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ANALYSIS OF AN $M/G/1$ QUEUE WITH EXPONENTIALLY DISTRIBUTED MULTIPLE AND SINGLE WORKING VACATIONS

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Abstract

Consider an $M/G/1$ queue with exponentially distributed multiple and single working vacations where the server cooks at a slower rate rather than completely stops service during vacation periods. Using supplementary variable technique, we derive the steady-state distributions for the number of customers in the system both at the arbitrary epoch and departure epoch. Further the expected system size and various probabilities are calculated and the results obtained are illustrated numerically.

0. Introduction

Queueing systems with server vacations have wide applicability in analyzing the performance of computer systems, data communication networks and product systems. During the last three decades, the queueing systems with vacations have been studied extensively. Various vacation policies provide more flexibility for

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optimal design. The explanations are seen in monograph of Takagi [11], Tian and Zhang [12] and the survey of Doshi [3]. In these studies, it is assumed that the server stops primary service completely during vacations. In 2002, Servi and Finn [10] introduced a class of semi vacation policies according to which a customer is served at a lower rate during vacation. Servi and Finn [10] analyzed an $M/M/1$ queue with working vacation, where the server works at a lower service rate rather than completely stopping the service during vacation period. Subsequently, Kim et al. [6], and Wu and Takagi [11] generalized this model as an $M/G/1$ queue with working vacations. Baba [1] provided a study of $GI/M/1$ queue with multiple working vacations by using matrix geometric method. Banik et al. [2] discussed $GI/M/1/N$ queue with working vacations. Liu et al. [7] proved the stochastic decomposition of the $M/M/1$ queue with working vacations and Li and Tian [8] considered two types of discrete-time $GI/Geo/1$ queues with working vacations. Later, Li et al. [9] in their paper considered the $M/G/1$ with exponentially distributed working vacations which is a special case of that in Wu and Takagi [13] and is the same as that in Kim et al. [6]. But they have utilized the matrix analytic approach and derived more results and properties about the system performance. Recently, Jemila Parveen et al. [4] analyzed $M/M/1$ queue with working vacations and derived the steady state solutions in a closed form by directly solving the difference differential equations. Later, they have discussed the waiting time distribution of an arbitrary customer for the model and verified the classical relation between the PGF of the system and the LST of the waiting time distribution. The steady state results of $M/M/1$ working vacation are also extended to $M^X/M/1$ working vacation queueing model for both multiple and single vacations by Julia et al. [5].

In this paper, we have analyzed a non Markovian $M/G/1$ queue with exponentially distributed working vacation. Many authors have paid attention on matrix geometric approach, where the results are not easily reachable. In this spirit, we have made an attempt to derive the probability generating functions of the steady state distribution using supplementary variable technique and presented in closed form. Finally the performance measures including expected system size, probability that the server is idle, on vacation and regular busy are calculated and numerical values are tabulated.

1. $M/G/1$ Multiple Working Vacations

Model description

Consider a classical $M/G/1$ queue with arrival rate λ and regular service rate μ_b. The server begins a working vacation of random length V at the instant, when the queue becomes empty. The vacation duration follows an exponential distribution with parameter η. During a working vacation, an arriving customer is served at the rate of μ_v. When a vacation ends if there are no customers yet in the queue, another vacation is taken. Otherwise the server switches the service rate from μ_v to μ_b and then regular busy period starts. It is assumed that the remaining regular service time $(S_b^o(t))$ and remaining service time during vacation $(S_v^o(t))$ are supplementary variables following general distributions with finite mean and variance and they are independent of each other and also independent of the arrivals. In addition, the service discipline is assumed to be FIFO.

System size distribution

In this section to derive the steady state system size equations, the following notations and probabilities are defined.

Let $N(t)$ denote the system size including the one in service at time t.

$Y(t) = \{0,$ if the server is idle in vacation at time t; 1, if the server is busy in vacation with lower service rate at time t; 2, if the server is busy with regular service rate at time $t\}$.

$$Q_0(t) = \Pr(N(t) = 0, Y(t) = 0),$$

$$Q_n(x, t) = \Pr(N(t) = n, Y(t) = 1, x \le S_v^0(t) < x + dt), \quad n \ge 1,$$

$$P_n(x, t) = \Pr(N(t) = n, Y(t) = 2, x \le S_b^0(t) < x + dt), \quad n \ge 1.$$

Thus, Q_0 gives the probability that the server is idle in vacation at time t. $Q_n(x, t)$ and $P_n(x, t)$ give the probability that there are n customers in the system while the server is serving at lower service rate and in regular service rate, respectively at time t.

Assuming the steady state probabilities $Q_0 = \underset{t\to\infty}{lt}\, Q_0(t)$, $Q_n(x, t) = \underset{t\to\infty}{lt}\, Q_n(x, t)$, $P_n(x, t) = \underset{t\to\infty}{lt}\, P_n(x, t)$, $n \geq 1$, $\frac{\partial}{\partial t} P_n(x, t) = \frac{\partial}{\partial t} Q_n(x, t) = 0$ exist and observing the changes of states in the interval $(t, t + \Delta t)$ at any time t, we obtain the following steady state system size equations.

Steady state equations:

$$(\lambda + \eta)Q_0 = P_1(0) + Q_1(0).$$

$$-\frac{d}{dt} Q_1(x) = -(\lambda + \eta)Q_1(x) + Q_2(0)s_v(x) + \lambda g_1 Q_0 s_v(x),$$

$$-\frac{d}{dt} Q_n(x) = -(\lambda + \eta)Q_n(x) + Q_{n+1}(0)s_v(x) + \lambda g_n Q_0 s_v(x) + \lambda Q_{n-1}(x), \quad n \geq 2,$$

$$-\frac{d}{dt} P_1(x) = -\lambda P_1(x) + P_2(0)s_b(x) + \int_0^{\infty} Q_1(y)dy\eta s_b(x),$$

$$-\frac{d}{dt} P_n(x) = -\lambda P_n(x) + P_{n+1}(0)s_b(x) + \int_0^{\infty} Q_n(y)dy\eta s_b(x) + \lambda P_{n-1}(x), \quad n \geq 2. \quad (1)$$

To obtain the probability generating functions we make use of the following LST:

$$Q_n^* = \int_0^{\infty} e^{-\theta x} Q_n(x)dx \text{ and } P_n^* = \int_0^{\infty} e^{-\theta x} P_n(x)dx.$$

Taking the LST on both sides of the steady state equations and using the LST properties, we have

$$\theta Q_1^*(\theta) - Q_1(0) = (\lambda + \eta)Q_1^*(\theta) - Q_2(0)S_v^*(\theta) - \lambda Q_0 g_1 S_v^*(\theta), \quad (2)$$

$$\theta Q_n^*(\theta) - Q_n(0) = (\lambda + \eta)Q_n^*(\theta) - Q_{n+1}(0)S_v^*(\theta)$$
$$- \lambda Q_0 g_n S_v^*(\theta) - \lambda Q_{n-1}^*(\theta), \quad n \geq 2, \quad (3)$$

$$\theta P_1^*(\theta) - P_1(0) = \lambda P_1^*(\theta) - P_2(0)S_b^*(\theta) - \int_0^{\infty} Q_1(y)dy\eta S_b^*(\theta), \quad (4)$$

$$\theta P_n^*(\theta) - P_n(0) = \lambda P_n^*(\theta) - P_{n+1}(0)S_b^*(\theta)$$
$$- \int_0^{\infty} Q_n(y)dy\eta S_b^*(\theta) - \lambda P_{n-1}^*(\theta), \quad n \geq 2. \quad (5)$$

Steady state solutions:

In order to derive the distribution of the system size probabilities, we define the following pgfs:

$$Q_1^*(z, \theta) = \sum_{n=1}^{\infty} Q_n^*(\theta) z^n, \quad Q_1(z, 0) = \sum_{n=1}^{\infty} Q_n(0) z^n, \quad P_V^*(z, \theta) = Q_1^*(z, \theta) + Q_0,$$

$$P_B^*(z, \theta) = \sum_{n=1}^{\infty} P_n^*(\theta) z^n \text{ and } P_B(z, 0) = \sum_{n=1}^{\infty} P_n(0) z^n.$$

Multiplying equations (2) and (3) by the proper powers of z and summing up over $n = 1$ to ∞, we get

$$(\theta - h(z)) Q_1^*(z, \theta) = Q_1(z, 0)\left(1 - \frac{S_v^*(\theta)}{z}\right) - S_v^*(\theta)(\lambda z Q_0 - Q_1(0)). \tag{6}$$

At

$$\theta = h(z) = \eta + \lambda(1 - z),\ Q_1(z, 0) = \frac{z S_v^*(h(z))}{z - S_v^*(h(z))}(\lambda z Q_0 - Q_1(0)).$$

The unique root z_1 of $z - S_v^*(h(z))$ lies inside $(0, 1)$ (by similar argument of Li et al. [9])

$$Q_1(0) = \lambda z_1 Q_0. \tag{7}$$

Substituting for $Q_1(0)$ in $Q_1(z, 0)$,

$$Q_1(z, 0) = \frac{\lambda z Q_0 S_v^*(h(z))(z - z_1)}{z - S_v^*(h(z))}. \tag{8}$$

Therefore substituting $Q_1(z, 0)$ in equation (6), we have

$$Q_1^*(z, \theta) = \frac{\lambda z Q_0 (z - z_1)(S_v^*(h(z)) - S_v^*(\theta))}{(\theta - h(z))(z - S_v^*(h(z)))}.$$

At

$$\theta = 0, \quad Q_1^*(z, 0) = \frac{\lambda z Q_0 (z - z_1)(1 - S_v^*(h(z)))}{h(z)(z - S_v^*(h(z)))}. \tag{9}$$

Similarly multiplying equations (4) and (5) by appropriate powers of z and then adding, we have

$$\theta P_B^*(z,\theta) - P_B(z,0) = \lambda P_B^*(z,\theta) - \frac{S_b^*(\theta)}{z}(P_B(z,0) - P_1(0)z)$$

$$- \eta S_b^*(\theta)\sum_{n=1}^{\infty}\int_0^{\infty} Q_n(y)dyz^n - \lambda z P_B^*(z,\theta).$$

Since $\sum_{n=1}^{\infty}\int_0^{\infty} Q_n(y)dyz^n = \sum_{n=1}^{\infty} Q_n^*(0)z^n = Q_1^*(z,0)$, we have

$$(\theta - w(z))P_B^*(z,\theta) = P_B(z,0)\left(1 - \frac{S_b^*(\theta)}{z}\right) - S_b^*(\theta)(\eta Q_1^*(z,0) - P_1(0)).$$

At

$$\theta = w(z) = \lambda(1-z),$$

$$P_B(z,0) = \frac{zS_b^*(w(z))}{z - S_b^*(w(z))}(\eta Q_1^*(z,0) - P_1(0)) \tag{10}$$

and

$$P_B^*(z,\theta) = \frac{z(S_b^*(w(z)) - S_b^*(\theta))}{(\theta - w(z))(z - S_b^*(w(z)))}(\eta Q_1^*(z,0) - P_1(0)).$$

At $\theta = 0$,

$$P_B^*(z,\theta) = \frac{Q_0 z(1 - S_b^*(w(z)))}{w(z)(z - S_b^*(w(z)))}\left[\frac{z\lambda\eta(z - z_1)(1 - S_v^*(h(z)))}{h(z)(z - S_v^*(h(z)))} - \lambda(1 - z_1)\right]. \tag{11}$$

Thus, the total PGF $P(z)$ of the system size probabilities is given by

$$P(z) = P_B^*(z,0) + P_v^*(z,0).$$

Using the normalizing condition $P(1) = 1$,

$$Q_0 = \frac{1 - \rho_b}{\dfrac{h(z)}{\eta} - \dfrac{\rho_b(1 - z_1)S_v^*(\eta)}{(1 - S_v^*(\eta))}}.$$

Mean system length

Let L_v and L_b denote the mean system size during the working vacation and regular busy, respectively. Then

$$L_v = \frac{d}{dz} P_v^*(z, 0)_{z=1}$$

$$= \frac{\rho_b}{1-\rho_b} Q_0 \left[\frac{h(z_1)}{\eta} - \frac{(1-z_1)S_v^*(\eta)}{(1-S_v^*(\eta))} \right],$$

$$L_b = \frac{d}{dz} P_b^*(z, 0)_{z=1}$$

$$= Q_0 \left\{ \left[\frac{-\rho_b}{1-\rho_b} + \frac{(\lambda E(X))^2 E(S_b^2)}{2(1-\rho_b)^2} \right] \left[\frac{h(z_1)}{\eta} + \frac{(z_1-1)S_v^*(\eta)}{1-S_v^*(\eta)} \right] \right.$$

$$\left. + \frac{\rho_b}{1-\rho_b} \left[\left(\frac{(1-z_1)(S_v^*(\eta) + \lambda S_v^{*'}(\eta))}{(1-S_v^*(\eta))^2} - \frac{S_v^*(\eta)h(z_1)}{\eta(1-S_v^*(\eta))} + \frac{h(z_1)}{\eta^2} \right) \right] \right\}.$$

Hence, the mean system size of the model L is given by $L = L_v + L_b$.

Other performance measures

- Probability that the server is on vacation (P_v) is given by

$$P_v = \underset{z \to 1}{lt}\, Q_1^*(z, 0) = \frac{Q_0 h(z_1)}{\eta}.$$

- Probability that the server is busy (P_b) is given by

$$P_b = \underset{z \to 1}{lt}\, P_b^*(z, 0) = \frac{Q_0 \rho_b}{(1-\rho_b)} \left[\frac{h(z_1)}{\eta} + \frac{\lambda(z_1-1)S_v^*(\eta)}{1-S_v^*(\eta)} \right].$$

2. $M/G/1$ Single Working Vacations

Model description

The model of this section differs from the model of Section 1 only in vacation pattern, i.e., the server who finds the system empty after a regular service takes a working vacation and after returning from vacation remains idle in the system. The other assumptions are same as in Section 1.

System size distribution

In this section to derive the steady state system size equations, the following notations and probabilities are defined.

Let $N(t)$ denote the system size including the one in service at time t.

$Y(t) = \{0$, if the server is idle in vacation at time t; 1, if the server is idle in the system at time t; 2, if the server is busy in vacation with lower service rate at time t; 3, if the server is busy with regular service rate at time $t\}$.

$$Q_0(t) = \Pr(N(t) = 0,\ Y(t) = 0),$$

$$P_0(t) = \Pr(N(t) = 0,\ Y(t) = 1),$$

$$Q_n(x,\ t) = \Pr(N(t) = n,\ Y(t) = 2,\ x \le S_v^0(t) \le x + dt), \quad n \ge 1,$$

$$P_n(x,\ t) = \Pr(N(t) = n,\ Y(t) = 3,\ x \le S_b^0(t) \le x + dt), \quad n \ge 1.$$

Thus, Q_0 and P_0 give the probability that the server is idle in vacation and in system, respectively at time t. $Q_n(x,\ t)$ and $P_n(x,\ t)$ give the probability that there are n customers in the system while the server is serving at lower service rate and in regular service rate, respectively at time t.

Assuming the steady state probabilities as in previous section the following are the steady state system size equations:

$$\lambda P_0 = \eta Q_0, \tag{12}$$

$$(\lambda + \eta)Q_0 = P_1(0) + Q_1(0),$$

$$-\frac{d}{dt} Q_1(x) = -(\lambda + \eta)Q_1(x) + Q_2(0)s_v(x) + \lambda Q_0 s_v(x),$$

$$-\frac{d}{dt} Q_n(x) = -(\lambda + \eta)Q_n(x) + Q_{n+1}(0)s_v(x) + \lambda Q_{n-1}, \quad n \ge 2,$$

$$-\frac{d}{dt} P_1(x) = -\lambda P_1(x) + P_2(0)s_b(x) + \lambda P_0 s_b(x) + \int_0^\infty Q_1(y)dy\eta s_b(x),$$

$$-\frac{d}{dt} P_n(x) = -\lambda P_n(x) + P_{n+1}(0)s_b(x)$$

$$+ \int_0^\infty Q_n(y)dy\eta s_b(x) + \lambda P_{n-1}(x), \quad n \ge 2. \tag{13}$$

For further simplification, we define the following LST:

$$Q_n^* = \int_0^\infty e^{-\theta x} Q_n(x)dx \text{ and } P_n^* = \int_0^\infty e^{-\theta x} P_n(x)dx.$$

Taking the LST on both sides of the above equations, we have

$$\theta Q_1^*(\theta) - Q_1(0) = (\lambda + \eta)Q_1^*(\theta) - Q_2(0)S_v^*(\theta) - \lambda Q_0 S_v^*(\theta), \tag{14}$$

$$\theta Q_n^*(\theta) - Q_n(0) = (\lambda + \eta)Q_n^*(\theta) - Q_{n+1}(0)S_v^*(\theta) - \lambda Q_{n-1}^*(\theta), \quad n \geq 2, \tag{15}$$

$$\theta P_1^*(\theta) - P_1(0) = \lambda P_1^*(\theta) - P_2(0)S_b^*(\theta) - \lambda P_0 S_b^*(\theta) - \int_0^\infty Q_1(y)dy\eta S_b^*(\theta), \tag{16}$$

$$\theta P_n^*(\theta) - P_n(0) = \lambda P_n^*(\theta) - P_{n+1}(0)S_b^*(\theta)$$
$$- \int_0^\infty Q_n(y)dy\eta S_b^*(\theta) - \lambda P_{n-1}^*(\theta), \quad n \geq 2. \tag{17}$$

By similar algebraic manipulation, the probability generating functions are obtained as follows.

The expressions for $Q_1(z, 0)$ and $Q_1^*(z, \theta)$ are obtained using equations (14) and (15) as

$$Q_1(z, 0) = \frac{\lambda z Q_0 S_v^*(h(z))(z - z_1)}{z - S_v^*(h(z))}$$

and

$$Q_1^*(z, \theta) = \frac{\lambda z Q_0 (z - z_1)(S_v^*(h(z)) - S_v^*(\theta))}{(\theta - h(z))(z - S_v^*(h(z)))}.$$

At $\theta = 0$, $Q_1^*(z, 0) = \dfrac{\lambda z Q_0 (z - z_1)(1 - S_v^*(h(z)))}{h(z)(z - S_v^*(h(z)))}$.

Using equations (13), (16) and (17), the pgf of corresponding to the busy period is found to be

$$P_B(z, 0) = \frac{zS_b^*(w(z))}{z - S_b^*(w(z))}(\eta Q_1^*(z, 0) + \lambda z P_0 - P_1(0))$$

and

$$P_B^*(z,\theta)$$

$$= \frac{Q_0 z(S_b^*(w(z)) - S_b^*(\theta))}{(\theta - w(z))(z - S_b^*(w(z)))}\left[\frac{z\lambda\eta(z-z_1)(1-S_v^*(h(z)))}{h(z)(z-S_v^*(h(z)))} - \eta(1-z) - \lambda(1-z_1)\right]$$

and at $\theta = 0$,

$$P_B^*(z,0) = \frac{Q_0 z(1 - S_b^*(w(z)))}{w(z)(z - S_b^*(w(z)))}\left[\frac{z\lambda\eta(z-z_1)(1-S_v^*(h(z)))}{h(z)(z-S_v^*(h(z)))} - \eta(1-z) - \lambda(1-z_1)\right].$$

Thus, the total PGF $P(z)$ of the system size probabilities is given by

$$P(z) = P_B^*(z,0) + P_v^*(z,0) + P_0.$$

Using the normalizing condition $P(1) = 1$, $Q_0 = \dfrac{1-\rho_b}{\dfrac{h(z)}{\eta} - \dfrac{\eta}{\lambda} - \dfrac{\rho_b(1-z_1)S_v^*(\eta)}{(1-S_v^*(\eta))}}$.

Mean system length

Let L_v and L_b denote the mean system size during the working vacation and regular busy, respectively. Then

$$L_v = \frac{d}{dz}P_v^*(z,0)_{z=1}$$

$$= \lambda Q_0\left[\frac{h(z_1)}{\eta^2} - \frac{(1-z_1)S_v^*(\eta)}{\eta(1-S_v^*(\eta))}\right],$$

$$L_b = \frac{d}{dz}P_b^*(z,0+P_0)_{z=1}$$

$$= Q_0\left\{\left[\frac{\rho_b}{1-\rho_b} + \frac{\lambda^2 E(S_b^2)}{2(1-\rho_b)^2}\right]\left[\frac{h(z_1)}{\eta} + \frac{(z_1-1)S_v'(\eta)}{1-S_v'(\eta)} + \frac{\eta}{\lambda}\right]\right.$$

$$\left. + \frac{\rho_b}{1-\rho_b}\left[\frac{(1-z_1)(S_v'(\eta) + \lambda S_v^{*'}(\eta))}{(1-S_v'(\eta))^2} - \frac{h(z_1)S_v^*(\eta)}{(1-S_v^*(\eta))} + \frac{h(z_1)\lambda}{\eta^2}\right]\right\}.$$

Hence, the mean system size of the model L is given by $L = L_v + L_b$.

Other performance measures

- Probability that the server is on vacation (P_v) is given by

$$P_v = \lim_{z \to 1} Q_1^*(z, 0) = \frac{Q_0 h(z_1)}{\eta}.$$

- Probability that the server is busy (P_b) is given by

$$P_b = \lim_{z \to 1} P_b^*(z, 0) = \frac{Q_0}{\mu_b(1 - \rho_b)}\left[\lambda\left(\frac{\eta}{\lambda} + \frac{h(z_1)}{\eta}\right) + \frac{\lambda(z_1 - 1)S_v^*(\eta)}{1 - S_v^*(\eta)}\right].$$

- Probability that the server is idle (P_I) is given by

$$P_I = \lim_{z \to 1} P_0 = \frac{\eta Q_0}{\lambda}.$$

3. Numerical Analysis

In this section, we present some numerical analysis for $M/G/1$ queue under working vacation (multiple (MWV) and single (SWV) vacations).to explain the influence of various parameters such as mean vacation time $(1/\eta)$, arrival rate λ, mean regular service time $(1/\mu_b)$ and mean working vacation service time $(1/\mu_v)$ on mean system size (L). Let us assume the service time distribution follow Erlang k distribution.

Values given in Table 1 and Table 2 show that the mean system size (L) decreases as the vacation parameter (η) increases for $M/G/1$ multiple working vacation (MWV) and $M/G/1$ single working vacation (SWV). The table values also show that the traffic intensity ρ_b also causes changes in mean system size. The graphical representations of Tables 1 and 2 are given in Figures 1 and 2, respectively.

Table 1. L for $M/G/1$-MWV

ρ_b	$\eta = 0.04$	0.05	0.06	0.1	0.55
0.60	8.172	6.793	6.044	4.148	1.832
0.65	8.397	7.108	6.364	4.471	2.179
0.70	8.827	7.541	6.799	4.911	2.646
0.75	9.444	8.159	7.424	5.539	3.303
0.80	10.392	9.11	8.349	6.5	4.293
0.85	15.275	13.999	13.279	11.414	9.272

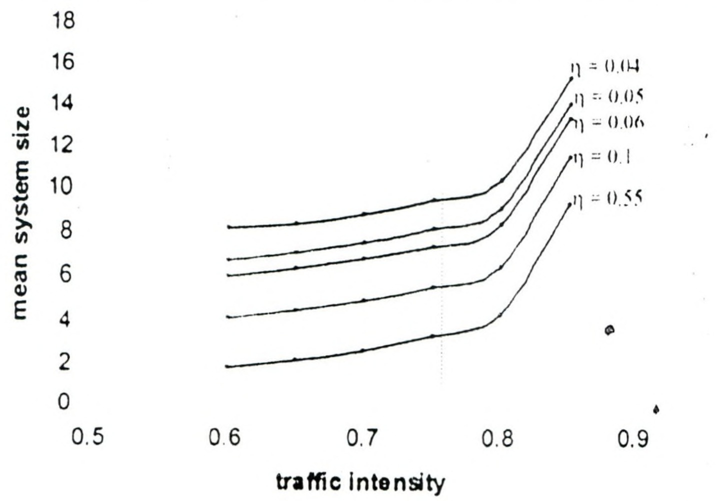

Figure 1. Mean system size vs traffic intensity for MWV.

Table 2. L for $M/G/1$-SWV

ρ_b	$\eta = 0.04$	0.05	0.06	0.1	0.55
0.60	8.083	6.793	5.924	4.148	1.832
0.65	8.397	7.108	6.241	4.471	2.179
0.70	8.827	7.541	6.675	4.911	2.646
0.75	9.444	8.159	7.297	5.539	3.303
0.80	10.392	9.11	8.249	6.5	4.293
0.85	12.002	10.723	9.865	8.125	5.949

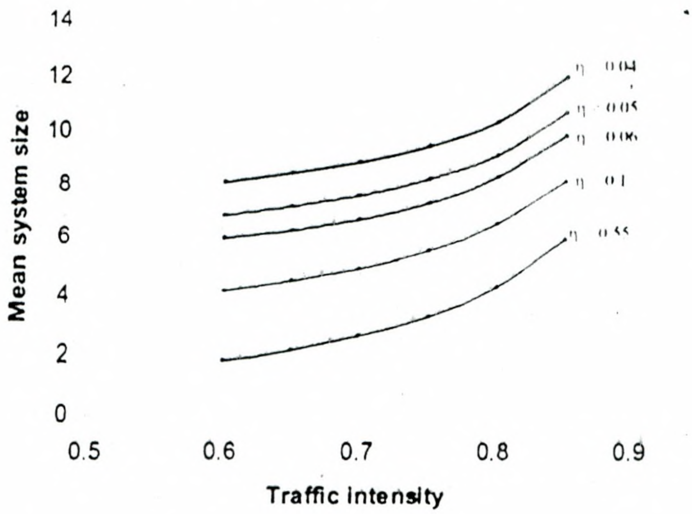

Figure 2. Mean system size vs traffic intensity for SWV.

Tables 3 and 4 indicate the effect of μ_b and μ_v on mean system size (L). It is shown that for a fixed value of η, L decreases as the server works faster and faster during vacation period or during regular busy period.

Table 3. L for $M/G/1$ MWV

μ_b	$\mu_v = 1$	2	3	4	5
0.5	8.87	8.59	8.328	8.081	7.847
0.6	3.906	3.673	3.465	3.279	3.11
0.7	2.272	2.079	1.915	1.775	1.6528
0.8	1.47	1.312	1.1831	1.077	0.987
1	1	0.872	0.772	0.691	0.625

Table 4. L for $M/G/1$ SWV

μ_b	$\mu_v = 1$	2	3	4	5
0.5	8.98	8.956	8.937	8.924	8.9145
0.6	3.987	3.962	3.946	3.935	3.926
0.7	2.325	2.303	2.882	2.278	2.2705
0.8	1.496	1.477	1.464	1.455	1.448
1	1	0.983	0.972	0.965	0.959

Conclusion

Till now working vacation models are discussed using matrix geometric method. In this paper, we have made an attempt to discuss the $M/G/1$ queue under working vacations (both multiple and single) through supplementary variable technique. The PGF of the steady state system size distribution is derived and presented in closed form. Various performance measures are calculated and numerical analysis is carried out for the corresponding measures obtained. The research can be extended to more general models such as bulk service, bulk arrival models, working vacation with startup, etc. in future.

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Analysis of The Batch Arrival $M^X/G/1$ Queue with Exponentially distributed Multiple Working Vacations

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Abstract :

In this paper, a batch arrival $M^X/G/1$ queueing system with exponentially distributed multiple working vacations is analyzed. The system size distribution is derived using the supplementary variable technique and various performance measures including expected system size are calculated. Finally some particular cases and numerical results for the system measures obtained are illustrated.

Key words : working vacations, batch arrival and $M^X/G/1$ queue.

2000 Mathematics subject classification: 60K25

Introduction:

Over the last two decades, the queueing system with vacations have been well studied because of their applications in modeling the computer networks, communication and manufacturing/ service systems. In the previous studies of classical vacation queueing models, it is assumed that the server stops the primary service completely and utilizes the vacation time for other secondary jobs. In 2002, Servi and Finn [9] introduced the concept of working vacation in M/M/1 queue, where the server works at a lower rate rather than completely stopping the primary service during the vacation period. He has considered multiple vacation policy and obtained simple explicit formula for the mean , variance, the probability generating function of number of customers in the system and the LST of waiting time distribution, and applied results to performance analysis of gateway router in fiber communication networks. Subsequently, Kim, Choi and Chae [5], Wu and Takagi[11] analyzed the Non Markovian M/G/1 queue and obtained the steady state results for multiple working vacations . Baba[1] provided a study of GI/M/1 N queue with working vacations. Liu et al [8] established the stochastic decomposition in the M/M/1 queue with working vacations. Li et al [7] considered two types of discrete time Geo/G/1 queues with working vacations and in 2008, Tian et al [10] studied M/M/1 queue under single working vacation. Using quasi birth and death process and matrix-geometric solution method, they derived the distributions for the number of customers, the virtual time in the system in steady state, expected busy period and expected busy cycle.

Recently Xu et al [12] analyzed the bulk input $M^X/M/1$ queue with multiple working vacations. They derived the PGF of the stationary queue length, mean queue length and the lower bounds of the mean waiting time. Jemila parveen et al [3] analyzed M/M/1 queue with working vacations and derived the steady state solutions in a closed form by directly solving the difference differential equations. Later they have discussed the waiting time distribution of an arbitrary customer for the model and verified the classical relation between PGF of queueing system and LST of the waiting time distribution. The steady state results of M/M/1 working vacations are also extended to $M^X/M/1$ working vacation queueing model for both multiple and single working vacations by Julia Rose Mary and Afthab Begum [4]. Later Li et al [6] analyzed M/G/1 queue with exponentially distributed multiple working vacations using matrix analytic approach. They obtained the distribution for the stationary queue length at departure epochs, the joint distributions for the stationary queue length and service status at the arbitrary epochs and the stationary waiting time and busy period. All these authors used matrix geometric method or embedded Markov chain technique to analyze the steady state distributions of the working vacation models.

In this paper, for the first time, we analyze the batch arrival Non Markovian $M^X/G/1$ queue with exponentially distributed multiple working vacations using supplementary variable technique and deduced the results in closed form. The method used in this paper to derive the stationary distribution of the model is very different and simple compared to the methods used in previous studies. Performance measures including expected system length, probabilities that the server is on vacation and busy are obtained. Finally numerical analysis is carried out for better understanding of the queue.

Model Description:

Consider a batch arrival $M^X/G/1$ queueing system where the arrival stream forms a Poisson process and the actual number of customers in any arriving stream is a random variable X, which may take on any positive integral value k less than ∞ with probability distribution $Pr(X = k) = g_k$, $k = 1, 2, 3, \ldots$ Let λ_k be the arrival rate of the Poisson process of batches of the size k then $g_k = \frac{\lambda_k}{\lambda}$, $k = 1,2,3\ldots$ where λ is the composite arrival rate of all batches and is equal to $\sum_{i=1} \lambda_i$. Total process, which arises from the overlap of the set of Poisson processes with rate $\{\lambda_i, i = 1,2..\}$ is called a Compound Poisson process. Further more, we assume that X has the pgf X(z), with finite mean E(X) and second order moments

$E(X^2)$. i.e.. $X(z) = \sum g_i z^i$. $|z| \le 1$ and $E(X^k) = \sum_{i=1}^{\infty} i^k g_k < \infty$, k=1.2,. Whenever the system becomes empty at a service completion instant, the server starts a working vacation during which the busy period S_b follows a general distribution with the mean $E(S_b) = \frac{1}{\mu_b}$

The service time during the working vacation period S_v also follows a general distribution with the mean $E(S_v) = \frac{1}{\mu_v}$. The vacation time is exponentially distributed with rate η. At a vacation completion instant, if there are customers in the systems, the server will start a new busy period. Otherwise, he takes another working vacation. Inter arrival times, service times and working vacation times are mutually independent of each other and the service discipline is FCFS.

System Size Distribution:

Let N(t) denote the system size at time t.

$$
Y(t) = \begin{cases} 0 & \text{if the server is idle in vacation} \\ 1 & \text{if the server is busy in vacation at } \mu_v \text{ rate} \\ 2 & \text{if the server is busy in vacation at } \mu_b \text{ rate} \end{cases}
$$

Let $S_b^0(t)$ and $S_v^0(t)$ denote the remaining regular service time and remaining service time during working vacation respectively. Then supplementary variables $S_v^0(t)$ and $S_b^0(t)$ make the state space {N(t), Y(t)} a bivariate Markov process. The system size probabilities used to describe the model are the following.

$Q_0(t)$ = Pr(N(t) = 0, Y(t) = 0), denotes the probability that the server is idle during vacation at time t.

$Q_n(x,t) = Pr(N(t) = n, Y(t) = 1, x \le S_v^0(t) \le x+dt), n \ge 1$ denotes the probability that there are n customers in the system and the server is doing service at a lower service rate and remaining service time lies in the interval $(x, x+dt)$.

And $P_n(x,t) = Pr(N(t) = n, Y(t) = 2, x \le S_b^0(t) \le x+dt), n \ge 1$ denotes the corresponding probability when the server is busy with regular service rate at time t.

Assuming that the steady state probabilities $Q_0 = \underset{t\to\infty}{lt}\, Q_0(t)$, $Q_n(x, t) = \underset{t\to\infty}{lt}\, Q_n(x,t)$,

$P_n(x, t) = \underset{t\to\infty}{lt}\, P_n(x,t)$, $n \ge 1$, exist and independent of time, $\frac{\partial}{\partial t} P_n(x,t) = \frac{\partial}{\partial t} Q_n(x,t) = 0$.

Following the arguments of Cox[2] and observing the changes of states in the interval $(t, t+\Delta t)$ at any time t, we obtain the following steady state system size equations.

Steady state equations:

$$\lambda Q_0 = P_1(0) + Q_1(0) \tag{1}$$

$$-\frac{d}{dx} Q_1(x) = -(\lambda + \eta) Q_1(x) + Q_2(0) s_v(x) + \lambda g_1 Q_0 s_v(x)$$

$$-\frac{d}{dx} Q_n(x) = -(\lambda + \eta) Q_n(x) + Q_{n+1}(0) s_v(x) + \lambda g_n Q_0 s_v(x) + \lambda \sum_{k=1}^{n-1} Q_{n-k}(x) g_k, n \ge 2$$

$$-\frac{d}{dx} P_1(x) = -\lambda P_1(x) + P_2(0) s_b(x) + \eta s_b(x) \int_0^\infty Q_1(y) dy$$

$$-\frac{d}{dx} P_n(x) = -\lambda P_n(x) + P_{n+1}(0) s_b(x) + \eta s_b(x) \int_0^\infty Q_n(y) dy + \lambda \sum_{k=1}^{n-1} P_{n-k}(x) g_k n \ge 2$$

To obtain the probability generating functions we make use of the following LST

$$Q_n^*(\theta)=\int_0^\infty e^{-\theta x}Q_n(x)dx \text{ and } P_n^*(\theta)=\int_0^\infty e^{-\theta x}P_n(x)dx.$$

Taking the LST on both sides of the steady state equations and using the LST properties, we have

$$\theta Q_1^*(\theta)-Q_1(0)=(\lambda+\eta)Q_1^*(\theta)-Q_2(0)S_v^*(\theta)-\lambda Q_0 g_1 S_v^*(\theta) \quad (2)$$

$$\theta Q_n^*(\theta)-Q_n(0)=(\lambda+\eta)Q_n^*(\theta)-Q_{n+1}(0)S_v^*(\theta)-\lambda Q_0 g_n S_v^*(\theta)$$

$$-\lambda\sum_{k=1}^{n-1}Q_{n-k}^*(\theta)g_k,\ n\geq 2 \quad (3)$$

$$\theta P_1^*(\theta)-P_1(0)=\lambda P_1^*(\theta)-P_2(0)S_b^*(\theta)-\int_0^\infty Q_1(y)dy\,\eta S_b^*(\theta) \quad (4)$$

$$\theta P_n^*(\theta)-P_n(0)=\lambda P_n^*(\theta)-P_{n+1}(0)S_b^*(\theta)-\int_0^\infty Q_n(y)dy\,\eta S_b^*(\theta)-\lambda\sum_{k=1}^{n-1}P_{n-k}^*(\theta)g_k,\ n\geq 2 \quad (5)$$

Steady state solutions:

To derive the system size distribution, the following partial pgf's are defined for $|z|\leq 1$.

$$Q_1^*(z,\theta)=\sum_{n=1}^\infty Q_n^*(\theta)z^n,\ Q_1(z,0)=\sum_{n=1}^\infty Q_n(0)z^n,\ P_V^*(z,\theta)=Q_1^*(z,\theta)+Q_0$$

$$P_B^*(z,\theta)=\sum_{n=1}^\infty P_n^*(\theta)z^n \text{ and } P_B(z,0)=\sum_{n=1}^\infty P_n(0)z^n$$

Multiplying equations (2) and (3) by the proper powers of z and summing up over n= 1 to ∞, we get

$$(\theta - h_X(z))Q_1^*(z,\theta) = Q_1(z,0)(1-\frac{S_v^*(\theta)}{z}) - S_v^*(\theta)(\lambda X(z)Q_0 - Q_1(0)) \qquad (6)$$

At $\theta = h_X(z) = \eta + \lambda(1 - X(z))$, $Q_1(z,0) = \dfrac{zS_v^*(h_X(z))}{z - S_v^*(h_X(z))}(\lambda X(z)Q_0 - Q_1(0))$.

By similar argument of Li et al[], $z - S_v^*(h_X(z)) = 0$ has unique root z_1 inside (0,1).

Therefore, $Q_1(0) = \lambda X(z_1)Q_0$ (7)

Substituting for $Q_1(0)$ in $Q_1(z, 0)$, $Q_1(z,0) = \dfrac{\lambda z Q_0 S_v^*(h_X(z))(X(z) - X(z_1))}{z - S_v^*(h_X(z))}$

Substituting the value of $Q_1(z,0)$ in equation (6), we have

$$Q_1^*(z,\theta) = \frac{\lambda z Q_0 (X(z) - X(z_1))(S_v^*(h_X(z)) - S_v^*(\theta))}{(\theta - h_X(z))(z - S_v^*(h_X(z)))}$$ and hence

at θ=0, $Q_1^*(z,0) = \dfrac{\lambda z Q_0 (X(z) - X(z_1))(1 - S_v^*(h_X(z)))}{h_X(z)(z - S_v^*(h_X(z)))}$ (8)

Similarly multiplying equations (4) and (5) by appropriate powers of z and then adding, we have

$$\theta P_B^*(z,\theta) - P_B(z,0) = \lambda P_B^*(z,\theta) - \frac{S_b^*(\theta)}{z}(P_B(z,0) - P_1(0)z) - \eta S_b^*(\theta)\sum_{n=1}^{\infty}\int_0^{\infty} Q_n(y)dyz^n$$

$$- \lambda X(z)P_B^*(z,\theta)$$

Since $\sum_{n=1}^{\infty}\int_0^{\infty} Q_n(y)dyz^n = \sum_{n=1}^{\infty} Q_n^*(0)z^n = Q_1^*(z,0)$, we have

$$(\theta - w_X(z))P_B^*(z,\theta) = P_B(z,0)(1 - \frac{S_b^*(\theta)}{z}) - S_b^*(\theta)(\eta Q_1^*(z,0) - P_1(0)) \qquad (9)$$

At $\theta = w_X(z) = \lambda(1 - X(z))$, $P_B(z,0) = \dfrac{zS_b^*(w_X(z))}{z - S_b^*(w_X(z))}(\eta Q_1^*(z,0) - P_1(0))$

Equation (1) imply $P_1(0) = \lambda Q_0 - Q_1(0) = \lambda Q_0(1 - X(z_1))$ (using (7)).

Thus by substituting the values of $P_B(z,0)$, $Q_1^*(z,0)$ and $P_1(0)$ in equation(9) and at $\theta = 0$,

$$P_B^*(z,0) = \frac{Q_0 z(1 - S_b^*(w_X(z)))}{w_X(z)(z - S_b^*(w_X(z)))}\left[\frac{z\lambda\eta(X(z) - X(z_1))(1 - S_v^*(h_X(z)))}{h_X(z)(z - S_v^*(h_X(z)))} - \lambda(1 - X(z_1))\right] \quad (10)$$

Thus the total PGF P(z) of the system size probabilities is given by

$$P(z) = P_B^*(z,0) + P_v^*(z,0)$$

Using the normalizing condition P(1)=1, $Q_0 = \dfrac{1 - \rho_b}{\dfrac{h_X(z_1)}{\eta} - \dfrac{\rho_b(1 - z_1)S_v^*(\eta)}{E(X)(1 - S_v^*(\eta))}}$

Mean system length:

Let L_v and L_b denote the mean system size during the working vacation and regular busy period respectively. Then

$$L_v = \frac{d}{dz}P_v^*(z,0)_{z=1} = \lambda Q_0\left[\frac{E(X)h_X(z_1)}{\eta^2} - \frac{(1 - X(z_1))S_v^*(\eta)}{\eta(1 - S_v^*(\eta))}\right]$$

$$L_b = \frac{d}{dz}P_b^*(z,0)_{z=1}$$

$$= Q_0\left\{\left[\frac{\lambda E(X(X-1))E(S_b) + (\lambda E(X))^2 E(S_b^2)}{2(1 - \rho_b)^2} - \frac{\rho_b}{1 - \rho_b}\right]\left[\frac{h_X(z_1)}{\eta} - \frac{(X(z_1) - 1)S_v(\eta)}{1 - S_v(\eta)}\right]\right.$$

$$\left. - \frac{\rho_b}{1 - \rho_b}\left[\frac{E(X(X-1))(X(z_1) - 1)S_v^*(\eta)}{2E(X)(1 - S_v^*(\eta))} + \frac{E(X)h_X(z_1)S_v^*(\eta)}{\eta(1 - S_v^*(\eta))}\right]\right.$$

$$-\frac{(1-X(z_1))(S_v^*(\eta)+\lambda E(X)S_v^{*'}(\eta))}{(1-S_v^*(\eta))^2}-\frac{h_X(z_1)(\lambda E(X))^2}{\eta^2}\Bigg]\Bigg\}$$

Hence the mean system size of the model L is given by $L=L_v+L_b$.

Other Performance measures:

- Probability that the server is on vacation (P_v) is given by

$$P_v = \lim_{z\to 1} Q_1^*(z,0) = \frac{Q_0 h_X(z_1)}{\eta}$$

- Probability that the server is busy (P_b) is given by

$$P_b = \lim_{z\to 1} P_b^*(z,0) = \frac{Q_0\rho_b}{(1-\rho_b)}\left[\frac{h_X(z_1)}{\eta}-\frac{\rho_b(1-X(z_1))S_v^*(\eta)}{E(X)(1-S_v^*(\eta))}\right]$$

Particular cases:

In this section, the steady state results of $M^X/M/1$ [4], M/M/1[3], M/G/1[6] are deduced as particular cases of the model.

1. $M^X/M/1$ MWV[4]

When both regular service time and service time during working vacation follow exponential distribution, the probability generating functions coincide with the corresponding generating functions of the $M^X/M/1$ MWV model.

i.e., $Q_1^*(z,0) = \dfrac{Q_0\mu_v(z-z_1)}{z_1(\mu_v(z-1)+zh_X(z))}$ and

$$P_b^*(z,0) = \frac{Q_0 z\mu_v[z_1(z-1)(X(z_1)-1)+z(X(z)-1)(1-z_1)]}{z_1(\mu_v(z-1)+zh_X(z))(\mu_b(z-1)+zw_X(z))}$$

2. M/M/1 MWV[3]

The results of M/M/1 MWV can be obtained by taking X(z)=z in the corresponding equations. Let $z_1 < 1$ and $z_2 > 1$ be the two roots of the equation

$\lambda z(1-z)+\mu_v(z-1)+\eta z=0$.

Then $\lambda z(1-z)+\mu_v(z-1)+\eta z=-\lambda(z-z_1)(z-z_2)$ and the relation $z_2=\dfrac{\mu_v}{\lambda z_1}=\dfrac{1}{r_v}$ implies

$\lambda z(1-z)+\mu_v(z-1)+\eta z=-\lambda(z-z_1)(z-1/r_v)$. Thus $Q_1^*(z,0)=\dfrac{Q_0}{1-r_v z}=\sum_{n=0}^{\infty} r_v^n z^n$ and

$$P_b^*(z,0)=\frac{Q_0\rho z(1-z_1)}{(1-\rho z)(1-zr_v)}=\frac{Q_0\eta r_v}{\mu(\rho-r_v)(1-r_v)}\sum_{n=0}^{\infty}(\rho^n-r_v^n)z^n$$

3. M/G/1 MWV[6]

Taking X(z)=z in equations (8) and (10), we have

$$Q_1^*(z,0)=\frac{Q_0[z(1-z_1)+b(z)(z-1)]}{z-b(z)} \text{ and } P_b^*(z,0)=\frac{Q_0 z[c(z)(z-z_1)+(z-1)(z-b(z))]}{(z-b(z))(z-a(z))}$$

where $a(z)=S_b^*(w(z)), b(z)=S_v^*(h(z)), v(z)=\dfrac{\eta(1-S_v^*(h(z)))}{h(z)}$ and c(z)=a(z)v(z). These results show that the PGF of system size during regular busy period and during working vacation busy period of M/G/1 model are obtained.

Numerical Analysis:

In this section, some numerical results are obtained to study the effect of various parameters such as mean vacation time ($1/\eta$), arrival rate λ, mean regular service time ($1/\mu_b$) and mean service rate during working vacation ($1/\mu_v$) on mean system size(L) and how the probability of a customer being served completely at a regular service rate change with μ_v. In classical vacation models, since the service rate is stopped completely during vacation, the system size increases notably as the mean vacation time increases. But. in working vacation, since the service is done with smaller rate μ_v ($< \mu_b$), the vacation rate η has little effect on the system size. The values of table 1 illustrate this. From table 1. The mean system size is found to be a decreasing function of η and the effect of η is smaller and turns to 0 when $\mu_v = \mu_b$ and the effect of η is larger when μ_v=0. Table 2 shows that the mean system size decreases as the vacation parameter increases and it is pictured in figure 1. The influence of vacation parameter η. service rate during working vacation (μ_v) and regular service rate (μ_b) on the mean system size is shown in figure 2 and 3. The values in table 3 gives the effect of traffic intensity on the probabilities including probability that the server is on vacation (Pv) and busy (Pb).

Table 1. Mean system size Vs η Vs μ_v

η / μ_v	0	0.5	1	1.5	2	2.5	3
0.5	1.0954	0.8333	0.6190	0.4773	0.3805	0.3115	0.2603
1	0.5791	0. 5013	0.4121	0.3399	0.2826	0.2367	0.1994
1.5	0.4069	0.3714	0.3217	0.2767	0.2379	0.2047	0.1762
2	0.3209	0.3012	0.2695	0.2384	0.2101	0.1847	0.1621

Table 2. Mean system size Vs Arrival rate for different values of η

λ	η=0.15	η=0.25	η=0.35	η=0.45	η=0.55
0.2	0.5644	0.4355	0.3571	0.3029	0.2629
0.25	0.7418	0.5571	0.4504	0.3788	0.3269
0.3	0.9360	0.6842	0.5456	0.4553	0.3906
0.35	1.1468	0.8169	0.6432	0.5323	0.4542
0.4	1.3736	0.9554	0.7429	0.6102	0.5178
0.45	1.6157	1.0996	0.8454	0.6891	0.5817
0.5	1.8726	1.2497	0.9505	0.7694	0.6461
0.55	2.1438	1.4061	1.0588	0.8513	0.7112
0.6	2.4292	1.5688	1.1706	0.9351	0.7774

Table 3. Probabilities Vs Traffic intensity

ρ	P_V	P_b
0.1	0.9507	0.0492
0.2	0.8671	0.1328
0.3	0.7669	0.2331
0.4	0.6608	0.3391

0.5	0.5523	0.4476
0.6	0.4337	0.5622
0.7	0.6675	0.3325
0.8	0.7782	0.2217
0.9	0.8889	0.1105

Figure 1. Mean system size, Vs Arrival rate for different values of η

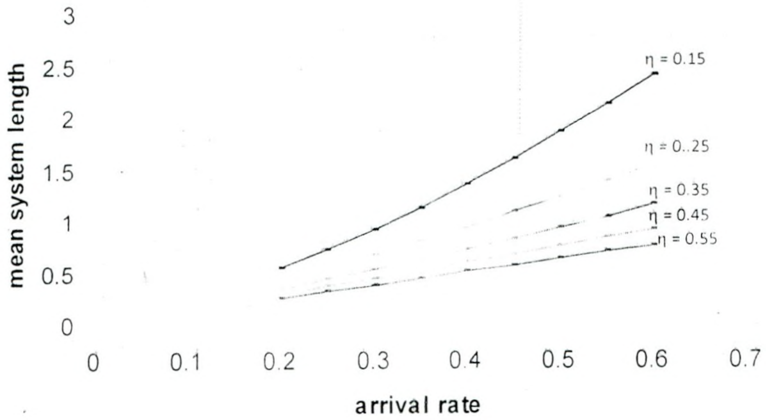


Figure 2. Mean system size Vs η Vs μ_v

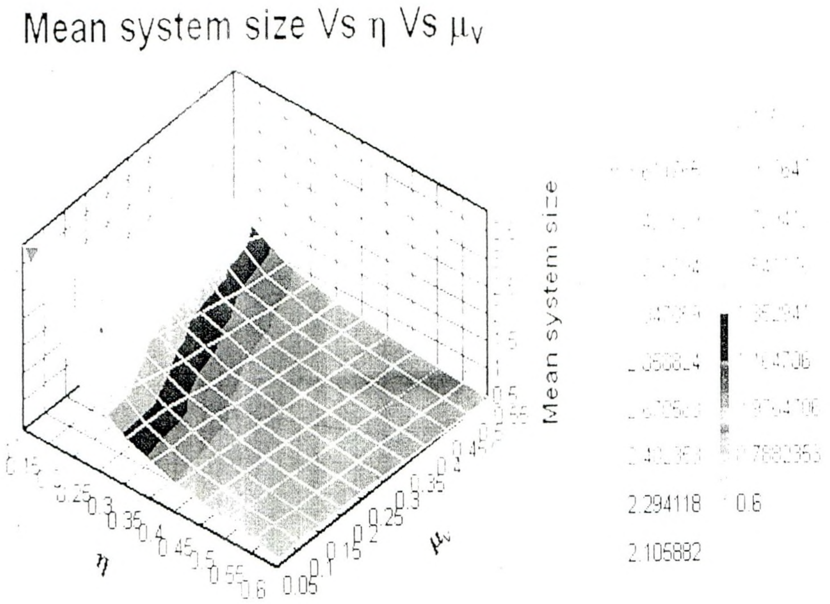
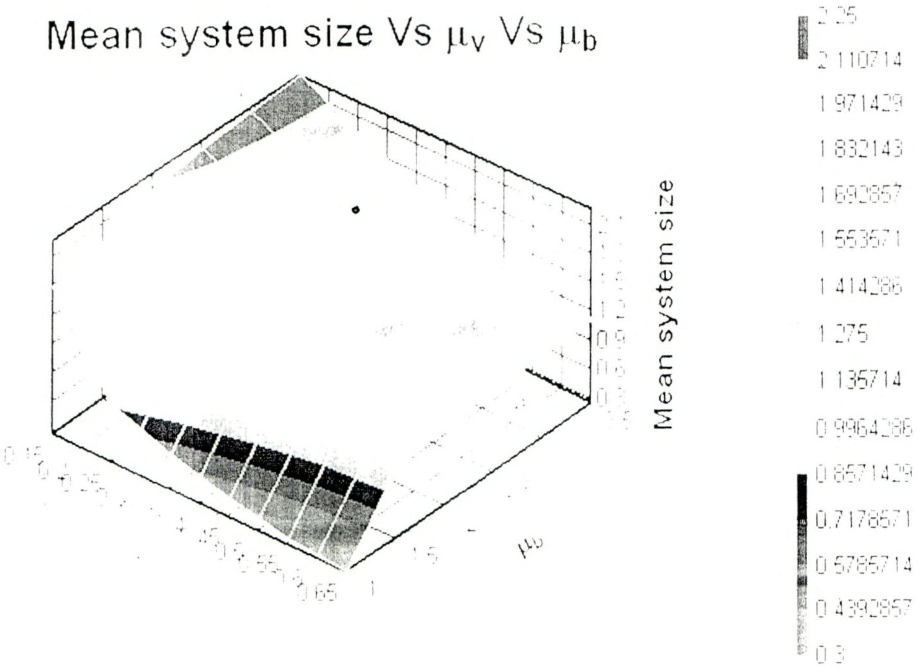


Figure 3. Mean system size Vs μ_v Vs μ_b



Conclusion:

Since 2002, few authors have worked on working vacation queues using matrix geometric method and Embedded Markov chain technique. However, the approach employed by them is not easy and sufficient to obtain the results in closed form. Concerning the Non Markovian queues, the most widely used tool is supplementary technique. In this paper, we have analyzed the batch arrival $M^X/G/1$ queue with multiple working vacations. The pgf of the system size probabilities are derived and presented in closed form. Further various performance measures including expected system length, probability that server is busy and on vacation are obtained and numerical analysis is carried out. The results obtained in this paper can be extended to unreliable server in future research.

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Bi-level threshold policy of $M^X/(G_1, G_2)/1$ queue with early setup and single vacation

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Abstract: In this paper, we analyse a queueing system with a second optional service channel under bi-level control policy and server's single vacation. There is a single server who provides the first essential service to all arriving customers. As soon as the first service of a customer is complete, then with probability r, the customer may opt for the second service in which case the second service will immediately commence, or else with probability $(1 - r)$ he may opt to leave the system, so that another customer at the head of the queue (if any) is taken up for his first essential service. The server operates an (m, N) policy with an early setup and takes a single vacation whenever the system becomes empty. For this model, the stationary probability generating function of the queue length distribution is obtained through supplementary variable technique and a decomposition property is discussed. The expected length of the cycle, various performance measures and the optimal values of thresholds m and N which minimise the total expected operating cost are also calculated.

Keywords: bi-level threshold policy; first essential service; second optional service; early setup; single vacation policy.

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1 Introduction

The batch arrival queueing system with double-threshold policy, setup time and vacation analysed by Lee et al. (2003) is among the most general queueing systems with threshold policies and include many previous works as special cases. The authors used the decomposition property of vacation queues to derive directly the probability generating function (p.g.f) of queue length. Also Lee et al. (2003) in their paper have assumed that the system has a single server who provides only one kind of service to the incoming customers.

But in everyday life there are queueing situations where all the arriving customers require the first essential service and some may require the second optional service provided by the same server. Madan (2000) has introduced the concept of second optional service, where the customers may depart from the system either with probability $(1 - r)$ or may immediately opt for a 2nd optional service with probability r. Choudhury and Paul (2006) have extended the second optional service results for batch arrival queue with N-policy and obtained the system size p.g.f using supplementary variable technique. Later Kalyana Raman and Pazhani Bala Murugan (2008) have considered the second optional service in batches with server vacation and derived the p.g.f for queue length. Recently, Gautam Choudhury et al. (2009) have dealt with an $M^X/G/1$ queueing system with an additional second phase of optional service (SOS) and unreliable server which consists of break down period and delay period under N-policy.

To the best of our knowledge the batch arrival queueing model with second optional service is not analysed with single vacation and early setup time. In this paper, we have analysed the second optional service batch arrival queueing model with early setup time and single vacation and presented the queue size distribution at random epoch as well as at the departure epoch in a closed form. Further, we have derived the stochastic decomposition property and various performance measures also in a closed form, so that one can obtain the numerical values of every measure directly. Moreover, we have developed a simple procedure to obtain the optimal value of m and N under suitable linear cost structure. Finally, graphical representation of the expected system length, total cost and the optimal cost need to run the system are given corresponding to a set of parametric values.

2 Literature review

The first study of batch arrival queueing system with N-policy was carried out by Lee and Srinivasan (1989).In their paper, they have discussed the mean waiting time of an arbitrary customer and a procedure to find the stationary optimal policy under a linear cost structure. Later, many authors including Lee et al. (1994, 1995) have analysed the N-policy of $M^X/G/1$ queueing models with servers having multiple and single vacation. Recently, Zhang and Xu (2008) have studied the N-policy of the $M/M/1$ queue with *multiple working vacation*. But these models, do not involve the server's setup time. In queueing models, server's setup time corresponds to the preparatory work of the server before starting his service. Hur and Park (1999) and Ke (2001) are some of the authors who analysed the N-policy of $M^X/G/1$ queueing models with server's setup time. Only few authors have analysed the N-policy of batch arrival queueing system with vacation and the setup time together.

As far as the setup models are concerned the server starts their setup work only when the queue length reaches N which is the minimum limit to start the service. But in practice, it is effective if the pre-service work (setup) starts at m where ($m \leq N$). This concept is considered as server's early start-up or bi-level control policy.

Lee and Park (1997) and Lee et al. (2003) examined the operating characteristic functions of the $M/G/1$ queueing system and $M^X/G/1$ queueing system under the bi-level control with early setup time and servers vacation and derived the stationary p.g.f of queue length by following the stochastic decomposition property. The system studied by them is the most general queueing system with threshold policies and include many previous work as special cases. Ke (2004) has derived the p.g.f for an unreliable server with a bi-level batch arrival queue. Later, Ke (2006) has also analysed the bi-level control for $M/G/1$ queueing system in which the server is unreliable and is characterised by a single vacation and an early setup time.

In all the above models mentioned above, the customers are provided only a single service. Madan (2000) has introduced the concept of second optional service for $M/G/1$ queue in which, the first service is essential and the second service is optional. Later, Al-Jararha and Madan (2003) has extended his results with general service time distribution and obtained the time-dependent p.g.f in terms of their Laplace transforms. Choudhury and Paul (2006) have extended the results Madan's model to batch arrival queue under N-policy and obtained the queue size distribution at random epoch as well as at a departure epoch and presented a simple procedure to obtain an optimal policy. Later, Madan and Choudhury (2006) have studied the steady state analysis of the $M^X/(G_1, G_2)/1$ queue with restricted admissibility and random setup time.

In a day-to-day life, one encounters numerous examples of second optional service queueing situations, thus second optional service queueing systems form an active research area. Recent work includes Geo/G/1 retrial queue with second optional service by Atencia and Moreno (2006), $M/G/1$ retrial queueing system with two phases of service subject to the server break down and repair by Choudhury and Deka (2008), on the single server batch arrival retrial queue with general vacation time under Bernoulli schedule and two phases of heterogeneous service by Senthil Kumar and Arumuganathan (2008), the N-policy for an unreliable server with delaying repair and two phases services by Choudhury et al. (2009) and an $M/G/1$ retrial queue with an additional phase of second service and general retrial time by Choudhury (2009).

3 Model description

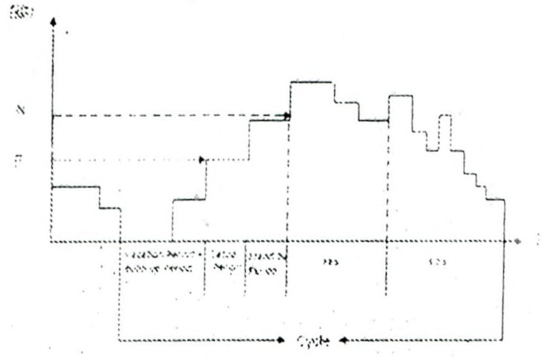
We consider $M^X/G/1$ queueing system in which the customers arrive according to the compound Poisson process with the random arrival size x. The server is turned off and leaves the system for a vacation of random length V, each time when the system becomes empty. After returning from the vacation if the server finds the system size is less than m then he remains idle (build up period) in the system until the queue length reaches at least m. On the other hand, if the server finds m or more customers in the system at the end of the vacation then he immediately starts the setup operation which takes a random length D (early setup).

After the setup period, if the number of customers in the queue is less than N $(N \geq m)$, then the server remains again idle (dormant) until the queue length reaches at least N. Instead, at the end of the set up period if the server finds more than N customers waiting in the system then the server immediately begins to serve the customers. Here, the idle period of the server is made up of vacation period, build up period, set up period and dormant period.

If the queue length reaches N or more either at the end of the setup period or at the end of the dormant period, then the server begins a busy period. During busy period, the server provides to each unit, two stages of heterogeneous service of which one is optional, that is, the server begins to serve the first phase of essential service (FES) for all the units. The service discipline is assumed to be FCFS. After the completion of FES of a unit, the customer may leave the system with probability$(1 - r)$ or may opt for a SOS in an additional channel by the same server with probability r $(0 \leq r \leq 1)$. The server continues this type of service until the system becomes empty and then turned off the system. Thus, a cycle is completed. The system will be turned on again for setup when at least m customers are present in the system. The service times S_1 and S_2 of two channels (FES and SOS) are assumed to be mutually independent of each other having general law of distribution $S_i(x)$ with its Laplace Stieltjes transform (LST) $S_i^*(\theta)$, $i = 1, 2$. Further, we note here that the same server serves both the channels. The system is depicted in Figure 1.

Notationally, this queueing system is denoted by $M^{[x]}_{(m,N)}/(G_1, G_2)/1/SV$, where (m, N) denotes the bi-level thresholds due to early setup and N-policy, G_1 denotes FES, G_2 denotes SOS and SV denotes single vacation. The service times S_1, S_2, vacation time (V) and setup time (D) are random variables with finite mean and variance and are independent of each other and follow general distribution.

Figure 1 The system



4 Queue size distribution at random epoch

In this section, we first setup the system state equations for the queue size distribution under the steady state conditions and then derive the p.g.f of it. To obtain it let us define:

λ:	group arrival rate
X:	group size random variable
g_k:	$\Pr(X = k)$ $k = 1, 2, 3$
$X(z)$:	p.g.f of X
$S_i(x)$ $(s_i(x))$:	probability distribution (density function) of the random variable S_i
$V(x)(v(x))$:	probability distribution (density function) of the random variable V
$D(x)(d(x))$:	probability distribution (density function) of the random variable D.

Let $N_s(t)$ denotes the system size at time t, $V^o(t)$, $D^o(t)$, $S_1^o(t)$ and $S_2^o(t)$ denote the remaining vacation time, setup time, FES and SOS time, respectively at time t also lie in the interval $[x, x + dt]$

$$y(t) = \begin{cases} 0 & \text{if the system is in vacation state at time } t \\ 1 & \text{if the system is in build-up state at time } t \\ 2 & \text{if the system is in setup state at time } t \\ 3 & \text{if the system is in dormant state at time } t \\ 4 & \text{if the system is in with FES state at time } t \\ 5 & \text{if the system is in with SOS state at time } t \end{cases}$$

The supplementary variables $V^o(t)$, $D^o(t)$, $S_1^o(t)$ and $S_2^o(t)$ make the state space $(N_s(t), y(t))$, a bi-variate Markov process. Next, we define the probabilities to derive the steady state system size equations.

$$Q_n(x, t) = \Pr(N_s(t) = n, x \leq V^o(t) \leq x + dt, y(t) = 0), n \geq 0$$

$$R_n(t) = \Pr(N_s(t) = n, y(t) = 1), \qquad 0 \le n \le m-1$$

$$D_n(x, t) = \Pr(N_s(t) = n, x \le D^o(t) \le x + \mathrm{d}t, y(t) = 2), \; n \ge m$$

$$U_n(t) = \Pr(N_s(t) = n, y(t) = 3), \qquad m \le n \le N-1$$

$$P_{n1}(x, t) = \Pr(N_s(t) = n, x \le S_1^o(t) \le x + \mathrm{d}t, y(t) = 4), \; n \ge 1$$

$$P_{n2}(x, t) = \Pr(N_s(t) = n, x \le S_2^o(t) \le x + \mathrm{d}t, y(t) = 0), \; n \ge 1$$

Thus, $R_n(t)$ and $U_n(t)$ give the probability that there are n customers in the system at time t when the system is in the build up and dormant state, respectively.

$Q_n(x, t)$, $D_n(x, t)$ and $P_{ni}(x, t)$, $i = 1, 2$ denote the probability that there are n customers in the system at time t when the system is in vacation, setup and busy state, respectively with the remaining time variables as a function of x.

Also let $p_{ni}(0)$, $i = 1, 2$ denote the probability that there are n customers in the system at the termination of service time.

By assuming that at the steady state, the probabilities are independent of time t, we have

$$\frac{\partial}{\partial x}\left(P_{ni}(x,t)\right) = \frac{\partial}{\partial x}\left(P_n(x)\right); \frac{\partial}{\partial t}\left(P_n(x,t)\right) = 0; P_{ni}(x,t) = P_{ni}(x)$$

$$\frac{\partial}{\partial x}\left(Q_{ni}(x,t)\right) = \frac{\partial}{\partial x}\left(Q_n(x)\right); \frac{\partial}{\partial t}\left(Q_n(x,t)\right) = 0; Q_{ni}(x,t) = Q_{ni}(x)$$

$$\frac{\partial}{\partial x}\left(D_{ni}(x,t)\right) = \frac{\partial}{\partial x}\left(D_n(x)\right); \frac{\partial}{\partial t}\left(D_n(x,t)\right) = 0; D_{ni}(x,t) = D_{ni}(x)$$

Similarly, the steady state probabilities for $R_n(t)$ and $U_n(t)$ are denoted by

$$\lim_{t\to\infty}\left(R_n(t)\right) = R_n; \; \lim_{t\to\infty}\left(U_n(t)\right) = U_n$$

4.1 Steady state equations

By using the above definitions and following the arguments of Cox (1955), the Kolmogrov-forward equations under the steady state condition are written as:

$$\lambda R_0 = Q_0(0) \tag{1}$$

$$\lambda R_n = Q_n(0) - \lambda \sum_{k=1}^{n} R_{n-k} g_k \qquad 1 \le n \le m-1 \tag{2}$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{11}(x)\right) = -\lambda P_{11}(x) + (1-r)P_{21}(0)S_1(x) + P_{22}(0)S_1(x) \tag{3}$$

$$-\frac{d}{dx}\left(P_{n-1}(x)\right) = -\lambda P_{n1}(x) + (1-r)P_{n+11}(0)s_1(x) + \lambda\sum_{k=1}^{n-1} P_{n-k}(x)g_k + P_{n+12}(0)s_1(x) \quad 2 \le n \le N-1 \tag{4}$$

$$-\frac{d}{dx}P_{n1}(x) = -\lambda P_{n1}(x) + (1-r)P_{n+11}(0)s_1(x) + \sum_{k=n-N+1}^{n-m} U_{n-k}g_k s_1(x) + \lambda\sum_{k=1}^{n-1} P_{n-k1}g_k + P_{n-12}(0)s_1(x) + D_n(0)s_1(x) \quad n \ge N \tag{5}$$

$$\frac{-d}{dx}P_{12}(x) = -\lambda P_{12}(x) + rP_{11}(0)s_2(x) \tag{6}$$

$$\frac{-d}{dx}P_{n2}(x) = -\lambda P_{n2}(x) + rP_{n1}(0)s_2(x) + \sum_{\lambda k=1}^{n-1} P_{n-k2}(x)g_k \quad n \ge 2 \tag{7}$$

$$\frac{-d}{dx}Q_0(x) = -\lambda Q_0(x) + \left[P_{11}(0)(1-r) + P_{12}(0)\right]v(x) \tag{8}$$

$$\frac{-d}{dx}Q_n(x) = -\lambda Q_n(x) + \lambda\sum_{k=1}^{n} Q_{n-k}(x)g_k \quad n \ge 1 \tag{9}$$

$$\frac{-d}{dx}D_m(x) = -\lambda D_m(x) + Q_m(0)dx + \lambda\sum_{k=1}^{m} R_{m-k}g_k dx \tag{10}$$

$$\frac{-d}{dx}D_n(x) = -\lambda D_n(x) + Q_n(0)dx + \lambda\sum_{k=n-m+1}^{n} R_{n-k}g_k dx + \lambda\sum_{k=1}^{n-m} D_{n-k}(x)g_k n \ge m-1 \tag{11}$$

$$\lambda U_m = D_m(0) \tag{12}$$

$$\lambda U_n = D_n(0) + \lambda\sum_{k=1}^{n-m} U_{n-k}g_k \quad m+1 \le n \le N-1 \tag{13}$$

The LST of Equations (3)–(10) are obtained by defining the following LST and thereby using their properties we get

$$Q_n^*(\theta) = \int_0^\infty e^{-\theta x}Q_n(x)dx;\ D_n^*(\theta) = \int_0^\infty e^{-\theta x}D_n(x)(dx),\ D^*(\theta) = \int_0^\infty e^{-\theta x}d(x)dx$$

$$P_n^*(\theta) = \int_0^\infty e^{-\theta x} P_n(x)\mathrm{d}x;\ V^*(\theta) = \int_0^\infty e^{-\theta x} V(x)\mathrm{d}x;\ S_i^*(\theta)$$
$$= \int_0^\infty e^{-\theta x} s_i(x)\mathrm{d}x \quad i = 1, 2$$

Thus, we have

$$\lambda R_0 = Q_0(0) \tag{14}$$

$$\lambda R_n = Q_n(0) + \lambda \sum_{k=1}^{n} R_{n-k} g_k \quad 1 \le n \le m-1 \tag{15}$$

$$\theta P_{11}^*(\theta) - P_{11}(0) = \lambda P_{11}^*(\theta) - (1-r)P_{21}(0)S_1^*(\theta) - P_{22}(0)S_1^*(\theta) \tag{16}$$

$$\begin{aligned} \theta P_{n1}^*(\theta) - P_{n1}(0) = \lambda P_{n1}^*(\theta) - (1-r)P_{n+11}^*(0)S_1^*(\theta) \\ - \lambda \sum_{k=1}^{n-1} P_{n-k1}^*(\theta) g_k - P_{n+12}(0)S_1^*(\theta) \quad 2 \le n \le N-1 \end{aligned} \tag{17}$$

$$\begin{aligned} \theta P_{n1}^*(\theta) - P_{n1}^* = \lambda P_{n1}^*(\theta) - (1-r)P_{n+11}(0)S_1^*(\theta) \\ - \lambda \sum_{k=1}^{n-1} P_{n-k1}^*(\theta) - D_n(0)S_1^*(\theta) - P_{n+12}(0)S_1^*(\theta) \\ - \lambda \sum_{k=n-N+1}^{n-m} U_{n-k} g_k S_1^*(\theta) \quad n \ge N \end{aligned} \tag{18}$$

$$\theta P_{12}^*(\theta) - P_{12}(0) = \lambda P_{12}^*(\theta) - rP_{11}(0)S_2^*(\theta) \tag{19}$$

$$\theta P_{n2}^*(\theta) - P_{n2}(0) = \lambda P_{n2}^*(\theta) - rP_{n1}(0)S_2^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k2}^*(\theta) g_k \quad n \ge 2 \tag{20}$$

$$\theta Q_0^*(\theta) - Q_0(0) = \lambda Q_0^*(\theta) - \left[P_{11}(0)(1-r) + P_{12}(0)\right]V^*(\theta) \tag{21}$$

$$\theta Q_n^*(\theta) - Q_n(0) = \lambda Q_n^*(\theta) - \lambda \sum_{k=1}^{n} Q_{n-k}^*(\theta) g_k \quad n \ge 1 \tag{22}$$

$$\theta D_m^*(\theta) - D_m(0) = \lambda D_m^*(\theta) - Q_m(0)D^*(\theta) - \lambda \sum_{k=1}^{m} R_{m-k} g_k D^*(\theta) \tag{23}$$

$$\begin{aligned} \theta D_n^*(\theta) - D_n(0) = \lambda D_n^*(\theta) - Q_n(0)D^*(\theta) - \lambda \sum_{k=n-m+1}^{n} R_{n-k} g_k D^*(\theta) \\ - \lambda \sum_{k=1}^{n-m} D_{n-k}^*(\theta) g_k \quad n \ge m+1 \end{aligned} \tag{24}$$

$$\lambda U_m = D_m(0) \tag{25}$$

$$\lambda U_n = D_n(0) + \lambda \sum_{k=1}^{n-m} U_{n-k} g_k \quad m+1 \le n \le N-1 \tag{26}$$

4.2 *Probability generating function of the steady state probabilities*

To obtain the queue size distribution under the steady state condition, we consider the following the p.g.f as:

$$P_i^*(z,\theta) = \sum_{n=1}^{\infty} P_{ni}^*(\theta) z^n; \quad P_i(z,0) = \sum_{n=1}^{\infty} P_{ni}(0) z^n; \quad U(z) = \sum_{n=m}^{N-1} U_n z^n$$

$$D^*(z,\theta) = \sum_{n=m}^{\infty} D_n(\theta) z^n; \quad D(z,0) = \sum_{n=m}^{\infty} D_n(0) z^n; \quad R(z) = \sum_{n=0}^{m-1} R_n z^n$$

$$Q^*(z,\theta) = \sum_{n=0}^{\infty} Q_n^*(\theta) z^n; \quad Q(z,0) = \sum_{n=0}^{\infty} Q_n(0) z^n$$

Now by multiplying Equations (21) and (22) by the proper powers of z and then adding both equations we get

$$\theta Q^*(z,\theta) - Q(z,0) = \lambda Q^*(z,\theta) - \left[P_{11}(0)(1-r) - P_{12}(0)\right] V^*(\theta) - \lambda \sum_{n=1}^{\infty} z^n \left(\sum_{k=1}^{n} Q_{n-k}^*(\theta) g_k \right)$$

According to the identity $\sum_{n=1}^{\infty} z^n \left(\sum_{k=1}^{n} Q_{n-k}^*(\theta) g_k \right) = Q^*(z,\theta) X(z)$ the above equation reduces to

$$(\theta - \lambda + \lambda x(z)) Q^*(z,\theta) = Q(z,0) - P_r(0) V^*(\theta) \quad (\text{where } P_r(0) = P_{11}(0)(1-r) + P_{12}(0)) \tag{27}$$

By letting $\theta = \lambda - \lambda X(Z) = W(Z)$, we get

$$Q(z,0) = P_r(0) V^*(W(z)) \tag{28}$$

Hence,

$$Q^*(z,\theta) = \frac{p_r(0)\left(v^*(w(z)) - v^*(\theta)\right)}{\theta - w(z)} \tag{29}$$

Next, we shall calculate the generating function of $D^*(z, \theta)$ and $P^*(z, \theta)$.

The generating function corresponding to setup probabilities can be calculated by multiplying Equations (23) and (24) by suitable powers of z and then adding over m to ∞ as

$$\theta D^*(z,\theta) - D(z,0) = \lambda D^*(z,\theta) - \sum_{n=m}^{\infty} Q_n(0) z^n D^*(\theta) - \lambda \sum_{n=m}^{\infty} z^n \left(\sum_{k=n-m+1}^{n} R_{n-k} g_k \right) D^*(\theta) - \lambda \sum_{n=m+1}^{\infty} z^n \left(\sum_{k=1}^{n-m} D^*_{n-k}(\theta) g_k \right) \tag{30}$$

After multiplying Equation (15) by z^n, $D^*(\theta)$ and adding with (14) over 0 to $(m-1)$ we have

$$\lambda R(z) D^*(\theta) = D^*(\theta) \sum_{n=0}^{m-1} Q_n(0) z^n + \lambda D^*(\theta) \sum_{n=1}^{m-1} z^n \left(\sum_{k=1}^{n} R_{n-k} g_k \right) \tag{31}$$

Further by subtracting Equation (31) from (30) we obtain

$$\theta D^*(z,\theta) - D(z,0) - \lambda R(z) D^*(\theta) = \lambda D^*(z,\theta) - D^*(\theta) Q(z,0) - \lambda D^*(\theta) \sum_{n=1}^{m-1} Z^n \left(\sum_{k=1}^{n} R_{n-k} g_k \right) - \lambda \sum_{n=m}^{\infty} z^n \left(\sum_{k=n-m+1}^{n} R_{n-k} g_k \right) D^*(\theta) - \lambda \sum_{n=m+1}^{\infty} z^n \left(\sum_{k=1}^{n-m} D^*_{n-k}(\theta) g_k \right)$$

According to the identities

$$\sum_{n=m+1}^{\infty} z^n \left(\sum_{k=1}^{n-m} D^*_{n-k}(\theta) g_k \right) = x(z) D^*(\theta)$$

and

$$\left(\sum_{n=m}^{\infty} z^n \left(\sum_{k=n-m+1}^{n} R_{n-k} g_k \right) \right) + \left(\sum_{n=1}^{m-1} z^n \left(\sum_{k=1}^{n} R_{n-k} g_k \right) \right) = x(z) R(z)$$

the above equation becomes,

$$(\theta - w(z)) D^*(z,\theta) = D(z,0) - D^*(\theta) \left(Q(z,0) - R(z) w(z) \right)$$

At

$$\theta = w(z),\ D(z,0) = D^*(w(z)) \left(Q(z,0) - R(z) w(z) \right) \tag{32}$$

By substituting for $D(z, 0)$ and $Q(z, 0)$ in Equation (32) we get,

$$D^*(z,\theta) = \frac{\left(D^*(w(z)) - D^*(\theta)\right)\left(p_r(0)v^*(w(z)) - R(z)w(z)\right)}{\theta - w(z)} \tag{33}$$

Similarly, multiplying Equations (19) and (20) by appropriate powers of z and then summing we have

$$\theta P_2^*(z,\theta) - P_2(z,0) = \lambda P_2^*(z,\theta) - rP_1(z,0)S_2^*(\theta) - \lambda\sum_{n=2}^{\infty} z^n\left(\sum_{k=1}^{n-1} P_{n-k2}^*(\theta)g_k\right)$$

But one can prove that $\sum_{(n=2)}^{\infty} z^n\left(\sum_{(k=1)}^{(n-1)} P_{(n-k2)}^*(\theta)g_k\right) = x(z)P_2^*(z,\theta)$

Hence,

$$(\theta - w(z))\ P_2^*(z,\theta) = P_2(z,0) - rP_1(z,0)S_2^*(\theta) \tag{34}$$

Now by substituting $\theta = w(z)$ in Equation (34) we get

$$P_2(z,0) = rP_1(z,0)S_2^*(w(z)) \tag{35}$$

Then

$$P_2^*(z,\theta) = \frac{rp_1(z,0)\left(s_2^*(w(z)) - s_2^*(\theta)\right)}{\theta - w(z)} \tag{36}$$

Next, we calculate the expansion for $P_1^*(z,\theta)$.

By multiplying Equations (16) (18) by the corresponding powers of z and summing over 1 to ∞ we get,

$$\begin{aligned}\theta P_1^*(z,\theta) - P_1(z,0) = \lambda P_1^*(z,\theta) &- \frac{(1-r)\left(s_1^*(\theta)\right)}{z}\left(P_1(z,0) - P_{11}(0)z\right)\\ &- \frac{s_1^*(\theta)}{z}\left(P_2(z,0) - P_{12}(0)z\right)\\ &- \lambda\sum_{n=2}^{\infty} z^n\left(\sum_{k=1}^{n-1} P_{n-k1}^*(\theta)g_k\right)\\ &- \lambda\sum_{n=N}^{\infty} D_n(0)z^n S_1^*(\theta) - \sum_{n=N}^{\infty} z^n\left(\sum_{k=n-N+1}^{n-m} U_{n-k}\right)g_k S_1^*(\theta)\end{aligned} \tag{37}$$

Further, multiplying Equations (25) by $z^n S_1^*(\theta)$ and (26) by $z^n S_1^*(\theta)$ it follows that,

$$\lambda U(z)S_1^*(\theta) = \sum_{n=m}^{N-1} D_n(0)z^n S_1^*(\theta) + \lambda S_1^*(\theta)\sum_{n=m}^{N-1} z^n\left(\sum_{k=1}^{n-m} U_{n-k}\right)g_k \tag{38}$$

Now by subtracting (38) from (37) and by using the following identities

$$\sum_{n=2}^{\infty} z^n\left(\sum_{k=1}^{n-1} P_{n-k1}^*(\theta)g_k\right) = x(z)P_1^*(z,\theta)x(z)$$

and

$$\sum_{n=m}^{N-1} z^n \left(\sum_{k=1}^{n-m} U_{n-k}\right) g_k + \sum_{n=N}^{\infty} z^n \left(\sum_{k=n-N+1}^{n-m} U_{n-k}\right) g_k = U(z)x(z)$$

we get,

$$\begin{aligned}(\theta - w(z))P_1^*(z,\theta) = {} & P_1(z,0) - \frac{(1-r)s_1^*(\theta)}{z}\left(P_1(z,0) - P_{11}^*(0)z\right)\\ & - \frac{s_1^*(\theta)}{z}\left(P_2(z,0) - P_{12}(0)z\right) - S_1^*(\theta)D(z,0)\\ & - \lambda S_1^*(\theta)D(z,0)\\ & - \lambda S_1^*(\theta)U(z)X(z) - \gamma U(z)S_1^*(\theta)\end{aligned}$$

Substituting the value of $P_2(z, 0)$ from (35) and after simplification it is found that

$$\begin{aligned}(\theta - w(z))P_1^*(z,\theta) = {} & P_1(z,0)\frac{\left(z - (1-r)s_1^*(\theta)S_2^*w(z)\right)}{z}\\ & - S_1^*(\theta)\left[D(z,0) - P_r(0) - U(z)w(z)\right]\end{aligned} \quad (39)$$

When $\theta = w(z)$ Equation (39) becomes

$$P_1(z,0) = \frac{zs_1^*(w(z))\left(D(z,0) - p_r(0) - U(z)w(z)\right)}{z - (1-r)\left(s_1^*(w(z))\right) - rs_1^*(w(z))s_2^*(w(z))\right)} \quad (40)$$

By using (40) in (39), Equation (39) reduces to

$$P_1^*(z,\theta) = \frac{z\left(s_1^*(w(z)) - s_1^*(\theta)\right)\left(D(z,0) - p_r(0) - U(z)w(z)\right)}{(\theta - (w(z)))\left(z - s_r^*(w(z))\right)} \quad (41)$$

where $S_r^*w(z) = S_1^*(w(z))\left((1-r) + rS_2^*(w(z))\right)$

Now by substituting $P_1(z, 0)$ in (36) we get,

$$P_2^*(z,\theta) = \frac{rzs_1^*(w(z))\left((s_2^*(w(z)) - s_1^*(\theta)\right)\left(D(z,0) - p_r(0) - U(z)w(z)\right)}{(\theta - (w(z)))\left(z - s_r^*(w(z))\right)} \quad (42)$$

The total p.g.f is given by

$$P(z) = P_1^*(z,0) + P_2^*(z,0) + D^*(z,0) + Q^*(z,0) + U(z) + R(z) \quad (43)$$

The expressions for $Q^*(z,0)$, $D^*(z,0)$, $P_1^*(z,0)$ and $P_2^*(z,0)$ can be obtained by substituting $\theta = 0$ in Equations (29), (33), (41) and (42). Moreover, to obtain the total p.g.f $P(z)$ in a more simplified form we need to calculate $R(z)$ and $U(z)$.

4.3 Computation of R(z) and U(z)

Let us first define $\pi_0 = 1$, and $\pi_n = \sum_{i=1}^{n} g_i \pi_{n-i}$ $(1 \le n \le m-1)$ (Lee et al., 1994) then Equations (1) and (2) together imply that $\lambda R_n = \left(\sum_{(k=0)}^{n} Q_k(0)\pi_{(n-k)}\right)$ (by recursion) where $Q_k(0)$ = The coefficient of z^k of $Q(z,0)$.

Equation (28) implies $Q(z,0) = P_r(0)V^*(w(z))$, where $V^*(w(z)) = \sum_{n=0}^{\infty} \alpha_n z^n$ and α_n gives the probability that n customers arrive during a vacation.

Hence,

$$Q(z,0) = P_r(0)\sum_{n=0}^{\infty} \alpha_n z^n$$

that is,

$$\lambda R_n = p_r(0)\sum_{k=0}^{n} a_k \pi_{n-k}$$

By defining $\psi_0 = \alpha_0$ and $\psi_n = \sum_{i=0}^{n} \alpha_i \pi_{n-i}$ we get,

$$\lambda R_n = p_r(0)\psi_n \quad \text{for } 0 \le n \le m-1 \quad \text{(Lee et al., 1995)}$$

and the p.g.f $R(z)$ of R_n is given by

$$R(z) = \frac{P_r(0)}{\lambda}\sum_{n=0}^{m-1} \psi_n z^n \tag{44}$$

Next, we calculate p.g.f $U(z)$ of U_n in the following manner: Equations (12) and (13) together imply by recursion that

$$\lambda U_n = \sum_{k=m}^{n} D_k(0)\pi_{n-k} \quad \text{for } m \le n \le N-1 \tag{45}$$

where $D_k(0)$ is the coefficient of z^k in $D(z,0)$ and $D(z,0) = D^* w(z)\,[Q(z,0) - R(z)\,w(z)]$ (from (32)) substituting for $Q(z,0)$ from (28) and for $R(z)$ from (44) we get

$$D(z,0) = D^*(w(z))P_r(0)\left[V^*(w(z)) - \sum_{n=0}^{m-1} \psi_n z^n (x(z)-1)\right] \tag{46}$$

To interpret the second term of Equation (46), we introduce $\mathrm{SI}(k)$ to denote the probability that there are k customers available at a set up initiation point. Then by conditioning the number of customers arrived during a vacation, we have

$$\mathrm{SI}(k) = pr(\mathrm{SI} = k) = \alpha_k + \sum_{j=0}^{m-1} \alpha_j\, pr\left(Q_{m-j} = k - j\right), \quad k \ge m$$

Here, SI is a random variable which denotes the queue length at the setup initiation point and Q_m denotes the number of customers at a busy period initiation point for the m policy $M^X/G/1$ queueing model with out vacation and the generating function $Q_m(z)$ of Q_m is given by

$$Q_m(z) = 1 - (x(z) - 1)\sum_{(n=0)}^{(m-1)} \pi_n z^n \text{ (Lee et al., 1994)}$$

Thus,

$$\mathrm{SI}(z) = \sum_{k=m}^{\infty}\left(\mathrm{SI}(k)z^k = \sum_{k=m}^{\infty} \alpha_k z^k + \sum_{k=m}^{\infty} z^k \sum_{j=0}^{m-1} \alpha_j pr\left(Q_{m-j} = k - j\right)\right.$$

$$= V^*(w(z)) + \sum_{j=0}^{m-1} \alpha_j z^j \left\{\sum_{k=m-j}^{\infty} \left(pr\left(Q_{m-j} = k\right) z^k\right) - 1\right\} \text{ (by simplification)}$$

$$= V^*(w(z)) + \sum_{j=0}^{m-1} \alpha_j z^j (Q_{m-j}(z) - 1)$$

$$= V^*(w(z)) + (x(z) - 1)\sum_{j=0}^{m-1} \alpha_j z_j \sum_{n=0}^{m-j-1} \pi_n z^n$$

By the definition of ψ,

$$\mathrm{SI}(z) = \sum_{i=0}^{n} \alpha_i \pi_{n-1}$$

Then, we get

$$\mathrm{SI}(z) = V^*(w(z)) + (x(z) - 1)\sum_{n=0}^{m-1} \psi_n z^n \tag{47}$$

Moreover,

$$D^*(w(z)) = \sum_{k=0}^{\infty} h_k z^k \tag{48}$$

where h_k represents the probability that k customers arrive during a setup time.

Then by using Equations (47) and (48) in (46) we get $D(z, 0)$ as

$$D(z,0) = p_r(0)\left(\sum_{(k=0)}^{\infty} h_k z^k\right)\left(\sum_{(r=m)}^{\infty} \mathrm{SI}(r) z^r\right)$$

Remark 1: *It is verified that the result of the model discussed here, coincides with the corresponding result of the*

- *model discussed by Lee et al. (2003) when $r = 0$*
- *model of Choudhury and Madan (2006) when $N = m$, $E(v) = 0$ and $E(D) = 0$.*

Remark 2 (Decomposition property): *The p.g.f of the system size of the (m, N) policy of $M^X/(G_1, G_2)/1$ queueing model with early setup time and single vacation is decomposed in the form,*

$$P(z) = P^x_{(M/G_1,G_2/1 \text{ without } N\text{-policy})}(z)\,\Psi(z) \tag{54}$$

where

$$\Psi(z) = \frac{1}{D_{m,N}}\left[\frac{1 - V^*w(z)D^*w(z)}{w(z)} + \frac{\sum_{n=0}^{m-1}\Psi_n z^n D^* w(z)}{\lambda} + \frac{\sum_{n=m}^{N-1}\phi_n^v z^n}{\lambda}\right]$$

Proof: The first term of Equation (53) corresponds to the p.g.f. of the second optional service $M^X/G_1,G_2/1$ queueing model without N-policy (Choudhury and Paul, 2006) which is denoted by

$$P_{(M^X/G_1,G_2/1 \text{without } N\text{-policy})}(z) = \frac{(1-\rho_r)(z-1)S_r^* w(z)}{z - S_r^* w(z)}$$

Also $\Psi(z)$ is the conditional p.g.f of the queue length during the idle period which includes (vacation period, building up period, setup period and dormant period).

Remark 3 (Mean system size): *Let L_{SOS}^r denote the expected system size of $M^X/G_1,G_2/1$ queueing system under the early setup and single vacation then,*

$$\begin{aligned} L_{SOS}^r &= L_{M^X/G_1,G_2/1} \\ &+ \frac{1}{D_{m,N}}\left(\frac{\lambda E(V)\left(E\left(D^2\right)\right) - 2E(D)E(V) - E\left(V^2\right)}{2}\right. \\ &\left. + \frac{\sum_{n=0}^{m-1} n\Psi_n}{\lambda} + \sum_{n=0}^{m-1}\Psi_n E(D)E(X) + \frac{\sum_{n=m}^{N-1} n\Phi_n^v}{\lambda}\right) \end{aligned} \tag{55}$$

Proof:

$$\begin{aligned} L_{SOS}^r &= \frac{d}{dz}(P(z))_{z=1} = \text{Expected system size of } M^X/G_1/G_2/1 \text{ without } N\text{-policy} \\ &+ \frac{d}{dz}(\Psi(z))_{z=1} \quad \text{(by Equation (54))} \end{aligned}$$

The expected system size of $M^X/(G_1G_2)/1$ without N-policy =

$$(\lambda E(X))^2\left(E\left(s_1^2\right)+2rE\left(s_1\right)E\left(s_2\right)+rE\left(s_2^2\right)\right)-\lambda(E(X(X-1))\left(rE\left(s_2\right)+E\left(s_1\right)\right)/2(1-\rho)$$

(by Choudhury and Madan, 2006)

$$\frac{\mathrm{d}}{\mathrm{d}z(\psi(z))_{z=1}}=\frac{1}{D_{m,N}}\left(\frac{\lambda E(V)(E(D^2)-2E(D)E(V)+E(V^2)}{2}-\frac{\sum_{n=0}^{m-1}n\Psi_n}{\lambda}\right.$$

$$\left.-\sum_{n=0}^{m-1}\Psi_n E(D)E(X)-\frac{\sum_{n=m}^{N-1}n\Phi_n^v}{\lambda}\right)$$

- Hence, we get the value of expected system size as in Equation (55).

5 Queue size distribution at departure epoch

By following the arguments of PASTA Wolff (1982), a departing customer will see j customers in the system just after a departure, if and only if there were $j+1$ customers in the system just before the departure. Thus, we have $\pi_j^* = K\left[(1-r)P_{j+1,1}(0)+P_{j+1,2}(0)\right]$, $j \geq 0$ where K is the normalising constant.

Let $\pi^*(z)$ be the p.g.f of the departure epoch probabilities $\{\pi_j^* : j \geq 0\}$

Then

$$\pi^*(z)=\sum_{(j=0)}^{\infty}\pi_j^* z^j=\frac{K}{Z}\left[(1-r)\sum_{j=1}^{\infty}P_{j1}(o)z^j-\sum_{j=1}^{\infty}P_{j2}(o)z^j\right]$$

that is,

$$\pi^*(z)=\frac{K}{Z}\left[P_1(z,0)[1-r]+P_2(z,0)\right].$$

Substituting, $P_2(z,0)$ from Equation (35), we get

$$\pi^*(z)=\frac{K}{Z}P_1(z,0)\left[(1-r)+rs_2^\bullet(W(z))\right.$$

$$=K\left(\frac{P_r(0)S_r^\bullet w(z)}{z-S_r^\bullet w(z)}\right)\left[D^\bullet(w(z))v^\bullet(w(z))-1-\frac{w(z)}{\lambda}D^\bullet(w(z))\right.$$

$$\left.\times\sum_{n=0}^{m-1}\psi_n z^n-\frac{w(z)}{\lambda}\sum_{n=m}^{N-1}\phi_n^v z^n\right]$$

Using the simplified form of (40)

$\pi^{+}(z)=\frac{Kw(z)}{(1-z)}P(z)$ here $P(z)$ is the total p.g.f given by Equation (53).

By using the normalising condition, K can be calculated as $\pi^{*}(1)=1$

It is found that $K=(1/\lambda E(X))$.

Remark 4: *If $\pi^{*}(z)$ denotes the p.g.f of queue size distribution at a departure epoch, then under the stability condition $\rho<1$, we have $\pi^{*}(z)=A(z)\ P(z)$ where $A(z)$ = p.g.f of number of customers placed before an arbitrary test customer in a batch in which the test customer arrives and $P(z)$ = p.g.f of queue size distributions at a random epoch (50)*

6. The performance measures

In this section, we present various steady state system size probabilities for the server is on vacation (P_v^r), build up period (P_{bu}^r), on set up period (P_{su}^r), on dormant period (P_d^r) and on busy period (P_{busy}^r), respectively. Further $Q^{*}(z,0)$, $R(z)$, $D^{*}(z,0)$, $U(z)$ and $P^{*}(z,0)=(P_1^{*}(z,0)+P_2^{*}(z,0))$ as discussed in section (4) represent the p.g.f of number of waiting customers in the system when the server is on vacation, idle, doing the pre-service work, in dormant and busy, respectively. Then the expression for the probabilities can be obtained as:

1 $P_v^r=\lim_{z\to 1}Q^{*}(z,0)$

$P_v^r=P_r(0)E(v)$ (by substituting $Q^{*}(z,0)$ from Equation (25)). But following long fraction time that the server is on vacation is given by $P_v^r=(E(v))/E(T_{cycle})$ (the total length of the cycle and its expected values are denoted by T_{cycle} and $E(T_{cycle})$ where T_{cycle} = vacation period + build up period + setup period + dormant period + busy period). Thus,

$$E\left(T_{cycle}\right)=\frac{E(v)}{P_v^r}=\frac{1}{P_r(0)}=\frac{D_{m,N}}{1-\rho_r}\quad\text{(from Equation (51))}$$

2 $P_{bu}^r=\lim_{z\to 1}R(z)$

By letting $R(z)$ from Equation (44) we get

$$P_{bu}^r=\frac{p_r(0)}{\lambda}\sum_{n=0}^{m-1}\psi_n$$

3 $P_{su}^r=\lim_{z\to 1}D^{*}(z,0)$

By substituting $D^{*}(z,0)$ from Equation (33) and applying the L 'Hospital rule we get, $P_{su}^r=P_r(0)E(D)$.

4 $P_d^r = \lim_{z \to 1} U(z)$

$$P_d^r = \frac{p_r(0)}{\lambda} \sum_{n=0}^{m-1} \phi_n^v \quad \text{(by using Equation (49))}$$

5 $P_{busy}^r = \lim_{z \to 1} \left(P_1^*(z,0) + P_2^*(z,0) \right)$

By letting for $P_1^*(z,0)$ and $P_2^*(z,0)$ from Equations (41) to (42) and then applying L' Hospital rule we have

$$P_{busy}^r = \lambda E(X)\left[E(S_1) + rE(S_2) \right] = \rho_r$$

7 Optimal operating policy

In this section, we develop a total expected cost function per unit time for $M^X/G_1G_2/1$ queue under bi-level threshold model with single vacation in which m and N are decision variables. Our objective is to find the optimal values m^* and $N^*(m)$ which minimise the linear cost function. For our optimal cost model, we consider the same cost structure that has been widely used by many authors as

C_s = turn on cost per cycle

C_h = holding cost per unit time

C_s = setup cost per unit time

C_v = reward per unit time due to vacation

C_d = dormant cost per unit time

C_b = build up cost per unit time

C_o = operating cost per unit time

$T_c(m, N)$ = average cost per unit time

Also by following the arguments of Lee and Srinivasan (1989) and Lee and Park (1997), the average cost per unit time is given by

$$T_c(m, N) = \frac{C_s}{E\left(T_{cycle}\right)} + C_h L_{SOS}^r + C_s P_{su}^r + C_d P_{bu}^r + C_0 \rho_r - C_v P_v^r \tag{56}$$

Substituting the values of $L_{SOS}^r, P_{s1}^r, P_d^r, P_{bu}^r$ and P_v^r from section (5) and (6), we have

$$T_c(m, N) = \frac{1}{D_{m,N}} \left[A - z_m - C_h \sum_{(n=m)}^{(N-1)} n\phi_n^v - C_d \sum_{(n=m)}^{(N-1)} \phi_n^v (1 - \rho_r) \right] - A' \tag{57}$$

where

$$A = \lambda(1-\rho_r)\left[C_y + C_s E(D) - C_s E(v)\right] + \frac{\lambda^2 E(X)\left[E(D^2) + 2E(D)E(v) + E(v^2)\right]}{2} C_h$$

$$z_m = C_h \sum_{n=0}^{m-1} n\psi_n + ((\lambda E(D)E(X)C_h + C_b(1-\rho_r)) \sum_{(n=0)}^{(m-1)} \psi_n$$

and $D_{m,N}$ as in Equation (52)

By letting $J_k^m = \sum_{k=m}^{\infty} \phi_k$ and $M_k^m = \sum_{k=m}^{\infty} k\phi_k^v$ and also using Equation (57) we get,

$$T_c(m,k+1) - T_c(m,k) = \frac{\phi_k^v}{c_{m\,k+1} c_{m\,k}} (h_m(k)) \tag{58}$$

where

$$h_m(k) = C_h\left(kL_m + kJ_k^m - M_k^m\right) + C_d(1-\rho_r)L_m - (A - z_m)$$

and

$$L_m = E(D) + E(v) - \frac{1}{\lambda}\sum_{n=0}^{m-1} \psi_n$$

Equation (58) implies that $T_c(m, k+1) - T_c(m, N) > 0$ whenever $h_m(k) > 0$ and also

$$h_m(k+1) = h_m(k) + C_h\left(L_m - J_k^m\right) > 0$$

whenever, $h_m(k) > 0$

If n is the first k for which $h_m(k) > 0$ then

$T_c(m, k) > T_c(m, N)$ for $k > n$.

This means that for any given m, the optimal value $N^*(m)$ of N is given by the first k such that $h_m(k) > 0$ and that once $T_c(m, N(m))$ increases with respect to N, it keeps on increasing thereafter. Therefore, we can say that for a given m, $T_c(m, N(m))$ is conditionally unimodal and thereby $N^*(m)$ is conditionally optimal. Thus

$$\begin{aligned} N^*(m) &= \min\left\{\frac{k \geq 1}{h_m(k)} > 0\right\} \\ &= \min\left\{\frac{k}{C_h}\left(kL_m + kJ_k^m - M_k^m\right) + C_d(1-\rho_r)L_m > (A - z_m)\right\} \end{aligned} \tag{59}$$

Now the search for the global optimum is in order. Let the pair $(m, N^*(m))$ be the optimal for the given m. It is difficult to prove mathematically that $T_c(m, N)$ as functions of both m and N is convex. But through numerical experiments, it is proved that the cost function is probably convex, which we assume in the following algorithm to find the joint optimal values of $(m^*, N^*(m))$.

Algorithm:

Step 1 Set $m = 1$. Determine $N^*(m)$ from Equation (59). Calculate $T_c(m, N^*(m))$. Go to step2.

Step 2 Calculate $N^*(m+1)$ and $T_c(m+1, N^*(m+1))$ Go to Step 3.

Step 3 If $T_c(m+1, N^*(m+1)) > T_c(m, N^*(m))$, stop. $T_c(m, N^*(m))$ is the optimal policy. Otherwise set $m = m + 1$ and go to Step 2.

8 Numerical analysis

In this section, we illustrate the results obtained in the previous sections numerically and discuss the effects of system parameters on system performance indices. Moreover, the demonstration of the sensitivity calculation is focused on four critical input parameters arrival rate (λ), service rate (μ), the mean vacation parameter $E(v)$ and the mean service parameter $E(s)$.

8.1 Parameter setting

For convenience of computation, we assume the following distributions for different random variables involved in the model as:

The batch size X follows the geometric distributions that is,

1 g_k $\Pr(X = k) = (1-p)p^{k-1},\ k \geq 1,$ with mean $E(X) = \dfrac{1}{1-p}$

2 The service times $S_i(i = 1, 2)$ follow two-stage Hyper-Exponential distributions with mean

$$E(S_1) = \frac{a_1}{\mu_{11}} + \frac{a_2}{\mu_{12}},\quad E(S_2) = \frac{b_1}{\mu_{21}} + \frac{b_2}{\mu_{22}} \quad \text{and} \quad E(s) = (1-r)E(S_1) + rE(S_2)$$

Further their second moments are

$$E\left(S_1^2\right) = 2\left(\frac{a_1}{\mu_{11}^2} + \frac{a_2}{\mu_{12}^2}\right)$$

and

$$E\left(S_2^2\right) = 2\left(\frac{b_1}{\mu_{21}^2} + \frac{b_2}{\mu_{22}^2}\right).$$

3 The setup time D and vacation time V follow Erlang three-type distributions with mean $E(D) = 1/\nu$, $E(v) = (1/\eta)$ further their second moments are $E(D^2) = (4/3\nu^2)$ and $E(v^2) = (4/3\eta^2)$.

Algorithm:

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$$E\left(S_1^2\right) = 2\left(\frac{a_1}{\mu_{11}^2} + \frac{a_2}{\mu_{12}^2}\right)$$

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By letting the following parametric values as $p = 0.01$, $\lambda = 0.2$, $\mu_{11} = 1.5$, $\mu_{12} = 2$, $\mu_{21} = 6$, $\mu_{22} = 0.5$, $a_1 = 0.1$, $a_2 = 1.5$, $b_1 = 2$ and $b_2 = 3$, the performance measures calculated are:

1 the expected queue length (L'_{sos}) and

2 the total cost function $T_c(m, N)$ and its optimal value $T_c(m^*, N^*)$

Figures 2 and 3 show that as the mean system size (L'_{sos}) decreases as the mean vacation $E(v)$ and the mean setup time $E(D)$ decrease.

The effect of $E(v)$ and $E(s)$ on the expected number customers in the system is presented in Figure 4. Figure 4 shows that as the service rate μ increases, the mean vacation time $E(v)$ and the mean system size (L'_{sos}) also decreases.

The total cost function $T_c(m, N)$ for various values of m and N and its optimal cost $T_c(m^*, N^*)$ are represented in Figure 5. For the convenience of computation, we consider the cost elements as

$C_h = 10$, $C_d = 0.1$, $Co = 100$, $C_r = 50$, $C_s = 1{,}000$, $C_i = 8$ and $C_b = 8$ and note that the optimal value cost per unit time of $ 67.567 is achieved at $m = 2$, $N = 11$.

Figure 2 Effect of L'_{SOS} with the traffic intensity ρ, for $v = 1$ (see online version for colours)

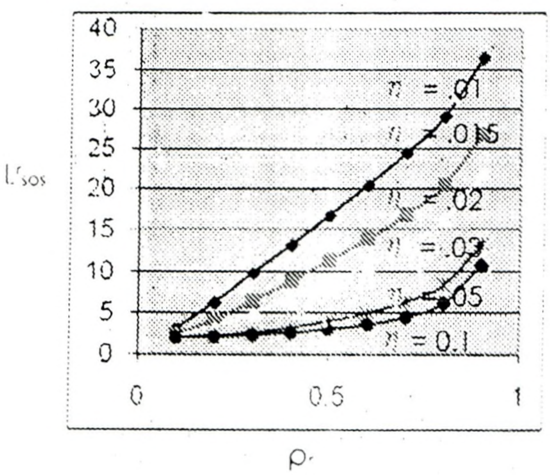


Figure 3 Effect of L^r_{SOS} with the traffic intensity ρ_r for $\eta = 0.03$ (see online version for colours)

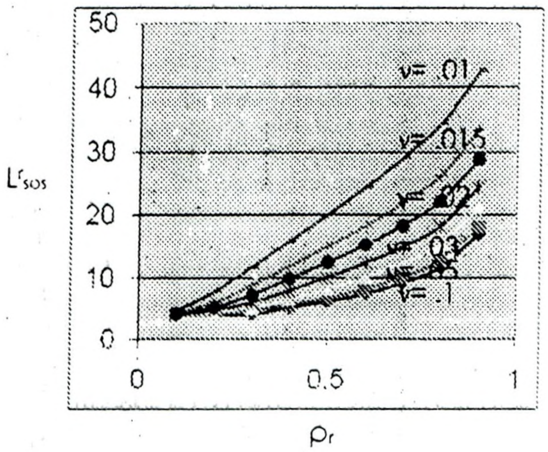


Figure 4 Effect of η and μ on L^r_{SOS}

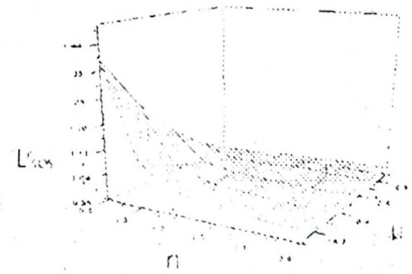
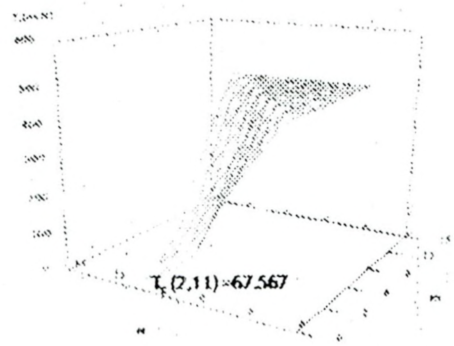


Figure 5 The expected cost $T_c(m, N)$ for different values of m and N



9 Conclusion

Lee et al. (2003) used the decomposition property in their paper for vacation queues to derive the queue length for the system. However, the approach employed by them is not sufficient to derive the busy period and waiting time distribution of the model. In most of the works, concerning the analysis of batch arrival queues, the supplementary variable technique is one of the most widely used tools. In this paper, we have analysed the most general second optional service queueing model $M^X/G_1\ G_2/1$ with early setup time and single vacation using supplementary variable technique and presented the p.g.f of the system size probabilities at random epoch, at setup initiation point and at departure epoch and various performance measures including the mean system length in a closed form. Moreover, a procedure to obtain N^* which minimises the long-run average cost under a suitable linear cost structure is also suggested. Further, we notice that the results of our model accommodate many previous models as particular cases. Thus, this model may become very useful in practical applications where more general situations may arise. The results of our model may be considered for unreliable server in future research.

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