPACES

In this section, the concepts of pairwise closed sets in biclosure spaces are studied.

Definition: 4.1.1

A subset A of a biclosure spaces (X,u_1,u_2) is called **Pairwise closed** if $u_1u_2A=A= u_2u_1A$. The complement of pairwise closed set is called **Pairwise open**.

Proposition: 4.1.2

Every closed set is pairwise closed.

Remark: 4.1.3

The converse of the above proposition need not be true.

Example: 4.1.4

Let $X = \{1,2\}$ and u_1 be a closure operator on X defined by $u_1\phi = \phi$ and $u_1\{1\} = u_1\{2\} = u_1X = X$. Let u_2 be a closure operator on X defined by $u_2\phi =\phi$, $u_2\{1\} = u_2\{2\} = u_2X = X$. Then $\{1\}$ is pairwise closed but it is not closed.

Proposition: 4.1.5

Let (X,u_1,u_2) be a biclosure space. If A and B are pairwise closed subsets of (X,u_1,u_2) , then $A \cap B$ is pairwise closed.

Proof

Let A and B be pairwise closed. Then $u_1u_2A=A= u_2u_1A$ and $u_1u_2B=B= u_2u_1B$. Therefore, $u_1u_2 (A\cap B) = u_1 (u_2(A \cap B)) = u_1(u_2A \cap u_2B)= u_1u_2A \cap u_1u_2B= A\cap B$ and $u_2u_1(A\cap B) = u_2(u_1A \cap u_1B)= u_2u_1A \cap u_2u_1B= A\cap B$.We have $u_1u_2 (A\cap B)= A\cap B= u_2u_1(A\cap B)$.

Hence, $A\cap B$ is pairwise closed.

Remark: 4.1.6

The union of two pairwise closed sets need not be a pairwise closed set as can be seen from the following example.

Example: 4.1.7

Let $X = \{1,2,3,4\}$ and u_1 be a closure operator on X defined by $u_1\phi = \phi$ and $u_1\{1\} =\{1\}$,$u_1\{2\} =\{2\}$, $u_1\{3\} =\{3\}$, $u_1\{4\} =\{4\}$, $u_1\{1,2\}=\{1,2,4\}$, $u_1\{1,3\}=\{1,3\}$, $u_1\{1,4\}=\{1,4\}$, $u_1\{2,3\}=\{2,3\}$, $u_1\{2,4\}=\{2,4\}$, $u_1\{3,4\}=\{3,4\}$, $u_1\{1,2,4\}=\{1,2,4\}$,and $u_1\{1,2,3\}= u_1\{1,3,4\}= u_1\{2,3,4\}= u_1X = X$. Let u_2 be a closure operator on X defined by $u_2\phi =\phi$, $u_2\{1\} =\{1\}$, $u_2\{2\} =\{2\}$, $u_2\{3\} =\{3\}$, $u_4\{4\} =\{4\}$, $u_2\{1,2\}=\{1,2,4\}$, $u_2\{1,3\}=\{1,3\}$, $u_2\{1,4\}=\{1,4\}$, $u_2\{2,3\}=\{2,3\}$, $u_2\{2,4\}=\{2,4\}$, $u_2\{3,4\}=\{3,4\}$ and $u_2\{1,2,4\}= u_2\{1,3,4\}= u_2\{2,3,4\}= u_2X = X$. Then $\{1\}$ and $\{2\}$ are pairwise closed but $\{1\}\cup\{2\}=\{1,2\}$ is not pairwise closed

Proposition: 4.1.8

Let (X,u_1,u_2) be a biclosure space and let u_1,u_2 be additive. If A and B are pairwise closed subsets of (X,u_1,u_2) , then $A\cup B$ is pairwise closed.

Proof

Let A and B be pairwise closed. Then $u_1u_2A=A= u_2u_1A$ and $u_1u_2B=B= u_2u_1B$. Since u_1 and u_2 are additive, $u_1u_2\ (A\cup B)= u_1(u_2A \cup u_2B)= u_1u_2A \cup u_1u_2B= A\cup B$ and $u_2u_1(A\cup B) = u_2(u_1A \cup u_1B)= u_2u_1A \cup u_2u_1B= A\cup B$.We have $u_1u_2\ (A\cup B) = A\cup B= u_2u_1(A\cup B)$.Hence, $A\cup B$ is pairwise closed.

Proposition: 4.1.9

Let (X,u_1,u_2) be a biclosure space and let (Y,v_1,v_2) be a closed subspace of (X,u_1,u_2). If F is a pairwise closed subset of (Y,v_1,v_2), then F is a pairwise closed subset of (X,u_1,u_2).

Proof

Let F be a a pairwise closed subset of (Y,v_1,v_2). Then $v_1v_2F=F$ and $v_2v_1F=F$. Since Y is both a closed subset of (X,u_1) and (X,u_2) , $u_1F=F$ and $u_2F=F$. Therefore, $F=v_1v_2F= v_1(u_2F\cap Y)= v_1(u_2(F\cap Y))= v_1(u_2F)= u_1(u_2F)\cap Y= u_1(u_2F\cap Y)= u_1(u_2(F\cap Y))= u_1u_2F$ and $F=v_1v_2F= v_2(u_2F\cap Y)= v_2(u_1(F\cap Y))= v_2(u_1F)= u_2(u_1F)\cap Y= u_2(u_1F\cap Y)= u_2(u_1(F\cap Y)) =u_2u_1F$. Consequently,$u_1u_2F=F=u_2u_1F$.

Hence, F is a pairwise closed subset of (X,u_1,u_2).

Proposition: 4.1.10

Let (X, u_1, u_2) be a biclosure space and let $A \subseteq X$. Then

(i) A is pairwise open if and only if $A = X - u_1u_2(X-A) = X - u_2u_1(X-A)$.

(ii) If G is pairwise open and $G \subseteq A$, then $G \subseteq X - u_1u_2(X-A) = X - u_2u_1(X-A)$

Proposition: 4.1.11

Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a pairwise closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$ if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a pairwise closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2)$.

Proof

Let $\beta \in I$ and let F be a pairwise closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

Then F is both a closed subset of (X_β, u_β^1) and (X_β, u_β^2).

Since $\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_\alpha^1) \rightarrow (X_\beta, u_\beta^1)$ is continuous, $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha^1)$.

Since $\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_\alpha^2) \rightarrow (X_\beta, u_\beta^2)$ is continuous $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha^2)$. Hence, $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of

$\prod_{\alpha\in I}(X_\alpha,u_\alpha^1,u_\alpha^2)$. Then $F\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in I}}X_\alpha$ is a pairwise closed subset of $\prod_{\alpha\in I}(X_\alpha,u_\alpha^1,u_\alpha^2)$.

Conversely, let $F\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in I}}X_\alpha$ be a pairwise closed subset of $\prod_{\alpha\in I}(X_\alpha,u_\alpha^1,u_\alpha^2)$. Then $F\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in I}}X_\alpha$ is both a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1)$ and $\prod_{\alpha\in I}(X_\alpha, u_\alpha^2)$.

Since π_β: $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1)\to(X_\beta,u_\beta^1)$ is closed, $\pi_\beta(F\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in I}}X_\alpha)=F$ is a closed subset of (X_β,u_β^1). Since π_β: $\prod_{\alpha\in I}(X_\alpha, u_\alpha^2)\to(X_\beta,u_\beta^2)$ is closed, $\pi_\beta(F\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in I}}X_\alpha)=F$ is a closed subset of (X_β,u_β^2). Hence, F is a closed subset of $(X_\beta,u_\beta^1,u_\beta^2)$ and F is a pairwise closed subset of $(X_\beta,u_\beta^1,u_\beta^2)$.

Definition: 4.1.12

Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. A map f: $(X,u_1,u_2)\to(Y,v_1,v_2)$ is said to be **preserve pairwise closed** if f(F) is a pairwise closed subset of (Y,v_1,v_2) whenever F is a pairwise closed subset of (X,u_1,u_2).

Definition: 4.1.13

Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. A map f: $(X,u_1,u_2)\to(Y,v_1,v_2)$ is said to be **preserve pairwise open** if f(F) is a pairwise open subset of (Y,v_1,v_2) whenever F is a pairwise open subset of (X,u_1,u_2).

Proposition: 4.1.14

Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces. If f: $(X,u_1,u_2) \to (Y,v_1,v_2)$ and g: $(Y,v_1,v_2) \to (Z,w_1,w_2)$ are be preserve pairwise closed maps, then $g \circ f$: $(X,u_1,u_2) \to (Z,w_1,w_2)$ is preserve pairwise closed.

Proposition: 4.1.15

Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Then for each $\beta \in I$, the projection map π_β: $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)$ is preserve pairwise closed.

Proof

Let F be a pairwise closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Then F is both a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$.

Since π_β: $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)$ is closed, $\pi_\beta(F)$ is a closed subset of (X_β, u^1_β).Since π_β: $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)$ is closed, $\pi_\beta(F)$ is a closed subset of (X_β, u^2_β).Hence $\pi_\beta(F)$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ and $\pi_\beta(F)$ is a pairwise closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.Hence, π_β is preserve pairwise closed.

Proposition: 4.1.16

Let (X,u_1,u_2) be a biclosure space, $\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and f: $X \to \prod_{\alpha\in I} Y_\alpha$ be a map. Then f: $(X,u_1,u_2) \rightarrow \prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$ is preserve pairwise closed if and only if $\pi_\alpha \circ$ f: $(X,u_1,u_2) \rightarrow \prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$ is preserve pairwise closed if and only if $\pi_\alpha \circ$ f : $(X,u_1,u_2) \rightarrow (Y_\alpha, v_\alpha^1, v_\alpha^2)$ is preserve pairwise closed for each $\alpha \in$ I.

Proof

Let f be preserve pairwise closed. Since π_α is preserve pairwise closed for each $\alpha \in$ I, also $\pi_\alpha \circ$ f is preserve pairwise closed for each $\alpha \in$ I.

Conversely, let $\pi_\alpha \circ$ f be preserve pairwise closed for each $\alpha \in$ I. Suppose that f is not preserve pairwise closed. Then, there exists a pairwise closed subset F of (X,u_1,u_2) such that $\prod_{\alpha\in I} v_\alpha^1 \pi_\alpha\left(\prod_{\alpha\in I} v_\alpha^2 \pi_\alpha(f(F))\right) \not\subseteq f(F)$ or

$\prod_{\alpha\in I} v_\alpha^2 \pi_\alpha\left(\prod_{\alpha\in I} v_\alpha^1 \pi_\alpha(f(F))\right) \not\subseteq f(F)$. If $\prod_{\alpha\in I} v_\alpha^1 \pi_\alpha\left(\prod_{\alpha\in I} v_\alpha^2 \pi_\alpha(f(F))\right) \not\subseteq f(F)$.

Then there exists $\beta \in$ I such that $v_\beta^1 v_\beta^2\ \pi_\beta\ (f(F)) \not\subseteq \pi_\beta(f(F))$.But $\pi_\beta \circ$ f is preserve pairwise closed, $\pi_\beta(f(F))$ is a pairwise closed subset of $\left(Y_\beta, v_\beta^1, v_\beta^2\right)$.

This is a contradiction. If $\prod_{\alpha\in I} v_\alpha^2 \pi_\alpha\left(\prod_{\alpha\in I} v_\alpha^1 \pi_\alpha(f(F))\right) \not\subseteq f(F)$. Then there $\beta \in$ I such that $v_\beta^2 v_\beta^1 \pi_\beta\ (f(F)) \not\subseteq \pi_\beta(f(F))$.But $\pi_\beta \circ$ f is preserve pairwise closed, $\pi_\beta(f(F))$ is a pairwise closed subset of $\left(Y_\beta, v_\beta^1, v_\beta^2\right)$. This is a contradiction.

Proposition: 4.1.17

Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ and $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in$ I, let $f_\alpha: X_\alpha \rightarrow Y_\alpha$ be a surjection and let f: $\prod_{\alpha\in I} X_\alpha \to \prod_{\alpha\in I} Y_\alpha$ be defined by $f((x_\alpha)_{\alpha\in I})=(f_\alpha(x_\alpha))_{\alpha\in I}$. Then f: $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2) \to \prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$ is preserve pairwise closed if and only if f_α: $(X_\alpha, v_\alpha^1, v_\alpha^2) \to (Y_\alpha, v_\alpha^1, v_\alpha^2)$ is preserve pairwise closed for each $\alpha \in I$.

Proof

Let $\beta \in I$ and let F be a closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$. Then $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a pairwise closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$. Since f is preserve pairwise closed, $f(F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha)$ is a pairwise closed subset of $\prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$.

But $f(F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha)=f_\beta(F)\times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} Y_\alpha$, hence $f_\beta(F)\times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} Y_\alpha$ is a pairwise closed subset of $\prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$. By Proposition 4.1.11, $f_\beta(F)$ is a pairwise closed subset of $(Y_\beta, v_\beta^1, v_\beta^2)$. Hence, f_β is preserve pairwise closed.

Conversely, let f_β be preserve pairwise closed for each $\beta \in I$. Suppose that f is not preserve pairwise closed.

Therefore, there exists a pairwise closed subset F of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ such that $\prod_{\alpha\in I} v_\alpha^1 \pi_\alpha(\prod_{\alpha\in I} v_\alpha^2 \pi_\alpha(f(F))) \not\subseteq f(F)$ or $\prod_{\alpha\in I} v_\alpha^2 \pi_\alpha(\prod_{\alpha\in I} v_\alpha^1 \pi_\alpha(f(F))) \not\subseteq f(F)$. If $\prod_{\alpha\in I} v_\alpha^1 \pi_\alpha(\prod_{\alpha\in I} v_\alpha^2 \pi_\alpha(f(F))) \not\subseteq f(F)$.

Then there exists $\beta \in I$ such that $v_\beta^1 v_\beta^2 \pi_\beta(f(F)) \nsubseteq \pi_\beta(f(F))$.But $\pi_\beta(F)$ is a pairwise closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$ and f_β is preserve pairwise closed, $f_\beta(\pi_\beta(F))$ is a pairwise closed subset of $(Y_\beta, v_\beta^1, v_\beta^2)$. This is a contradiction.

If $\prod_{\alpha\in I} v_\alpha^2 \pi_\alpha(\prod_{\alpha\in I} v_\alpha^1 \pi_\alpha(f(F))) \nsubseteq f(F)$. Then there $\beta \in I$ such that $v_\beta^2 v_\beta^1 \pi_\beta(f(F)) \nsubseteq \pi_\beta(f(F))$.But $\pi_\beta(F)$ is a pairwise closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$ and f_β is preserve pairwise closed, $f_\beta(\pi_\beta(F))$ is a pairwise closed subset of $(Y_\beta, v_\beta^1, v_\beta^2)$. This is a contradiction.

SECTION 4.2
BICONTINUOUS MAPS

In this section, the concepts of bicontinuous maps in biclosure spaces are studied.

Definition: 4.2.1

Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and let $i \in \{1,2\}$. A map $f: (X,u_1,u_2) \to (Y,v_1,v_2)$ is called **i-continuous** if the map $f: (X,u_i) \to (Y,v_i)$ is continuous. A map f is called **continuous** if f is i-continuous for each $i \in \{1, 2\}$.

Definition: 4.2.2

Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. A map $f :(X,u_1,u_2) \to (Y,v_1,v_2)$ is called **bicontinuous** if the map $f: (X,u_1) \to (Y,v_2)$ is continuous.

Proposition: 4.2.3

Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. Then $f: (X,u_1,u_2) \to (Y,v_1,v_2)$ is bicontinuous if and only if $u_1 f^{-1}(B) \subseteq f^{-1}(v_2 B)$ for every $B \subseteq Y$.

Proof

Let $B \subseteq Y$. Then $f^{-1}(B) \subseteq v_2 B$. Since f is bicontinuous, $f(u_1 f^{-1}(B)) \subseteq v_2 f(f^{-1}(B)) \subseteq v_2 B$. Therefore, $u_1 f^{-1}(B) \subseteq f^{-1}(v_2 B)$.

Conversely, let A $\subseteq$ X. Then f(A) $\subseteq$ Y.Thus $u_1 f^{-1}(f(A)) \subseteq f^{-1}(v_2 f(A))$. Consequently, $f(u_1 A) \subseteq f(u_1 f^{-1}(f(A))) \subseteq f(f^{-1}(v_2 f(A))) \subseteq v_2 f(A)$.

Hence, F is bicontinuous.

Proposition: 4.2.4

Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces. If f: $(X,u_1,u_2) \rightarrow (Y,v_1,v_2)$ is bicontinuous and g: $(Y,v_1,v_2) \rightarrow (Z,w_1,w_2)$ is 2-continuous, then $g \circ f : X \rightarrow Z$ is bicontinuous.

Proof

Let A $\subseteq$ X. Since $g \circ f(u_1 A) = g(f(u_1 A))$ and f is bicontinuous, $g(f(u_1 A)) \subseteq g(v_2 f(A))$. Since g is 2-continuous, $g(v_2 f(A)) \subseteq w_2 g(f(A))$. Thus $g \circ f(u_1 A) \subseteq w_2 g \circ f(A)$. Consequently, $g \circ f$ is bicontinuous.

Definition: 4.2.5

Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and let i $\in\{1,2\}$. A map f: $(X,u_1,u_2) \rightarrow (Y,v_1,v_2)$ is called **i-closed** if the map f: $(X, u_i) \rightarrow (Y, v_i)$ is closed. A map f is called closed if f is i-closed for each i $\in\{1,2\}$.

Definition: 4.2.6

Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces and let i $\in\{1,2\}$. A map f: $(X,u_1,u_2) \rightarrow (Y,v_1,v_2)$ is called **i-open** if the map f: $(X, u_i) \rightarrow (Y, v_i)$ is open. A map f is called open if f is i-open for each i $\in\{1,2\}$.

Definition: 4.2.7

Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. A map f: $(X,u_1,u_2) \rightarrow (Y,v_1,v_2)$ is called **biclosed** if the map f: $(X, u_1) \rightarrow (Y, v_2)$ is closed.

Definition: 4.2.8

Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. A map f: $(X,u_1,u_2) \rightarrow (Y,v_1,v_2)$ is called **biopen** if the map f: $(X, u_1) \rightarrow (Y, v_2)$ is open.

Proposition: 4.2.9

Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces. If f: $(X,u_1,u_2) \rightarrow (Y,v_1,v_2)$ is 1-closed and g: $(Y,v_1,v_2) \rightarrow (Z,w_1,w_2)$ is biclosed, then $g \circ f : (X,u_1,u_2) \rightarrow (Z,w_1,w_2)$ is biclosed.

Proof

Let F be a closed subset of (X,u_1). Since f is 1-closed, f(F)is a closed subset of (Y,v_1). Since g is biclosed, g(f(F)) is a closed subset of (Z,w_2). Hence, $g \circ f(F)$ is a closed subset of (Z,w_2). Consequently, $g \circ f$ is biclosed.

Proposition: 4.2.10

Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces. If f: $(X,u_1,u_2) \rightarrow (Y,v_1,v_2)$ and g: $(Y,v_1,v_2) \rightarrow (Z,w_1,w_2)$ is maps. Then

(*i*) If $g \circ f$ is biclosed and f is surjective 1-continuous, then g is biclosed.

(*ii*) If $g \circ f$ is biclosed and g is injective 2-continuous, then f is biclosed.

Proof

(i) Let F be a closed subset of (Y, v_1). Since f is 1-continuous, $f^{-1}(F)$ is a closed subset of (X, u_1). Since g o f is biclosed and f is surjective $g \circ f(f^{-1}(F))=g(F)$ is a closed subset of (Z, w_2). Hence, g is biclosed.

(ii) Let F be a closed subset of (X, u_1). Since g o f is biclosed, $g \circ f(F)$ is a closed subset of (Z, w_2). Since g is 2-continuous and injective $g^{-1}(g \circ f(F))=f(F)$ is a closed subset of (Y, v_2). Therefore, f is biclosed.

Proposition: 4.2.11

Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Then for each $\beta \in I$, the projection map $\pi_\beta: \prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha) \rightarrow (X_\beta, u^1_\beta, u^2_\beta)$ is closed.

Proposition: 4.2.12

Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ and $\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in I$, let $f_\alpha: (X_\alpha, u^1_\alpha, u^2_\alpha) \rightarrow (Y_\alpha, v^1_\alpha, v^2_\alpha)$ be a surjection and let f: $\prod_{\alpha \in I} (X_\alpha, u_\alpha) \rightarrow \prod_{\alpha \in I} (Y_\alpha, v^1_\alpha, v^2_\alpha)$ be defined by $f((x_\alpha)_{\alpha \in I})=(f_\alpha(x_\alpha))_{\alpha \in I}$. Then f is biclosed if and only if f_α is biclosed for each $\alpha \in I$.

Proof

Let $\beta \in I$ and let F be a closed subset of (X_β, u^1_β). Then $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha)$. Since f is biclosed, $f(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha)$ is a closed subset of $\prod_{\alpha\in I}(Y_\alpha, v_\alpha^2)$. But $f\left(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\right) = f_\beta(F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$,

hence $f_\beta(F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$ is a closed subset of $\prod_{\alpha\in I}(Y_\alpha, v_\alpha^2)$.

By Proposition 1.1.16 $f_\beta(F)$ is a closed subset of (Y_β, v_β).Hence, f_β is biclosed.

Conversely, let $f_\beta$be biclosed for each $\beta \in I$. Suppose that f is a not biclosed. Then there exists a closed subset F of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1)$ such that $\prod_{\beta\in I} v_\beta^2$ $\pi_\beta(f(F)) \not\subseteq f(F)$.

Therefore, there exists $\beta \in I$ such that $v_\beta^2 f_\beta(\pi_\beta(F)) \not\subseteq f_\beta(\pi_\beta(F))$. But $\pi_\beta(F)$ is a closed subset of (X_β, u_β^1) and f_β is biclosed, $f_\beta(\pi_\beta(F))$ is a closed subset of (Y_β, v_β^2). This is a contradiction.

Proposition: 4.2.13

Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. Then for each $\beta \in I$, the projection map $\pi_\beta: \prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2) \to (X_\beta, u_\beta^1, u_\beta^2)$ is continuous.

Proposition: 4.2.14

Let (X, u_1, u_2) be a biclosure space, $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in I\}$ be a family of biclosure space and f: $(X, u_1, u_2) \to \prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$ be a map. Then f is bicontinuous if and only if $\pi_\alpha \circ f$ is bicontinuous for each $\alpha \in I$.

Proof

Let f be bicontinuous. Since π_α is 2-continuous for each $\alpha \in I$, π_α of is bicontinuous for each $\alpha \in I$.

Conversely, let π_α of be bicontinuous for each $\alpha \in I$. Suppose that f is not bicontinuous. Then there exists a subset A of X such that $f(u_1 A) \not\subseteq \prod_{\alpha \in I} v_\alpha^2\ \pi_\alpha\ (f(A))$. Therefore, there exists $\beta \in I$ such that $\pi_\beta(f(u_1 A)) \not\subseteq v_\beta^2\ \pi_\beta(f(A))$. This is contradicts the bicontinuity of π_β of. Hence, f is bicontinuous.

Proposition: 4.2.15

Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ and $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in I$, let $f_\alpha: (X_\alpha, u_\alpha) \rightarrow (Y_\alpha, v_\alpha)$ be a map and let $f: \prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2) \rightarrow \prod_{\alpha \in I} (Y_\alpha, v_\alpha^1, v_\alpha^2)$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. Then f is bicontinuous if and only if f_α is bicontinuous for each $\alpha \in I$.

Proof

Let f be bicontinuous, let $\beta \in I$ and let $A \subseteq X_\beta$. Then

$$f_\beta(u_\beta^1 A) = \pi_\beta\Big(f_\beta(u_\beta^1 A) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} f_\alpha(u_\alpha^1 X_\alpha)\Big)$$

$$= \pi_\beta\Big(f_\beta\big(u_\beta^1 A \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} u_\alpha^1 X_\alpha\big)\Big)$$

$$= \pi_\beta\Big(f\big(\prod_{\alpha \in I} u_\alpha^1 \pi_\alpha\big(A \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\big)\big)\Big)$$

$$\subseteq \pi_\beta\Big(\prod_{\alpha\in I} v_\alpha^2\pi_\alpha\Big(f\Big(A\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)\Big)\Big)$$

$$= \pi_\beta\Big(\prod_{\alpha\in I} v_\alpha^2\pi_\alpha\Big(f_\beta(A)\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} f_\alpha(X_\alpha)\Big)\Big)$$

$$= \pi_\beta\Big(v_\beta^2\, f_\beta(A)\times \prod_{\alpha\in I} v_\alpha^2 f_\alpha(X_\alpha)\Big)$$

$$= v_\beta^2\, f_\beta(A).$$

Hence, f_β is bicontinuous.

Conversely, let f_α be bicontinuous for each $\alpha \in I$ and let $A \subseteq \prod_{\alpha\in I} X_\alpha$. Then

$$f\Big(\prod_{\alpha\in I} u_\alpha^1\pi_\alpha(A)\Big) = \prod_{\alpha\in I} f_\alpha\Big(\prod_{\alpha\in I} u_\alpha^1\pi_\alpha(A)\Big)$$

$$= \prod_{\alpha\in I} f_\alpha(u_\alpha^1\,\pi_\alpha(A))$$

$$\subseteq \prod_{\alpha\in I} v_\alpha^2 f_\alpha(\pi_\alpha(A))$$

$$= \prod_{\alpha\in I} v_\alpha^2\,\pi_\alpha(f(A))$$

Therefore f is bicontinuous.