

Some Interesting Generalizations
Of The Concept - Normal

By

Mercy Jecyntha. J

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND
HIGHER EDUCATION FOR WOMEN - DEEMED UNIVERSITY, COIMBATORE - 641 043
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

MAY - 1999

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May, 1999

Certified as bonafide research work

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Acknowledgement

ACKNOWLEDGEMENT

The investigator is grateful to **Dr. (Tmt.) RAJAMMAL P. DEVADAS**, M.A., M.Sc., Ph.D. (Ohio State), D.Sc. (Madras), Hon. D.H.L. (Oregon State), Hon. D.H.L. (Ohio State), Hon. D.Sc., (C.A. Agri., Kanpur), Hon. D.Sc. (University of Vlster, Northern Ireland), Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for providing the opportunity to conduct her investigation in this esteemed university.

The author expresses her gratitude to **Dr. Mrs. LAKSHMI SANTA RAJAGOPAL**, M.S., (Tennessee), Ph.D. (Madras), Vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for providing facilities to carry out the study.

The author extends her deep sense of gratitude to **Dr. Mrs. SAROJA PRABHAKARAN**, M.A., Diploma in Education (Madras), Ph.D. (Mother Teresa), Registrar and **Dr. (Mrs.) S. SIVAKAMASUNDARI**, M.Sc. (Annamalai), M.Phil, Ph.D. (Madras), Dean, Faculty of Science, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for facilitating the conduct of the study.

The author wishes to express her sincere gratitude to **Dr. (Miss.) K.N. MEENAKSHI**, M.Sc., Ph.D., Professor and Head of

the Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for the encouragement extended by her, during the entire course of the study.

*The author wishes to place on record her indebtedness to **Mrs. A. KALAICHELVI**, M.Sc., B.Ed., M.Phil., Lecturer (Senior Scale), Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for her constant benevolent help, invaluable advice and illuminating guidance. She gratefully acknowledges her persistent efforts and affectionate endeavours.*

*The author would like to extend her thanks to other members of the **STAFF** who were responsible for the good finish of this dissertation.*

*The author is greatly indebted to her loving **PARENTS, SISTERS, BROTHERS** and dear **FRIENDS** and all family members for being the motivating forces behind this dissertation and providing moral support in carrying out the work.*

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Introduction

INTRODUCTION

Mathematics is of profound significance in the universe, not because it exhibits principles that we obey, but because it exhibits principles that we impose.

- J.N.N. Sullivan

The study of Normal Spaces has attracted the attention of many famous topologists, Takashi Noiri, J.C. Kelley, C. Wenjen, E.P. Lane, N. Levine, M.K. Singal, Asha Rani Singal and S.P. Arya. The aim of this thesis is to study some interesting generalizations of the concept – Normal. Some interesting generalizations discussed in this thesis are :

1. Mildly normal, M.K.Singal and Asha Rani Singal, 1973, (2.1.3)
2. Quasi-normal, Zaitsev, 1968, (2.2.3)
3. x -normal, E.V. Shchepin, 1972, (2.2.4)
4. s -normal, S.N. Maheswari and R. Prasad, 1981, (2.3.1)
5. δ -normal, John Mack, 1970, (2.4.4)
6. δ -normally separated, John Mack, 1970, (2.4.5)
7. Weakly δ -normally separated, John Mack, 1970, (2.4.6)
8. Lightly normal, Asha Rani Singal and S.B. Nimse, 1997, (2.4.8)
9. Weakly normal, Dugundji, 1966, (2.4.15)

Section 1 of the first chapter deals with preliminary definitions. In the second section of this chapter, some fundamental and interesting results regarding normal spaces are studied.

The second chapter is devoted to the study of different generalizations of normal spaces. The definitions of the concepts listed above are collected and their properties are studied.

The first section of this chapter deals with Mildly Normal Spaces. In 1973, M.K. Singal and Asha Rani Singal [19] introduced the concept of Mildly Normal Spaces and obtained several properties of such a space. The important theorems are:

THEOREM : I

Every closed, continuous and open image of a mildly normal space is mildly normal.

THEOREM : II

For a space X , the following are equivalent:

- a) X is mildly normal.
- b) For any $A \in RC(X)$ and any $V \in RO(X)$ such that $A \subset V$, there exists an open set U such that $A \subset U \subset Cl U \subset V$.
- c) For any $A \in RC(X)$ and $V \in RO(X)$ such that $A \subset V$, there exists $U \in RO(X)$, such that $A \subset U \subset Cl U \subset V$.

THEOREM : III

An almost regular almost compact space is mildly normal.

Takashi Noiri [21] improved the above Theorem III as follows :

“The almost – continuous almost – closed image of a normal space is mildly normal”. He also proved that “Mild normality is invariant under θ -continuous almost – open and almost – closed surjections”.

These results are discussed in this section in detail.

Sunder Lal and M.S. Rahman [20] improved the above 3 theorems for Quasi-normal spaces.

1. An almost continuous almost closed image of a normal space is quasi-normal.
2. For a topological space X , the following are equivalent:
 - a) X is quasi-normal
 - b) For every π -closed set A and every π -open set B containing A , there exists an open set U such that $A \subset U \subset Cl U \subset B$.
 - c) For every π -closed set A and every π -open set B containing A , there exists a regular open set V such that $A \subset V \subset Cl V \subset B$.
 - d) For every π -closed set A and every open set B containing A , there exists a π -open set V such that $A \subset V \subset Cl V \subset B$.
 - e) For every pair of disjoint π -closed sets A and B there exists a continuous function $f : X \rightarrow [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

3. If X is an almost regular space in which every π -closed set is almost compact, then X is quasi-normal.

These results are discussed in detail in Section 2 of the second chapter. Katetov and Tong characterized normal spaces in terms of insertion of a continuous function between two comparable semi-continuous functions. The same way Sunder Lal and M.S. Rahman characterized quasi-normal spaces in terms of insertion of a continuous function between 2 comparable π -semi continuous functions. This characterization is also studied in this section.

The third section of this chapter is devoted to the study of s -normal spaces. A result similar to the above Theorem II is proved for s -normal spaces.

“For a space X , the following are equivalent :

- a) X is s -normal
- b) For any pair of disjoint closed sets A and B , there exist disjoint gs - open sets U and V such that $A \subseteq U$ and $B \subseteq V$.
- c) For every closed set A and an open set B containing A , there exists a gs - open set U such that $A \subseteq U \subseteq scl U \subseteq B$.
- d) For every closed set A and every g -open set B containing A , there exists a semi-open set U such that $A \subseteq U \subseteq scl U \subseteq Int B$.
- e) For every g -closed set A and every open set B containing A , there exists a semi-open set U such that $A \subseteq scl A \subseteq U \subseteq scl U \subseteq B$.

The fourth section deals with lightly normal spaces. Light normality is a simultaneous generalization of δ -normality and δ -normally separation of John Mack [11]. Every Tychonoff, lightly normal space is weakly normal. Light normality characterizes countable paracompact spaces in as much as space X is countably paracompact iff $X \times I$ is lightly normal. A characterization similar to Mildly normal, quasi-normal, s -normal are also obtained for lightly normal spaces by Asha Rani Singal and S.B. Nimse [2]. Minimal- P and P -closed spaces (2.4.21) for $P = \delta$ -normally separation + lightly normal + Tychonoff are also studied. It is shown that for these spaces, minimal - P spaces are precisely the compact Hausdorff spaces.

The last chapter is devoted to the study of the concept – Normal, in bitopological spaces. The first section deals with pairwise normal spaces and the second section deals with pairwise perfectly normal spaces. A characterization of pairwise normal spaces is obtained by establishing some results about the extension of semi-continuous functions on a closed subset of a topological space. A characterization of pairwise perfectly normal spaces for T_1 topologies is obtained.

Review of Literature

REVIEW OF LITERATURE

The concept – Normal plays an important role in Topology. Many authors have worked in this topic. Some important contributions to the study of normal spaces are published by Kelley, Noiri, Levine, Lane, Arya, M.K. Singal and Asha Rani Singal. Since the number of papers published on this concept is numerous we restrict ourselves to a few papers in this review of literature. We have given a brief review of nine important papers, dealing with different types of generalizations of normality, products of normal spaces and normality in bitopological spaces.

1. DORSETT, CHARLES

“Semi Normal Spaces [7]”

In this paper, semi normal spaces are defined and characterized. A topological space (X, τ) is said to be semi normal if for each pair of disjoint semi closed sets A and B , disjoint semi open sets U and V exist such that $A \subset U$ and $B \subset V$. Define an equivalence relation on (X, τ) by xRy iff $Cl \{x\} = Cl \{y\}$ [respectively, xRy iff $scl\{x\} = scl\{y\}$]; let X_0 [respectively X_3] and $Q(X_0)$ [respectively $Q(X_3)$] denote the sets of equivalence classes and the corresponding decomposition topologies.

Here, it is shown that (X, τ) semi normal is equivalent to saying that

- i) For each semi closed set A and each semi open U , $A \subset U$, there exists a semi open, semi closed V such that $A \subset V \subset U$.

- ii) For each α -set A , (A, T_A) is semi normal.
- iii) $(X_3, Q(X_3))$ is semi normal

Other equivalent formulations are also given. For a semi compact and R_0 space (X, τ) , (X, τ) semi normal is equivalent to saying that (X, τ) is:

- (a) s-normal; (b) s-regular (c) R_1 ;
- (d) regular; (e) completely regular; (f) Normal

Continuous, closed, open images of a semi normal [semi regular] space need not be such.

2. *MUNSHI, B.M.*

“Separation Axioms [15]”

N. Levine defined a set A to be g -closed if $Cl A \subseteq 0$ whenever $A \subseteq 0$ and 0 is open and a set A to be g -open if $F \subseteq Int A$ whenever F is closed and $F \subseteq A$. In this note, the author uses these ideas to define g -regular, g -normal and completely g -regular spaces and obtained results regarding these concepts.

3. *DI BARI, CRISTINA M; GIOVANNELLI, NICOLO*

“Weakly normal topological spaces and upper semi continuous functions [6]”.

In the first part of this paper, the authors introduced the notion of weakly normal topological spaces and characterized these spaces in terms of multi functions and subsets of product topological spaces. In the second part, the authors established an analogous characterization of normal topological spaces.

4. **BESLAGIC, AMER; CHIBA, KEIKO**

“Normality of product spaces [4]”

The authors have studied the normality of product spaces $X \times Y$, where X and Y satisfy special conditions (M-space, paracompact P-space). They have also dealt with Lashnev spaces (A Hausdorff space which is the closed image of a metric space).

The two interesting results are :

- (1) Assume X is hereditarily normal, Y is Lashnev and G is an open subset of $X \times Y$. If G is countably paracompact, then G is normal.
- (2) Assume X is hereditarily collection wise normal, Y is Lashnev and G is an open subset of $X \times Y$. If G is normal, then G is collection wise normal.

5. **PRZYMUSINSKI, TEODOR, C.**

“Products of normal spaces[17]”

Topological properties such as normality, paracompactness and the Lindelof property are not inherited by finite products. In this paper, the author gives a useful survey of results on the product of normal spaces.

CONTENTS

- (i) Products of normal spaces need not be normal
- (ii) Products with a compact factor
- (iii) Products with a metric factor
- (iv) Hereditarily normal products

- (v) Infinite products
- (vi) Σ -products
- (vii) Some open problems

Most of the results are presented with complete proofs and a variety of examples are discussed.

6. GUTIERREZ, ANGEL; ROMAGUERA, SALVADOR

“C-binary relations and pairwise normality [10]”

In this paper, the following result of Katetov and Tong was generalized to a characterization of pairwise normal spaces.

Result :

A space X is normal iff for real valued functions g, f on X with $g \leq f$ such that g is upper semi continuous and f is lower semi continuous, there exists a continuous function h on X such that $g \leq h \leq f$.

7. RANI, ASHA ; JAIN, S.C.

“Pairwise perfectly normal spaces [18]”

A bitopological space is said to be pairwise perfectly normal if it is pairwise normal and every pairwise closed subset is pairwise G_δ . The following results are proved here :

- (1) Every pairwise perfectly normal space is pairwise completely normal

- (2) (X, P_1, P_2) is pairwise perfectly normal iff for each non-empty pairwise continuous mapping $f : (X, P_1, P_2) \rightarrow (I, \mathcal{U}_I, \mathcal{U}_I)$ such that $f^{-1}(0) = A$ and $f(b) = 1$ (I is the unit interval and \mathcal{U} is the usual topology on \mathbb{R})
- (3) A pairwise regular space with a σ -pairwise locally finite pairwise base is pairwise perfectly normal.
- (4) A pairwise closed pairwise continuous image of a pairwise perfectly normal space is pairwise perfectly normal.

8. TAKASHI NOIRI

“Mildly normal spaces and some functions [22]”

Using regular g -closed sets, the author obtained a characterization of mildly normal spaces.

9. YAKIVCHIK, ANDREW, N.

“Weakly normal topological spaces and products [23]”

The behaviour of the property of weak normality with respect to topological products is examined versus normality. The following generalization of Tamano’s theorem is proved :

If $X \times \beta X$ is weakly normal, then X is paracompact. Some versions of Katetov’s theorem are obtained. In particular, it is proved that if $X \times Y$ is hereditarily weakly normal then either each countable subset of X is closed or each convergent free sequence in Y has countable cofinality.

Chapter I

CHAPTER I

SECTION 1.1

PRELIMINARY DEFINITIONS

DEFINITION : 1.1.1

A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X , there exist neighbourhoods U_1 and U_2 of x_1 and x_2 respectively that are disjoint.

DEFINITION : 1.1.2

A space X is said to be **compact** if every open covering \mathcal{A} of X contains a finite sub collection that also covers X .

DEFINITION : 1.1.3

A space for which every open covering \mathcal{A} contains a countable sub covering is called a **Lindelof space**.

DEFINITION : 1.1.4

Suppose that one point sets are closed in X . Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from x , there exist disjoint open sets containing x and B , respectively.

DEFINITION : 1.1.5

A topological space is said to be a **T_0 space** if for $x \neq y$, there exist neighbourhoods of one of the points not containing the other.

DEFINITION : 1.1.6

A space which is completely regular and T_0 is called **Tychonoff space**.

DEFINITION : 1.1.7

Let X be a topological space. A collection \mathcal{A} of subsets of X is said to be **locally finite** if every point of X has a neighbourhood that intersects only finitely many elements of \mathcal{A} .

DEFINITION : 1.1.8

A collection \mathcal{B} of subsets of X is said to be **σ -locally finite base** if \mathcal{B} can be written as countable union of collections \mathcal{B}_n each of which is locally finite.

DEFINITION : 1.1.9

A space X is **paracompact** if it is Hausdorff and if every open covering \mathcal{A} of X has a locally finite open refinement \mathcal{B} that covers X .

DEFINITION : 1.1.10

A space X is called **Countably paracompact** if every countable open covering of X has a locally finite refinement.

DEFINITION : 1.1.11

A space X is called **Countably compact** if every countable open covering of X has a finite subcovering.

DEFINITION : 1.1.12

A space X is said to be **almost compact** if each open covering of X has a finite sub family, the closures of whose members cover X .

DEFINITION : 1.1.13

In a topological space X , a set A is said to be **semi-open** if there exists an open set U such that $U \subseteq A \subseteq \text{Cl } U$.

DEFINITION : 1.1.14

The complement of a semi-open set is **semi-closed**.

DEFINITION : 1.1.15

The **semi-closure** of a set A is the intersection of all semi-closed sets containing A .

DEFINITION : 1.1.16

The **semi-interior** of a set A is the union of all semi-open sets contained in A .

SECTION 1.2

NORMAL SPACES

In this section, we give some fundamental and interesting results regarding normal spaces.

DEFINITION : 1.2.1

The space X is said to be **normal** if for each pair A, B of disjoint closed sets of X there exist disjoint open sets containing A and B respectively.

EXAMPLE : 1.2.2

A discrete space is normal.

An indiscrete space is vacuously normal.

EXAMPLE : 1.2.3

The space R_l (the set of real numbers with the lower limit topology) is normal.

RESULT : 1.2.4

1. Subspace of a normal space need not be normal
2. Closed subspace of a normal space is normal
3. A product of normal spaces need not be normal (R_l^2 is not normal).
4. Every metrizable space is normal
5. Every compact Hausdorff space is normal
6. Every regular Lindelof space is normal.
7. Every well ordered set X is normal in the ordered topology.
8. Every regular space with a countable basis is normal.

9. If X is normal every pair of disjoint closed sets of X have neighbourhoods whose closures are disjoint
10. A finite regular space is normal
11. Let X be a topological space. Let one point sets in X be closed. Then X is normal if given a closed set A and an open set U containing A , there is an open set V containing A such that $\text{Cl } V \subset U$.
12. A compact regular space is normal.
13. Urysohn theorem
A topological space X is normal iff for any pair of disjoint closed sets F and K there is a continuous function $f : X \rightarrow [0,1]$ such that $f \equiv 0$ on F and $f \equiv 1$ on K .
14. Tietze's theorem
A topological space X is normal iff any continuous function $f:A \rightarrow [-1,1]$ for any closed set $A \subset X$, can be continuously extended to X .
15. A continuous image of a normal space need not be normal.
For example, let X be a space which is not normal and let X_d be the same set with discrete topology. Then X_d is normal and the identity map $\text{id} : X_d \rightarrow X$ is continuous.
16. Let X be a normal space. Let $f : X \rightarrow Y$ be a continuous, closed onto map. Then Y is normal.
17. Every paracompact space X is normal.

18. Let $X = \bigcup_{i=1}^n X_i$, where each X_i is clopen and normal. Then X is normal.
19. Let A be a closed subset of a normal space X . Then X/A is normal.

Chapter II

CHAPTER II

SECTION 2.1

MILDLY NORMAL SPACES

DEFINITION : 2.1.1

Let A be a subset of a topological space X . A is said to be **regularly open** if $\text{Int Cl } A = A$.

DEFINITION : 2.1.2

A is said to be **regularly closed** if $\text{Cl Int } A = A$.

$\text{RO}(X)$ and $\text{RC}(X)$ will denote the family of all regularly open and regularly closed sets in a topological space X , respectively.

DEFINITION : 2.1.3

A space X is said to be **mildly normal** if for every pair of disjoint F_1 and $F_2 \in \text{RC}(X)$ there exist disjoint open sets U_1 and U_2 such that $F_1 \subset U_1$ and $F_2 \subset U_2$.

Next we prove some interesting results regarding mildly normal spaces. For this purpose, we give the following definitions.

DEFINITION : 2.1.4

A mapping $f : X \rightarrow Y$ is said to be **almost continuous** if for each point $x \in X$ and each neighbourhood V of $f(x)$ in Y , there exists a neighbourhood U of x such that $f(U) \subset \text{Int Cl } V$.

DEFINITION : 2.1.5

A mapping $f : X \rightarrow Y$ is said to be **θ -continuous** if for each point $x \in X$ and each neighbourhood V of $f(x)$ in Y , there exists a neighbourhood U of x such that $f(Cl U) \subset Cl V$.

REMARK : 2.1.6

Continuity \Rightarrow Almost – continuity $\Rightarrow \theta$ – continuity.

REMARK: 2.1.7

The almost continuity of a mapping $f : X \rightarrow Y$ is characterized by the following statements:

- (1) For each $V \in RO(Y)$, $f^{-1}(V)$ is open in X .
- (2) For each $B \in RC(Y)$, $f^{-1}(B)$ is closed in X .

DEFINITION : 2.1.8

A mapping $f : X \rightarrow Y$ is said to be **almost-open** if for each $U \in RO(X)$, $f(U)$ is open in Y .

DEFINITION : 2.1.9

A mapping $f : X \rightarrow Y$ is said to be **almost-closed** if for each $U \in RC(X)$, $f(U)$ is closed in Y .

REMARK : 2.1.10

Every open (closed) mapping is almost-open (almost-closed) but the converse is not necessarily true.

EXAMPLE : 2.1.11

Let R be the set of real numbers and τ consist of ϕ , R and the complements of all countable subsets of R .

Let \mathcal{U} denote the usual topology on R . Let i be the identity mapping from (R, τ) to (R, \mathcal{U}) . Then i is almost-open and almost-closed but it is neither open nor closed.

THEOREM : 2.1.12

The almost continuous almost-closed image of a normal space is mildly normal.

PROOF :

To prove this theorem we need the following lemma.

LEMMA:

A surjective mapping $f : X \rightarrow Y$ is almost - closed iff for any subset $S \subset Y$ and any $U \in RO(X)$ containing $f^{-1}(S)$, there exists an open set V in Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

PROOF OF LEMMA:

NECESSITY:

Assume f is almost-closed. For any subset $S \subset Y$ and any $U \in RO(X)$ containing $f^{-1}(S)$, let us put $V = Y - f(X-U)$. Then, since $f^{-1}(S) \subset U$, we have $S \subset V$. Since f is almost-closed and $U \in RO(X)$, V is open in Y . Hence we obtain $f^{-1}(V) \subset U$.

SUFFICIENCY:

Assume $A \in RC(X)$ and $y \in Y - f(A)$.

Then we have $f^{-1}(Y) \subset X - A \in RO(X)$. By the hypothesis, there exists an open set V in Y such that $y \in V$ and $f^{-1}(V) \subset X - A$.

Thus we obtain $y \in V \subset Y - f(A)$. This implies that $Y - f(A)$ is open in Y . Hence $f(A)$ is closed. Consequently, f is almost-closed.

PROOF OF THE THEOREM:

Let X be a normal space (not necessarily T_1) and $f : X \rightarrow Y$ be an almost-continuous and almost-closed surjection. Suppose B_1 and B_2 are disjoint regularly closed sets in Y . Since f is almost-continuous, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are disjoint closed sets in X . By the normality of X , there exist disjoint open sets U_1 and U_2 such that $f^{-1}(B_j) \subset U_j$ for $j=1,2$. Since U_1 and U_2 are disjoint open, $\text{Int Cl } U_1$ and $\text{Int Cl } U_2$ are disjoint regularly open sets such that $f^{-1}(B_j) \subset U_j \subset \text{Int Cl } U_j$ for $j=1,2$. Since f is almost-closed, by the above Lemma, there exists an open set V_j in Y such that $B_j \subset V_j$ and $f^{-1}(V_j) \subset \text{Int Cl } U_j$ for $j=1,2$. Since f is surjective, V_1 and V_2 are disjoint. This implies that Y is mildly normal.

THEOREM : 2.1.13

The mildly normality is invariant under θ -continuous, almost-open and almost-closed surjections.

PROOF:

To prove this theorem, we need the following 3 lemmas.

LEMMA 1:

If $f : X \rightarrow Y$ is almost-open and θ -continuous then f is almost-continuous.

PROOF:

For each point $x \in X$ and each neighbourhood V of $f(x)$ in Y , there exists an open neighbourhood U of x such that $f(\text{Cl } U) \subset \text{Cl } V$ because f is θ -continuous. Since f is almost-open and $\text{Int } \text{Cl } U \in \text{RO}(X)$, $f(\text{Int } \text{Cl } U)$ is open and hence we have $f(U) \subset f(\text{Int } \text{Cl } U) \subset \text{Int}(f(\text{Cl } U))$. Thus we obtain $f(U) \subset \text{Int } \text{Cl } V$. This shows that f is almost-continuous.

LEMMA 2:

If a mapping $f : X \rightarrow Y$ is almost-continuous and almost-open, then,

(1) For each $V \in \text{RO}(Y)$, $f^{-1}(V) \in \text{RO}(X)$;

(2) For each $B \in \text{RC}(Y)$, $f^{-1}(B) \in \text{RC}(X)$.

PROOF:

(1) If $V \in \text{RO}(Y)$, then $f^{-1}(V)$ is open and hence we have $f^{-1}(V) \subset \text{Int } \text{Cl } f^{-1}(V)$.

On the other hand, since f is almost-continuous and $\text{Cl } V \in \text{RC}(Y)$, $f^{-1}(\text{Cl } V)$ is closed and hence we have $\text{Int } \text{Cl } f^{-1}(V) \subset \text{Cl } f^{-1}(V) \subset f^{-1}(\text{Cl } V)$.

Moreover, since f is almost-open and $\text{Int } \text{Cl } f^{-1}(V) \in \text{RO}(X)$, $f[\text{Int } \text{Cl } f^{-1}(V)]$

is open. Hence we have $f[\text{Int Cl } f^{-1}(V)] \subset \text{Int Cl } V = V$. Thus we obtain $\text{Int Cl } f^{-1}(V) \subset f^{-1}(V)$.

The proof of (2) follows easily from (1) and the following two facts:

- (a) $f^{-1}(Y-V) = X - f^{-1}(V)$ for each subset $V \subset Y$
- (b) $V \in \text{RO}(Y)$ iff $Y-V \in \text{RC}(Y)$.

LEMMA 3:

For a space X , the following are equivalent:

- (a) X is mildly normal.
- (b) For any $A \in \text{RC}(X)$ and any $V \in \text{RO}(X)$ such that $A \subset V$, there exists an open set U such that $A \subset U \subset \text{Cl } U \subset V$.
- (c) For any $A \in \text{RC}(X)$ and any $V \in \text{RO}(X)$ such that $A \subset V$, there exists $U \in \text{RO}(X)$ such that $A \subset U \subset \text{Cl } U \subset V$.

PROOF OF THE THEOREM :

Let X be a mildly normal space and $f : X \rightarrow Y$ be a θ -continuous, almost-open and almost-closed surjection. Let us suppose $A \in \text{RC}(Y)$, $V \in \text{RO}(Y)$ and $A \subset V$. Since f is θ -continuous and almost-open, by Lemma 1 and Lemma 2 we have $f^{-1}(A) \in \text{RC}(X)$, $f^{-1}(V) \in \text{RO}(X)$ and $f^{-1}(A) \subset f^{-1}(V)$. Since X is mildly normal, by Lemma 3, there exists $W \in \text{RO}(X)$ such that $f^{-1}(A) \subset W \subset \text{Cl } W \subset f^{-1}(V)$. Hence we have $A \subset f(W) \subset f(\text{Cl } W) \subset V$ because f is surjective. Since f is almost-open and almost-closed, $f(W)$ is open

and $f(Cl W)$ is closed. Therefore, let us put $U=f(W)$ and we have $A \subset U \subset Cl U \subset V$. By Lemma 3, we observe that Y is mildly normal.

COROLLARY : 2.1.14

The mildly normality is invariant under continuous, open and closed surjections.

SECTION 2.2

QUASI-NORMAL SPACES

DEFINITION : 2.2.1

A finite union of regular open sets is called a π -open set.

DEFINITION : 2.2.2

The complement of a π -open set is called a π -closed set.

DEFINITION : 2.2.3

A regular space is **quasi-normal**, if any two disjoint π -closed sets are separated by open sets.

DEFINITION : 2.2.4

A regular space is **x-normal**, if any two disjoint regular closed sets are separated by open sets.

REMARK : 2.2.5

For a topological space X , normal \Rightarrow quasi-normal \Rightarrow mildly normal.

THEOREM 2.2.6

For a topological space X , the following are equivalent:

- (a) X is quasi-normal.
- (b) For every π -closed set A and every π -open set B containing A , there exists an open set U such that $A \subset U \subset \text{Cl } U \subset B$.
- (c) For every π -closed set A and every π -open set B containing A there exists a regular open set V such that $A \subset V \subset \text{Cl } V \subset B$.

- (d) For every π -closed set A and every open set B containing A there exists a π -open set V such that $A \subset V \subset \overline{V} \subset B$.
- (e) For every pair of disjoint π -closed sets A and B , there exists a continuous function $f: X \rightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

PROOF :

- (a) \Rightarrow (b) is obvious
- (b) \Rightarrow (c) follows by taking $V = \text{Int } \overline{A} \cup B$.
- (c) \Rightarrow (d) follows since a regular open set is π -open
- (d) \Rightarrow (e) and (e) \Rightarrow (a) are obvious.

DEFINITION : 2.2.7

A real valued function f defined on a space X is said to be **π -lower semi continuous** (π -lsc) if for each real number r , the set $\{x: f(x) > r\}$ is a π -open set.

DEFINITION : 2.2.8

A real-valued function f is defined to be **π -upper semi continuous** (π -usc) if $-f$ is π -lower semi continuous.

DEFINITION : 2.2.9

Let ρ be a binary relation in a set S . Let $\bar{\rho}$ denote the relation defined by setting $x \bar{\rho} y$ iff $y \rho v \Rightarrow x \rho v$ and $u \rho x \Rightarrow u \rho y$ for all u, v in S .

DEFINITION : 2.2.10

If X is a topological space, then ρ defined on $P(X)$, power set of X is said to be a **strong, binary relation** if it satisfies the following conditions:

- (a) If $A_i \rho B_j$ for each i in $\{1, 2, \dots, m\}$ and for each j in $\{1, 2, \dots, n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for each i in $\{1, 2, \dots, m\}$ and for each j in $\{1, 2, \dots, n\}$.
- (b) If $A \subseteq B$, then $A \bar{\rho} B$.
- (c) If $A \rho B$ then $Cl A \subset Int B$.

DEFINITION : 2.2.11

If f is a real-valued function and l is a real number, then any set $A(f, l)$ such that $\{x : f(x) < l\} \subset A(f, l) \subset \{x : f(x) \leq l\}$ is called a **lower indefinite cut set** in the domain of f at the level l .

Now we state E.P. Lane's theorem [13] which is used to prove the following theorem.

THEOREM : 2.2.12

Let g and f be real-valued functions on X such that $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $r < s$, then $A(f, r) \rho A(g, s)$, then there exists a continuous function h defined on X such that $g \leq h \leq f$.

THEOREM : 2.2.13

A topological space X is quasi-normal iff for each π -lsc function f and each π -usc function g with $g \leq f$, there exists a continuous function h such that $g \leq h \leq f$.

PROOF :

Suppose X is quasi-normal. We define a binary relation ρ on $P(X)$ by setting, $A \rho B$ iff $Cl A \subset F \subset G \subset Int B$ for some π -closed set F and some π -open set G .

We show that ρ is a strong, binary relation. Lane proved that if X is mildly normal, then ρ^* defined on $P(X)$ by setting $A \rho^* B$ iff $Cl A \subset F \subset G \subset Int B$ for some regular closed set F and a regular open set G is a strong binary relation. In view of Theorem 2.2.6 (c), the sets F and G in the definition of ρ may be taken as regular closed and regular open, respectively. Also, every quasi-normal space is mildly normal and every regular closed set is π -closed and every regular open set is π -open. Thus on a quasi-normal space ρ coincides with ρ^* and is therefore a strong binary relation.

Let f and g be functions such that $g \leq f$, g is π -usc and f is π -lsc.

For each rational number t , let $A(f, t)$ and $A(g, t)$ be lower indefinite cut sets for f and g .

If r and s are 2 rational numbers such that $r < s$ then $\overline{A(f, r)} \subset \{x : f(x) \leq r\} \subset \{x : g(x) < s\} \subset Int(A(g, s))$. Also $\{x : f(x) \leq r\}$ is a

π -closed set and $\{x : g(x) < s\}$ is a π -open set. By the definition of ρ we have $A(f, r) \rho A(g, s)$. By the above mentioned theorem of Lane, Theorem 2.2.12, there exists a continuous function h such that $g \leq h \leq f$.

Conversely, let X satisfy the given condition and let A and B be two disjoint π -closed sets in X . If g denotes the characteristic function of A and f that of $X-B$, then g is π -usc, f is π lsc and $g \leq f$. If h be a continuous function such that $g \leq h \leq f$, then $G = \{x : h(x) > 2^{-1}\}$. By Theorem 2.2.6 (b), X is quasi-normal.

DEFINITION : 2.2.14

A space is defined to be **almost regular** if a point and a regular closed set not containing the point can be separated by open sets.

DEFINITION : 2.2.15

A space is said to be **almost compact** if every open cover of the space has a finite sub family closures of whose members cover the space.

RESULT : 2.2.16

A regular closed subset of an almost compact space is almost compact.

THEOREM : 2.2.17

If X is an almost regular space in which every π -closed set is almost compact then X is quasi-normal.

PROOF :

Let A and B be two disjoint π -closed sets and let $x \in A$.

Then $x \notin B = \bigcap_{i=1}^n B_i$ where B_i are regular closed sets. Thus $x \notin B_j$ for

some regular closed set B_j . Since X is almost regular, there exists open sets G_x and H_x such that $x \in G_x$, $B \subset B_j \subset H_x$ and $\text{Cl } G_x \cap \text{Cl } H_x = \phi$.

Now $\{G_x \cap A : x \in A\}$ is a cover of A by sets open in A . Since A is almost compact, there exists a finite subfamily $\{G_{x_i} \cap A : i=1, 2, \dots, m\}$ closures

of whose members cover A . But in this case $A \subset \bigcup_{i=1}^m \text{Cl } G_{x_i}$.

Also, if $H = \bigcap_{i=1}^m H_{x_i}$ and $G = X - \bigcap_{i=1}^m \text{Cl } H_{x_i}$ then G and H are disjoint

open sets enclosing A and B respectively. If A is a π -closed set and $x \notin A$, then there exists a regular closed set B such that $A \subset B$ and $x \notin B$. Hence X is quasi-normal.

THEOREM : 2.2.18

X is almost regular iff every π -closed set A and every point $x \notin A$ are separated by open sets.

Using this theorem we prove,

THEOREM : 2.2.19

Every almost-regular Lindelof space is quasi-normal.

PROOF:

Let X be an almost regular Lindelof space and let A and B be two disjoint π -closed subsets of X . If $x \in A$, then $x \notin B$. By Theorem 2.2.18 there exists an open set U_x such that $x \in U_x$ and $\text{Cl } U_x \cap B = \phi$.

Then $\hat{U} = \{U_x : x \in A\}$ is an open cover of A . Since A being a closed subset of X is Lindelof, \hat{U} admits of a countable sub covering $\{U_n : n=1,2, \dots\}$. Similarly, we have for each y in B an open subset V_y such that $y \in V_y$ and $\text{Cl } V_y \cap A = \phi$. Again $\hat{V} = \{V_y : y \in B\}$ is an open cover of B which admits of a countable sub cover $\{V_n : n = 1, 2, \dots\}$. Let $A_n = U_n - \cup \{\text{Cl } V_m : m \leq n\}$ and $B_n = V_n - \cup \{\text{Cl } U_m : m \leq n\}$ for each $n=1, 2, \dots$. Then $A_n \cap B_m = \phi$ for all m, n . If $G = \bigcup_{n=1}^{\infty} A_n$ and $H = \bigcup_{n=1}^{\infty} B_n$ then G and H are disjoint open sets containing A and B respectively. Therefore X is quasi-normal.

In a similar manner as above, we can prove the following theorems.

THEOREM : 2.2.20

An almost-regular space with a σ -locally finite base is quasi-normal.

THEOREM : 2.2.21

A nearly paracompact Hausdorff space is quasi-normal.

In Section 2.1 it has been proved that an almost-continuous almost-closed image of a normal space is mildly normal.

Now we prove the image is in fact, quasi-normal.

THEOREM : 2.2.22

An almost-continuous almost-closed image of a normal space is quasi-normal.

PROOF :

Let X be normal and let $f : X \rightarrow Y$ be an almost-continuous almost-closed surjection. Suppose that A and B are two disjoint π -closed sets in Y . Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets in X . By the normality of X , we find disjoint regular open sets U and V in X , such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let $W = Y - f(X - U)$. Since $f^{-1}(A) \subset U$ we have $A \subset W$. Also f is almost-closed. Hence W is open in Y . Similarly, $Z = Y - f(X - V)$ is an open set in Y containing B . Since f is surjective, W and Z are disjoint. Thus Y is quasi-normal.

SECTION 2.3

s-NORMAL SPACES

DEFINITION : 2.3.1

A space X is said to be **s-normal** if for any pair of disjoint closed subsets A and B of X , there exist disjoint semi-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

We define *gs*-open and *gs*-closed sets to obtain an interesting characterization of *s*-normal spaces.

DEFINITION : 2.3.2

A subset A of X is said, to be **generalized open** (*g*-open) if $F \subset \text{Int } A$, whenever $F \subseteq A$ and F is closed in X .

DEFINITION : 2.3.3

The complement of a generalized open set is called **generalized closed** (*g*-closed).

DEFINITION : 2.3.4

A subset A of a space X is said to be **generalized semi-open** (*gs*-open) if $F \subset \text{sint } A$ whenever $F \subset A$ and F is closed in X .

DEFINITION : 2.3.5

A subset A of a space X is said to be **generalized semi-closed** (*gs*-closed) iff $X-A$ is *gs*-open.

REMARK : 2.3.6

g-open sets are gs-open

Also, semi-open sets are gs-open.

But it is interesting to note that a set is gs-open and semi-open but not g-open. Also, a set is gs-open and g-open but not semi-open.

EXAMPLE : 2.3.7

Let $X = \{a, b, c\}$ and $T = \{x, \phi, \{a\}, \{a, b\}\}$. Then $\{a, c\}$ is semi-open, hence gs-open but not g-open. Also $\{b\}$ is g-open, hence gs-open but not semi-open.

THEOREM : 2.3.8

The union of an arbitrary collection of pairwise separated gs-open sets is again gs-open.

PROOF :

Let A and B be two separated gs-open subsets of X . Let F be a closed set such that $F \subseteq A \cup B$.

Since A and B are separated sets $Cl A \cap B = A \cap Cl B = \phi$.

Now $F \cap Cl A \subseteq Cl A \cap (A \cup B) \subseteq A \cup \phi = A$.

Similarly, $F \cap Cl B \subseteq B$. Hence by definition, $F \cap Cl A \subseteq \text{sint } A$ and $F \cap Cl B \subseteq \text{sint } B$.

$F = F \cap (A \cup B) = (F \cap A) \cup (F \cap B) \subseteq (F \cap Cl A) \cup (F \cap Cl B) \subseteq \text{sint } A \cup \text{sint } B \subseteq \text{sint}(A \cup B)$.

Hence $A \cup B$ is gs-open.

The following example shows that in general the union of two disjoint gs-open sets is not gs-open.

EXAMPLE : 2.3.9

Let $X = \{a, b, c\}$ and $T = \{x, \phi, \{a\}\}$. Then $\{b\}$ and $\{c\}$ are both g-open, hence, gs-open but $\{b,c\}$ is not a gs-open set.

THEOREM : 2.3.10

A subset A of a space X is gs-closed iff $scl A \subseteq U$ whenever $A \subseteq U$ and U is open in X .

The proof is obvious.

THEOREM : 2.3.11

If A is gs-closed in X and $A \subseteq B \subseteq scl A$, then B is gs-closed.

PROOF :

Let $B \subseteq U$, where U is open in X . Since A is gs-closed and $A \subseteq U$, $scl A \subseteq U$. $B \subseteq scl A \Rightarrow scl B \subseteq scl A$. Hence $scl B \subseteq U$ and so B is gs-closed.

THEOREM : 2.3.12

For a space X , the following are equivalent.

- (a) X is s-normal.
- (b) For any pair of disjoint closed sets A and B , there exist disjoint gs-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

- (c) For every closed set A and an open set B containing A , there exists a gs -open set U such that $A \subseteq U \subseteq scl U \subseteq B$.
- (d) For every closed set A and every g -open set B containing A , there exists a semi-open set U such that $A \subseteq U \subseteq scl U \subseteq Int B$.
- (e) For every g -closed set A and every open set B containing A , there exists a semi-open set U such that $A \subseteq scl A \subseteq U \subseteq scl U \subseteq B$.

PROOF :

(a) \Rightarrow (b)

Let A and B be two disjoint closed subsets of X . Since X is s -normal, there exist disjoint semi-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Since semi-open sets are gs -open it follows that U and V are gs -open sets.

(b) \Rightarrow (c)

Let A be a closed subset of X and B is an open set such that $A \subset B$. Then A and $X-B$ are disjoint closed subsets of X . Therefore, there exist disjoint gs -open sets U and V such that $A \subseteq U$ and $X-B \subseteq V$. Thus $A \subseteq U \subseteq X-V \subseteq B$. Since B is open and $X-V$ is gs -closed, $scl(X-V) \subseteq B$.

Hence $A \subseteq U \subseteq scl U \subseteq B$.

(c) \Rightarrow (d)

Let A be a closed subset of X and B be a g -open set such that $A \subseteq B$. Since B is g -open and A is closed, $A \subseteq Int B$. There exists a semi-open set U such that $A \subseteq U \subseteq scl U \subseteq Int B$.

(d) \Rightarrow (e)

Let A be any g -closed subset of X and B be an open set such that $A \subseteq B$.
 $A \subseteq B \Rightarrow Cl A \subseteq B$. There exists a semi-open set U such that
 $Cl A \subseteq U \subseteq scl U \subseteq B$.

Hence $A \subseteq scl A \subseteq Cl A \subseteq U \subseteq scl U \subseteq B$.

(e) \Rightarrow (a)

Let A and B be two disjoint closed subsets of X . Then A is g -closed and
 $A \subseteq X - B$. Therefore, there exists a semi-open set U such that
 $A \subseteq scl A \subseteq U \subseteq scl U \subseteq B$. Thus $A \subseteq U$, $B \subseteq X - scl U$, which is semi-open and
 $U \cap (X - scl U) = \phi$. Hence X is s -normal.

SECTION 2.4

LIGHTLY NORMAL SPACES

DEFINITION : 2.4.1

A subset A of a topological space X is called a **regular G_δ - set** if it is the intersection of a countable number of closed sets whose interiors contain A .

DEFINITION : 2.4.2

A subset Z of a space X is called a **zero-set** if $Z = f^{-1}(0)$ for some continuous real-valued function f on X . The complement of a **zero-set** is called a **cozero-set**.

REMARK : 2.4.3

Every zero-set is a regular G_δ set.

DEFINITION : 2.4.4

A topological space is called **δ -normal** if each pair of disjoint closed sets, one of which is a regular G_δ set, have disjoint neighbourhoods.

DEFINITION : 2.4.5

A topological space is called **δ -normally separated** if each pair of a disjoint closed set and a zero-set can be completely separated.

DEFINITION : 2.4.6

A topological space is called **weakly δ -normally separated** if each pair of a regularly closed set and a zero-set can be completely separated.

REMARK : 2.4.7

Every δ -normally separated space is weakly δ -normally separated.

DEFINITION : 2.4.8

A topological space is called **lightly normal** if each pair of disjoint closed sets, one of which is a zero set, have disjoint neighbourhoods.

The following theorem characterizes light normality.

THEOREM : 2.4.9

For a topological space X , the following are equivalent:

- (a) X is lightly normal
- (b) For every closed set A and every cozero-set G containing A , there exists an open set U such that $A \subset U \subset \text{Cl } U \subset G$.
- (c) For every zero-set A and every open set U containing A , there exists an open set V such that $A \subset V \subset \text{Cl } V \subset U$.
- (d) For every pair consisting of disjoint closed sets A and B , one of which is a zero-set, there exist open sets U and V such that $A \subset U$, $B \subset V$ and $\text{Cl } U \cap \text{Cl } V = \phi$.

REMARK : 2.4.10

Every δ -normally separated space is lightly normal.

REMARK : 2.4.11

Since δ -normality and δ -normally separation are independent concepts, light normality is strictly weaker than both of them.

DEFINITION : 2.4.12

A space X is an O_z -space iff every regularly closed set is a zero-set.

REMARK : 2.4.13

In O_z -spaces the concepts δ -normality, light normality and δ -normally separation are equivalent.

THEOREM : 2.4.14

A topological space X is δ -normally separated iff it is lightly normal and weakly δ -normally separated.

PROOF:

Suppose X is lightly normal and weakly δ -normally separated. Let A and B be disjoint closed sets and let B be a zero-set. By light normality, there exists a regularly closed set F such that $A \subset F$ and $F \cap B = \emptyset$. Since X is weakly δ -normally separated, F and B can be completely separated. Hence A and B can be completely separated. Thus X is δ -normally separated.

The converse is obvious.

DEFINITION : 2.4.15

A Tychonoff space X is called **weakly normal** if any two disjoint closed sets, one of which is countable, have disjoint neighbourhoods.

THEOREM : 2.4.16

A Tychonoff lightly normal space is weakly normal.

PROOF:

Let X be a Tychonoff lightly normal space. Let A and B be disjoint closed sets and suppose B is countable. Now for each $b \in B$, there exists a cozero-set G_b such that $b \in G_b$ and $A \cap G_b = \phi$. Since B is countable, $\bigcup \{G_b: b \in B\} = G$ is a cozero-set containing B and disjoint from A . Now $X - G$ is a zero-set and $A \subset X - G$, $B \cap X - G = \phi$. Since X is lightly normal, B and $X - G$ have disjoint open neighbourhoods. Hence B and A can be separated by disjoint open sets. $\therefore X$ is weakly normal.

THEOREM : 2.4.17

A topological space X is countably paracompact iff $X \times I$ is lightly normal.

To prove this theorem we need the following lemma.

LEMMA :

For any topological space X , the following are equivalent:

- (a) X is countably paracompact.
- (b) If A is a closed subset of $X \times I$ and K is closed in I such that A and $X \times K$ are disjoint, then A and $X \times K$ have disjoint neighbourhoods.

PROOF OF THE THEOREM:

Suppose X is countably paracompact. Then $X \times I$ is δ -normal and hence lightly normal.

Conversely, suppose $X \times I$ is a lightly normal space. If K is a closed set in I , then K is a zero-set. $\therefore X \times K$ is a zero-set in $X \times I$. In view of the above lemma, the light normality of $X \times I$ implies that X is countably paracompact.

THEOREM : 2.4.18

A closed continuous image of a lightly normal space is lightly normal.

PROOF:

We have a continuous inverse image of a zero-set is a zero-set. Also a closed continuous image of a normal space is normal. Applying these facts, we get a closed continuous image of a lightly normal space is lightly normal.

DEFINITION : 2.4.19

A mapping $f: X \rightarrow Y$ is called ***z-closed*** if it maps zero-sets of X to closed sets in Y .

Using this definition, it can be easily proved that a ***z-closed*** continuous image of a δ -normally separated space is lightly normal.

THEOREM : 2.4.20

A clopen subset of a lightly normal space is lightly normal.

PROOF:

If Y is a clopen subset of X , then every zero-set in Y is a zero-set in X . Assume Y is a clopen subset of a lightly normal space X . Let A and B be two disjoint closed sets in Y . Assume A is a zero-set in Y . \therefore By the above result,

A is a zero-set in X. Since A and B are closed in Y and Y is a clopen subset of X, A and B are closed in X. Since X is lightly normal, there exist disjoint neighbourhoods U and V in X containing A and B respectively.

$$\text{Let } U^1 = U \cap Y \text{ and } V^1 = V \cap Y.$$

Then U^1 and V^1 are disjoint open neighbourhoods in Y, containing A and B respectively. \therefore Y is lightly normal.

DEFINITION : 2.4.21

Given a topological property P and a set X, we let $P(X)$ denote the set of topologies on X with property P and note that $P(X)$ is partially ordered by set inclusion. A topological space (X, τ) is minimal-P provided τ is a minimal element in $P(X)$. A topological space (X, τ) with property P is **P-closed** if it is closed in every space having the property P in which it can be embedded.

DEFINITION : 2.4.22

A space X is **completely Hausdorff** if any two distinct points of X can be completely separated.

THEOREM : 2.4.23

A space (X, τ) is minimal δ -normally separated completely Hausdorff iff it is compact Hausdorff.

PROOF:

Suppose (X, τ) is compact Hausdorff. Then (X, τ) is minimal Hausdorff and hence minimal δ -normally separated completely Hausdorff.

Let (X, τ) be a minimal δ -normally separated completely Hausdorff space. We first observe that (X, τ) is completely regular. To prove this, consider the complete regularization space (X, τ^*) of (X, τ) generated by cozero-sets in (X, τ) .

Since $C(X, \tau) = C(X, \tau^*)$, (X, τ^*) is δ -normally separated completely Hausdorff space and $\therefore \tau = \tau^*$. Hence (X, τ) is completely regular. If (X, τ) is not compact, then there exists a z-filter base of zero-sets of (X, τ) with empty adherence, i.e., a family \mathcal{F} of zero-sets such that any finite intersection of sets of \mathcal{F} is a member of \mathcal{F} and whose intersection is empty while the intersection of every finite subfamily is non-empty. Let p be a fixed point of X . Let $\tau^1 = \{U \in \tau: p \notin U\} \cup \{V \in \tau: \{p\} \cup F \subset V \text{ for some } f \in \mathcal{F}\}$.

From the finite intersection property of filter base \mathcal{F} it follows that τ^1 is a topology. The space (X, τ^1) is strictly coarser than (X, τ) . For if $\tau = \tau^1$, then each $G \in \tau$ containing the point p , contains some $F \in \mathcal{F}$. Hence $G \cap \mathcal{F} \neq \emptyset$ for all $F \in \mathcal{F}$.

$\therefore p \in \text{Cl}_\tau F = F$, for each $F \in \mathcal{F}$. Hence p is an adherent point of the filter base \mathcal{F} which is a contradiction. The topology τ^1 is T_1 , since for any $y \in X$, $X \setminus \{y\}$ is open in τ and contains some $F \in \mathcal{F}$ (since $y \notin F$ for some $F \in \mathcal{F}$) and so is open in τ^1 . Let x and y be two distinct points of X . Then one of them, say x , is not p . Also $x \notin F$ for some $F \in \mathcal{F}$. Hence $x \notin \{y, p\} \cup F = A$ (say). Also A is τ -closed set. \therefore There exists $f \in C(X, \tau)$ such that $f(x) = 0$ and $f(A) = 1$. By the construction of τ^1 , f is continuous at each point $z \neq p$ in (X, τ^1) .

Since $f(\{p\} \cup F) = 1$, f is continuous at p , also in (X, τ^1) .

Hence $f \in C(X, \tau^1)$. As $f(x) = 0$ and $f(y) = 1$, (X, τ^1) is a completely Hausdorff space.

Now we claim that (X, τ^1) is a δ -normally separated space. Let A and B be disjoint closed sets in τ^1 and let A be a zero-set in τ^1 . Since $\tau^1 \subset \tau$, A is a zero-set in τ also.

CASE 1:

Let $p \notin A$. Then for some $F \in \mathcal{F}$, $\{p\} \cup F \subset X \setminus A$. Now A and $B \cup F \cup \{p\}$ are disjoint closed sets in τ and A is a zero-set in τ . Since (X, τ) is δ -normally separated, there exists $f \in C(X, \tau)$ such that $f(A) = 0$ and $f(B \cup F \cup \{p\}) = 1$. Since $f(F \cup \{p\}) = 1$, we can show that $f \in C(X, \tau^1)$.

CASE 2:

Let $p \in A$. Then $p \notin B$. Hence there exists $F \in \mathcal{F}$ such that $F \cap B = \emptyset$. Now $A \cup F$ and B are disjoint closed sets in τ and $A \cup F$ is a zero-set in τ . Since (X, τ) is δ -normally separated, there exists $f \in C(X, \tau)$ such that $f(B) = 0$ and $f(A \cup F) = 1$. As $f(\{p\} \cup F) = 1$, $f \in C(X, \tau^1)$. Thus, (X, τ^1) is a δ -normally separated completely Hausdorff space. This is a contradiction to the minimality of the space (X, τ) with respect to this property. Thus (X, τ) must be compact.

REMARK : 2.4.24

Let X be a topological space and let P be a property which implies Tychonoff property and is possessed by compact Hausdorff spaces. Then the space X is P -closed iff X is compact Hausdorff.

PROOF:

Let X be P -closed. Then X should be a closed subset in its Stone-Cech compactification βX and hence $X = \beta X$ and therefore compact.

The converse is obvious since compact Hausdorff spaces are Hausdorff closed (H-closed).

THEOREM : 2.4.25

A space (X, τ) is minimal lightly normal Tychonoff iff it is compact Hausdorff.

PROOF:

As a compact Hausdorff space (X, τ) is minimal Hausdorff, every compact Hausdorff space is minimal-P for $P = \text{lightly normal} + \text{Tychonoff}$.

Conversely, let (X, τ) be a minimal lightly normal Tychonoff space. Let βX be the Stone-Cech compactification of X . Suppose that $p \in \beta X - X$. Then the open neighbourhood filter base in βX at p has a unique adherent point and if \mathcal{F} denotes the trace of this filter base in X , then \mathcal{F} has no adherent point in X as $p \notin X$. It is easy to see that for each $U \in \mathcal{F}$ there exist a cozero-set G and a zero-set F such that $G \subset F \subset U$ and $G \in \mathcal{F}$. Let q be a fixed point of X . Let $u(x)$ denote the neighbourhood base at any point x of X in (X, τ) .

$$\text{Define } u^1(x) = \begin{cases} \{U : U \in u(x)\} & \text{if } x \neq q \\ \{U \cup V : U \in u(x), V \in \mathcal{F}\} & \text{if } x = q \end{cases}$$

Then $u^1(x)$ defines a neighbourhood base at each point x of X . Let τ^1 be the topology generated by this system. Then τ^1 is strictly coarser than τ . It is shown by Berri [3] that (X, τ^1) is Tychonoff. Now we prove that (X, τ^1) is also lightly normal. Let A and B be two disjoint closed sets in τ^1 and let A be a zero-set in τ^1 .

CASE 1:

Let $q \notin A$. Since $p \notin A$, there exists $U \in \mathcal{F}$ such that $G \subset F \subset U$, $G \in \mathcal{F}$, G and F are a cozero-set and a zero set in τ respectively and

$U \cap A = \phi$. Now A and $B \cup \{q\} \cup F$ are two disjoint, a zero-set and a closed set in τ respectively. Hence there exist τ -open sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$. Since $\{q\} \cup G \subset V$, U and V are τ^1 -open also.

CASE 2:

Let $q \in A$. Since $p \notin B$, there exists $U \in \mathcal{F}$ such that $G \subset F \subset U$, $G \in \mathcal{F}$, G and F are a cozero and a zero-set in τ respectively and $U \cap B = \phi$. Now $A \cup F$ and B are two disjoint, a zero set and a closed set in τ respectively. Hence there exist τ -open sets U and V such that $A \cup F \subset U$, $B \subset V$ and $U \cap V = \phi$. Since $\{q\} \cup G \subset U$, U and V are τ^1 -open also.

Thus (X, τ^1) is a lightly normal Tychonoff space. This is a contradiction to the minimality of the space (X, τ) with respect to the property $P = \text{lightly normal} + \text{Tychonoff}$. Hence $\beta X = X$. So X is compact.

THEOREM : 2.4.25

Let (X, τ) be a given space with property $P = \text{lightly normal} + \text{Tychonoff}$. Then the following are equivalent.

- (a) (X, τ) is minimal $-P$.
- (b) (X, τ) is P -closed.
- (c) (X, τ) is compact Hausdorff.

PROOF:

The proof of this theorem follows from Theorem 2.4.25 and Remark 2.4.24.

Chapter III

CHAPTER – III

SECTION 3.1

PAIRWISE NORMAL SPACES

DEFINITION : 3.1.1

A bitopological space (X, P, L) is a set X together with two topologies P and L on X . A space (X, P, L) is said to be **pairwise normal** if given a P -closed set A and a L -closed set B with $A \cap B = \phi$, there exists a L -open set U and a P -open set V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$. Equivalently, (X, P, L) is pairwise normal if, given a L -closed set C and a P -open set D such that $C \subset D$, there are a P -open set G and a L -closed set F such that $C \subset G \subset F \subset D$.

DEFINITION : 3.1.2

A subspace A of a bitopological space (X, P, L) is **sc-embedded** in X if every real valued P -lsc and L -usc function on A can be extended to a P -lsc and L -usc function on X , where lsc and usc are used for lower and upper semi continuous functions, respectively.

We will give some results about the extension of semi-continuous functions on a closed subset of a topological space, which will be used to characterize pairwise normal spaces.

RESULT : 3.1.3

Let A be closed set of the topological space X . Let f be a real-valued function defined on A which is lsc (usc) and upper (lower) bounded on the

boundary $b(A)$ of A . Then there exists an extension F of f on X such that F is lsc (usc) and $F(X)$ is in the closure $Cl(f(A))$ of $f(A)$.

The converse is false.

EXAMPLE : 3.1.4

Let R be the real line with the usual topology and let Z be the set of integers. The continuous function $f : Z \rightarrow R$ defined by $f(n) = n$ for $n \in Z$ has the lsc extension $F : R \rightarrow R$ defined by $f(x) = n + 1$ if $x \in]n, n+1]$ for some $n \in Z$; then $F(R) = f(Z)$, but $f(z)$ is not upper bounded on the boundary $b(Z)$ of Z , since $b(Z)=Z$.

RESULT : 3.1.5

Let A be a G_δ closed subset of the topological space X . Let f be a function defined on A which is lsc(usc). Then there exists an extension F of f on X , such that F is lsc (usc) and $F(x)$ is in the closure $Cl(f(A))$ of $f(A)$.

THEOREM : 3.1.6

Let (X, P, L) be a pairwise normal bitopological space. Let $A \subset X$ be P -closed and L -closed. Let f be a bounded real function defined on A which is a P -usc and L -lsc function. Then there exists an extension F of f to the whole of x such that F is a P -usc and L -lsc function. The extension F can be chosen so that $F : X \rightarrow [a, b]$ with $a = \text{Inf} \{ F(t) : t \in x \} = \text{Inf} \{ f(t) : t \in A \}$ and $b = \text{Sup} \{ F(t) : t \in X \} = \text{Sup} \{ f(t) : t \in A \}$.

PROOF:

Let n be a positive integer. For each integer k ,
let $U\mathcal{R} = \{x : f(x) \geq k/n\}$ and $L\mathcal{R} = \{x : f(x) \leq (k-1)/n\}$.

Then for every integer k , $U\mathcal{R}$ and $L\mathcal{R}$ are respectively P -closed and L -closed subsets of X . Also, $U\mathcal{R} \cap L\mathcal{R} = \emptyset$, for every $k \in \mathbb{Z}$.

For each $k=1,2,\dots$ if $U\mathcal{R} \neq \emptyset$, there is a function u_k defined on X which is a P -usc and L -lsc function on X and such that $u_k(L\mathcal{R})=0 \leq u_k(x) \leq 1/n = u_k(U\mathcal{R})$ for every $x \in X$.

If $U\mathcal{R} = \emptyset$, choose $u_k(x)=0$ for every $x \in X$. Also, for each $k = 0,-1,-2,\dots$ if $L\mathcal{R} \neq \emptyset$ there is a function v_k defined on X , which is a P -usc and L -lsc function on X and such that $v_k(L\mathcal{R}) = -1/n \leq v_k(x) \leq 0 = v_k(U\mathcal{R})$ for every $x \in X$.

If $L\mathcal{R} = \emptyset$, choose $v_k(x)=0$, for every $x \in X$. Since f is bounded, there exists $k \geq k_n \in \mathbb{Z}^+$ such that $U\mathcal{R} = \emptyset = L\mathcal{R}_k$, for every $k \geq k_n$, $k \in \mathbb{Z}$. Therefore, for every, $k \geq k_n$, we have $u_k(x) = v_{-k}(x) = 0$, for every $x \in X$ and so

$f_n = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_{-k}$ is a functional series with only a finite number of non-zero

terms. Now it is obvious that $f_n(x) = \sum_{k=1}^{\infty} u_k(x) + \sum_{k=1}^{\infty} v_{-k}(x)$ is a bounded real

function defined on X which is a P -usc and L -lsc function for each $n \in \mathbb{Z}^+$. The

restrictions $f_n|_A (n=1,2,\dots)$ converge uniformly to f on A and form a Cauchy sequence. f has an extension F to X which satisfies the theorem conditions implicitly including that the range of the extension F is a subset of $[a,b]$.

THEOREM : 3.1.7

Let (X, P, L) be pairwise normal. Each P -closed and L -closed subset A of X such that the P -boundary $b_P(A)$ of A is L -countably compact and the L -boundary $b_L(A)$ of A is P -countably compact is sc -embedded in X .

PROOF:

Let f be a P -usc and L -lsc function on the P -closed and L -closed subsets A of X . Suppose the boundaries $b_P(A)$ and $b_L(A)$ are non-empty. Then f is a L -lsc function $b_P(A)$, which is L -countably compact and f has a lower bound m on $b_P(A)$. In the same way, f has an upper bound M on $b_L(A)$ and we will suppose $M > m$.

From Result 3.1.3 we can consider the extension F of f on X such that $F(x) = M$ if $x \in X - A$ which is L -lsc and the extension G of f on X such that $G(x) = m$ if $x \in X - A$ which is P -usc.

Therefore, we have $G \leq F$. There exists a function h on x which P -usc and L -lsc such that $G \leq h \leq F$, but $F(x) = G(x) = f(x)$. If $x \in A$ and then h is the required extension.

Suppose now that only one boundary, namely $b_p(A)$ is empty. Then $X-A$ is P -closed and the above function G is a P -usc function on X for each $m \in \mathbb{R}$.

So we can repeat the last argument with some real m such that $m \leq M$.

Finally, if both boundaries are empty we can obtain the same result with 2 reals, m and M such $m \leq M$.

SECTION 3.2

PAIRWISE PERFECTLY NORMAL SPACES

DEFINITION : 3.2.1

The space (X, P, L) is **pairwise perfectly normal** in case X is pairwise normal and every P -closed (L - closed) subset of X is a $L - G_\delta$ ($P - G_\delta$).

DEFINITION : 3.2.2

A **P-zero set** in a bitopological space (X, P, L) is a set of the form $A : h^{-1}(0)$, where h is P -lsc and L -usc on X and $h \geq 0$.

When P and L are T_1 topologies we obtain the following generalization of perfectly normal T_1 spaces.

THEOREM: 3.2.3

Let P and L be two T_1 topologies on X . Then (X, P, L) is pairwise perfectly normal iff for A P -closed and B L - closed with $A \cap B = \emptyset$, there is a function $f : X \rightarrow [0, 1]$ which is P -lsc and L -usc such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

PROOF:

Suppose (X, P, L) is pairwise perfectly normal. If A is P -closed it is $L - G_\delta$ and hence P -zero set. So there exists a P -lsc and a L -usc function h on X , such that $A = h^{-1}(0)$ and $h \geq 0$. We can consider $0 \leq h \leq 1$ (Otherwise choose $h / (1+h)$ which is P -lsc and L -usc too).

Let B be a L -closed subset with $A \cap B = \emptyset$. Since X is pairwise normal there exists a function $g : X \rightarrow [0, 1]$ which is P -lsc and L -usc and such that $g(A) = 0$ and $g(B) = 1$.

Now consider the function $f_A : X \rightarrow [0, 1]$ defined by $f_A = \text{Sup } \{h, g\}$ which is P -lsc and L -usc and satisfies $f_A^{-1}(0) = A$ and $B \subset f_A^{-1}(1)$.

Let f_B be L -lsc and P -usc function with $f_B(X) \subset [0, 1]$, $f_B^{-1}(0) = B$ and $A \subset f_B^{-1}(1)$.

Now we consider the function $F = f_A - f_B$ which is P -lsc and L -lsc and satisfies $F^{-1}(-1) = A$ and $F^{-1}(1) = B$.

Finally, $F = (F+1)/2$ is a P -lsc and L -usc function from X to $[0, 1]$ and such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. For the converse, it is obvious that (X, P, L) is pairwise normal from the hypothesis.

Now let A be a P -closed set ($A \neq X$). Then there exists $p \in X - A$ such that $B = \{p\}$ is L -closed with $A \cap B = \emptyset$. Let f be the function $f : X \rightarrow [0, 1]$ which is P -lsc and L -usc such that $f^{-1}(0) = A$; then A is a P -zero set and hence A is G_δ .

Summary and Conclusion

SUMMARY AND CONCLUSION

Some interesting generalizations of the concept – Normal studied in this thesis are :

1. Mildly normal
2. Quasi – normal
3. x – normal
4. s – normal
5. δ - normal
6. δ - normally separated
7. Weakly δ - normally separated
8. Lightly normal
9. Weakly normal

Chapter I deals with fundamental definitions and results regarding normal spaces.

Chapter II is devoted to the study of different generalizations of normal spaces.

Chapter III is devoted to the study of the concept – Normal, in bitopological spaces.

For further research, one can study different combinations of these generalizations. It is a good research problem to study all these concepts for bitopological spaces and fuzzy topological spaces.

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