Chapter 6

Δ^*-Homeomorphisms in Topological Spaces

6.1 Introduction

If A and B are subsets of Euclidean spaces then a homeomorphism from A to B is a bijection map $f : A \longrightarrow B$ such that both f and its inverse map are continuous. When such f exists, A and B are said to be homeomorphic to each other. A property of an object which is invariant under homeomorphism is said to be topological in character. The concept of generalised homeomorphisms and gc- homeomorphisms were introduced by Maki.et.al.,(1991). In this chapter, a new class of closed maps namely, Δ^*-closed maps is introduced and the dependency relations with various existing closed maps are obtained. It is proved that the composition of two Δ^*-closed maps need not be a Δ^*-closed map. Further using Δ^*-open maps, a new class of maps called Δ^*-homeomorphisms is introduced and their properties are studied. In continuation of these maps $\Delta^*\mathcal{C}$-homeomorphisms via Δ^*-irresolute maps are introduced and their group property is also analyzed in this chapter.

6.2 Δ^*-Closed Maps

Definition 6.2.1 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a Δ^***-closed map** if the image of each closed set in (X, τ) is a Δ^*-closed set in (Y, σ).

Relationship with other maps :

Proposition 6.2.2 Every δ-closed map (resp., δg^*-closed map) is a Δ^*-closed map but not conversely.

Proof : Let f : (X, τ) → (Y, σ) be a δ-closed map (resp., δg^*-closed map). Let V be a closed set in (X, τ). Then its image f(V) is δ-closed (resp., δg^*-closed) in (Y, σ). Since every δ-closed set (resp., δg^*-closed set) is Δ^*-closed, f(V) is Δ^*-closed in (Y, σ). Hence f is a Δ^*-closed map.

Counter Example 6.2.3 Let X = {a, b, c} = Y with τ = {ϕ , X, {a}}, and σ = {ϕ , Y, {a}, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = c, f(c) = b. Then f is a Δ^*-closed map but not a δ-closed map since for the closed set {b, c} in (X, τ), f [{b, c}] = {b, c} is not δ-closed in (Y, σ).

Proposition 6.2.4 a) Every Δ^*-closed map is a $\delta g^‡$-closed map.

b) Every Δ^*-closed map is a gδ-closed map.

c) Every Δ^*-closed map is a πg-closed map.

d) Every Δ^*-closed map is a πgs-closed map.

e) Every Δ^*-closed map is a πgα-closed map.

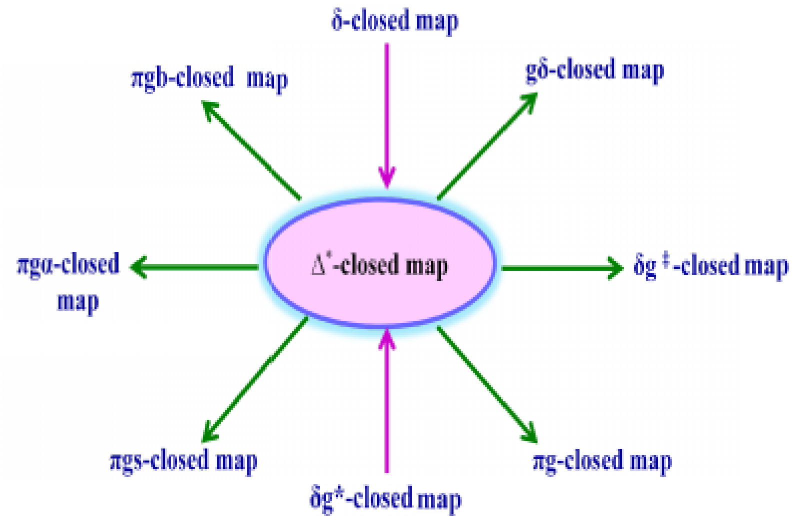
f) Every Δ^*-closed map is and a πgb-closed map.

Proof : Follows from the fact that every Δ^*-closed set is $\delta g^‡$-closed, gδ-closed, πg-closed, πgs-closed, πgα-closed and πgb-closed. (Proposition 2.2.6, 2.2.8, 2.2.19, 2.2.25, 2.2.27, 2.2.29).

Remark 6.2.5 The converse of the above Proposition 6.2.4 is not true as seen from the following example.

Counter example 6.2.6 Let X ={a, b, c} = Y with τ = {ϕ , X, {a, b}}, and σ = {ϕ , Y, {a}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is a $\delta g^‡$-closed map, a gδ-closed map, a πg-closed map, a πgs-closed map, a πgα-closed map and a πgb-closed map but not a Δ^*-closed map as the image of the closed set {c} in (X, τ) is not a Δ^*-closed in (Y, σ).

Remark 6.2.7 The above observations are represented by the following diagram.



Remark 6.2.8 The following counter examples show that the Δ^*-closed map is independent from a g-closed map, a δg-closed map, a δg -closed map, a αg-closed map and a αg -closed map.

Counter Example 6.2.9 Let X = {a, b, c} = Y with τ = {ϕ , X, {a}}, and σ = {ϕ , Y, {a}, {a, b}, {a, c}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = b, f(c) = a. Then f is a Δ^*-closed map but not g-closed, δg-closed, δg -closed, αg-closed and αg -closed maps, since for the closed set {b, c} in (X, τ), f [{b, c}] = {a, b} is not g-closed, δg-closed, δg -closed, αg-closed and αg -closed in (Y, σ).

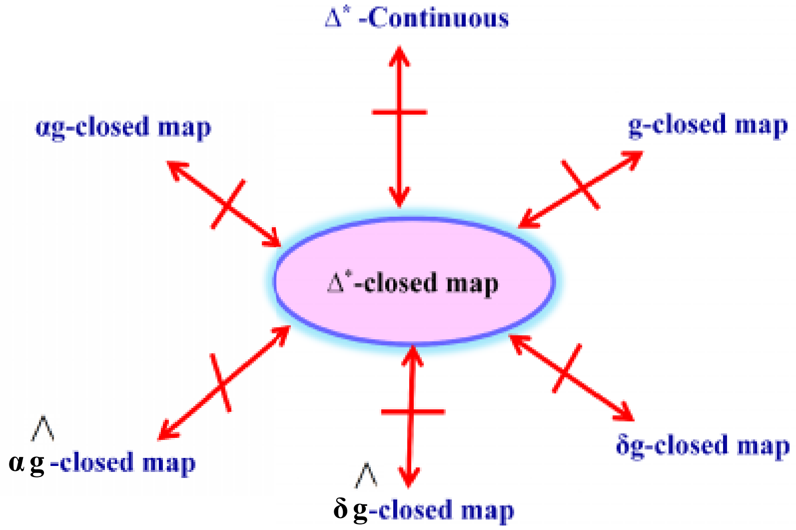
Counter Example 6.2.10 Let X = {a, b, c} = Y with τ = {ϕ , X, {a}}, and σ = {ϕ , Y, {a}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = c, f(c) = a. Then f is g-closed, δg-closed, δg -closed, αg-closed and αg -closed maps but not a Δ^*-closed map, since for the closed set {b, c} in (X, τ), f [{b, c}] = {a, b} is not Δ^*-closed in (Y, σ).

Remark 6.2.11 The Δ^*-closed map and Δ^*-continuity are independent as shown by the following examples.

Counter Example 6.2.12 Let X = {a, b, c}= Y with τ = {ϕ , X, {a}, {a, b}, {a, c}} and σ = {ϕ , Y, {a}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = a, f(c) = c. Then f is Δ^*-continuous but not a Δ^*-closed map since for the closed set {b, c} in (X, τ), f [{b, c}] = {a, c} is not Δ^*-closed in (Y, σ).

Counter Example 6.2.13 Let X = {a, b, c} = Y with τ = {ϕ , X, {a}} and σ = {ϕ , Y, {a}, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = c, f(c) = b . Then f is a Δ^*-closed map but not Δ^*-continuous since for the closed set {c} in (Y, σ), f^{-1}{c} = {b} is not Δ^*-closed in (X, τ).

Remark 6.2.14 The above results are depicted by the following diagram.



Properties of Δ^*-closed maps

Theorem 6.2.15 A map f : (X, τ) → (Y, σ) is Δ^*-closed if and only if for each subset U of (Y, σ) and for each open set V of (X, τ) containing f^{-1}(U) there exists a Δ^*-open set G of (Y, σ) such that U □ G and f^{-1}(G) □ V.

Proof : (Necessity) : Suppose that f : (X, τ) → (Y, σ) be a Δ^*-closed map and U be a subset of (Y, σ). Let V be an open subset of (X, τ) containing f^{-1}(U). Then (X – V) is closed in (X, τ). Since f is Δ^*-closed, f(X − V) is Δ^*-closed in (Y, σ). Hence Y − f(X − V) is a Δ^*-open set in (Y, σ). Take G = Y − f(X − V). Then G is Δ^*-open in (Y, σ) containing U such that f^{-1}(G) □ V.

(Sufficiency) : Let H be a closed subset of (X, τ). Then f^{-1}[Y − f(H)] □ (X – H) and (X – H) is open. By hypothesis there is a Δ^*-open set G of (Y, σ) such that (Y − f(H)) □ G and f^{-1}(G) □ (X – H). Therefore H □ (X − f^{-1}(G)). Hence (Y – G) □ f(H) □ f[X − f^{-1}(G)] □ (Y – G) which implies that f(H) = (Y – G) and f(H) is Δ^*-closed in (Y, σ). Thus f is a Δ^*-closed map.

Theorem 6.2.16 A bijection mapping f : (X, τ) → (Y, σ) is a Δ^*-closed map if and only if f(U) is a Δ^*-open set in (Y, σ) for every open set U in (X, τ).

Proof : Let f : (X, τ) → (Y, σ) be a Δ^*-closed map and U be any open set in (X, τ). Then U^C is a closed set in (X, τ). Therefore by the hypothesis, f(U^C) is Δ^*-closed in (Y, σ). Since f is bijective, f(U^C) = $[f(U)]^c$ is Δ^*-closed in (Y, σ). Hence f(U) is Δ^*-open in (Y, σ).

Conversely, let U be a closed subset of (X, τ). Then U^C is a open set in (X, τ). By the hypothesis, f(U^C) is Δ^*-open in (Y, σ). Since f is a bijection map, f(U^C) = $[f(U)]^c$. Thus f(U) is Δ^*-closed in (Y, σ). Hence f is a Δ^*-closed map.

Remark 6.2.17 In the above proposition, bijection condition on f is necessary which is proved in the following example.

Counter example 6.2.18 Let X = {a, b, c} = Y with τ = {ϕ , X, {a}}, σ = {ϕ , Y, {a}, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = b, f(b) = a, f(c) = a. Then for the only open set {a} in (X, τ) , f({a}) is Δ^*-open but not Δ^*-closed as for the closed set {b, c} in (X, τ), $f(\{b, c\}) = \{a\}$ is not Δ^*-closed in (Y, σ).

Proposition 6.2.19 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is δg-irresolute and Δ^*-closed map then $f(A)$ is Δ^*-closed subset of (Y, σ) where A is a Δ^*-closed subset of (X, τ).

Proof : Let U be a δg-open set in (Y, σ) such that $f(A)$ □ U. Since f is δg-irresolute, $f^{-1}(U)$ is a δg-open set containing A. That is A □ $f^{-1}(U)$. Hence $\delta cl(A)$ □ $f^{-1}(U)$. Since every δ-closed set is closed. Therefore $\delta cl(A)$ is closed. Since f is a Δ^*-closed map, $f(\delta cl(A))$ is Δ^*-closed contained in the δg-open set U which implies that $\delta cl[f(\delta cl(A)]$ □ U and hence $\delta cl[f(A)]$ □ U. Thus $f(A)$ is a Δ^*-closed subset of (Y, σ).

Theorem 6.2.20 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a Δ^*-closed map and A is a closed subset of X then $f|A : (A, \tau_A) \rightarrow (Y, \sigma)$ is Δ^*-closed.

Proof : Let B □ A be closed in (A, τ_A). Since A is closed in (X, τ), B is closed in (X, τ). Since f is a closed map, $f(B) = (f|A)(B)$ is Δ^*-closed in (Y, σ). Hence $f|A$ is Δ^*-closed.

Composition of Δ^*-closed maps

Remark 6.2.21 The composition of two Δ^*-closed maps need not be a Δ^*-closed map as seen from the following example.

Counter Example 6.2.22 Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\sigma = \{\phi, Y, \{a, b\}\}$ and $\eta = \{\phi, Z, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map such that $f(a) = a$, $f(b) = a$, $f(c) = c$. Let $g : (Y, \sigma) \rightarrow (Z, \eta)$ be a map such that $g(a) = c$, $g(b) = b$, $g(c) = a$. Then both f and g are Δ^*-closed maps. But their composition map $(g \circ f) : (X, \tau) \rightarrow (Z, \eta)$ is not a Δ^*-closed map since for the closed set $\{b, c\}$ in (X, τ), $(g \circ f)[\{b, c\}] = \{a, c\}$ is not Δ^*-closed in (Z, η).

Proposition 6.2.23 Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be Δ^*-closed maps. If (Y, σ) is a ${}_{\Delta^*}T_{\delta}$-space then their composition $(g \circ f) : (X, \tau) \to (Z, \eta)$ is a Δ^*-closed map.

Proof : Let A be a closed set in (X, τ). Since f is a Δ^*-closed map, f(A) is Δ^*-closed in (Y, σ). Since (Y, σ) is a ${}_{\Delta^*}T_{\delta}$-space, f(A) is δ-closed in (Y, σ). Since every δ-closed set is closed, f(A) is closed in (Y, σ). Since g is a Δ^*-closed map, $g[f(A)] = (g \circ f)(A)$ is Δ^*-closed in (Z, η). Hence $(g \circ f)$ is a Δ^*-closed map.

Proposition 6.2.24 Let $f : (X, \tau) \to (Y, \sigma)$ be a closed map and $g : (Y, \sigma) \to (Z, \eta)$ be a Δ^*-closed map. Then their composition map $(g \circ f) : (X, \tau) \to (Z, \eta)$ is a Δ^*-closed map.

Proof : Let U be any closed subset of X. Then f(U) is closed in (Y, σ). Therefore $g[f(U)]$ is Δ^*-closed in (Z, η). Hence $(g \circ f)$ is a Δ^*-closed map.

Remark 6.2.25 If $f : (X, \tau) \to (Y, \sigma)$ is a Δ^*-closed map and $g : (Y, \sigma) \to (Z, \eta)$ is a closed map then their composition need not be a Δ^*-closed map as seen from the following example.

Counter Example 6.2.26 Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\sigma = \{\phi, Y, \{a, b\}\}$ and $\eta = \{\phi, Z, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map such that $f(a) = a$, $f(b) = a$, $f(c) = c$. Let $g : (Y, \sigma) \to (Z, \eta)$ be a map such that $g(a) = c$, $g(b) = b$, $g(c) = a$. Then f is a Δ^*-closed map and g is a closed map. But their composition map $(g \circ f) : (X, \tau) \to (Z, \eta)$ is not a Δ^*-closed map since for the closed set $\{b, c\}$ in (X, τ), $(g \circ f)[\{b, c\}] = \{a, c\}$ is not Δ^*-closed in (Z, η).

Proposition 6.2.27 Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be Δ^*-closed maps. If (Y, σ) is a ${}_{\Delta^*}T_{\delta g^*}$-space and ${}_{\delta g^*}T_{\delta}$-space then their composition $(g \circ f) : (X, \tau) \to (Z, \eta)$ is a Δ^*-closed map.

Proof : Let A be a closed set in (X, τ). Then f(A) is Δ^*-closed in (Y, σ). Since (Y, σ) is a

${}_{\Delta^*}T_{\delta g^*}$-space and ${}_{\delta g^*}T_{\delta}$-space, f(A) is δ$g^*$-closed and hence it is δ-closed in (Y, σ). Since every δ-closed set is closed, f(A) is closed in (Y, σ). Since g is a Δ^*-closed map, g[f(A)] = (g ∘ f)(A) is Δ^*-closed in (Z, η). Hence the composition map (g ∘ f) is Δ^*-closed.

Theorem 6.2.28 Let f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η) be any two maps. Then the following results are true.

a) If (g ∘ f) : (X, τ) → (Z, η) is a Δ^*-closed map and g is a Δ^*-irresolute injective map then f is a Δ^*-closed map.

b) If (g ∘ f) : (X, τ) → (Z, η) is a Δ^*-irresolute map and g is a Δ^*-closed injective map then f is a Δ^*-continuous map.

Proof : a) Let U be any closed set in (X, τ). Since (g ∘ f) is Δ^*-closed, (g ∘ f)(U) is Δ^*-closed in (Z, η). Therefore g[f(U)] is Δ^*-closed in (Z, η). Since g is Δ^*-irresolute, g^{-1}[g(f(U)] is Δ^*-closed in (Y, σ). Since g is injective, g^{-1}[g(f(U)] = f(U) is Δ^*-closed in (Y, σ). Hence f is a Δ^*-closed map.

b) Let V be a closed set in (Y, σ). Since g is Δ^*-closed, g(V) is Δ^*-closed in (Z, η). Since (g ∘ f) is Δ^*-irresolute, $(g \circ f)^{-1}$[g(V)] is Δ^*-closed in (X, τ). Therefore $f^{-1}g^{-1}$[g(V)] is Δ^*-closed in (X, τ). Since g is injective, g^{-1}[g(V)] = V and hence g^{-1}(V) is Δ^*-closed in (X, τ). Thus f is a Δ^*-continuous map.

Theorem 6.2.29 Let f : (X, τ) → (Y, σ) and g : (Y, σ)→(Z, η) be any two maps. Let (g ∘ f) : (X, τ) → (Z, η) be a Δ^*-closed map and f be a continuous map. Then g is a Δ^*-closed map.

Proof : Let V be a closed set in (Y, σ). Since f is continuous, f^{-1}(V) is closed in (X, τ). Since (g ∘ f) is Δ^*-closed, (g ∘ f)[f^{-1}(V)] = g(V) is Δ^*-closed in (Z, η). Hence g is a Δ^*-closed map.

Theorem 6.2.30 Let f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η) be any two maps such that their composition map $(g \circ f)$: (X, τ) → (Z, η) is a Δ^*-closed map. Then the following statements are true.

a) If f is a surjective continuous map then g is a Δ^*-closed map.

b) If f is a surjective g-continuous map and (X, τ) is a $T_{1/2}$-space then g is a Δ^*-closed map.

c) If f is a quasi Δ^*-continuous and injective map then f is a closed map.

Proof : a) Let U be any closed set in (Y, σ). Since f is continuous, $f^{-1}(U)$ is closed in (X, τ). Since $(g \circ f)$ is a Δ^*-closed map, $(g \circ f)(f^{-1}(U))$ is Δ^*-closed in (Z, η). Since f is surjective, $(g \circ f)(f^{-1}(U)) = g(U)$. Hence g(U) is Δ^*-closed in (Z, η). Therefore g is a Δ^*-closed map.

b) Let V be any closed set in (Y, σ). Since f is g-continuous, $f^{-1}(V)$ is g-closed in (X, τ). Since (X, τ) is a $T_{1/2}$-space, $f^{-1}(V)$ is closed in (X, τ). Since $(g \circ f)$ is Δ^*-closed and f is surjective, $(g \circ f)(f^{-1}(V)) = g(V)$ is Δ^*-closed in (Z, η). Therefore g is a Δ^*-closed map.

c) Let V be any closed set in (X, τ). Since $(g \circ f)$ is a Δ^*-closed map, $(g \circ f)(V)$ is Δ^*-closed in (Z, η). Since g is quasi Δ^*-continuous and injective, $g^{-1}[(g \circ f)(V)] = f(V)$ is closed in (Y, σ). Hence f is a closed map.

Δ^*-Open Maps

Definition 6.2.31 A map f : (X, τ) → (Y, σ) is called a **Δ^*-open map** if the image of each open set in (X, τ) is a Δ^*-open set in (Y, σ).

Proposition 6.2.32 For any bijection map f : (X, τ) → (Y, σ), the following statements are equivalent.

a) f^{-1} : (Y, σ) → (X, τ) is Δ^*-continuous.

b) f is a Δ^*-open map.

c) f is a Δ^*-closed map.

Proof : (a) ⟹ (b) : Let U be any open set in (X, τ). Then by the assumption $[f^{-1}]^{-1}(U) = f(U)$ is Δ^*-open in (Y, σ). Therefore f is a Δ^*-open map.

(b) ⟹ (c) : Let V be a closed set in (X, τ). Then V^C is open in (X, τ). By the assumption $f(V^C)$ is Δ^*-open in (Y, σ). That is $f(V^C) = [f(V)]^C$ is Δ^*-open in (Y, σ) and therefore f(V) is Δ^*-closed in (Y, σ). Hence f is Δ^*-closed.

(c) ⟹ (a) : Let V be a closed set in (X, τ). By the assumption f(V) is Δ^*-closed in (Y, σ). But $f(V) = [f^{-1}]^{-1}(V)$ and therefore f^{-1} is Δ^*-continuous on (Y, σ).

Proposition 6.2.33 Let f : (X, τ) → (Y, σ) be any map and g : (Y, σ) → (Z, η) be injective and Δ^*-irresolute map. If their composition map (g ∘ f) : (X, τ) → (Z, η) is Δ^*-open then f is Δ^*-open in (Y, σ).

Proof : Let V be any open set in (X, τ). Since (g ∘ f) is Δ^*-open, (g ∘ f)(V) is Δ^*-open in (Z, η). Since g is Δ^*-irresolute and injective, $f^{-1}[(g \circ f)(V)] = f(V)$ is Δ^*-open in (Y, σ). Hence f is Δ^*-open in (Y, σ).

6.3 Δ^*-Homeomorphism

Definition 6.3.1 A bijection map f : (X, τ) → (Y, σ) is called a **Δ^*-homeomorphism** if f is both Δ^*-continuous and Δ^*-open map.

Proposition 6.3.2 Every Δ^*-homeomorphism is a gδ-homeomorphism but not conversely.

Proof : Let f : (X, τ) → (Y, σ) be a Δ^*-homeomorphism. Then f is bijective, Δ^*-continuous and Δ^*-open map. Let V be a closed set in (Y, σ). Then $f^{-1}(V)$ is Δ^*-closed in (X, τ). Since every Δ^*-closed set is gδ-closed (Proposition 2.2.8), $f^{-1}(V)$ is gδ-closed in (X, τ). This implies that f is gδ-continuous. Let U be an open set in (X, τ). Then f(U) is Δ^*-open set and hence it is gδ-open in (Y, σ). Thus f is a gδ-open map. Hence f is a gδ-homeomorphism.

Counter Example 6.3.3 Let X = {a ,b, c} = Y with τ = {ϕ , X, {a}} and σ = {ϕ , Y, {a}, {a, b}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is gδ-homeomorphism but not Δ^*-homeomorphism as f is not a Δ^*-continuous function since for the closed set {c} in (Y, σ), f^{-1}\{c\} = {c} is not Δ^*-closed in (X, τ).

Remark 6.3.4 Every Δ^*-homeomorphism is a $\delta g^{\ddagger}$-homeomorphism but not conversely.

Counter Example 6.3.5 Let X = {a ,b, c} = Y with τ = {ϕ , X, {a}} and σ = {ϕ , Y, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = b, f(b) = c, f(c) = a. Then f is $\delta g^{\ddagger}$-continuous but not Δ^*-continuous since for the closed set {c} in (Y, σ), f^{-1}\{c\} = {b} is not Δ^*-closed in (X, τ).

Remark 6.3.6 a) Every δ-homeomorphism is a Δ^*-homeomorphism but not Conversely.

b) Every δg^*-homeomorphism is a Δ^*-homeomorphism but not conversely.

Counter Example 6.3.7 a) Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\phi, Y, \{b\}, \{c\}, \{b, c\}\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a Δ^*-homeomorphism but not a δ-homeomorphism as f is not δ-continuous, since for the closed set $\{b\}$ in (Y, σ), $f^{-1}\{b\} = \{b\}$ is not δ-closed in (X, τ).

b) Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\phi, Y, \{b\}, \{c\}, \{b, c\}\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a Δ^*-homeomorphism but not a δg^*-homeomorphism as f is not δg-continuous, since for the closed set $\{b\}$ in (Y, σ), $f^{-1}\{b\} = \{b\}$ is not δg^*-closed in (X, τ).

Remark 6.3.8 The Δ^*-homeomorphism map and $\delta\hat{g}$-homeomorphism map are independent as seen from the following examples.

Counter Example 6.3.9 Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a $\delta\hat{g}$-homeomorphism but not a Δ^*-homeomorphism as f is not Δ^*-continuous, since for the closed set $\{c\}$ in (Y, σ), $f^{-1}\{c\} = \{c\}$ is not Δ^*- closed in (X, τ).

Counter Example 6.3.10 Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$, and $\sigma = \{\phi, Y, \{a\}, \{a, b\}, \{a, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a Δ^*-homeomorphism but not a $\delta\hat{g}$-homeomorphism map since for the closed set $\{b\}$ in (Y, σ), $f^{-1}\{b\} = \{b\}$ is not $\delta\hat{g}$-closed in (X, τ).

Remark 6.3.11 The Δ^*-homeomorphism map and αg-homeomorphism map are independent as seen from the following examples.

Counter Example 6.3.12 Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a αg-homeomorphism but not a Δ^*-homeomorphism as f is not Δ^*-continuous, since for the closed set $\{c\}$ in (Y, σ), $f^{-1}\{c\} = \{c\}$ is not Δ^*- closed in (X, τ).

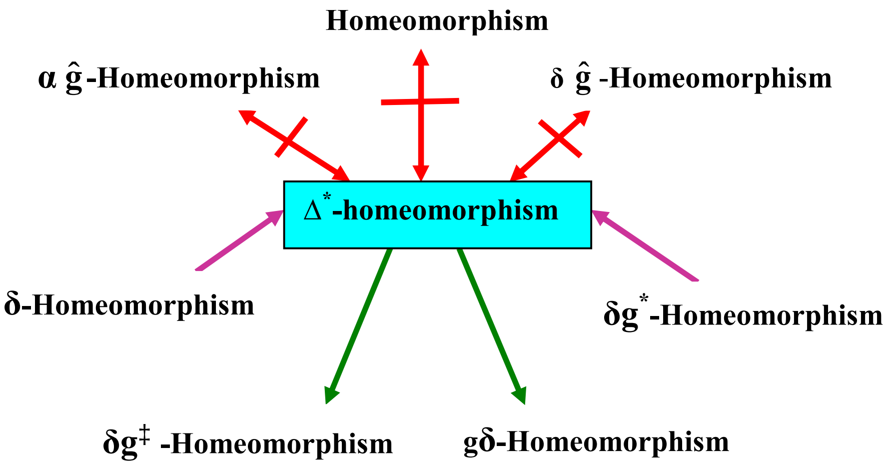
Counter Example 6.3.13 Let X = {a, b, c} = Y with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{a, b\}, \{a, c\}\}$. Let f : (X, τ) → (Y, σ) be the identity map. Then f is a Δ^*-homeomorphism but not a αg-homeomorphism as f is not a αg-open map, since the image of the open set {b} in (X, τ) is not αg-open in (Y, σ).

Remark 6.3.14 The Δ^*-homeomorphism and homeomorphism are independent as seen from the following examples.

Counter Example 6.3.15 Let X = {a, b, c} = Y with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}\}$. Let f : (X, τ) → (Y, σ) be the identity map. Then f is a Δ^*-homeomorphism but not a homeomorphism as f is not continuous, since for the closed set {a, c} in (Y, σ), $f^{-1}\{a, c\} = \{a, c\}$ is not closed in (X, τ).

Counter Example 6.3.16 Let X = {a, b, c, d} = Y with $\tau = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Let f : (X, τ) → (Y, σ) be the identity map. Then f is a homeomorphism but not a Δ^*-homeomorphism as f is not a Δ^*-open map, since the image of the open set {a, c, d} in (X, τ), f({a, c, d}) = {a, c, d} is not Δ^*-open in (Y, σ).

Remark 6.3.17 The above results are depicted by the following diagram.



Theorem 6.3.18 Every Δ^{*}-homeomorphism from a ${}_{\Delta^{*}}T_{\delta}$-space into another ${}_{\Delta^{*}}T_{\delta}$-space is a homeomorphism.

Proof : Let $f : (X, \tau) \to (Y, \sigma)$ be a Δ^{*}-homeomorphism. Then f is bijective, Δ^{*}-open and Δ^{*}-continuous. Let U be an open set in (X, τ). Since f is Δ^{*}-open and (Y, σ) is a ${}_{\Delta^{*}}T_{\delta}$-space, f(U) is δ-open which implies that f(U) is open in (Y, σ). Therefore f is an open map. Let V be a closed set in (Y, σ). Since f is Δ^{*}-continuous and (X, τ) is a ${}_{\Delta^{*}}T_{\delta}$-space, $f^{-1}(V)$ is δ-closed in (X, τ) which implies that $f^{-1}(V)$ is closed in (X, τ). Therefore f is continuous. Hence f is a homeomorphism.

Proposition 6.3.19 For the bijective Δ^{*}-continuous map $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent.

a) f is a Δ^{*}-open map.

b) f is a Δ^{*}-homeomorphism.

c) f is a Δ^{*}-closed map.

Proof : The proof follows from the definition 6.3.1 and Proposition 6.2.16

Theorem 6.3.20 If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are Δ^{*}-homeomorphisms where (Y, σ) is a ${}_{\Delta^{*}}T_{\delta}$-space then the composition map $(g \circ f) : (X, \tau) \to (Z, \eta)$ is a Δ^{*}-homeomorphism.

Proof : Let U be any open set in (X, τ). Since f is Δ^{*}-open, f(U) is Δ^{*}-open set in (Y, σ). Since (Y, σ) is a ${}_{\Delta^{*}}T_{\delta}$-space, f(U) is δ-open in (Y, σ). Since every δ-open is open, f(U) is open in (Y, σ). Since g is Δ^{*}-open g[f(U)] is Δ^{*}-open in (Z, η). Hence $(g \circ f)$ is a Δ^{*}-open map.

Let V be a closed set in (Z, η). Since g is Δ^{*}-continuous, $g^{-1}(V)$ is Δ^{*}-closed in (Y, σ). Since (Y, σ) is a ${}_{\Delta^{*}}T_{\delta}$-space, $g^{-1}(V)$ is Δ^{*}-closed and hence it is closed in (Y, σ). Since f is Δ^{*}-continuous, $f^{-1}[g^{-1}(V)] = (g \circ f)^{-1}(V)$ is Δ^{*}-closed in (X, τ). Therefore $(g \circ f)$ is Δ^{*}-continuous. Hence $(g \circ f)$ is a Δ^{*}-homeomorphism.

Definition 6.3.21 A bijection map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be a **$\Delta^*\mathcal{C}$-homeomorphism** if both f and f^{-1} are Δ^*-irresolute.

The family of all $\Delta^*\mathcal{C}$-homeomorphism of a topological space (X, τ) onto itself is denoted by $\Delta^*\mathcal{C}h(X, \tau)$.

Remark 6.3.22 The $\Delta^*\mathcal{C}$-homeomorphism and $\delta g^*\mathcal{C}$-homeomorphism are independent as seen from the following examples.

Counter example 6.3.23 Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{a, b\}, \{a, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a $\delta g^*\mathcal{C}$-homeomorphism since $\delta g^*\tau^\# = \delta g^*\sigma^\#$ but not a $\Delta^*\mathcal{C}$-homeomorphism as f is not a Δ^*-irresolute since for the Δ^*-open set $\{a, c\}$ in (Y, σ), $f^{-1}\{a, c\} = \{a, c\}$ is not Δ^*-open in (X, τ).

Counter example 6.3.24 Let $X = \{a, b, c, d\} = Y$ with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\phi, Y, \{a, b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a $\Delta^*\mathcal{C}$-homeomorphism since $\Delta^*\tau^\# = \Delta^*\sigma^\#$ but not a $\delta g^*\mathcal{C}$-homeomorphism as f is not a δg^*-irresolute since for the δg^*-closed set $\{d\}$ in (Y, σ), $f^{-1}\{d\} = \{d\}$ is not δg^*-closed in (X, τ).

Proposition 6.3.25 The composition of two $\Delta^*\mathcal{C}$-homeomorphisms is a $\Delta^*\mathcal{C}$-homeomorphism.

Proof : Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be $\Delta^*\mathcal{C}$-homeomorphisms. Let V be a Δ^*-open set in (Z, η). Since g is Δ^*-irresolute, $g^{-1}(V)$ is Δ^*-open in (Y, σ). Since f is Δ^*-irresolute and bijective, $f^{-1}[g^{-1}(V)]$ is Δ^*-open in (X, τ). That is $(g \circ f)^{-1}(V) = f^{-1}[g^{-1}(V)]$ is Δ^*-open in (X, τ). This implies that $(g \circ f)$ is Δ^*-irresolute. Let G be Δ^*-open in (X, τ). Since f^{-1} is Δ^*-irresolute, $[f^{-1}]^{-1}(G)$ is Δ^*-open in (Y, σ). That is $f(G)$ is Δ^*-open in (Y, σ). Since g^{-1} is Δ^*-irresolute, $[g^{-1}]^{-1}(f(G))$ is Δ^*-open in (Z, η). That is $g[f(G)]$ is Δ^*-open in (Z, η). Therefore $(g \circ f)(G)$ is Δ^*-open in (Z, η). This implies that $[(g \circ f)^{-1}]^{-1}(G)$ is Δ^*-open in (Z, η).

Thus the mapping $(g \circ f)^{-1} : (Z, \eta) \to (X, \tau)$ is Δ^*-irresolute. Hence the composition map $(g \circ f)$ is a $\Delta^*\mathcal{C}$–homeomorphism.

Theorem 6.3.26 The set $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X,\tau)$ is a group under the composition of maps.

Proof : Let us define a binary operation □ : $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau) \times \Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau) \to \Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau)$ by (f □ g) = $(g \circ f)$ for every f, g □ $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau)$ and $\circ$ is the usual operation of composition of maps. Then by the Proposition 6.3.25, $(g \circ f)$ □ $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau)$. We know that the composition of maps is associative and the identity map $I : (X, \tau) \to (X, \tau)$ belongs to $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau)$ serves as the identity element. If f □ $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau)$ then f^{-1} □ $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau)$ such that $(f \circ f^{-1}) = f^{-1} \circ f = I$. So the inverse exists for each element of $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau)$. Therefore $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X,\tau)$ is a group under the operation of composition of maps.

Theorem 6.3.27 Let $f : (X, \tau) \to (Y, \sigma)$ be a $\Delta^*\mathcal{C}$-homeomorphism. Then f induces an isomorphism from the group $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau)$ onto the $\Delta^*\mathcal{C}\text{-}\mathcal{h}(Y, \sigma)$.

Proof : Using the map f, let us define a map $\theta_f : \Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau) \to \Delta^*\mathcal{C}\text{-}\mathcal{h}(Y, \sigma)$ by $\theta_f(\mathcal{h}) = f \circ \mathcal{h} \circ f^{-1}$ for every $\mathcal{h}$ □ $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau)$. Then θ_f is a bijective map. Further for every $\mathcal{h}_1, \mathcal{h}_2$ □ $\Delta^*\mathcal{C}\text{-}\mathcal{h}(X, \tau)$, $\theta_f(\mathcal{h}_1, \mathcal{h}_2) = f \circ(\mathcal{h}_1 \circ \mathcal{h}_2)\circ f^{-1} = (f \circ \mathcal{h}_1 \circ f^{-1}) \circ (f \circ \mathcal{h}_2 \circ f^{-1}) = \theta_f(\mathcal{h}_1) \circ \theta_f(\mathcal{h}_2)$. Therefore θ_f is a homomorphism. Hence θ_f is an isomorphism induced by f.

Theorem 6.3.28 The $\Delta^*\mathcal{C}$-homeomorphism is an equivalence relation in the collection of all topological spaces.

Proof : The reflexivity and symmetric relations are immediate and the transitivity follows from the Proposition 6.3.25