Various Concepts Using J-Closed Sets

§ 3.1. Introduction

Topological structures on the assortment of information are reasonable mathematical models for mathematizing not just quantitative information yet in addition qualitative ones. The concept of g-openness assume a significant part in General Topology and they are presently the research topics of numerous topologists around the world. One of the most notable thoughts and furthermore a motivation source is the idea of J-open sets are presented. Dual operators like J-Closure of a subset and J-interior of a subset are established.

The intention of this chapter is to propose Neighbourhood using J-open sets namely J-Neighborhood of a point and J-Neighborhood of a subset in topological spaces. Also a few essential relations and fascinating characterizations of J-Neighborhood are discussed.Likewise,J-Limit point of a subset, J-Derived set,J-Frontier of a subset, J-Border of a subset, J-Exterior of a subset are introduced using the concept of J-closed sets here. Some exciting features of J-Closure, J-interior and J-Derived set, J-Saturated sets are obtained. Moreover, the interrelations between J-Border,J-Frontier,J-Exterior sets are analysed.

§ 3.2. J-Closure Operator

In this section, the notion of J-closure of a set is introduced and some of its properties are studied.

Definition 3.2.1. The **J-closure of D** (briefly **JCl(D)**) of a topological space (Y,ζ) is defined as follows. **JCl(D) =∩{F⊆ Y:D⊆F and F∈JC(Y,ζ)}**

Proposition 3.2.2. Let D be any subset of (Y,ζ).If D is J-closed in (Y,ζ), then JCl(D)=D.

Proof Let D be J-closed in (Y,ζ). By definition, JCl(D)=∩{F⊆ Y:D⊆ F and F∈JC(Y,ζ)}.Since D is J -closed, the smallest F in the above collection is D itself and hence JCl(D)=D.

Counter Example 3.2.3. Let Y ={p,q,r,s},ζ={ϕ, Y, {p},{p,q}}. Here JCl(Y, ζ) = P(Y) −{ {p},{q},{p,q}}.Let D = {p,q}.Then JCl(D) ={p,q}= D ≠ a J-closed set.

Remark 3.2.4. For a subset D of (Y,ζ), D⊆ JCl(D) ⊆ Cl(D).

Proposition 3.2.5. Let D and B be subsets of (Y, ζ). Then the following statements are true:

(a) JCl(∅) = ∅ and JCl(Y)=Y

(b) If D⊆ B, then JCl(D)⊆JCl(B)

(c) D⊆JCl(D)

(d) JCl(D)∪JCl(B) = JCl(D∪B)

(e) JCl(D∩B)⊆ JCl(D)∩JCl(B)

(f) JCl(JCl(D)) = JCl(D).

Proof (a), (b), (c) follow from **Definition 3.2.1.**

(d) Since D⊆ D∪B and B ⊆ D ∪ B. By (b), JCl(D)⊆JCl(D∪B) and JCl(B)⊆JCl(D∪B). Hence JCl(D)∪JCl(B)⊆JCl(D∪B).To prove the reverse inequality, let x∉JCl(D)∪JCl(B), then x∉JCl(D) and x∉JCl(B). Therefore, there exist J-closed sets U and V in Y such that D⊆ U and B ⊆ V and x ∉ U and x ∉ V. Hence we have D∪B⊆U∪V and x∉U∪V.By **Theorem 2.3.48.**, U∪V is J-closed and hence x∉JCl(D∪B).

(e) Since D∩B⊆D and D∩B⊆ B. By **(b)**, JCl(D ∩ B)⊆ JCl(D) and JCl(D ∩ B)⊆ JCl(B). Hence JCl(D ∩ B)⊆ JCl(D)∩JCl(B).

(f) Follows from the definition of J-closure.

The converse of the above Proposition 3.2.5.(e) is not true from the following Counter Example.

Counter Example 3.2.6. Let Y ={p,q,r},ζ={ϕ, Y, {p},{q},{p,q}}.Then JC(Y, ζ)={ϕ,Y, {r},{q,r},{p,r}}.Take D = {p} and B = {q},D∩B = ϕ, JCl(D) = {p,r}, JCl(B) = {q,r},JCl(D)∩JCl(B) = {r} but JCl(D∩ B) = ϕ.Hence JCl(D) ∩JCl(B)⊈JCl(D∩ B).

Result 3.2.7. From Proposition 3.2.5.(a),(b),(c) and (d),(f),J-closure operator is a Kuratowski's closure operator.

Theorem 3.2.8. For each y∈ Y, y∈JCl(D) if and only if U ∩ D ≠ ϕ for every J -open set U in (Y,ζ) containing y.

Proof(Necessity) Let y∈JCl(D). Suppose that there exists a J-open set U in (Y,ζ) containing y such that U ∩ D = ϕ. Hence D⊆Y − U is J-closed in (Y,ζ) which implies that JCl(D)⊆Y−U. Hence y∉JCl(D) which is a contradiction. Hence U ∩ D ≠ ϕ.

(Sufficiency) Let us assume that U ∩ D ≠ ϕ for every J-open set U in (Y,ζ) containing y. Suppose that y∉JCl(D). By definition of J-closure, there exists a J-closed set U in (Y,ζ) containing D such that y∉ U. Hence Y − U is J-open in (Y,ζ) containing y. Therefore (Y−U) ∩ D = ϕ, which is a contradiction. Hence y∈ JCl(D).

§ 3.3. J-Neighbourhood

Definition 3.3.1. A subset M of a topological space (Y,ζ) is said to be a **J-Neighbourhood of** x∈Y if there exists a J-open set D such that x ∈ D ⊆ M.The set of all J-Neighbourhoods of x is denoted by **JNr(x)**.

Example 3.3.2. Let Y={p,q,r,s}, ζ={ϕ, Y, {p,q}}.Then JO(Y,ζ) = {ϕ, Y,{p},{q},{r},{s},{p,q}{q,r},{p,r},{p,s},{q,s},{p,q,r},{p,q,s}}.Here {q,r,s} is a J-Neighbourhood of q as q∈{q,r}⊆{q,r,s}.

Theorem 3.3.3. A J-open set N is a J-Neighbourhood of each of its points.

Proof Let N be a J-open set and x ∈ N. Then x ∈ N ⊆ N satisfying the condition of N being a J-Neighbourhood.Since x is an arbitrary point of N, N is a J-Neighbourhood of each of its points.

Corollary 3.3.4. Every J-open set containing a point x belongs to JNr(x).

Remark 3.3.5. A J-Neighbourhood of some point in Y need not be a J-open set as observed from the upcoming example.

Example 3.3.6. Let Y={p,q,r,s}, ζ={ϕ,Y,{p},{q,r},{p,q,r}}.Then JO(Y,ζ) = {ϕ, Y,{p},{q},{r},{p,q}{q,r},{p,r},{p,q,r}}.A subset {p,q,s} is J-Neighbourhood of p as p∈{p,q}⊆{p,q,s}.But it is not a J-open set.

Definition 3.3.7. A subset M is said to be a **J-Neighbourhood of N** ⊆ Y if there exists a J-open set A such that N ⊆ A⊆ M.The set of all J-Neighbourhoods of N is denoted by **JNr(N)**.

Example 3.3.8. In the above **Example 3.3.2.**, {p,q,s} is J-Neighbourhood of {p} as {p}∈{p,q}⊆{p,q,s}.

Theorem 3.3.9. If M is a J-closed subset of a topological space(Y,ζ) and x ∈ Y – M, then there exists a J-Neighbourhood N of x such that N ∩M =∅.

Proof Let M be a J-closed subset of Y then Y – M is J-open in Y. By **Theorem 3.3.3.,** Y–M is a J-Neighbourhood of each of its points. Therefore there exists a J-open set N of x such that N⊆ Y – M which in turn implies that N ∩ M =∅.

Theorem 3.3.10. In a topological space (Y,ζ) with x ∈ Y, the following results are true.

(i) JNr(x)≠ ∅

(ii) If M ∈JNr(x), then x ∈ M

(iii) If M ∈JNr(x) and M⊆ N, then N ∈JNr(x)

(iv) If M ∈JNr(x) and N ∈JNr(x), then M ∩ N∈JNr(x)

(v) If M ∈JNr(x) then there exists a N ∈JNr(x) such that N⊆ M and N∈ JNr(y), for every y ∈ N.

Proof (i) Since Y itself is a J-open set by **Theorem 3.3.3.**,it is a J-Neighbourhood for every x ∈ Y. That is Y∈JNr(x), for all x ∈Y.Hence JNr(x)≠ ∅ for all x ∈ Y.

(ii) Follows from the **Definition 3.3.1.**

(iii) Let M ∈JNr(x) and M⊆ N. Since M is a J-Neighbourhood of x then there exists a J-open set A such that x∈ A ⊆M.Since M⊆ N,we get x∈ A ⊆ N.Hence N is a J-Neighbourhood of x.

(iv) Let M ∈JNr(x) and N ∈JNr(x) then there exists J-open sets A and B such that x∈ A ⊆ M and x∈ B ⊆ N. This implies x∈ A∩B ⊆ M∩N. Now in order to prove M∩N is a J-Neighbourhood of x, it is enough to prove that A∩B is J-open. Since finite intersection of J-open sets is J-open (by **Theorem 2.3.88**),A∩B is J-open and hence M ∩ N is a J-Neighbourhood of x. Therefore M ∩ N∈ JNr(x).

(v) Let M ∈JNr(x) then there exists a J-open set N such that x∈ N ⊆ M. Since N is J-open, it is a J-Neighbourhood of all its points (by **Theorem 3.3.3.**).Thus N∈ JNr(y), for every y ∈ N.

Lemma 3.3.11. Nr(x) ⊆ JNr(x).

Proof Let A ∈ Nr(x).Then ∃ B ∈ ζ such that x∈ B⊆A.Since every open set is a J-open set (by **Theorem 2.3.75.),**A∈JNr(x).

Lemma 3.3.12. JNr(x) satisfies

(i) M ∈ JNr(x) such that x ∈ M

(ii) N,M ∈ JNr(x) implies N ∩M ∈ JNr(x) then $\mathcal{B}$ = {∅} ∪ {G ⊆ Y / x ∈ G implies N ∈ JNr(x) such that x∈ N ⊆ G} forms a basis for the topological space (Y,ζ).

Proof Obvious.

§ 3.4. J-Derived Set

Definition 3.4.1. Let D ⊆Y and a point y ∈ Y is known as a **J-limit point** of D if every J-Neighbourhood of y intersects D in some point other than y itself.

Example 3.4.2. Let Y={p,q,r}, ζ={ϕ,Y,{p},{q},{p,q}}.Then JO(Y,ζ) ={ϕ,Y, {p},{q}, {p,q}}.Here r is the J-limit point of {p,r}.

Definition 3.4.3. The set of all J-limit points of D ⊆Y is called **J-Derived set** of D and is denoted by **JDr(D)**.

Theorem 3.4.4. Let A,B⊆Y.Then the following statements are valid in (Y,ζ).

- **(i)** **JDr**(∅) = ∅
- **(ii)** **JDr(A)⊆Dr(A)**
- **(iii)** **If A ⊆ B,then JDr(A)⊆JDr(B)**
- **(iv)** **JDr(A∪B) = JDr(A) ∪ JDr(B)**
- **(v)** **JDr(A∩B)⊆JDr(A)∩JDr(B)**
- **(vi)** **[JDr(JDr(A))]– A ⊆JDr(A)**
- **(vii)** **JDr(A∪JDr(A))⊆A∪JDr(A).**

Proof (i) Follows from the **Definition 3.4.1.**

(ii) Let x∈JDr(A).Then every J-Neighbourhood N of x is such that N∩(A−{x})≠ ∅ ----- (1).Consider N′ is a neighbourhood of x. By the above **Lemma 3.3.11.,** N′is J-Neighbourhood of x. Hence by (1), N′ ∩(A−{x})≠ ∅.Hence x ∈Dr(A).

(iii) Let x∈JDr(A).Then for every J-Neighbourhood N of x is such that N∩(A−{x})≠ ∅.Since A ⊆ B, N∩(B−{x})≠ ∅. Therefore x∈JDr(B).

(iv) Since A⊆A∪B,JDr(A)⊆JDr(A∪ B).Similarly B⊆A∪B,JDr(B)⊆JDr(A∪B).Therefore JDr(A)∪JDr(B)⊆JDr(A∪B).Suppose x∉ JDr(A)∪JDr(B).Then x∉JDr(A) or x∉JDr(B),that is x is neither a limit point of A nor of B. Therefore there exist J-Neighbourhoods N_1 and N_2,then N_1∩(A−{x}) = ∅ and N_2∩(B−{x}) = ∅.Here to prove N_1∩ N_2 is a J-Neighbourhood containing x. It is enough to prove A∩B is J-open. Since finite intersection of J-open sets is J-open (by **Theorem 2.3.88.**),A∩B is J-open. Hence N_1∩ N_2 is a J-Neighbourhood containing x is such that (N_1∩ N_2)∩((A∪B)-{x}) = ∅ implies x∉ JDr(A∪B) gives JDr(A∪B)⊆ JDr(A)∪JDr(B).

(v) Since A∩B⊆ A,B, the proof follows.

(vi) Let x∈JDr(JDr(A)) – A.Then N∩(JDr(A)−{x})≠ ∅ for each J-Neighbourhood N of x. Now let y∈ N ∩(JDr(A)-{x}) implies y∈N and y∈JDr(A).Here y∈JDr(A) gives N∩(A−{y}) ≠ ∅ for each J-Neighbourhood N of y. So that take z∈ N∩(A−{y}).Then z≠x as z∈A and x∉A.Therefore N∩(A−{x})≠ ∅ for each J-Neighbourhood N of x. Hence x∈JDr(A).

(vii) Let x∈ JDr(A∪JDr(A)). If x∈A,then the result is obvious. Suppose x∈ JDr(A∪JDr(A)) – A. Then N∩(A∪JDr(A)) −{x})≠ ∅ for each J-Neighbourhood N of x implies N∩(A−{x})≠ ∅ and N∩(JDr(A)−{x})≠ ∅. Now let y∈ N ∩ (JDr(A) − {x}) implies y∈N and y∈JDr(A). So N∩(A−{y})≠ ∅ for each J-Neighbourhood N of y. So that take z∈ N∩(A−{y}). Then z≠x as z∈A and x∉A.Therefore N∩(A−{x})≠ ∅ for each J-Neighbourhood N of x. Hence x∈JDr(A)and thus x∈ A∪JDr(A).

Theorem 3.4.5. Let A⊆ Y.If A is J-closed then JDr(A) ⊆A.

Proof Let x∈ JDr(A) then N ∩(A−{x})≠ ∅ whenever N is a J-Neighbourhood of x implies N ∩A≠ ∅ whenever N is a J-Neighbourhood of x ----- (1).Consider N′ ∩A where

N′ is J-open,then by **Theorem 3.3.3.**,every J-open is J-Neighbourhood of x and from (1),N′ ∩A≠ ∅.Therefore x∈JCl(A) = A,since A is J-closed.Hence the proof.

Corollary 3.4.6. JDr(A) ⊆ JCl(A).

Theorem 3.4.7. For any subset A of a topological space (Y,ζ),JCl(A) = A ∪ JDr(A).

Proof By the above **Corollary 3.4.6.**,JDr(A) ⊆ JCl(A),A∪ JDr(A)⊆ JCl(A).On the other hand,let x ∈ JCl(A).If x ∈ A,then the proof is complete.If x ∉ A,each J-open set U containing x intersects A at a point distinct from x ; So x ∈ JDr(A).Thus JDr(A) ⊆ A ∪ JDr(A) which completes the proof.

Lemma 3.4.8. JCl(A) – A ⊆JDr(A).

Proof Let x ∈JCl(A)– A.Then x∈ JCl(A) and x ∉A.For every J-open set N containing x intersects A and x ∉A.By **Theorem 3.3.3.**,for every J-Neighbourhood N containing x intersects A and x ∉A.Hence N∩(A−{x})≠ ∅ and x ∈ JDr(A).Hence the proof.

§ 3.5. J-Interior Operator

In this section, the notion of J-Interior of a set is introduced and some of its properties are studied.

Definition 3.5.1. Let D be a subset of Y.A point y∈ D is said to be **J-interior point** of D if D is a J-Neighbourhood of y. The set of all J-interior points of D is called the J-interior of D and is denoted by Jint(D).

Lemma 3.5.2. If D is a subset of Y, then Jint(D) = ∪{G:G ⊆ D and G∈JO((Y,ζ))}.

Proof Let D be a subset of Y. Let y ∈ Jint(D) ⟺ y is a J-interior point of D ⟺ D is a J-Neighbourhood of y ⟺ ∃ a J-open set G such that y ∈ G ⊆ D ⟺y∈∪{G:G ⊆ D and G∈JO((Y,ζ))}.Hence Jint (D) =∪{G ⊆Y:G ⊆ D and G∈JO((Y,ζ))}.

Lemma 3.5.3. If D is a subset of Y, then int(D) ⊆ Jint(D).

Proof Let D be a subset of Y. Let y ∈ int(D) ⇒ y∈∪{G:G ⊆ D and G∈ ζ ⇒ ∃ an open set G such that y∈ G⊆ D.Since every open set is a J-open set (by **Theorem 2.3.75.**) in Y ⇒ y∈ ∪{G:G ⊆ D and G∈JO((Y,ζ))}⇒ y∈ Jint(D).Hence int(D)⊆ Jint(D).

The converse of the above Lemma 3.5.3. is not true from the following counter example.

Counter Example 3.5.4. Let Y={p,q,r},ζ ={ϕ, Y, {p,q}}.Here JO((Y,ζ)) = P(Y).Let D={r},Jint(D) = {r},int(D) = ϕ.Hence Jint(D) ⊈ int(D).

Result 3.5.5. In general intD is an open set. But JintD need not be a J-open set. It can be proved by the following counter example.

Counter Example 3.5.6. Let Y={p,q,r,s},ζ ={ϕ, Y, {p}}.Here JO((Y,ζ))= P(Y) – {q,r,s}.Let D = {q,r,s},Jint(D) = {q,r,s} but it is not a J-open set.

Theorem 3.5.7. For any two subsets D and B of (Y,ζ), the following statements are true:

- **(a) Jint(Y) = Y and Jint(ϕ) = ϕ**
- **(b) Jint(D) ⊆ D**
- **(c) If B is any J-open set contained in D, then B ⊆ Jint(D)**
- **(d) If D⊆ B, then Jint(D)⊆ Jint(B).**

Proof (a) It is obvious. **(b)** Let y∈Jint(D).Then y is a J-interior point of D ⇒ D is a J-Neighbourhood of y ⇒ y∈ D. Hence Jint(D) ⊆ D. **(c)** Let B be any J-open set such that B ⊆ D. Let y∈ B. Since B is a J-open set contained in D, y is a J-interior point of D. That is y∈Jint(D). Hence B ⊆Jint(D). **(d)** Let D and B be subsets of Y such that D⊆ B. Let y∈Jint(D). Then y is a J-interior point of D and so D is a J-Neighbourhood of y.(ie) there exists a J-open set G such that y∈ G⊆ D.Now D⊆ B implies y∈ G⊆ B.Hence B is also J-Neighbourhood of y then y∈Jint(B). Hence Jint(D) ⊆ Jint(B).

Remark 3.5.8. For a subset D of (Y,ζ), int(D)⊆ Jint(D) ⊆ D.

Lemma 3.5.9. Jint(D) = D – JDr(Y –D) .

Proof Let x ∈ D – JDr(Y –D),then x∉ JDr(Y –D).Therefore ∃ a J-open set G containing x such that G ∩(Y –D) = ∅ implies x∈ G ⊆D and hence x ∈Jint(D).Now to prove Jint(D)⊆ D –JDr(Y –D).Let x ∈ Jint(D) then JintD is a J-Neighbourhood of x. Since JintD ∩(Y−D)= ∅,there is a J-Neighbourhood of x does not intersect Y –D implies x∉ JDr (Y –D). Therefore x ∈ D – JDr(Y –D). Hence Jint(D) = D – JDr(Y –D) .

Proposition 3.5.10. Let D be any subset of (Y,ζ). If D is J-open in (Y,ζ) then Jint(D) = D.

Proof Let D be J-open in (Y,ζ).We know that Jint(D) ⊆ D.Also D is a J-open set contained in D. From above **Theorem 3.5.7.(c),** D⊆ Jint(D). Hence Jint(D) = D.

The converse of Proposition 3.5.10. need not be true.

Counter Example 3.5.11. Same as in **Counter Example 3.5.6.** Here Jint(D) = D but D is not a J-open set.

Corollary 3.5.12. Jint(Jint(D)) =Jint(D).

Theorem 3.5.13. If D and B are subsets of Y, then Jint(D) ∪Jint(B) ⊆Jint(D∪B).

Proof Since D⊆D∪B and B ⊆D∪B, by **Theorem 3.5.7.(d)**, Jint(D) ⊆Jint(D∪B) and Jint(B) ⊆Jint(D∪B). Hence Jint(D) ∪Jint(B) ⊆Jint(D∪B).

The converse of Theorem 3.5.13. need not be true as seen from the following Counter example.

Counter Example 3.5.14. Let Y = {p,q,r,s}, ζ = {Y,ϕ, {r}, {p,q}, {p,q,r}. Here JO(Y,ζ) = { Y,ϕ,{p},{q},{r},{p,r},{q,r},{p,q}, {p,q,r}}.Let D = {p,q,r}, B = {p,q,s} and D∪B = {p,q,r,s } =Y then Jint{D} = {p,q,r}, Jint{B} = {p,q} and Jint{D∪B} = Y. Hence Jint(D∪B) = Y ⊈Jint{D} ∪Jint{B} = {p,q,r}.

Theorem 3.5.15. If D and B are subsets of Y, then Jint(D∩B) = Jint(D) ∩Jint(B).

Proof Since D∩B⊆D and D∩B⊆B, by **Theorem 3.5.7.(d),** Jint(D∩B)⊆Jint(D) and Jint(D∩B)⊆Jint(B). Hence Jint(D∩B) ⊆ Jint(D)∩Jint(B).In other way,to prove Jint(D) ∩Jint(B) ⊆ Jint(D∩B). Let y ∈ Jint(D) and y ∈ Jint(B).Then y is a J-interior point of D and y is a J-interior point of B. That implies D is a J-Neighbourhood of y and B is a J-Neighbourhood of y. Therefore ∃ a J-open set G such that y ∈ G⊆ D and ∃ a J-open set M such that y∈ M ⊆ D.By **Theorem 2.3.88.**, ∃ a J-open set such that y ∈ G∩M⊆D∩B. Hence y ∈ Jint(D∩B).

Note 3.5.16. From Theorem 3.5.7. (a),(b) and Corollary 3.5.12.,Theorem 3.5.15., J-interior operator is a Kuratowski's interior operator.

Theorem 3.5.17. Let D be any subset of (Y,ζ), then

(a) $(Jint(D))^c = JCl(D^c)$.

(b) $(JCl(D))^c = Jint(D^c)$.

Proof (a) Let $x \in (Jint(D))^c$ .Then $x \notin Jint(D)$.That is every J-open set U containing x is such that $U \nsubseteq D$.That is every J-open set U containing x is such that $U \cap D^c \neq \emptyset$.By **Theorem 3.2.8.**, $x \in JCl(D^c)$ and therefore $(Jint(D))^c \subseteq JCl(D^c)$.Conversely, let $x \in JCl(D^c)$.Then by **Theorem 3.2.8.**, every J-open set U containing x is such that $U \cap D^c \neq \emptyset$. Then $x \notin Jint(D)$. Hence $x \in (Jint(D))^c$ and $JCl(D^c) \subseteq (Jint(D))^c$. Thus $(Jint(D))^c = JCl(D^c)$. **(b)** Follows by replacing int by Cl and Cl by int in **(a)**.

Remark 3.5.18. In other notation, the above Theorem 3.5.17. can be stated as follows :

(a) Y−Jint(D) = JCl(Y –D).

(b) Y−JCl(D) = Jint(Y –D).

§ 3.6. J-Saturated Set

Definition 3.6.1. A subset D of a topological space (Y,ζ) is said to be **J-saturated set** if $JCl(\{x\}) \subseteq D$ for each $x \in D$. The set of all J-saturated sets in (Y,ζ) is denoted by **JSr(Y)**.

Example 3.6.2. Consider Y={m,n,o},ζ ={ϕ, Y,{m},{n},{m,n}}.Then JC(Y,ζ) = {ϕ, Y,{n,o},{m,o},{o}}. Let D = {n,o}.Then $JCl(\{x\}) \subseteq D$ for each $x \in D$, it is a J-saturated set.

Theorem 3.6.3. Every J-closed set is a J-saturated set but not conversely.

Proof Let D be a J-closed set and $x \in D$. Then $\{x\} \subseteq D$.Take J-closure on both sides of $\{x\} \subseteq D$, we get $JCl(\{x\}) \subseteq JCl(D) = D$, as D is J-closed. Therefore D is J-saturated.

Counter Example 3.6.4. Let Y={p,q,r,s}, ζ={ϕ, Y,{p},{q},{p,q},{p,q,s},{p,q,r}}.Then JC(Y,ζ) = {ϕ, Y,{r},{s},{q,r},{p,r},{p,s},{r,s},{q,s},{p,q,r},{p,r,s},{p,q,s},{q,r,s}}.Let D = {p,q}.Then $JCl(\{x\}) \subseteq D$ for each $x \in D$,it is J-saturated but D is not J-closed.

§ 3.7. J-Frontier

Definition 3.7.1. Let $D \subseteq Y$. A subset D of (Y,ζ) is known as the **J-Frontier of D** is defined as JCl(D)−Jint(D) and is denoted by **JFr(D)**.

Example 3.7.2. In the above **Example 3.3.6.**, JC(Y,ζ) = {ϕ, Y, {s},{p,s},{r,s},{q,s},{p,r,s},{p,q,s}{q,r,s}}.Let D = {r},JCl(D) = {r,s}.Here Jint(D) = {r}.Therefore JFr(D) = JCl(D) –Jint(D) = {r,s} –{r} = {s}.

Proposition 3.7.3. Let D ⊆ Y. Then the upcoming results are perfect.

- **(a)** **JFr(D) ⊆Fr(D)**
- **(b)** **JCl(D) = Jint(D) ∪ JFr(D)**
- **(c)** **Jint(D) ∩ JFr(D) = ∅**
- **(d)** **If D is a J-open set then JFr(D) ⊆ JDr(D)**
- **(e)** **JFr(D) = JCl(D) ∩ JCl(Y− D)**
- **(f)** **JFr(D) is J-closed**
- **(g)** **JFr(D) = JFr(Y−D)**
- **(h)** **JFr(JFr(D)) ⊆ JFr(D)**
- **(i)** **JFr(Jint(D)) ⊆ JFr(D)**
- **(j)** **JFr(JCl(D)) ⊆ JFr(D)**
- **(k)** **Jint(D) = D– JFr(D).**

Proof

(a) Since JCl(D) ⊆ Cl(D) (by **Proposition 2.3.2.**) and int(D) ⊆ Jint(D) (by **Remark 3.5.8.**).It gives JCl(D) – Jint(D) ⊆ Cl(D) –int(D) implies JFr(D) ⊆ Fr(D).

(b) RHS = Jint(D) ∪ JFr(D) = Jint(D) ∪ [JCl(D)−Jint(D)] =Jint(D) ∪ [JCl(D)∩ (Y −Jint(D))] = (Jint(D) ∪ JCl(D)) ∩(Jint(D) ∪ (Y −Jint(D))) = JCl(D) ∩ Y = JCl(D) [By Jint(D) ⊆ D ⊆ JCl(D)] =LHS.

(c) Jint(D) ∩ JFr(D) = Jint(D) ∩[JCl(D) –Jint(D)] = Jint(D) ∩[JCl(D)∩(Y −Jint(D))] = Jint(D) ∩(Y −Jint(D)) ∩ JCl(D) = JCl(D) ∩ ∅ = ∅.

(d) Given D is J-open implies that Jint(D) = D.JFr(D) = JCl(D) – Jint(D) = JCl(D) – D ⊆ JDr(D).(By **Lemma 3.4.8.**)

(e) RHS = JCl(D) ∩ JCl(Y− D) =JCl(D) − Jint(D) = JFr(D) = LHS.

(f) JCl(JFr(D)) = JCl(JCl(D) –Jint(D)) = JCl(JCl(D)∩JCl(Y−D)) ⊆JCl(JCl(D)) ∩JCl(JCl(Y−D)) =JCl(D) ∩JCl(Y−D) (by **Proposition 3.2.5. (f)**) =JCl(D) −Jint(D) =JFr(D).Hence JFr(D) is J-closed.

(g) RHS = JFr(Y−D) = JCl(Y −D) –Jint(Y−D) = (Y –Jint(D)) – (Y – JCl(D)) = (Y –Jint(D)) ∩JCl(D) = JCl(D) –Jint(D) = JFr(D) = LHS.

(h) JFr(JFr(D)) = JCl(JFr(D)) ∩JCl(Y−JFr(D))⊆ JCl(JFr(D)) = JFr(D) by **(e)** and **(f).**

(i) JFr(Jint(D)) = JCl(Jint(D))–Jint(Jint(D)) = JCl(Jint(D))–Jint(D)⊆ JCl (D)–Jint(D) = JFr(D) (Since JCl(Jint(D))⊆JCl (D)).

(j) JFr(JCl(D)) = JCl(JCl(D)) –Jint(JCl(D))⊆JCl (D) –Jint (D) =JFr(D) (Since Jint(JCl (D))⊇Jint (D)).

Proposition 3.7.4. Let A⊆B and Jint(B) = ∅ then JFr(A)⊆JFr(B).

Proof Let A⊆B and Jint(B) = ∅.Let x ∈JFr(A) = JCl(A) –Jint(B) as A⊆B and since Jint(B) = ∅,we get x ∈ JCl(A)⊆JCl(B) = JCl(B)− Jint(B) = JFr(B).Hence x∈JFr(B).

§ 3.8. J-Border

Definition 3.8.1. Let D be a subset of a topological space (Y,ζ). The **J-Border of D** is defined as D −Jint(D) and is denoted by **JBr(D)**.

Example 3.8.2. In the above **Example 3.3.6.**, let D = {q,r,s},then Jint(D) = {q,r}.Therefore JBr(D) = {s}.

Theorem 3.8.3. Let A be a subset of a topological space (Y,ζ).Then the following results hold:

(a) JBr(A) ⊆Br(A)

(b) JBr(∅) = ∅

(c) JBr(Y) = ∅

(d) JBr(A) ⊆ A

(e) A = Jint(A) ∪JBr(A)

(f) If A is J-open, then JBr(A) = ∅

(g) Jint(A) ∩JBr(A) = ∅

(h) JBr(Jint(A)) = ∅

(i) Jint(JBr(A)) = ∅

(j) JBr(JBr(A)) = JBr(A)

(k) JBr(A) = A ∩ JCl(Y−A)

(l) JBr(A) = JDr(Y –A)

(m)Jint(A) = A –JBr(A)

(n) JBr(A) ⊆ JFr(A).

Proof

(a) JBr(A) = A – JintA ⊆ A –int A =Br(A) (By **Lemma 3.5.3.**).

(b) It is a trivial one.

(c) By definition JBr(Y) = Y−Jint(Y) = Y – Y = ∅.

(d) It follows from the **Definition of Border of A**.

(e) RHS = Jint(A) ∪JBr(A) =Jint(A) ∪[A−Jint(A)] = Jint(A) ∪ (A∩(Y−JintA)) = (Jint(A) ∪ A) ∩ (Jint(A) ∪ (Y−JintA)) = A∩ Y = A=LHS.

(f) Let A be J-open.Then Jint(A) = A.By definition of JBr(A) = A –Jint(A) = A−A = ∅.

(g) LHS = Jint(A) ∩JBr(A) = Jint(A) ∩[A−Jint(A)] = Jint(A) ∩ [A∩(Y –Jint A)] =∅ =RHS.

(h) JBr(Jint(A)) = Jint(A) – Jint(Jint(A)) = Jint(A) – Jint(A) = ∅.

(i) Let x ∈ Jint(JBr(A)).Then x ∈JBr(A) .In other way JBr(A) ⊆ A by **(d)**.Hence x ∈Jint(JBr(A)) ⊆ Jint(A).This implies that x ∈ Jint(A) ∩JBr(A) which contradicts (g).Therefore Jint(JBr(A)) = ∅.

(j) LHS = JBr(JBr(A)) = JBr(A −Jint(A)) = A –Jint(Jint(A)) = A –Jint(A) =JBr(A) = RHS.

(k) JBr(A) = A – Jint(A) = A – (Y – JCl(Y –A)) = A ∩ JCl(Y−A).

(l) JBr(A) = A –Jint(A) = A –(A –JDr(Y−A)) (by **Lemma 3.5.9.**) =JDr(Y –A).

(m) RHS = A –JBr(A) = A – (A –Jint(A)) = Jint(A) = LHS.

(n) Direct Proof.

Remark 3.8.4. The converse of the above Theorem 3.8.3. (f) is not true as seen from the following Counter Example.

Counter Example 3.8.5. In **Counter Example 3.5.6.**, JBr(D) = ∅.But D is not J-open.

§ 3.9. J-Exterior

Definition 3.9.1. Let D be a subset of a topological space (Y,ζ). The **J-Exterior** of D is defined as Y−JCl(D) and is denoted by **JEr(D).**

Example 3.9.2. In the above **Example 3.7.2.**,take D = {p}.Then JCl(D) = {p,s}.Therefore JEr(D) = Y−JCl(D) = Y−{p,s} = {q,r}.

Theorem 3.9.3. Let A be a subset of a topological space (Y,ζ). Then the following results hold:

(a) JEr(A) ⊆Er(A)
(b) JEr(Y) = ∅
(c) JEr(∅) = Y
(d) JEr(A) = Jint(Y –A) = Y –JCl(A)
(e) If A ⊆B then JEr(A) ⊇ JEr(B)
(f) JEr(A∪B) = JEr(A) ∩JEr(B)
(g) JEr(A∩B) ⊇JEr(A) ∩JEr(B)
(h) JEr(JEr(A)) = Jint(JCl(A))
(i) JEr(A) = JEr(Y – JEr(A))
(j) Y = Jint(A) ∪ JEr(A) ∪ JFr(A)
(k) Jint(A) ⊆ JEr(JEr(A))
(l) JEr(A) ∪ JEr(B) ⊆JEr(A ∩ B).

Proof

(a) JEr(A) = Y –JCl(A) ⊆Y–Cl(A) = Er(A).(By **Proposition 2.3.2.**).

(b) JEr(Y) = Y –JCl(Y) = Jint(Y –Y) =Jint**(∅) = ∅.**

(c) JEr(∅) = Y – JCl(∅) = Y.

(d) From this JEr(A) = Y – JCl(A) = Jint(Y – A)(by **Remark 3.5.18(b)),**we got the proof.

(e) JEr(A) = Y− JCl(A) = Jint(Y−A) ⊇ Jint(Y−B) as A ⊆B.Hence JEr(A) ⊇ JEr(B).

(f) LHS = JEr(A∪B) = Y –JCl(A ∪ B) = Y – (JCl(A) ∪ JCl(B))(by **Proposition 3.2.5. (d)**) = (Y – JCl(A)) ∩ (Y – JCl(B)) = JEr(A) ∩ JEr(B) = RHS.

(g) Since A ∩ B ⊆ A and A ∩ B ⊆ B.By **(e),** JEr(A ∩ B) ⊇ JEr(A) and JEr(A ∩ B) ⊇ JEr(B).Hence JEr(A ∩ B) ⊇ JEr(A) ∩ JEr(B).

(h) JEr(JEr(A)) =JEr(Y –JCl(A)) = Jint(Y –(Y –JCl(A))) (by **(d)**) =Jint(JCl(A)).

(i) JEr(Y – JEr(A)) = JEr(Y –(Y−JCl(A))) = JEr(JCl(A)) = (Y –JCl(JCl(A))) (by **(d)**)= Y− JCl(A) (by **Proposition 3.2.5.(f)**)= JEr(A).

(j) Jint(A) ∪ JEr(A) ∪ JFr(A) = (Jint(A) ∪ JFr(A)) ∪JEr(A) = JCl(A) ∪ (Y −JCl(A)) = Y.(By **Proposition 3.7.3.(b)**).

(k) Jint(A) ⊆ Jint(JCl(A)) = JEr(JEr(A)),by **(h)**.

(l) JEr(A) ∪ JEr(B) = Jint(Y−A) ∪ Jint(Y−B) ⊆ Jint((Y−A)∪ (Y−B)) (by **Theorem 3.5.13.**) = Jint(Y −(A∩B)) = JEr(A∩B), by **(d)**.

In Latif (2014), Theorem 1.13.(2) states that Ext_{δ}(A) is δ-open but in J-closed sets, the result does not hold good.

Result 3.9.4. JEr(A) need not be J-open.

Counter Example 3.9.5. Let Y={p,q,r}, ζ={ϕ, Y,{p}}.Take A = {p}.Then JCl(A) = {p}.Therefore JEr(A) = Y −JCl(A) = {q,r} which is not J-open in (Y,ζ).