On Separation Axioms Through Gradation Of Openness

By

Revathi N.

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN - DEEMED UNIVERSITY, COIMBATORE - 641 043

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

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Certified as bonafide research work

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Introduction

INTRODUCTION

"THE ESSENCE OF MATHEMATICS IS ITS FREEDOM"

– George Cantor.

In 1969, **Chang** introduced the concept of fuzzy topology on a set X. In the definition of Chang fuzzy topology, fuzziness in the concept of openness of fuzzy subset is absent. Keeping this in mind **Hazra, Samanta** and **Chattopadhya** introduced the concept of gradation of openness of fuzzy subsets of X and gave a new definition of fuzzy topology on X.

The aim of this thesis is to study the concept of separation axioms in this new set-up as introduced by **R. Srivastava** [16]. Our study is based on the following articles.

(1) "Fuzzy topology redefined" by **Hazra, Samanta, Chattopadhyay** [9]

(2) "On separation axioms in a newly defined fuzzy topology" by **R. Srivasata** [16]

This thesis is split into three chapters. In the first chapter, fundamental concepts on fuzzy topological spaces, separation axioms in a fuzzy topological space are introduced.

This chapter deals with the results established mainly in the following articles.

(1) "Fuzzy topological spaces" by **Chang** [2]

(2) "Fuzzy topological spaces and fuzzy compactness" by **Lowen** [11]

(3) "Fuzzy Hausdorff topological spaces" **R. Srivastava, S.N. Lal** and **A.K. Srivastava** [13]

The second chapter is devoted to the study of fuzzy topological spaces, subspaces of fuzzy topological spaces and fuzzy continuous mappings through the concept of gradation of openness.

The results presented in this chapter are contained in the article "Fuzzy topology redefined" [9].

By a gradation of openness τ on a set X, we mean a map $\tau : I^X \to I$ satisfying the following conditions:

(i) $\tau(o) = \tau(1) = 1$

(ii) $\tau(\mu_i) > 0,\ i = 1,2 \Rightarrow \tau(\mu_1 \wedge \mu_2) > 0$

(iii) $\tau(\mu_i) > 0,\ i \in \Delta \Rightarrow \tau(\underset{i\in\Delta}{\vee} \mu_i) > 0$

With every gradation τ, a Chang fuzzy topology δ is associated as follows:

$$\delta = \{\mu \,/\, \tau(\mu) > 0\}.$$

The concept of gradation of closedness is also introduced. A simple relation between these two concepts is established. The closure of a fuzzy set is obtained in terms of gradation of closedness. It is proved that the intersection of a finite family of gradation of openness is again a gradation of openness. It is shown that arbitrary intersection of gradation of openness need not be a gradation of openness by constructing an interesting example.

With every Chang fuzzy topology δ a collection of gradation of openness τ_r is constructed.

Let (X, τ) be a fuzzy topological space and $Y \subset X$. The mapping $\tau_Y : I^Y \to I$ defined by

$\tau_Y(\mu) = \vee\} \tau(\lambda) / \lambda \in I^X; \lambda \mid y = \mu\}$ is a gradation of openness on Y.

The fuzzy topological space (Y, τ_Y) is called a subspace of the fuzzy topological space (X, τ) and τ_Y is called the induced gradation of openness on Y from (X, τ).

The author has obtained a relation between the closure of a fuzzy set in (X, τ) and its closure in the subspace (Y, τ_Y).

Let (X, τ) and (Y, τ^1) be two fts and f be a function from X into Y. The map f is called

(i) a **gradation preserving** (gp-) map, if

$\tau^1(\mu) \leq \tau(f^{-1}(\mu))$, for each $\mu \in I^Y$.

(ii) a **strongly gradation preserving** (sgp -) map if

$\tau^1(\mu) = \tau(f^{-1}(\mu))$, for each $\mu \in I^Y$.

(iii) a **weakly gradation preserving** (wgp -) map if

$\tau^1(\mu) > 0 \Rightarrow \tau(f^{-1}(\mu)) > 0$, for each $\mu \in I^Y$.

The author has constructed an example to show that weakly gradation preserving map need not be a gradation preserving map and gradation preserving map need not be a strongly gradation preserving map.

The third chapter is devoted to the study of separation axioms through gradation of openness. Starting with the fuzzy topology in terms of gradation of openness the concept of base, subbase, product etc. are defined. The separation axioms T_o, R_o, T_1, T_2 and regularity are introduced and the following results are proved.

(1) Let (X, τ) be a fts. Then the following statements are equivalent :

(a) (X, τ) is Hausdorff

(b) $\Delta x = \{(x,x) \in X \times X\}$ has positive grade of closedness in $(X \times X, \tau_{X \times X})$ where $\tau_{X \times X}$ is the product fuzzy topology on $X \times X$.

(c) For any two weakly gradation preserving maps f, g from a fts (X^1, τ^1) to fts (X,τ), the set $A = \{x^1 \in X^1 / f(x^1) = g(x^1)\}$ has positive grade of closedness in (X^1, τ^1)

(d) If $f : (X^1, \tau^1) \rightarrow (X, \tau)$ is a weakly gradation preserving map then the graph G of f, that is $\{(x^1, f(x^1)) / x^1 \in X^1\}$ has positive grade of closedness in $(X \, xX, \tau_{X' xX})$

(2) Let $\{(X_i, \tau_i) / i \in \Delta\}$ be a family of fts. Then their product fts (X, τ) is Hausdorff iff each (X_i, τ_i) is Hausdorff.

(3) Hausdorffness is a hereditary property

(4) The separation properties T_o, R_o, T_1 and regularity are productive and hereditary.

Review of Literature

REVIEW OF LITERATURE

Now we present abstracts of some important papers on fuzzy gradation of openness published by **K.C. Chattopadhyay**, **R.N. Hazra**, **S.K. Samanta**.

1. **Gradation of openness : "Fuzzy topology"** by **K.C. Chattopadhyay, R.N. Hazra**, **S.K.Samanta** [3]

In [2], the authors have given a new definition of fuzzy topology by introducing a concept of gradation of openness of fuzzy subsets. In order to make the concept more appropriate, the authors modify the definition of gradation function and then study subspace of fuzzy topological spaces, gradation preserving maps and category of fuzzy topological spaces and gradation preserving maps.

2. **"Fuzzy topology : Fuzzy closure operator, fuzzy compactness** and **fuzzy connectedness"** by **K.C.Chattopadhyay, S.K.Samanta** [4]

In this paper, definitions of fuzzy closure operator, fuzzy compactness and fuzzy connectedness are given in a redefined fuzzy topological space and then some characteristic properties of fuzzy closure operator, product theorems of fuzzy compactness, invariance of fuzzy compactness and fuzzy connectedness under gradation preserving maps and invariance of fuzzy connectedness under the fuzzy closure operator are studied.

Now we present abstracts of some important papers on fuzzy separation axioms published by **R. Srivastava, A.K. Srivastava, A. Kandil,**

3. **"On fuzzy Hausdorffness concepts"** by **R. Srivastava, A.K. Srivastava,** (1985) [14]

In this paper, the authors compare several existing definitions of fuzzy Hausdorffness with their definitions.

4. **"On separation axioms in fuzzy topological spaces"** by **A.Kandil,** (1987) [10]

The author uses the concept of fuzzy points and Q-relations introduced by **P.M.Pu** and **Y.M.Liu** [12]. In this paper, some separation axioms for fuzzy topological spaces, all of which are a natural generalization of $T_i(i = 0,1,2,3,4)$ in topological spaces are studied. The author generalizes many theorems concerning separation axioms for topological spaces.

5. **"Some weaker separation axioms in fuzzy topological spaces"** by **Dewan Muslim Ali,** (1988) [6]

In this paper, the author gives an elementary discussion of the counterparts of some separation axioms, such as R_0, R_1, T_0, T_1 and regularity, in the setting of fuzzy topological spaces. In the literature on fuzzy topology, some related research, for example on the investigation of T_2, is already rather in-depth. Nevertheless the discussion on R_0 and R_1 seems to be new.

6. **"A comparison of some Hausdorff notions in fuzzy topological space"** by **D.R. Cutler, I.L. Reilly**, (1990) [5]

There exist several equivalent definitions of Hausdorff topological space. Their fuzzification leads to different definitions of a fuzzy Hausdorff space. In the paper under review the authors analyse 12 such definitions and find out when one is a corollary of the other and when it is not. In this connection some interesting counter examples are provided.

7. **"Separation axioms, subspaces and product spaces in fuzzy topology"** by **Ali Ahmad Fora,** (1990) [1]

The author introduces some fuzzy separation axioms and study their hereditary and productive properties. Further they establish the relation between spaces having the fixed point property and fuzzy connected space.

Chapter I

CHAPTER I

PRELIMINARY DEFINITIONS AND RESULTS

DEFINITION : 1.1

Let X be an arbitrary non-empty set. Let $I = [0,1]$. A **fuzzy set** in X is a mapping from X into I (i.e.) a fuzzy set is an element of I^X. For any two fuzzy sets μ, λ in X,

$$\mu = \lambda \Leftrightarrow \mu(x) = \lambda(x), \text{ for all } x \in X.$$

$$\mu \leq \lambda \Leftrightarrow \mu(x) \leq \lambda(x), \text{ for all } x \in X.$$

The **union** $\mu \vee \lambda$ and the **intersection** $\mu \wedge \lambda$ are defined respectively by

$$(\mu \vee \lambda)(x) = \text{Max}\{\mu(x), \lambda(x)\}, \text{ for all } x \in X.$$

and

$$(\mu \wedge \lambda)(x) = \text{Min}\{\mu(x), \lambda(x)\}, \text{ for all } x \in X.$$

The **complement** μ^1 of a fuzzy set μ is defined by $\mu^1(x) = 1 - \mu(x)$, for all $x \in X$. For a family $\{\mu_i\}_{i\in\Delta}$ of fuzzy sets, the union $\underset{i\in\Delta}{\vee} \mu_i$ and the **intersection** $\underset{i\in\Delta}{\wedge} \mu_i$ are defined respectively by

$$\underset{i\in\Delta}{\vee} \mu_i(x) = \underset{i\in\Delta}{\text{Sup}} \{\mu_i(x)\}, \text{ for all } x \in X$$

and

$$\underset{i\in\Delta}{\wedge} \mu_i(x) = \text{Inf} \{\mu_i(x)\}, \text{ for all } x \in X.$$

DEFINITION : 1.2

A fuzzy set α is said to be **constant** if for all $x \in X$, $\alpha(x) = \alpha$.

DEFINITION : 1.3

Let X and Y be any two non-empty sets. Let θ be a function from X to Y and λ be a fuzzy set in Y. Then the **inverse of** λ under θ written as $\theta^{-1}(\lambda)$ is the fuzzy set in X defined by

$$\theta^{-1}(\lambda)(x) = \lambda(\theta(x)), \text{ for } x \in X$$

For every fuzzy set μ in X, the **image of** μ under θ, written as $\theta(\mu)$ is the fuzzy set defined by

$$\theta(\mu)(y) = \sup_{z \in \theta^{-1}(y)} \mu(z), \text{ if } \theta^{-1}(y) \text{ is not empty}$$

$$= 0, \quad \text{otherwise.}$$

DEFINITION : 1.4

An ordinary subset A of X can be considered as a fuzzy set by identifying it with its characteristic function χ_A, such fuzzy sets are called **crisp sets**.

DEFINITION : 1.5

The crisp sets corresponding to ϕ and X are denoted by 0 and 1. That is

$$\chi_{\phi} = 0 \text{ and } \chi_X = 1.$$

DEFINITION : 1.6

Let $\mu : X \to [0,1]$ be a fuzzy et. Then the **support of** μ is $\{x / \mu(x) > 0\}$.

DEFINITION : 1.7

Let X be a non-empty set. Then a fuzzy point denoted by x_α, is defined as a fuzzy set in X which takes value $\alpha \in (0,1)$ at x and 0, elsewhere. x and α are respectively called the support and value of the **fuzzy point** $\mathbf{x_\alpha}$. Fuzzy points with distinct supports are called distinct.

Let $\mu \in I^X$. Then x_α is said to belong to μ if $\alpha < \mu(x)$ and denoted by $x_\alpha \in \mu$.

PROPOSITION : 1.8

Let $\{\mu_i \,/\, i \in \Delta\} \subseteq I^X$. Then $x_\alpha \in \underset{i\in\Delta}{\vee} \mu_i$ iff $x_\alpha \in \mu_i$, for some $i \in \Delta$.

PROOF

Assume $x_\alpha \in \mu_i$, for some $i \in \Delta$

Then $\alpha < \mu_i(x)$, for some $i \in \Delta$

$\Rightarrow \quad \alpha < \underset{i\in\Delta}{\vee} \mu_i(x)$

$\Rightarrow \quad x_\alpha \in \underset{i\in\Delta}{\vee} \mu_i$

Conversely, assume $x_\alpha \in \underset{i\in\Delta}{\vee} \mu_i$

Then $\alpha < \underset{i\in\Delta}{\vee} \mu_i(x)$

$\Rightarrow \quad \alpha < \mu_i(x)$, for some $i \in \Delta$

$\Rightarrow \quad x_\alpha \in \mu_i$, for some $i \in \Delta$

Hence the result.

DEFINITION : 1.9 [Chang] [2]

A fuzzy topology [Chang] is a family δ of fuzzy sets in X which satisfies the following conditions:

(a) $0,1 \in \delta$

(b) If $\mu, \lambda \in \delta$ then $\mu \wedge \mu \in \delta$

(c) If $\mu_i \in \delta$ for each $i \in \Delta$ then $\vee_{i \in \Delta} \mu_i \in \delta$

δ is called a **fuzzy topology** for X and the pair (X, δ) is called a **fuzzy topological space**. Every member of δ is called a δ-open fuzzy set.

DEFINITION : 1.10

A fuzzy set is δ-closed iff its complement is δ-open.

DEFINITION : 1.11 [Lowen] [11]

$\delta \subset I^X$ is a **fuzzy topology [Lowen]** on X iff

(i) Every constant function α belongs to δ

(ii) $\mu, \lambda \in \delta \Rightarrow \mu \wedge \lambda \in \delta$.

(iii) $(\mu_j)_{j \in \Delta} \in \delta \Rightarrow \operatorname{Sup}_{j \in \Delta} \mu_j \in \delta$

The fuzzy sets in δ are called **open fuzzy sets** and the pair (X,δ) is called a **fuzzy topological space**.

DEFINITION : 1.12 [2]

A function θ from a fuzzy topological space (X, δ) to a fuzzy topological space (Y, δ^1) is **fuzzy continuous** iff the inverse of each δ^1 – open fuzzy set is δ-open.

DEFINITION : 1.13

The **closure** and **interior** of a fuzzy set $\mu \in I^X$ are defined respectively as

$$Cl\ \mu = Inf\ \{\lambda / \lambda \geq \mu, \lambda^1 \in \delta\}$$

$$Int\ \mu = Sup\ \{\lambda / \lambda \leq \mu, \lambda \in \delta\}$$

NOTE

Cl μ is the smallest closed fuzzy set larger than μ and Int μ is the largest open fuzzy set smaller than μ.

DEFINITION : 1.14 [11]

An operator $\varphi : I^X \rightarrow I^X$ is a **fuzzy *closure operator*** iff

(1) $\varphi(\alpha) = \alpha$, for every α constant

(2) $\varphi(\mu) \geq \mu$, for every $\mu \in I^X$

(3) $\varphi(\mu) \vee \varphi(\lambda) = \varphi(\mu \vee \lambda)$, for every $\mu, \lambda \in I^X$.

(4) $\varphi(\varphi(\mu)) = \varphi(\mu)$, for every $\mu \in I^X$.

DEFINITION : 1.15 [11]

Let (X, δ) be a fuzzy topological space. A subset $\sigma \subset \delta$ is a **base for** δ iff for every $\mu \in \delta$, there exists a collection $(\mu_j)_{j \in \Delta} \subset \sigma$ such that $\mu = \operatorname{Sup}_{j \in \Delta} \mu_j$.

DEFINITION : 1.16

Let (X, δ) be a fuzzy topological space. A subset $\sigma^1 \subset \delta$ is **a subbase** for δ iff the family of finite infima of members of σ^1 is a base for δ.

DEFINITION : 1.17

Let (X, δ) be a fuzzy topological space and $Y \subset X$. Then the family δ_Y defined by

$$\delta_Y = \{\mu|Y \,/\, \mu \in \delta\},$$

where $\mu|Y$ denotes the restriction of the function μ to the set Y, is a fuzzy topology for Y called the **relative fuzzy topology of δ to Y**. The fuzzy topological space (Y, δ_Y) is called a subspace of (X,δ).

DEFINITION : 1.18

Let $\{(X_\alpha, \delta_\alpha) \,/\, \alpha \in \Delta\}$ be a family of fuzzy topological spaces.

Let $X = \prod_{\alpha \in \Delta} X_\alpha$ be the usual product set and let p_α be the projection from X onto X_α. Let $\mu \in \delta_\alpha$. The $p_\alpha^{-1}(\mu)$ is a fuzzy set in X. The family of fuzzy sets $\sigma = \{ p_\alpha^{-1}(\mu) \,/\, \mu \in \delta_\alpha, \alpha \in \Delta\}$ will be a subbase for a topology δ on X, called the **product topology** on X. The pair (X, δ) is called the **product fuzzy topological space**.

DEFINITION : 1.19 (Rekha Srivastava) [13]

A fuzzy topological space (X, δ) is said to be **fuzzy Hausdorff** if for any two distinct fuzzy points x_α, y_β in X, there exist disjoint fuzzy sets λ, μ ∈ δ with $x_\alpha \in \lambda$ and $y_\beta \in \mu$.

THEOREM 1.20 [13]

Let (X,δ) be a fuzzy topological space. Then the following are equivalent.

(i) (X, δ) is fuzzy Hausdorff

(ii) $\Delta_X = \{(x,x) \in X \times X\}$ is fuzzy closed in X x X.

(iii) For any two fuzzy continuous functions f, g : $(Y, \delta^1) \rightarrow (X, \delta)$ the set $A = \{ y \in Y / f(y) = g(y) \}$ is fuzzy closed in the fuzzy topological space(Y, δ^1)

(iv) If f : $(Y, \delta^1) \rightarrow (X, \delta)$ is a fuzzy continuous function, then the graph of f^1 (i.e.) $\{(y, f(y)) / y \in Y\}$ is fuzzy closed in $(YxX, \delta^1 x\delta)$

THEOREM 1.21 [13]

a) A fuzzy subspace (A, δ_A) of a fuzzy Hausdorff topological space (X, δ) is fuzzy Hausdorff.

b) If $\{(X_i, \delta_i) / i \in \Delta\}$ is a family of fuzzy Hausdorff topological spaces, then their product (X, δ) is also fuzzy Hausdorff.

DEFINITION : 1.22 [Rekha Srivastava] [15]

Let (X, δ) be a fuzzy topological space. Then (X, δ) is called a **fuzzy T_1 – topological space** iff for every pair of distinct fuzzy points x_α, y_β in X, there exist fuzzy open sets λ and μ in (X, δ) such that $x_\alpha \in \lambda$, $y_\beta \notin \mu$ and $y_\beta \in \mu$, $x_\alpha \notin \lambda$.

DEFINITION : 1.23 [16]

A fuzzy topological space (X, δ) is said to be **fuzzy T_o** if $\forall$ x, y $\in$ X, $x \neq y$, there exists $\mu \in \delta$ such that either $\mu(x) = 1$ and $\mu(y) = 0$ or $\mu(y) = 1$ and $\mu(x) = 0$.

DEFINITION : 1.24 [16]

A fuzzy topological space (X, δ) is said to be **R_0** if each fuzzy open set can be written as a supremum of fuzzy closed sets.

THEOREM : 1.25

Fuzzy T_o, T_1 and R_0 are productive and hereditary.

DEFINITION : 1.26

A fuzzy set μ in a fuzzy topological space (X, δ) is a **neighbourhood** of a fuzzy set μ_1 iff there exists an open fuzzy set λ such that $\mu_1 < \lambda < \mu$.

THEOREM 1.27

A fuzzy set μ is open iff for each fuzzy set λ contained in μ, μ is a neighbourhood of λ.

Chapter II

CHAPTER II

FUZZY TOPOLOGY REDEFINED

R.N. Hazra, S.K. Samanta and K.C. Chattopadhyay have approached the concept of fuzzy topological space by introducing the concept of gradation of openness [9]. In this chapter, fuzzy topological spaces, subspaces of fuzzy topological spaces and gradation preserving mappings are discussed.

SECTION 2.1

NOTATIONS AND PRELIMINARIES

DEFINITION : 2.1.1

Let Y be a subset of X and $\mu \in I^X$. For each $\mu \in I^Y$ the extension of μ on X, denoted by μ_X is defined by

$$\mu_X = \begin{cases} \mu \text{ on } Y \\ 0 \text{ on } X - Y. \end{cases}$$

R.N. Hazra, S.K. Samanta, K.C. Chattopadhyay [9] introduced definition of fuzzy topological space as follows :

DEFINITION : 2.1.2

A fuzzy topological space (fts) is a pair (X, τ) where X is a non-empty set and $\tau : I^X \to I$ is a mapping satisfying the following properties :

(01) $\tau(0) = \tau(1) = 1$;

(02) If $\tau(\mu_i) > 0$, $i = 1,2$, then $\tau(\mu_1 \wedge \mu_2) > 0$;

(03) If $\tau(\mu_i) > 0$, $i \in \Delta$, then $\tau(\underset{i\in\Delta}{\vee} \mu_i) > 0$.

Then τ is called a **gradation of openness on X** or a **fuzzy topology on X.**

DEFINITION : 2.1.3

Let X be a set. A mapping $\mathfrak{F} : I^X \rightarrow I$, satisfying

(C1) $\mathfrak{F}(0) = \mathfrak{F}(1) = 1$

(C2) If $\mathfrak{F}(\mu_i) > 0$, for $i = 1, 2$, then $\mathfrak{F}(\mu_1 \vee \mu_2) > 0$

(C3) If $\mathfrak{F}(\mu_i) > 0$, for $i \in \Delta$, then $\mathfrak{F}(\underset{i\in\Delta}{\wedge} \mu_i) > 0$,

is called a **gradation of closedness** on X.

PROPOSITION : 2.1.4

Let τ be a gradation of openness on X and $\mathfrak{F}_\tau : I^X \rightarrow I$ be a mapping defined by $\mathfrak{F}_\tau(\mu) = \tau(\mu^1)$. Then $\mathfrak{F}_\tau$ is a gradation of closedness on X.

PROOF

Given τ is a gradation of openness on X and $\mathfrak{F}_\tau : I^X \rightarrow I$ is defined by

$$\mathfrak{F}_\tau(\mu) = \tau(\mu^1).$$

CLAIM : $\mathfrak{F}_\tau$ is a gradation of closedness on X.

(C1) $\mathfrak{F}_\tau(0) = \tau(1) = 1$

$\mathfrak{F}_\tau(1) = \tau(0) = 1$

(C2) Assume $\mathfrak{F}_\tau(\mu_1) > 0$, and $\mathfrak{F}_\tau(\mu_2) > 0$

Then $\tau(\mu_1{}^1) > 0$ and $\tau(\mu_2{}^1) > 0$.

Since τ is a gradation of openness on X,

$$\tau(\mu_1{}^1 \wedge \mu_2{}^1) > 0$$

$$\Rightarrow \tau[(\mu_1 \vee \mu_2)^1] > 0$$

$$\Rightarrow \mathfrak{F}_\tau(\mu_1 \vee \mu_2) > 0$$

(C3) If $\mathfrak{F}_\tau(\mu_i) > 0$, for $i \in \Delta$, then $\tau(\mu_i{}^1) > 0$ for $i \in \Delta$..

Since τ is a gradation of openness on X, $\tau\left(\underset{i\in\Delta}{\vee} \mu_i{}'\right) > 0$

$$\Rightarrow \tau\left[\left(\underset{i\in\Delta}{\wedge} \mu_i\right)'\right] > 0$$

$$\Rightarrow \mathfrak{F}_\tau\left(\underset{i\in\Delta}{\wedge} \mu_i\right) > 0$$

Hence the result is proved.

PROPOSITION: 2.1.5

Let $\mathfrak{F}$ be a gradation of closedness on X and $\tau_{\mathfrak{F}}: I^X \rightarrow I$ be a mapping defined by $\tau_{\mathfrak{F}}(\mu) = \mathfrak{F}(\mu^1)$. Then $\tau_{\mathfrak{F}}$ is a gradation of openness on X.

The proof is straight forward as previous proposition 2.1.4.

COROLLARY: 2.1.6

Let $\mathfrak{F}, \tau$ be gradation of closedness and openness respectively on X.

Then

$$\tau_{\mathfrak{F}_\tau} = \tau \text{ and } \mathfrak{F}_{\tau_{\mathfrak{F}}}.$$

PROOF

For $\mu \in I^X$,

$$\tau_{\mathfrak{F}_\tau}(\mu) = \mathfrak{F}_\tau(\mu) = \tau((\mu^1)^1) = \tau(\mu)$$

$$\therefore \tau_{\mathfrak{F}_\tau} = \tau$$

and

$$\mathfrak{F}_{\tau_{\mathfrak{F}}}(\mu) = \tau_{\mathfrak{F}}(\mu^1) = \mathfrak{F}((\mu^1)^1) = \mathfrak{F}(\mu)$$

$$\therefore \mathfrak{F}_{\tau_{\mathfrak{F}}} = \mathfrak{F}.$$

Hence the corollary.

REMARK :2.1.7

Let X be a set and let GO(X) (CO(X)) be the set of all gradations of openness (closedness) on X. From the preceding results it follows that the mapping $\tau \to \mathfrak{F}_\tau$ is a bijection of GO(X) onto CO(X), and $\tau \to \mathfrak{F}_\tau$ and $\tau \to \tau_{\mathfrak{F}}$ are the inverse of each other. In view of this fact, each of the pair (X, τ) and (X, $\mathfrak{F}$), where τ ($\mathfrak{F}$)is a gradation of openness (closedness) on X, will be referred to as a fuzzy topological space. Henceforth τ($\mathfrak{F}$), with or without suffix, will be referred to as a gradation of openness (closedness).

DEFINITION : 2.1.8

Let (X,τ) be a fuzzy topological space and $\mu \in I^X$. Then **τ-closure of μ**, denoted by $\overline{\mu}$, is defined by

$$\overline{\mu} = \wedge\{ \eta \in I^X / \mathfrak{F}_\tau(\eta) > 0,\ \eta \geq \mu\}$$

PROPOSITION : 2.1.9

(i) $\mathfrak{F}_\tau(\overline{\mu}) > 0$

(ii) $\mu \geq \eta$ implies that $\overline{\mu} \geq \overline{\eta}$, for all $\mu, \eta \in I^X$.

PROPOSITION : 2.1.10

Let (X, τ) be a fuzzy topological space, then

(i) $\overline{0} = 0$

(ii) $\overline{\mu} \geq \mu$

(iii) $\overline{\mu_1 \vee \mu_2} = \overline{\mu_1} \vee \overline{\mu_2}$

(iv) $\overline{\overline{\mu}} = \mu$

PROOF

From the definition 2.1.8, (i) and (ii) are obvious.

(iii) TO PROVE: $\overline{\mu_1 \vee \mu_2} = \overline{\mu_1} \vee \overline{\mu_2}$

$$\mu_1 \vee \mu_2 \geq \mu_1$$

$$\Rightarrow \quad \overline{\mu_1 \vee \mu_2} = \overline{\mu}_1 \tag{1}$$

and

$$\mu_1 \vee \mu_2 \geq \mu_2$$

$$\Rightarrow \quad \overline{\mu_1 \vee \mu_2} = \overline{\mu}_2 \tag{2}$$

Using (1) and (2)

$$\overline{\mu_1 \vee \mu_2} \geq \overline{\mu}_1 \vee \overline{\mu}_2 \tag{3}$$

Since $\mathfrak{F}_\tau(\overline{\mu}_i) > 0$ for i = 1, 2, $\mathfrak{F}_\tau(\overline{\mu}_1 \vee \overline{\mu}_2) > 0$

But $\mu_1 \leq \overline{\mu}_1$ and $\mu_2 \leq \overline{\mu}_2$

$$\Rightarrow \mu_1 \vee \mu_2 \leq \overline{\mu}_1 \vee \overline{\mu}_2$$

Hence $\mathfrak{F}_\tau(\overline{\mu}_1 \vee \overline{\mu}_2) > 0$ such that $\mu_1 \vee \mu_2 \leq \overline{\mu}_1 \vee \overline{\mu}_2$.

$\Rightarrow \overline{\mu}_1 \vee \overline{\mu}_2 \in \overline{\mu_1 \vee \mu_2}$

$\therefore \overline{\mu}_1 \vee \overline{\mu}_2 \geq \overline{\mu_1 \vee \mu_2}$ (4)

From (3) and (4)

$$\overline{\mu}_1 \vee \overline{\mu}_2 = \overline{\mu_1 \vee \mu_2}$$

(iv) TO PROVE: $\overline{\overline{\mu}} = \mu$

We know that

$$\overline{\mu} = \wedge\{\eta \in I^X / \mathfrak{F}_\tau(\eta) > 0, \eta \geq \mu\}$$

$$\therefore \quad \overline{\overline{\mu}} = \wedge\{\eta \in I^X / \mathfrak{F}_\tau(\eta) > 0, \eta \geq \overline{\mu}\}$$

$$\Rightarrow \quad \overline{\mu} \leq \overline{\overline{\mu}}$$

But $\overline{\overline{\mu}} \leq \overline{\mu}$.

Hence, $\overline{\overline{\mu}} = \overline{\mu}$.

PROPOSITION : 2.1.11

Let (X, τ) be a fuzzy topological space. Then for each $\mu \in I^X$,

$$\mathfrak{F}_\tau(\mu) > 0 \Leftrightarrow \mu = \overline{\mu}.$$

PROOF

The proof follows from the above results.

PROPOSITION : 2.1.12

Let $\{\tau_K : K = 1, 2, \ldots\ldots\ldots, n\}$ be a finite family of gradation of openness on X. Then $\tau = \bigwedge_{k=1}^{n} \tau_k$ is a gradation of openness on X.

PROOF

Let $\{\tau_K : K = 1, 2, \ldots\ldots\ldots, n\}$ be a finite family of gradation of openness on X.

Since each τ_K is a gradation of openness,

(01) $\tau_K(0) = \tau_K(1) = 1$ for every $k = 1, 2, \ldots n$

$\therefore \tau(0) = \tau(1) = 1$

(02) Let $\tau(\mu_i) > 0$, for $i = 1,2$. Then

$$\bigwedge_{k=1}^{n} \tau_k(\mu_i) > 0, \text{ for } i = 1,2.$$

$\Rightarrow$ $\tau_k(\mu_i) > 0$, for $i = 1, 2$ and $k = 1,2, \ldots\ldots n$.

$\Rightarrow$ $\tau_k(\mu_1 \wedge \mu_2) > 0$, for $k = 1,2,\ldots\ldots n$.

$\Rightarrow$ $\bigwedge_{k=1}^{n} \tau_k(\mu_1 \wedge \mu_2) > 0$

$\Rightarrow$ $\tau(\mu_1 \wedge \mu_2) > 0$

(03) If $\tau(\mu_i) > 0$ for all $i \in \Delta$, then

$$\bigwedge_{k=1}^{n} \tau_k(\mu_i) > 0, \text{ for all } i \in \Delta$$

$\Rightarrow \quad \tau_k(\mu_i) > 0$, for $i \in \Delta$ and $k = 1,2,\ldots..n$

$\Rightarrow \quad \tau_k(\bigvee_{i\in\Delta} \mu_i) > 0$, for $k = 1,2,\ldots..n$

$\Rightarrow \quad \bigwedge_{k=1}^{n} \tau_k(\bigvee_{i\in\Delta} \mu_i) > 0$

$\Rightarrow \quad \tau(\bigvee_{i\in\Delta} \mu_i) > 0$

Hence, τ is a gradation of openness on X.

REMARK: 2.1.13

There is a difference between fuzzy topology defined by R.N. Hazra, S.K. Samanta, K.C. Chattopadhyay [9] and the fuzzy topology defined by Chang [2] in the sense that arbitrary intersection of fuzzy topologies of Chang type is again a fuzzy topology, but this is not necessarily true in this case which is illustrated in the following example.

EXAMPLE : 2.1.14

Let X = N, the set of all natural numbers.

Define

$\mu_0 = \chi_O$, where O is the set of all odd numbers.

$\mu_n = \chi_{\{n\}}$, $\mu_n^* = \chi_{A_n}$, where $A_n = \{1, 3, \ldots\ldots 2n-1\}$ for all $n \in N$.

Clearly, for each $n \in N$, μ_0, μ_n, μ_n^* are fuzzy sets on X.

Now for each $i \in N$, we define a mapping

$\tau_i : I^X \to I$, by the rule

$$\tau_i(\mu_0) = \frac{1}{i}$$

$$\tau_i(\mu_n^*) = \text{Max}\left\{\frac{1}{i}, \frac{1}{2n-1}\right\}$$

$\tau_i(\mu_n) = 1$, for other fuzzy sets on X.

Then for each $i \in N$, τ_i is a fuzzy topology on X.

Take $\tau = \underset{i \in N}{\wedge} \tau_i$

Then $\tau(\mu_0) = 0$

Since for each n = 1, 3, 5, ... $\tau(\mu_n) = 1$, but $\tau(\vee\mu_n) = \tau(\mu_0) = 0$, it follows that τ is not a fuzzy topology on X.

REMARKS : 2.1.15

If τ is a graduation of openness on X, then supp $\tau = \{ \mu \in I^X / \tau(\mu) > 0\}$ is a Chang fuzzy topology on X.

PROOF

(01) As τ is a gradation of openness on X,

$\tau(0) = \tau(1) = 1$

$\therefore$ 0,1 $\in$ supp τ.

(02) Assume $\mu_i \in$ supp τ, for i = 1,2.

Then $\tau(\mu_1) > 0$ and $\tau(\mu_2) > 0$.

Since τ is a gradation of openness on X,

$$\tau(\mu_1 \wedge \mu_2) > 0$$

$\Rightarrow \mu_1 \wedge \mu_2 \in$ supp τ.

(03) Assume $\mu_i \in$ supp, τ for $i \in \Delta$

Then $\tau(\mu_i) > 0$, for $i \in \Delta$

Since τ is a gradation of openness on X,

$$\tau(\bigvee_{i \in \Delta} \mu_i) > 0$$

$\Rightarrow \bigvee_{i \in \Delta} \mu_i \in$ supp τ.

Hence supp τ is a Chang fuzzy topology on X.

DEFINITION : 2.1.16

Let δ be a Chang fuzzy topology on X. Then a gradation τ of openness on X is said to be **compatible with** δ if supp $\tau = \delta$.

PROPOSITION : 2.1.17

Let δ be a Chang fuzzy topology on X. Then for each $r \in (0,1]$, there is a gradation of openness τ_r on X compatible with δ.

PROOF

For each $r \in (0,1]$, define a mapping

$\tau_r : I^X \rightarrow I$ by the rule

$$\tau_r(0) = \tau_r(1) = 1$$

$$\tau_r(\mu) = \begin{cases} r, \text{ for all } \mu\ (\neq 0,1) \in \delta \\ 0, \text{ elsewhere} \end{cases}$$

Now,

(01) Given $\tau_r(0) = \tau_r(1) = 1$

(02) $\tau_r(\mu_1) = r > 0$ and $\tau_r(\mu_2) = r > 0$

$\therefore \tau_r\ (\mu_1 \wedge \mu_2) = r > 0$

(03) $\tau_r(\mu_i) = r > 0,\ i \in \Delta$

$\therefore \tau_r(\underset{i\in\Delta}{\wedge} \mu_i) = r > 0$

Hence τ_r is a gradation of openness on X.

Supp $\tau_r = \{\mu \in I^X / \tau_r(\mu) > 0\}$

$= \delta$

$\therefore \tau_r$ is compatible with δ.

Hence the proof.

PROPOSITION : 2.1.18

Let δ be a Chang fuzzy topology on X, then the set of all gradation of openness on X compatible with δ is equipotent to the set $I_0^{\overline{\delta}}$ where $I_0 = (0,1]$ and $\overline{\delta} = \delta - \{0,1\}$.

PROOF

Let δ be a Chang fuzzy topology on X.

Let $f \in I_0^{\bar{\delta}}$. Define a mapping

$\tau_f : I^X \rightarrow I$ by the rule

$\tau_f(0) = \tau_f(1) = 1$

and $\tau_f(\mu) = \begin{cases} f(\mu), & \text{for all } \mu \in \bar{\delta} \\ 0, & \text{elsewhere} \end{cases}$

(01) Obviously $\tau_f(0) = \tau_f(1) = 1$

(02) Assume $\tau_f(\mu_1) > 0$ and $\tau_f(\mu_2) > 0$ for $\mu_1, \mu_2 \in \bar{\delta}$

$\Rightarrow f(\mu_1) > 0$ and $= f(\mu_2) > 0$, for $\mu_1, \mu_2 \in \bar{\delta}$

$\Rightarrow f(\mu_1 \wedge \mu_2) > 0$, for $\mu_1, \mu_2 \in \bar{\delta}$

$\Rightarrow \tau_f(\mu_1 \wedge \mu_2) > 0$, for $\mu_1, \mu_2 \in \bar{\delta}$

(03) Assume $\tau_f(\mu_i) > 0$, for all $i \in \Delta$

$\Rightarrow f(\mu_i) > 0$, for all $i \in \Delta$ and $\mu_i \in \bar{\delta}$

$\Rightarrow f(\vee_{i \in \Delta} \mu_i) > 0$, $\mu_i \in \bar{\delta}$

$\Rightarrow \tau_f(\vee_{i \in \Delta} \mu_i) > 0$.

$\therefore$ τ_f is a gradation of openness on X compatible with δ.

Now, suppose that τ is a gradation of openness on X compatible with δ.

Define $f_\tau : \overline{\delta} \to I_0$ such that

$$f_\tau(\mu) = \tau(\mu),\ \mu \in \overline{\delta}$$

Clearly, $f_\tau \in I_0^{\overline{\delta}}$

Consider $\tau_{f_\tau}(0) = 1$ and $\tau_{f_\tau}(1) = 1$

Also,

$$\tau_{f_\tau}(\mu) = \begin{cases} f_\tau(\mu) = \tau(\mu), & \text{for all } \mu \in \overline{\delta} \\ 0, & \text{elsewhere} \end{cases}$$

$\therefore \tau_{f_\tau} = \tau.$

Similarly, it can be proved that, for $f \in I_0^{\overline{\delta}}$

$$f_{\tau_f} = f$$

Hence the proof.

SECTION 2.2

FUZZY SUBSPACES AND GRADATION PRESERVING MAP

PROPOSITION : 2.2.1

Let (X, τ) be a fuzzy topological space and $Y \subset X$. Define a mapping $\tau_Y : I^Y \to I$ by the rule $\tau_Y(\mu) = \vee\{\tau(\lambda) / \lambda \in I^X ; \lambda|Y = \mu\}$

Then τ_Y is a gradation of openness on Y.

PROOF

Given $\tau_Y(\mu) = \vee\{\tau(\lambda) / \lambda \in I^X ; \lambda|Y = \mu\}$

(01) Clearly $\tau_Y(0) = \tau_Y(1) = 1$

(02) Assume $\tau_Y(\mu_1) > 0$ and $\tau_Y(\mu_2) > 0$

Then $\vee \{\tau(\lambda_1) / \lambda_1 \in I^X; \lambda_1|Y = \mu_1\} > 0$ and

$\vee \{\tau(\lambda_2) / \lambda_2 \in I^X; \lambda_2|Y = \mu_2\} > 0$

$\Rightarrow \vee \{\tau(\lambda_1\wedge \lambda_2) / \lambda_1\wedge\lambda_2 \in I^X; \lambda_1\wedge\lambda_2|Y = \mu_1\wedge \mu_2\} > 0$

$\Rightarrow \tau_Y \{\mu_1\wedge \mu_2\} > 0$

(03) Assume $\tau_Y(\mu_i) > 0$, for $i \in \Delta$

Then $\vee \{\tau(\lambda_i) / \lambda_i \in I^X, \lambda_1|Y = \mu_i\} > 0$, for $i \in\Delta$

$\Rightarrow \vee \{\tau(\underset{i\in\Delta}{\vee} \lambda_i) / \underset{i\in\Delta}{\vee} \lambda_i \in I^X, \underset{i\in\Delta}{\vee} \lambda_i \,|\, Y = \underset{i\in\Delta}{\vee} \mu_i \} > 0$

$\Rightarrow \tau_Y (\underset{i\in\Delta}{\vee} \mu_i) > 0$

Hence τ_Y is a gradation of openness on Y.

DEFINITION : 2.2.2

The fuzzy topological space (Y, τ_Y) is called a **subspace** of the fuzzy topological space (X, τ) and τ_Y is called the **induced gradation of openness** on Y from (X, τ).

Let (Y, τ_Y) be a subspace of the fuzzy topological space (X, τ) and $\mu \in I^Y$. μ_{τ_Y} means that τ_Y – closure of μ.

PROPOSITION : 2.2.3

Let (Y,τ_Y) be a fuzzy subspace of the fuzzy topological space (X, τ) and $\mu \in I^Y$, then

(i) $\mathcal{F}_{\tau_Y}(\mu) = \vee\{\mathcal{F}_\tau(\eta) / \eta \in I^X, \eta|Y = \mu\}$

(ii) $\overline{\mu}_{\tau_Y} = (\overline{\mu}_X)|Y$

(iii) If $Z \subset Y \subset X$ then $\tau_Z = (\tau_Y)_Z$

PROOF

(i)

$$\begin{aligned}
\mathcal{F}_{\tau_Y}(\mu) &= \tau_Y(\mu^1) \\
&= \vee\{\tau(\lambda) / \lambda \in I^X, \lambda|Y = \mu^1\} \\
&= \vee\{\tau(\lambda) / \lambda^1 \in I^X, \lambda^1|Y = \mu\} \\
&= \vee\{\mathcal{F}_\tau(\lambda^1) / \lambda^1 \in I^X, \lambda^1|Y = \mu\} \\
&= \vee\{\mathcal{F}_\tau(\eta) / \eta \in I^X, \eta|Y = \mu\}
\end{aligned}$$

$$\therefore \mathcal{F}_{\tau_Y}(\mu) = \vee\{\mathcal{F}_\tau(\eta) / \eta \in I^X, \eta|Y = \mu\}$$

(ii)

$$\begin{aligned}
\overline{\mu}_{\tau_Y} &= \wedge\{\eta \in I^X / \mathcal{F}_{\tau_Y}(\eta) > 0, \eta \geq \mu\} \\
&= \wedge\{\eta \in I^Y / \exists\, \lambda \in I^X, \lambda|Y = \eta \geq \mu, \mathcal{F}_\tau(\lambda) > 0\} \\
&= \wedge\{\lambda|Y / \lambda \in I^X, \mathcal{F}_\tau(\lambda) > 0, \lambda \geq \mu_X\}
\end{aligned}$$

(since $\lambda \geq \mu_X$ iff $\lambda|Y \geq \mu$)

$$= (\overline{\mu}_x)|Y$$

$$\therefore \overline{\mu}_{\tau_Y} = (\overline{\mu}_x)|Y$$

(iii) We have, for each $\mu \in I^Z$

$(\tau_Y)_Z(\mu) \quad = \vee\{\tau_Y(\mu)/\eta \in I^Y, \eta\,|Z = \mu\}$

$= \vee[\vee\{\tau(\lambda)/\lambda \in I^Y, \lambda \mid Y = \eta\}$ such that $\eta \in I^Y, \eta|Z = \mu]$

$= \vee\{\tau(\lambda)/\lambda \in I^Y, \lambda|Z = \mu\}$

(Since $Z \subset Y \subset X$ and $\lambda \in I^X$, $(\lambda|Y)|Z = \lambda|Z$)

$(\tau_Y)_Z(\mu) = \tau_Z(\mu)$

Hence, $\tau_Z = (\tau_Y)_Z$

DEFINITION : 2.2.4

Let (X, τ) and (Y, τ^1) be two fuzzy topological space and f be a function from X to Y. The map f is called

(i) a **gradation preserving** (gp-) map,

if $\tau^1(\mu) \leq \tau(f^{-1}(\mu))$, for each $\mu \in I^Y$,

(ii) a **strongly gradation preserving** (sgp-) map,

if $\tau^1(\mu) \leq \tau(f^{-1}(\mu))$, for each $\mu \in I^Y$,

(iii) a **weakly gradation preserving** (wgp-) map,

if $\tau^1(\mu) > 0 \Rightarrow \tau(f^{-1}(\mu)) > 0$, for each $\mu \in I^Y$

REMARK : 2.2.5

Clearly sgp- map $\Rightarrow$ gp- map $\Rightarrow$ wgp- map. However, the converse is not necessarily true as is evident from the following example.

EXAMPLE : 2.2.6

Let X = N, the set of all natural numbers, Define

$\mu_O = \chi_O$, where O is the set of all odd numbers in N.

$\mu_n = \chi_{A_n}$, where $A_n = \{1, 3, ..., 2n-1\}$

For each i = 1, 2, define $\tau_i : I^X \to I$ by the rule

$$\tau_i(\mu_O) = \frac{1}{i}$$

$$\tau_i(\mu_n) = \text{Max}\left\{\frac{1}{i}, \frac{1}{2n-1}\right\}$$

$\tau_i(\mu) = 1$ for all other fuzzy sets μ in X.

Then (X, τ_1) and (X, τ_2) are fuzzy topological space. The identity mapping $i : (X, \tau_2) \to (X, \tau_1)$ is a wgp- map, but not a gp- map.

Therefore wgp-map $\not\Rightarrow$ gp- map.

Also, the identity map,

$i : (X, \tau_1) \to (X, \tau_2)$ is a gp- map, but not a sgp- map.

$\therefore$ gp-map $\not\Rightarrow$ sgp- map

Hence, wgp- map $\not\Rightarrow$ gp- map $\not\Rightarrow$ sgp- map

Hence the result.

Next, we prove that the composition of two gradation preserving maps is again a gradation preserving. A similar result holds in case of two weakly gradation preserving maps or two strongly gradation preserving maps.

PROPOSITION : 2.2.7

Let (X, τ_1), (Y, τ_2) and (Z, τ_3) be three fuzzy topological space and fuzzy topological space $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two gp(resp. sgp, wgp) maps. Then g o f is also a gp(resp. sgp, wgp) map.

PROOF

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two gp maps

TO PROVE : $g \circ f : X \rightarrow Z$ is a gp map.

Take $\mu \in I^Z$

Since g is gp map, $\tau_3(\mu) \leq \tau_2(g^{-1}(\mu))$

Since f is gp map,

$$\tau_2(g^{-1}(\mu) \leq \tau_1[f^{-1}(g^{-1}(\mu))]$$

$$= \tau_1[(f^{-1} o\, g^{-1})(\mu)]$$

$$= \tau_1[(g \circ f)^{-1}(\mu)]$$

$\therefore \tau_3(\mu) \leq \tau_1[(g \circ f)^{-1}(\mu)]$

Hence g o f is gradation preserving.

The proof for the other cases, (i.e.,) when f, g are sgp or when f, g are wgp are similar.

PROPOSITION: 2.2.8

Let (X, τ), (Y, τ^1) be two fuzzy topological space and $f : X \rightarrow Y$ be a function. Then the following are equivalent :

(i) f is a wgp- map

(ii) $f(\overline{\eta}) \leq \overline{f(\eta)}$, for $\eta \in I^X$

PROOF

Assume that f is a wgp- map.

Then, for each $\eta \in I^X$,

$$f^{-1}(\overline{f(\eta)}) = f^{-1}[\wedge\{\lambda \in I^Y / \mathfrak{F}_\tau(\lambda) > 0, \lambda \geq f(\eta)\}]$$

$$\geq \wedge\{ f^{-1}(\lambda) \in I^X / \mathfrak{F}_\tau(f^{-1}(\lambda)) > 0, f^{1}(\lambda) \geq f^{-1}(f(\eta)) \geq \eta\}$$

$$\geq \wedge\{ \mu \in I^X / \mathfrak{F}_\tau(\mu) > 0, \mu \geq \eta\}$$

$$\geq \overline{\eta}$$

i.e., $f^{-1}(\overline{f(\eta)}) \geq \overline{\eta}$

So $\overline{f(\eta)} \geq f(\overline{\eta})$

Hence (i) $\Rightarrow$ (ii)

Now, Assume $f(\overline{\eta}) \leq \overline{f(\eta)}$, for $\eta \in I^X$

For each $\mu \in I^Y$, $\tau^1(\mu) > 0$

$\Rightarrow \mathfrak{F}\tau^1(\mu^1) > 0$

$\Rightarrow \mu^1 = \overline{\mu}^1$

Since $f(\overline{f^{-1}(\mu^1)}) \leq \overline{f(f^{-1}(\mu^1))}$

$\leq \mu^1$

We have $f(\overline{f^{-1}(\mu^1)}) \leq f^{-1}(\mu^1))$

Hence,

$$\mathfrak{F}_\tau(f^{-1}(\mu^1)) > 0$$

$$\Rightarrow \quad \mathfrak{F}_\tau(f^{-1}(\mu))^1 > 0$$

$$\Rightarrow \quad \tau(f^{-1}(\mu)) > 0$$

$\therefore \tau^{-1}(\mu) > 0 \Rightarrow \tau(f^{-1}(\mu)) > 0$, for each $\mu \in I^Y$

$\therefore$ f is a weakly gradation preserving map.

Hence (ii) $\Rightarrow$ (i)

$\therefore$ (i) and (ii) are equivalent.

Chapter III

CHAPTER III

ON SEPARATION AXIOMS IN A NEWLY DEFINED FUZZY TOPOLOGY

In this chapter, separation axioms, T_2, T_1, T_0, R_0 and regularity concepts introduced by Rekha Srivastava [15,16] in a newly defined fuzzy topology [17].

SECTION 3.1

BASE SUBBASE AND PRODUCT FUZZY TOPOLOGY

DEFINITION : 3.1.1

An fuzzy topological space (X,τ) is called **discrete** if $\tau(\mu) = ,1$ for every $\mu \in I^X$.

DEFINITION : 3.1.2

Let (X, τ') be an fuzzy topological space and $x \in X$. Then $N \in I^X$ is called an **α-neighbourhood of x** for $\alpha \in (0,1)$ if there exists $\mu \in I^X$ such that $\tau(\mu) > 0$, $\alpha < \mu(X)$and $\mu \leq N$.

N is called a closed **α-neighbourhood** if in addition $\mathfrak{F}_\tau(N) > 0$.

DEFINITION : 3.1.3

A family L of closed α-neighbourhoods of x, $\alpha \in (0,1)$ is called a **local base of closed α-neighbourhoods** if for each α-neighbourhood N of x, there exists some $M \in L$ with $\alpha < M(x)$ and $M \leq N$. A family S of closed α-neighbourhoods is called a **local subbase of closed α-neighbourhoods** if the family of finite intersection of members of S forms a local base of closed α-neighbourhoods of X.

DEFINITION : 3.1.4

Let (X, τ) be a fuzzy topological space. Then a family $\mathcal{B} = \{\lambda \in I^X / \tau(\lambda) > 0\}$ is called a **base** of (X, τ) iff for every $\mu \in I^X$ with $\tau(\mu) > 0$ and for every $x_\alpha \in \mu$, there exists $\lambda \in \mathcal{B}$ such that $x_\alpha \in \lambda \leq \mu$.

THEOREM : 3.1.5

A family $\mathcal{B} = \{\lambda \in I^X / \tau(\lambda) > 0\}$ is a base of (X, τ) iff every $\mu \in I^X$ with $\tau(\mu) > 0$ can be expressed as a union of members of $\mathcal{B}$.

PROOF

Let $\mathcal{B}$ be a base of (X, τ) and $\tau(\mu) > 0$ for some $\mu \in I^X$. Then for each $x_\alpha \in \mu$, there exists $\lambda_{x_\alpha} \in \mathcal{B}$ such that $x_\alpha \in \lambda_{x_\alpha} \leq \mu$.

$$\Rightarrow \bigvee_{x_\alpha \in \mu} \lambda_{x_\alpha} = \mu$$

Conversely, assume that every $\mu \in I^X$ such that $\tau(\mu)>0$ can be written as a union of members of $\mathcal{B}$. That is $\mu = \vee \lambda_i$ for some subfamily $\{\lambda_i / \lambda_i \in \mathcal{B}\}$. Let $x_\alpha \in \mu$. Then $X_\alpha \in \lambda_i$ for some i.

$$\Rightarrow x_\alpha \in \lambda_i \leq \mu$$

Hence $\mathcal{B}$ is a base of (X, τ).

THEOREM : 3.1.6

Let $\mathcal{B} \subseteq I^X$, which contains 0 and 1 satisfy the following property : For every $\mu, \eta \in \mathcal{B}$ and $x_\alpha \in \mu \wedge \eta$ there exists $\lambda_{x_\alpha} \in \mathcal{B}$ such that $x_\alpha \in \lambda_{x_\alpha} \leq \mu \wedge \eta$. Then any map $\tau : I^X \to I$ such that $\tau(0) = \tau(1) = 1$ and $\tau(\lambda) > 0$ iff λ is a union of members of $\mathcal{B}$, defines a gradation of openness on X, with $\mathcal{B}$ as a base.

PROOF

We first show that (X, τ) is a fuzzy topological space.

(01) $\tau(0) = \tau(1) = 1$

(02) Let for each $i \in \Delta$, $\mu_i \in I^X$ be such that $\tau(\mu_i) > 0$. By theorem 3.1.5. μ_i's are union of members of $\mathcal{B}$.

$\Rightarrow$ $\underset{i\in\Delta}{\vee} \mu_i$ is also a union of members of $\mathcal{B}$

$\Rightarrow$ $\tau(\underset{i\in\Delta}{\vee} \mu_i) > 0$

(03) Let $\mu, \eta \in I^X$ such that $\tau(\mu) > 0$, $\tau(\eta) > 0$. Then μ and η must be unions of members of $\mathcal{B}$, say $\mu = \underset{i\in\Delta}{\vee} \mu_i$ and $\eta = \underset{j\in\Delta}{\vee} \eta_j$.

Let $x_\alpha \in \mu \wedge \eta$

Then $x_\alpha \in \mu$ and $x_\alpha \in \eta$

$\Rightarrow$ $x_\alpha \in \mu_i$ for some i and $x_\alpha \in \eta_j$ for some j

$\Rightarrow$ $x_\alpha \in \mu_i \wedge \eta_j \leq \mu \wedge \eta$.

Since $\mu_i, \eta_j \in \mathcal{B}$ and $x_\alpha \in \mu_i \wedge \eta_j$, there exists $\lambda_{x_\alpha} \in \mathcal{B}$ such that $x_\alpha \in \lambda_{x_\alpha} \leq \mu \wedge \eta$. (Since $\mu_i \wedge \eta_j \leq \mu \wedge \eta$)

$\therefore$ $\mu \wedge \eta = \vee \lambda_{x_\alpha}$

$\Rightarrow$ $\tau(\mu \wedge \eta) > 0$

Hence τ is a gradation of openness on X. Further, $\mathcal{B}$ is a base since, whenever $\mu \in I^X$ is such that $\tau(\mu)>0$, μ is a union of members of $\mathcal{B}$, by the definition of τ itself.

PROPOSITION:3.1.7

Let $\mathcal{B}$ be a base on a fuzzy topological space (X, τ). Then for a fuzzy set $\mu \in I^X$, $\tau(\mu) > 0$ iff for every $x_\alpha \in \mu$, there exists $\lambda_{x_\alpha} \in \mathcal{B}$ such that $x_\alpha \in \lambda_{x_\alpha} \leq \mu$.

PROOF

Let $\mathcal{B}$ be a base on a fuzzy topological space (X, τ). Assume the fuzzy set $\mu \in I^X$, such that $\tau(\mu) > 0$.

From the definition of $\mathcal{B}$, we get the result that for every $x_\alpha \in \mu$ there exists $\lambda_{x_\alpha} \in \mathcal{B}$ such that $x_\alpha \in \lambda_{x_\alpha} \leq \mu$.

Conversely, let $\mu \in I^X$ be such that for every $x_\alpha \in \mu$, there exists a $\lambda_{x_\alpha} \in \mathcal{B}$ such that $x_\alpha \in \lambda_{x_\alpha} \leq \mu$.

$$\mu = \bigvee_{x_\alpha \in \mu} \lambda_{x_\alpha}$$

Since $\tau(\lambda_{x_\alpha}) > 0$ for every x_α and τ is a gradation of openness on X,

$$\tau(\mu) > 0.$$

Hence the proof.

DEFINITION : 3.1.8

A family $\mathcal{G} \subseteq I^X$ of a fuzzy topological space is called a **subbase** of (X, τ) iff the family $\mathcal{B}_{\mathcal{G}}$ of finite intersections of members of $\mathcal{G}$ is a base of (X, τ).

THEOREM : 3.1.9

Let $\mathcal{G} \subseteq I^X$, contain 0 and 1. Let τ be any map from $\mathcal{G}$ to I such that $\tau(0) = \tau(1) = 1$ and $\tau(\mu) > 0$, for every $\mu \in \mathcal{G}$. Then the extension $\tau_{\mathcal{G}} : I^X \to I$ given as follows: For each $\mu \in I^X$

$$\tau_{\mathcal{G}}(\mu) = \begin{cases} \text{Inf}\,\{\tau(\mu_1), \tau(\mu_2)\} \text{ if } \mu = \mu_1 \wedge \mu_2 \text{ where } \mu_1, \mu_2 \in \mathcal{G} \\ \text{Sup}\;\tau(\mu_i), \text{ if } \mu = \underset{i}{\vee}\lambda_i \text{ where each } \lambda_i \in \mathcal{B}_{\mathcal{G}} \\ 0, \quad \text{otherwise} \end{cases}$$

defines a gradation of openness on X.

PROOF

Since for every $\mu, \eta \in \mathcal{B}_{\mathcal{G}}$, $\mu \wedge \eta$ itself is a member of $\mathcal{B}_{\mathcal{G}}$. $\mathcal{B}_{\mathcal{G}}$ satisfies the conditions of theorem : 3.1.7.

Further, $\tau_{\mathcal{G}} : I^X \to I$ satisfies the condition that $\tau_{\mathcal{G}}(\mu) > 0$ whenever μ is a union of members of $\mathcal{B}_{\mathcal{G}}$ and zero otherwise and also $\tau_{\mathcal{G}}(0) = \tau_{\mathcal{G}}(1) = 1$.

Hence, from theorem 3.1.7

$\mathcal{B}_{\mathcal{G}}$ is a base of (X, $\tau_{\mathcal{G}}$) and so $\mathcal{G}$ clearly turns out to be a subbase of (X, $\tau_{\mathcal{G}}$).

NOTE

The fuzzy topological space (X, $\tau_{\mathcal{G}}$) described in the above theorem will be called a **fuzzy topological space generated by** $\mathcal{G}$**.**

REMARK : 3.1.10

Since any map τ satisfying the conditions of theorem 3.1.9 gives a gradation of openness on X, the fuzzy topology on X generated by $\mathcal{G}$ is not unique.

DEFINITION : 3.1.11

Let $\{(X_i, \tau_i) \ / \ i \in \Delta\}$ be a family of fuzzy topological space and $p_i : X = \prod_{i\in\Delta} X_i \to X_i$ denote the i^{th} projection map. Consider the family $\mathcal{G} = \{p_i^{-1}(\mu_i) \ / \ \tau_i(\mu_i) > 0, i \in \Delta\}$, and define $\tau : \mathcal{G} \to I$ by $\tau(p_i^{-1}(\mu_i)) = \tau(\mu_i)$. Then τ_g is called the **product of τ_I's** and (X, τ) is called the **product of the fuzzy topological spaces** $\{(X_i, \tau_i) \ / \ i\in\Delta\}$.

REMARK: 3.1.12

(i) $\mathcal{G}$ as defined above, and the corresponding $\mathcal{B}_{\mathcal{G}}$, shall be referred to as the 'Standard subbase' and the 'standard base' for the product fuzzy topology.

(ii) The product fuzzy topology is unique. Further more, **projections are wgp mapping**

SECTION 3.2

HAUSDORFFNESS IN A FUZZY TOPOLOGICAL SPACE

DEFINITION : 3.2.1

A Fuzzy topological space (X, τ) is said to be **Hausdorff** or $\mathbf{T_2}$ if for every x_α, $y_\beta \in X$, $x \neq y$, there exists disjoint $\mu, \eta \in I^X$ with $\tau(\mu) > 0$, $\tau(\eta) > 0$, $x_\alpha \in \mu$ and $y_\beta \in \eta$.

THEOREM : 3.2.2

Let (X, τ) be a fuzzy topological space (fts). Then the following statements are equivalent :

a) (X, τ) is Hausdroff

b) $\Delta_X = \{(x,x) \in X \times X\}$ has positive grade of closedness in $(X \times X, \tau_{X \times X})$ where $\tau_{X \times X}$ is the product fuzzy topology on $X \times X$.

c) For any two weakly gradation preserving maps f, g from a fts (X^1, τ^1) to fts (X, τ), the set $A = \{x^1 \in X^1 / f(x^1) = g(x^1)\}$ has positive grade of closedness in (X^1, τ^1)

d) If $f : (X^1, \tau^1) \to (X, \tau)$ is a weakly gradation preserving map then the graph G of f, that is $\{(x^1, f(x^1)) / x^1 \in X^1\}$ has positive grade of closedness in $(X^1 \times X, \tau_{X^1 x X})$

PROOF

(a) $\Rightarrow$ (b)

Let (X, τ) be Hausdorff

CLAIM : $\tau_{XxX}(X \times X - \Delta_x) > 0$

Let $(x,y)_\alpha \in X \times X - \Delta_x$ Then $x \neq y$

Now x_α and y_α are distinct fuzzy points in X.

Since (X, τ) is Haudorff, there exist $\mu, \eta \in I^X$ such that $x_\alpha \in \mu$ and $y_\alpha \in \eta$,

$\tau(\mu) > 0$, $\tau(\eta) > 0$ and $\mu \wedge \eta = \phi$.

Clearly, $(x, y)_\alpha \in \mu \times \eta$ and $\mu \times \eta \leq X \times X - \Delta_x$.

Further, $\tau_{X \times X}(\mu \times \eta) = \text{Inf}\{\tau(\mu), \tau(\eta)\}$ and hence is positive.

Therefore, for all fuzzy points $(x,y)_\alpha$ in $X \times X - \Delta_x$, we have found a member $\mu \times \eta$ of the standard base of $\tau_{X \times X}$ such that $(x,y)_\alpha \in \mu \times \eta \leq X \times X - \Delta_x$

By proposition 3.1.7., we have, $\tau_{X \times X}(X \times X - \Delta_x) > 0$

(b) $\Rightarrow$ (c)

Let us consider the function $(f, g) : (X^1, \tau^1) \rightarrow (X \times X, \tau_{X \times X})$

given by $(f, g)(x^1) = (f(x^1), g(x^1))$ for every $x^1 \in X^1$.

Also we know,

$$(f, g)^{-1}(\mu \times \eta) = f^{-1}(\mu) \wedge g^{-1}(\eta), \text{ for every } \mu, \eta \in I^X \qquad (1)$$

Now let $\mu \times \eta$ be such that $\tau_{X \times X}(\mu \times \eta) > 0$

$\Rightarrow \tau(\mu) > 0,\ \tau(\eta) > 0$

Since f and g are wgp maps, we get

$\tau^1(f^{-1}(\mu)) > 0$ and $\tau^1(g^{-1}(\eta)) > 0$

$\Rightarrow \quad \tau^1(f^{-1}(\mu) \wedge g^{-1}(\eta)) > 0$

Now using (1) we get,

$\tau^1\{(f,g)^{-1}(\mu \times \eta)\} > 0$

Thus $(f, g)^{-1}$ is also a wgp mapping.

Now, as $A = (f, g)^{-1} \Delta_x$, $X^1 - A = (f, g)^{-1} (X \times X - \Delta_x)$

Hence by (b), we have $\tau^1(X^1 - A) > 0$

(c) $\Rightarrow$ (d)

Consider the projection maps,

$$p_{X^1} : (X^1 \times X, \tau_{X^1 x X}) \rightarrow (X^1, \tau^1)$$

and $p_x : (X^1 \times X, \tau_{X^1 x X}) \rightarrow (X, \tau)$

Then the map $f \circ p_{X^1} : (X^1 \times X, \tau_{X^1 x X}) \rightarrow (X, \tau)$.

Since composition of two wgp maps is a wgp mapping, we get $f \circ p_{X^1}$ is wgp map.

Also,

$$\{(x^1, x) \in X^1 \times X / p_x(x^1,x) = f \circ p_{X^1}, (x^1, x)\}$$

$$= \{(x^1, x) \in X^1 \times X / x = f(x^1)\}$$

$$= \{(x^1, f(x^1)) / x^1 \in X^1\}$$

$$= G$$

Hence, using (c),

we get $\tau_{X^1 x X}(X^1 \times X - G) > 0$

(d) $\Rightarrow$ (a)

In condition (d), put $X^1 = X$ and f = identity map.

Then the set $\{(x^1, f(x^1) / x^1 \in X^1\}$ reduces to Δ_X.

Therefore we have $\tau_{X x X}(X \times X - \Delta x) > 0$

Take any two distinct fuzzy points x_α and y_β in X

Then $x \neq y$ and further, without any loss of generality it can be assumed that $\alpha \leq \beta$. Then $(x, y)_\beta$ is a fuzzy point in $(X \times X - \Delta x)$.

Since $\tau_{X \times X}(X \times X - \Delta x) > 0$, there exists a member, say $\mu \times \eta$ of the standard base of $\tau_{X \times X}$ such that $(x, y)_\beta \in \mu \times \eta \leq X \times X - \Delta x$

Also it is clear that $x_\alpha \in \mu$, $y_\beta \in \eta$ and $\mu \wedge \eta = \phi$

Further $\tau(\mu) > 0$ and $\tau(\eta) > 0$. For, otherwise Inf $\{\tau(\mu), \tau(\eta)\}$ will be zero.

$\Rightarrow \tau_{X \times X}(\mu \times \eta) > 0$, a contradiction.

Thus (X, τ) is Hausdorff.

THEOREM 3.2.3

Let $\{(X_i, \tau_i) / i \in \Delta\}$ be a family of fts. Then their product fts (X, τ) is Hausdorff iff each (X_i, τ_i) is Hausdorff.

PROOF

Let (X_i, τ_i) be Hausdorff for every i.

Let x_α and y_β be any two distinct fuzzy points in X.

Since $x \neq y$, they differ atleast in one co-ordinate, say the i th co-ordinate.

(i.e.) $x_i \neq y_i$

Since (X_i, τ_i) is Hausdorff and $(x_i)_\alpha$ and $(y_i)_\beta$ are distinct fuzzy points in X_i, there exist $\mu_i, \eta_i \in I^X$ such that $\tau(\mu_i) > 0$, $\tau_i(\eta_i) > 0$,

$(x_i)_\alpha \in \mu_i$, $(y_i)_\beta \in \eta_i$ and $\mu_i \wedge \eta_i = \phi$

Now, consider the fuzzy sets $\prod_j \mu_j^1$ and $\prod_j \eta_j^1$ where $\mu_j{}^1 = X_j$ for $j \neq i$ and $\mu_i{}^1 = \mu_i$; $\eta_j{}^1 = X_j$ for $j \neq i$ and $\eta_i{}^1 = \eta_i$

Then it can be easily seen that

$$x_\alpha \in \prod_j \mu_j^1, \ y_\beta \in \prod_j \eta_j^1 \text{ and } (\prod_j \mu_j^1) \cap (\prod_j \eta_j^1) = \phi$$

Further $\tau(\prod_j \mu_j^1) > 0$ and $\tau(\prod_j \eta_j^1) > 0$

Hence, (X, τ) is Hausdorff.

Conversely, let (X, τ) be Hausdorff.

Take any particular factor (X_i, τ_i) and let any $x, y \in X$ with $x_i \neq y_i$

Then the two fuzzy points x_α and y_β in X such that all the co-ordinates in x and y except the i^{th} ones are the same, and i^{th} co-ordinates are x_i and y_i respectively, are distinct.

As (X, τ) is Hausdorff, there exist μ, η in I^X such that $\tau(\mu) > 0$, $\tau(\eta) > 0$, $x_\alpha \in \mu$, $y_\beta \in \eta$ and $\mu \wedge \eta = \phi$.

Now $\tau(\mu) > 0$ and $x_\alpha \in \mu$

Hence there exists a member of the standard base for τ, say $\prod_j \mu_j$ such that $x_\alpha \in \prod_j \mu_j \leq \mu$ (1)

Similarly, there exists a member $\prod_j \eta_j$ in the standard base for τ such that

$$y_\beta \in \prod_j \eta_j \leq \eta \qquad (2)$$

Now $(\prod_j \mu_j) \wedge \prod_j \eta_j = \phi$, since $\mu \wedge \eta = \phi$

Further, $(x_i)_\alpha \in \mu_i$ and $(y_i)_\beta \in \eta_i$

CLAIM: $\mu_i \wedge \eta_i = \phi$

Suppose if $\mu_i \wedge \eta_i \neq \phi$, then there exists $z_i \in X_i$ such that $\mu_i(z_i) > 0$, $\eta_i(z_i) > 0$.

Now consider the point $z \in X$ whose i^{th} co-ordinate is z_i and all the other co-ordinates are the same as those of x or y.

By using (1) and (2)

$\mu_i(x_j) > \alpha$ and $\eta_j(y_j) > \beta$, $i \neq j$

Hence, $\Pi\mu_j(z) > 0$ and $\Pi(\eta_j(z) > 0$, contradicting the disjointness of $\Pi\mu_i$ and $\Pi\eta_i$.

Hence $\mu_i \wedge \eta_i = \phi$

Next, since $\tau_i(\Pi\mu_j) > 0$ and $\tau(\Pi\eta_j) > 0$,

we get that $\tau_i(\mu_i) > 0$ and $\tau_i(\eta_i) > 0$, as $\tau(\Pi\ \mu_j) = \text{Inf}_j\ \tau_j\ (\mu_j)$ and $\tau(\Pi\eta_j) = \text{Inf}_j\ \tau_j(\eta_j)$.

Thus $(X_i\ \tau_i)$ is Hausdorff.

THEOREM: 3.2.4

Hausdorffness is hereditary

PROOF

Let (X, τ) be a Hausdorff fts and (Y, τ_Y) be its subspace.

Let x_α, y_β be two distinct fuzzy points in Y. Consider x_α and y_β as two distinct fuzzy points in X. Then there exist $\mu, \eta \in I^X$, such that $x_\alpha \in \mu$, $y_\beta \in \eta$, $\tau(\mu)>0$, $\tau(\eta) > 0$ and $\mu \wedge \eta = \phi$.

Now let $\mu_Y = \mu \mid Y$ and $\eta_Y = \eta \mid Y$. Then it can easily be seen that,

$$\tau_Y(\mu_Y) > 0,\ \tau_Y(\eta_Y) > 0,\ x_\alpha \in \mu_Y,\ y_\beta \in \eta_Y \text{ and } \mu_Y \wedge \eta_Y = \phi$$

Hence (Y, τ_Y) is Hausdorff.

DEFINITION : 3.2.5

A fuzzy topological space (X, τ) is said to be

i) **T_0** if for every $x, y \in X$, $x \neq y$ there exists $\mu \in I^X$ such that $\tau(\mu) > 0$ and either $\mu(x) = 1$, $\mu(y) = 0$ or $\mu(x) = 0$, $\mu(y) = 1$.

ii) **R_0** if whenever, there exists $\mu \in I^X$ such that $\tau(\mu) > 0$, $\mu(x) = 1$ and $\mu(y)=0$ then there exists $\eta \in I^X$ with $\tau(\eta) > 0$, $\eta(y) = 1$ and $\eta(x) = 0$.

iii) **T_1** if for every $x, y \in X$, $x \neq y$ there exist $\mu, \eta \in I^X$ such that $\tau(\mu) > 0$, $\tau(\eta) > 0$, $\mu(x) = 1$, $\mu(y) = 0$ and $\eta(x) = 0$, $\eta(y) = 1$.

iv) **Regular** if for every $\alpha \in (0,1]$, $\lambda \in I^X$ with $\mathfrak{F}_\tau(\lambda) > 0$, $x \in X$ and $\alpha < 1 - \lambda(x)$, there exist $\mu, \eta \in I^X$ with $\tau(\mu) > 0$, $\tau(\eta) > 0$, $\alpha < \mu(x)$, $\lambda \leq \eta$ and $\mu + \eta \leq X$.

THEOREM : 3.2.6

The separation properties T_0, R_0, T_1 and regularity are productive and hereditary.

The productivity of T_0 R_0, T_1 and regularity properties can be proved in the framework of the above definitions on the parallel lines as in [15, 14] and [7] respectively and hereditary properties for these axioms can be proved as it was done earlier in the case of Hausdorffness.

THEOREM : 3.2.7

In a fuzzy topological space (X, τ) the following statements are equivalent :

a) Δx, the diagonal of X, has positive grade of closedness in $(X \times X, \tau \times d)$ where d is the discrete fuzzy topology on X.

b) $\{x\}$, for every $x \in X$, has positive grade of closedness in (X, τ).

c) (X, τ) is T_1

PROOF

It can be proved on the same lines as in [14].

A nice characterisation of regularity in terms of closed α - neighbourhoods is given in this following theorem which can be proved as Dewan Muslim Ali [8].

THEOREM : 3.2.8

The following statements are equivalent in a fuzzy topological space (X, τ)

i) (X, τ) is regular

ii) For each $x \in X$, $\alpha \in (0,1)$ and $\mu \in I^X$ with $\tau(\mu) > 0$ and $\alpha < \mu(x)$, there exist $\eta \in I^X$ with $\tau(\eta) > 0$ such that $\alpha < \eta(x)$ and $\overline{\eta} \leq \mu$.

iii) For each $x \in X$, and $\alpha \in (0,1)$, x has a local base of closed α-neighbourhoods

iv) For each $x \in X$ and $\alpha \in (0,1)$, x has a local subbase of closed α-neighbourhoods.

Summary and Conclusion

SUMMARY AND CONCLUSION

In 1992, **Hazra Samanta, Chattopadhyay** observed that fuzziness is absent in the concept of openness of a fuzzy subset in the definition of Chang fuzzy topology. They felt that this is a draw back in fuzzyfying the concept of topological spaces. With this in mind, they introduced a new concept called gradation of openness and developed the fundamental concepts of fuzzy topological spaces.

In this thesis we have made an attempt to give a brief survey of various developments in the study of fuzzy topological spaces through this new concept "Gradation of openness" [3].

It is very interesting to redefine and investigate many concepts like base, subbase, separation axioms, etc., [3] using gradation of openness. There is a lot of scope for further research in the study of fuzzy topological spaces through gradation of openness.

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