Chapter 2

LMI Optimization Approach to New Stability Results for Uncertain Stochastic Neural Networks with Discrete Interval and Distributed Time-Varying Delays

2.1 Introduction

Neural networks have great potential applications in various areas such as signal processing, pattern recognition, static image processing, associative memory and combinatorial optimization, see Cichoki and Unbehauen (1993) and Haykin (1998). The achieved applications heavily depend on the dynamic behaviors of the equilibrium point of neural networks. Since stability is one of the most important issues related to such behavior, the problem of stability analysis of neural networks has attracted considerable attentions in recent years. However, in the neural networks, the interaction between neurons are generally asynchronous, which inevitably results in time-varying delays. Furthermore, time delay is frequently a source of instability and deterioration of a systems performance; therefore, the stability analysis of time delayed systems has received considerable attention over the past several years. In electronic implementation of artificial neural networks, time delay is often time variant and sometimes varies dramatically with time because of the finite switch speed of amplifiers and faults in the electrical circuits. Recently, many investigators have examined the stability of delayed recurrent neural networks by various efforts directed for obtaining delay-independent and delay-dependent stability conditions, see Arik (2000 a, 2004), Cao (2001), Chen and Rong (2003), Chen et al.(2006), Gao et al. (2008), Hua et al.(2006), Li and Feng (2009), Li and Chen (2009), Liao and

Wang (2000), Samidurai et al. (2010), Singh (2004), Xu et al.2006) , Zeng and Wang (2006) and Zhang et al. (2007 b). The earlier works of these efforts are devoted to assessing stability of Delayed Recurrent Neural Networks for arbitrary delays including zero, while the latter ones are devoted to determining the delay-intervals such that stability is preserved when the network is not delay-independently stable. In the literature, the term delay-dependent is often used in stability analysis of a delay recurrent neural networks with time delay τ which belongs to the delay interval, $0 \leq \tau \leq \bar{\tau}$, refer, Chen et al (2006), Hua et al.(2006), Li and Feng (2009), Samidurai et al. (2010), Xu et al. (2006), Zeng and Wang (2006) and Zhang et al.(2007 b). On the other hand, interval time-varying delays $0 \leq h_1 \leq \tau(t) \leq h_2$ are used to indicate that the propagated speed of signals is finite and uncertain in systems. Also, there are systems which are with some nonzero delays, but they are unstable without delay, see Gu (2001),Gu et al.(2001) and Zhao and Wang (2004). Therefore, it is important to perform the stability analysis for systems with non-zero delays as in Tian Peng (2006) and the non-zero delay can be placed into a given interval.

It is worth noting that the synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes in real nerves systems. Therefore, it is of practical importance to study the stochastic effects on the stability property of delayed neural networks, see for example Li et al. (2008 a) and Wan and Sun (2005). On the other hand, the connection weights of the neurons depend on certain resistance and capacitance values that include uncertainties. In addition, the stability criteria for neural networks with delays can be classified into two categories, namely, delay-independent criteria and delay-dependent criteria, and the former is more conservative than the latter. As discussed in Balasubramaniam et al.(2009),Li et al.(2008 a) and Rakkiyappan et al. (2008) distributed delays should be incorporated into the model due to the fact that there may exist a distribution of propagation delays over a period of time in some cases. Recently, some results on stability of stochastic neural networks with delays have been reported in Chen et al. (2011),Chen and Lu (2008),Feng et al.(2009 a, 2009 b), Huang and Feng (2007, 2008), Li and Fu (2010), Li (2010), Li and Xu (2012), Meng et al. (2011), Pan and Cao(2012), Yu et al. (2009) and Zhang et al. (2007 a). To the best of authors knowledge, very few results on the delay/interval dependent robust exponential stability analysis for uncertain stochastic neural networks with discrete interval and distributed time-varying delays are available in the literature.

Based on the above discussions, a class of uncertain stochastic neural networks with discrete interval and distributed time-varying delays is considered in this chapter. The main purpose of this chapter is to study the global robust stability in the mean square for uncertain stochastic neural networks with discrete interval and distributed time-varying delays. The parameter uncertainties are assumed to be norm bounded. By using the new Lyapunov-Krasovskii functional technique, global robust stability conditions for the considered uncertain stochastic neural networks are given in terms of LMIs, which can be easily calculated by MATLAB LMI control toolbox. Numerical examples are given to illustrate the effectiveness and less conservativeness of the proposed methods.

Notations: Throughout this chapter, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times n$ real matrices. The superscript T denotes the transposition and the notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I_n is the $n \times n$ identity matrix. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). The notation $*$ always denotes the symmetric block in one symmetric matrix. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

2.2 Problem Description and Preliminaries

Consider the following Hopfield neural networks with both discrete and distributed time-varying delays described by

$$\begin{aligned} y_i'(t) &= -a_i(y_i(t)) + \sum_{j=1}^{n} b_{ij}^0 g_j(y_j(t)) + \sum_{j=1}^{n} b_{ij}^1 g_j(y_j(t-\tau(t))) \\ &\quad + \sum_{j=1}^{n} b_{ij}^2 \int_{t-r(t)}^{t} g_j(y_j(s))ds + I_i, \quad i = 1, 2, ..., n \end{aligned} \tag{2.2.1}$$

or equivalently the vector form

$$y'(t) = -Ay(t) + B_0 g(y(t)) + B_1 g(y(t-\tau(t))) + B_2 \int_{t-r(t)}^{t} g(y(s))ds + I \tag{2.2.2}$$

where $y(t) = [y_1(t), y_2(t), \cdots, y_n(t)]^T \in \mathbb{R}^n$ denotes the state vector associated with n neurons. The matrix $A = diag(a_1, a_2, \cdots, a_n)$ is a diagonal matrix with positive entries $a_i > 0$. $B_0 = (b^0_{ij})_{n\times n}$, $B_1 = (b^1_{ij})_{n\times n}$ and $B_2 = (b^2_{ij})_{n\times n}$ are connection weights, the discrete delayed connection weights and the distributed delayed connection weights of the j neuron on the i neuron respectively. $g(x) = [g_1(x_1), g_2(x_2), \cdots, g_n(x_n)]^T \in \mathbb{R}^n$ is the activation function with $g(0) = 0$. $I = [I_1, I_2, \cdots, I_n]$ is a constant external input.

In order to obtain our main results, the assume the following condition hold.

Assumption $\mathbf{A_1}$: The activation function g is bounded and satisfy the Lipschitz condition

$$|g(x_1) - g(x_2)| \le l_i|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R} \;\; i = 1, 2, \cdots, n.$$

Then, by ($\mathbf{A_1}$) we can have

$$|g(x)| \le l_i|x|, \quad \forall x \in \mathbb{R}.$$

Assumption $\mathbf{A_2}$: The time-varying delays $\tau(t)$ satisfy

$$0 \le h_1 \le \tau(t) \le h_2, \quad \dot{\tau}(t) \le \mu < 1,$$

where h_1, h_2, μ are constants.

Assuming $y^* = (y^*_1, y^*_2, ...y^*_n)^T$ is an equilibrium point of equation (2.2.2), one can derive from (2.2.2) that the transformation $x(t) = y(t) - y^*$ transforms system (2.2.2) into the following system:

$$x'(t) = -Ax(t) + B_0 f(x(t)) + B_1 f(x(t - \tau(t))) + B_2 \int_{t-r(t)}^{t} f(x(s))ds \qquad (2.2.3)$$

where $x(t)$ is the state vector of the transformation system, $f_j(x(t)) = g_j(x_j(t) + y^*_j) - g_j(y^*_j)$ with $f_j(x(0)) = 0$ for $j = 1, 2, \cdots, n$.

In reality, it is often the case that the connection weights of the neurons include uncertainties, and the network is distributed by environmental noises that affect the stability of the equilibrium. In this chapter, as in Li et al.(2008 a), the Hopfield neural network with parameter uncertainties and stochastic perturbations can be described as follows:

$$\begin{aligned} dx(t) &= \Big[- A(t)x(t) + B_0(t)f(x(t)) + B_1(t)f(x(t - \tau(t))) + B_2(t) \int_{t-r(t)}^{t} f(x(s))ds \Big] dt \\ &\quad + \Big[C(t)x(t) + D_0(t)x(t - \tau(t)) + D_1(t)f(x(t)) + D_2(t)f(x(t - \tau(t))) \\ &\quad + D_3(t) \int_{t-r(t)}^{t} f(x(s))ds \Big] dw(t) \qquad (2.2.4) \\ x(t) &= \phi(t), \quad \forall t \in [-2\bar{h}, 0], \quad \bar{h} = \max\{h_2, \bar{r}\}, \quad \bar{r} = \max\{r(t)\}, \qquad (2.2.5) \end{aligned}$$

where $w(t)$ denotes a one-dimensional Brownian motion satisfying $\mathbb{E}\{dw(t)\} = 0$ and
$\mathbb{E}\{dw(t)^2\} = dt$. The matrices $A(t) = A + \Delta A(t)$, $B_0(t) = B_0 + \Delta B_0(t)$, $B_1(t) = B_1 + \Delta B_1(t)$, $B_2(t) = B_2 + \Delta B_2(t)$, $C(t) = C + \Delta C(t)$, $D_0(t) = D_0 + \Delta D_0(t)$, $D_1(t) = D_1 + \Delta D_1(t)$, $D_2(t) = D_2 + \Delta D_2(t)$ and $D_3(t) = D_3 + \Delta D_3(t)$, where $A = diag(a_1, a_2, \cdots, a_n)$ has positive entries $a_i > 0$,$B_0, B_1, B_2, C, D_0, D_1, D_2$ and D_3 are connection weight matrices with appropriate dimensions. In this system, the parameter uncertainties are assumed to be of the form:

$$\begin{aligned}&[\Delta A(t), \Delta B_0(t), \Delta B_1(t), \Delta B_2(t), \Delta C(t), \Delta D_0(t), \Delta D_1(t), \Delta D_2(t), \Delta D_3(t)]\\ &= HF(t)[T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9],\end{aligned} \tag{2.2.6}$$

where $\Delta A(t)$ is a diagonal matrix, H and $T_i, (i = 1, 2, \cdots, 9)$ are real known constant matrices of appropriate dimensions. The matrix $F(t)$, which may be time-varying, is unknown and satisfies

$$F^T(t)F(t) \leq I. \tag{2.2.7}$$

It is assumed that all the elements of $F(t)$ are Lebesque measurable. The matrices $\Delta A(t), \Delta B_0(t)$, $\Delta B_1(t), \Delta B_2(t)$, $\Delta C(t)$, $\Delta D_0(t)$, $\Delta D_1(t)$, $\Delta D_2(t)$, $\Delta D_3(t)$ are said to be admissible if both (2.2.6) and (2.2.7) hold. $\phi(t) \in C([-2\bar{h}, 0]; \mathbb{R}^n)$ is the initial function.
$f(x) = [f_1(x_1), f_2(x_2), \cdots, f_n(x_n)]^T \in \mathbb{R}^n$ is the activation function with $f(0) = 0$.

Lemma 2.2.1. *[Schur Complement] Given constant matrices Ω_1, Ω_2 and Ω_3 with appropriate dimensions, where $\Omega_1^T = \Omega_1$ and $\Omega_2^T = \Omega_2 > 0$, then*

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ * & -\Omega_2 \end{bmatrix} < 0, \quad or, \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{bmatrix} < 0.$$

Lemma 2.2.2. *For any constant matrix $M > 0$, any scalars a and b with $a < b$, and a vector function $x(t) : [a, b] \to \mathbb{R}^n$ such that the integrals concerned as well defined, then the following holds*

$$\Big[\int_a^b x(s)ds\Big]^T M \Big[\int_a^b x(s)ds\Big] \leq (b-a) \int_a^b x^T(s)Mx(s)ds.$$

Lemma 2.2.3. *Let M, E and $F(t)$ be the real matrices of appropriate dimensions with $F(t)$ satisfying $F^T(t)F(t) \leq I$. Then, the following inequality holds for any $\epsilon > 0$,*

$$MF(t)E + E^T F^T(t) M^T \leq \epsilon M M^T + \epsilon^{-1} E^T E.$$

2.3 Main Results

In the following Theorem, the delay-dependent robust stability results are derived for the following uncertain stochastic neural networks with interval discrete and distributed time-varying delays

$$\begin{aligned} dx(t) &= \Big[-A(t)x(t) + B_0(t)f(x(t)) + B_1(t)f(x(t-\tau(t))) + B_2(t)\int_{t-r(t)}^{t} f(x(s))ds\Big]dt \\ &\quad +\Big[C(t)x(t) + D_0(t)x(t-\tau(t)) + D_1(t)f(x(t)) + D_2(t)f(x(t-\tau(t))) \\ &\quad +D_3(t)\int_{t-r(t)}^{t} f(x(s))ds\Big]dw(t) \\ x(t) &= \phi(t), \quad \forall t \in [-2\bar{h}, 0], \quad \bar{h} = \max\{h_2, \bar{r}\} \end{aligned}$$

Defining two new state variables for the stochastic neural networks (2.2.4),

$$y(t) = -A(t)x(t) + B_0(t)f(x(t)) + B_1(t)f(x(t-\tau(t))) + B_2(t)\int_{t-r(t)}^{t} f(x(s))ds \tag{2.3.1}$$

$$\begin{aligned} g(t) &= C(t)x(t) + D_0(t)x(t-\tau(t)) + D_1(t)f(x(t)) + D_2(t)f(x(t-\tau(t))) \\ &\quad +D_3(t)\int_{t-r(t)}^{t} f(x(s))ds, \end{aligned} \tag{2.3.2}$$

then the above can be represented as

$$dx(t) = y(t)dt + g(t)dw(t). \tag{2.3.3}$$

Moreover, the following equality holds,

$$x(t) - x(t-\tau(t)) = \int_{t-\tau(t)}^{t} dx(s) = \int_{t-\tau(t)}^{t} y(s)ds + \int_{t-\tau(t)}^{t} g(s)dw(s). \tag{2.3.4}$$

Theorem 2.3.1. *For given scalars $h_2 > h_1 \geq 0$ and μ, the equilibrium solution of stochastic neural networks (2.2.4) is globally asymptotically stable in the mean square* *if there exist matrices* $P_1 > 0, \quad Y > 0, \quad Q_l = Q_l^T \geq 0,\ l = 1, 2, \cdots, 4,\ R_i = R_i^T > 0,\ i = 1, 2,$ *diagonal matrices* $K_1 > 0, K_2 > 0,\ K = diag\{k_1, k_2, \cdots, k_n\} \geq 0,$ *and positive scalars* $\epsilon_i > 0\ i = 1, 2, \cdots, 4$ *such that*

$$\mathit{\Pi}_1 = \begin{bmatrix} \mathit{\Pi}_{1,1} & -N & -M & -S & UB_2 + VD_3 & UH & VH & UH & VH \\ * & \mathit{\Pi}_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \mathit{\Pi}_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \mathit{\Pi}_{4,4} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \mathit{\Pi}_{5,5} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\epsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\epsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & * & -\epsilon_3 I & 0 \\ * & * & * & * & * & * & * & * & -\epsilon_4 I \end{bmatrix} < 0, \quad (2.3.5)$$

with

$$\mathit{\Pi}_{1,1} = (\varphi_{i,j})_{8\times 8} + \epsilon_1 \bar{T}_1^T \bar{T}_1 + \epsilon_2 \bar{T}_2^T \bar{T}_2, \quad \mathit{\Pi}_{2,2} = -\frac{1}{h_2} R_1, \quad \mathit{\Pi}_{3,3} = -\frac{1}{(h_2 - h_1)}(R_1 + R_2),$$

$$\mathit{\Pi}_{4,4} = -\frac{1}{(h_2 - h_1)} R_2, \quad \bar{\mathit{\Pi}}_{5,5} = -\frac{1}{\bar{r}} Y + \epsilon_3 T_4^T T_4 + \epsilon_4 T_9^T T_9,$$

where

$$
\begin{aligned}
\varphi_{1,1} &= Q_1 + Q_2 + Q_3 + P_2 + P_2^T, \quad \varphi_{1,2} = -P_2^T + P_3^T - P_4^T - P_5 A + P_9 C, \\
\varphi_{1,3} &= P_4^T - P_6 A + P_{10} C, \quad \varphi_{1,4} = -P_3^T - P_7 A + P_{11} C, \quad \varphi_{1,5} = P_1 - A^T P_8 + C^T P_{12}, \\
\varphi_{1,6} &= -A^T P_{16} + C^T P_{17} + P_{13}^T, \quad \varphi_{1,7} = L^T K_1, \quad \varphi_{1,8} = 0, \quad \varphi_{2,2} = -(1-\mu) Q_1 + P_9 D_0 + \\
&\quad D_0^T P_9^T, \quad \varphi_{2,3} = D_0^T P_{10}^T, \quad \varphi_{2,4} = D_0^T P_{11}^T, \quad \varphi_{2,5} = -P_5 + D_0^T P_{12}^T, \\
\varphi_{2,6} &= -P_{13}^T + P_{14}^T - P_{15}^T + D_0^T P_{17} - P_9, \varphi_{2,7} = P_5 B_0 + P_9 D_1, \varphi_{2,8} = P_5 B_1 + P_9 D_2 + L^T K_2, \\
\varphi_{3,3} &= -Q_2, \quad \varphi_{3,4} = 0, \quad \varphi_{3,5} = -P_6, \quad \varphi_{3,6} = P_{15}^T - P_{10}, \quad \varphi_{3,7} = P_6 B_0 + P_{10} D_1, \\
\varphi_{3,8} &= P_6 B_1 + P_{10} D_2, \quad \varphi_{4,4} = -Q_3, \quad \varphi_{4,5} = -P_7, \quad \varphi_{4,6} = -P_{14}^T - P_{11}, \\
\varphi_{4,7} &= P_7 B_0 + P_{11} D_1, \quad \varphi_{4,8} = P_7 B_1 + P_{11} D_2, \quad \varphi_{5,5} = -2P_8 + h_1 R_1 + (h_2 - h_1) R_2, \\
\varphi_{5,6} &= -P_{16}^T - P_{12}, \quad \varphi_{5,7} = K^T + P_8 B_0 + P_{12} D_1, \quad \varphi_{5,8} = P_8 B_1 + P_{12} D_2, \\
\varphi_{6,6} &= P_1 + K + h_1 R_1 + (h_2 - h_1) R_2 - 2P_{17}, \varphi_{6,7} = P_{16} B_0 + P_{17} D_1, \varphi_{6,8} = P_{16} B_1 + P_{17} D_2, \\
\varphi_{7,7} &= Q_4 + \bar{r} Y - 2K_1, \quad \varphi_{7,8} = 0, \varphi_{8,8} = -(1-\mu) Q_4 - 2K_2, \quad \bar{T}_1 = [-T_1, 0, 0, 0, 0, 0, T_2, T_3], \\
\bar{T}_2 &= [T_5, T_6, 0, 0, 0, 0, T_7, T_8], \quad M^T = [P_2, 0, 0, 0, 0, P_{13}, 0, 0], \quad N^T = [P_3, 0, 0, 0, 0, P_{14}, 0, 0], \\
S^T &= [P_4, 0, 0, 0, 0, P_{15}, 0, 0], U = [0, P_5^T, P_6^T, P_7^T, P_8^T, P_{16}^T, 0, 0], \\
V &= [0, P_9^T, P_{10}^T, P_{11}^T, P_{12}^T, P_{17}^T, 0, 0].
\end{aligned}
$$

Proof. Consider the Lyapunov-Krasovskii functional as follows

$$
\begin{aligned}
V(x_t, t) &= V_1(x_t, t) + V_2(x_t, t) + V_3(x_t, t) + V_4(x_t, t) \\
&\quad + V_5(x_t, t) + V_6(x_t, t)
\end{aligned}
\tag{2.3.6}
$$

where

$$
\begin{aligned}
V_1(x_t,t) &= \xi_0^T(t)EP\xi_0(t), \quad V_2(x_t,t) = 2\sum_{i=1}^{n} k_i \int_0^{x_i} f_i(s)ds, \\
V_3(x_t,t) &= \int_{t-\tau(t)}^{t} x^T(s)Q_1x(s)ds + \int_{t-h_1}^{t} x^T(s)Q_2x(s)ds \\
&\quad + \int_{t-h_2}^{t} x^T(s)Q_3x(s)ds + \int_{t-\tau(t)}^{t} f^T(x(s))Q_4f(x(s))ds \\
V_4(x_t,t) &= \int_{-r(t)}^{0}\int_{t+\theta}^{t} f^T(x(s))Yf(x(s))dsd\theta \\
V_5(x_t,t) &= \int_{-h_2}^{0}\int_{t+\theta}^{t} y^T(s)R_1y(s)dsd\theta + \int_{-h_2}^{0}\int_{t+\theta}^{t} g^T(s)R_1g(s)dsd\theta \\
V_6(x_t,t) &= \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} y^T(s)R_2y(s)dsd\theta + \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} g^T(s)R_2g(s)dsd\theta,
\end{aligned}
$$

with

$$
E = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, P = \begin{bmatrix} P_1 & 0 & 0 & 0 & 0 & 0 \\ P_2 & 0 & 0 & 0 & 0 & P_{13} \\ P_3 & 0 & 0 & 0 & 0 & P_{14} \\ P_4 & 0 & 0 & 0 & 0 & P_{15} \\ 0 & P_5 & P_6 & P_7 & P_8 & P_{16} \\ 0 & P_9 & P_{10} & P_{11} & P_{12} & P_{17} \end{bmatrix}, \xi_0(t) = \begin{bmatrix} x(t) \\ x(t-\tau(t)) \\ x(t-h_1) \\ x(t-h_2) \\ y(t) \\ g(t) \end{bmatrix}
$$

and

$$
EP = P^TE^T \geq 0,
$$

it is noted that $\xi_0^T(t)EP\xi_0(t)$ is actually $x^T(t)P_1x(t)$.

On the other hand, from the stochastic theory, the following equations are true

$$
\begin{aligned}
\eta_1 &= 2\xi^T(t)N\Big[x(t)-x(t-\tau(t))-\int_{t-\tau(t)}^{t}y(s)ds-\int_{t-\tau(t)}^{t}g(s)dw(s)\Big]\\
\eta_2 &= 2\xi^T(t)M\Big[x(t-\tau(t))-x(t-h_2)-\int_{t-h_2}^{t-\tau(t)}y(s)ds-\int_{t-h_2}^{t-\tau(t)}g(s)dw(s)\Big]\\
\eta_3 &= 2\xi^T(t)S\Big[x(t-h_1)-x(t-\tau(t))-\int_{t-\tau(t)}^{t-h_1}y(s)ds-\int_{t-\tau(t)}^{t-h_1}g(s)dw(s)\Big]\\
\eta_4 &= 2\xi^T(t)U\Big[-A(t)x(t)+B_0(t)f(x(t))+B_1(t)f(x(t-\tau(t)))\\
&\quad +B_2(t)\int_{t-r(t)}^{t}f(x(s))ds-y(t)\Big]\\
\eta_5 &= 2\xi^T(t)V\Big[C(t)x(t)+D_0(t)x(t-\tau(t))+D_1(t)f(x(t))+D_2(t)f(x(t-\tau(t)))\\
&\quad +D_3(t)\int_{t-r(t)}^{t}f(x(s))ds-g(t)\Big].
\end{aligned}
$$

Then, it can be obtained by Ito's formula that

$$
dV(x_t,t)=\mathcal{L}V(x_t,t)dt+2x^T(t)Pg(t)dw(t), \tag{2.3.7}
$$

where

$$
\mathcal{L}V_1(x_t,t) = 2x^T(t)P_1y(t)+g^T(t)P_1g(t)+2\xi_0^T(t)P^T\begin{bmatrix}0\\ \eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{bmatrix} \tag{2.3.8}
$$

Denote

$\xi^T(t) = [x^T(t)\ x^T(t-\tau(t))\ x^T(t-h_1)\ x^T(t-h_2)\ y^T(t)\ g^T(t)\quad f^T(x(t))\ f^T(x(t-\tau(t)))]$,

then

$$2\xi_0^T(t)\begin{bmatrix} P_2^T \\ 0 \\ 0 \\ 0 \\ 0 \\ P_{13}^T \end{bmatrix}\eta_1 \;=\; 2\xi^T(t)M\Big[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} y(s)ds - \int_{t-\tau(t)}^{t} g(s)dw(s)\Big] = 0,$$

$$2\xi_0^T(t)\begin{bmatrix} P_3^T \\ 0 \\ 0 \\ 0 \\ 0 \\ P_{14}^T \end{bmatrix}\eta_2 \;=\; 2\xi^T(t)N\Big[x(t-\tau(t)) - x(t-h_2) - \int_{t-h_2}^{t-\tau(t)} y(s)ds - \int_{t-h_2}^{t-\tau(t)} g(s)dw(s)\Big] = 0,$$

$$
2\xi_0^T(t)\begin{bmatrix} P_4^T \\ 0 \\ 0 \\ 0 \\ 0 \\ P_{15}^T \end{bmatrix}\eta_3 = 2\xi^T(t)S\Big[x(t-h_1)-x(t-\tau(t))-\int_{t-\tau(t)}^{t-h_1}y(s)ds-\int_{t-\tau(t)}^{t-h_1}g(s)dw(s)\Big]=0,
$$

$$
\begin{aligned}
h_1(t) &= 2\xi_0^T(t)\begin{bmatrix} 0 \\ P_5^T \\ P_6^T \\ P_7^T \\ P_8^T \\ P_{16}^T \end{bmatrix}\eta_4 \\
&= 2\xi^T(t)U\Big[-A(t)x(t)+B_0(t)f(x(t))+B_1(t)f(x(t-\tau(t))) \\
&\quad +B_2(t)\int_{t-r(t)}^{t}f(x(s))ds-y(t)\Big]=0,
\end{aligned}
$$

$$
\begin{aligned}
h_2(t) &= 2\xi_0^T(t)\begin{bmatrix} 0 \\ P_9^T \\ P_{10}^T \\ P_{11}^T \\ P_{12}^T \\ P_{17}^T \end{bmatrix}\eta_5 \\
&= 2\xi^T(t)V\Big[C(t)x(t)+D_0(t)x(t-\tau(t))+D_1(t)f(x(t))+D_2(t)f(x(t-\tau(t))) \\
&\quad +D_3(t)\int_{t-r(t)}^{t} f(x(s))ds - g(t)\Big] = 0.
\end{aligned}
$$

$$
\mathcal{L}V_2(x_t,t) \leq 2f^T(x(t))Ky(t)+g^T(t)Kg(t), \tag{2.3.9}
$$

$$
\begin{aligned}
\mathcal{L}V_3(x_t,t) &= x^T(t)Q_1x(t)-(1-\mu)x^T(t-\tau(t))Q_1x(t-\tau(t))+x^T(t)Q_2x(t)-x^T(t-h_1) \\
&\quad \times Q_2x(t-h_1)+x^T(t)Q_3x(t)-x^T(t-h_2)Q_3x(t-h_2) \qquad (2.3.10) \\
&\quad +f^T(x(t))Q_4f(x(t))-(1-\mu) \\
&\quad \times f^T(x(t-\tau(t)))Q_4f(x(t-\tau(t))) \qquad (2.3.11)
\end{aligned}
$$

$$
\mathcal{L}V_4(x_t,t) \leq \bar{r}f^T(x(t))Yf(x(t)) - \int_{t-r(t)}^{t} f^T(x(s))Yf(x(s))ds \tag{2.3.12}
$$

$$
\mathcal{L}V_5(x_t,t) = h_2y^T(t)R_1y(t) - \int_{t-h_2}^{t} y^T(s)R_1y(s)ds + h_2g^T(t)R_1g(t) - \int_{t-h_2}^{t} g^T(s)R_1g(s)ds \tag{2.3.13}
$$

$$\mathcal{L}V_6(x_t,t) = (h_2-h_1)y^T(t)R_2y(t)+(h_2-h_1)g^T(t)R_2g(t)-\int_{t-h_2}^{t-h_1}y^T(s)R_2y(s)ds - \int_{t-h_2}^{t-h_1}g^T(s)R_2g(s)ds \tag{2.3.14}$$

Then by Lemma 2.2.2 and using $0 \le h_1 \le \tau(t) \le h_2$ and $0 < r(t) \le \bar{r}$, we have

$$-h_2\int_{t-\tau(t)}^{t}y^T(s)R_1y(s)ds \le -\Big[\int_{t-\tau(t)}^{t}y(s)ds\Big]^T R_1\Big[\int_{t-\tau(t)}^{t}y(s)ds\Big], \tag{2.3.15}$$

$$-(h_2-h_1)\int_{t-\tau(t)}^{t-h_1}y^T(s)(R_1+R_2)y(s)ds \le -\Big[\int_{t-\tau(t)}^{t-h_1}y(s)ds\Big]^T(R_1+R_2)\Big[\int_{t-\tau(t)}^{t-h_1}y(s)ds\Big], \tag{2.3.16}$$

$$-(h_2-h_1)\int_{t-h_2}^{t-\tau(t)}y^T(s)R_2y(s)ds \le -\Big[\int_{t-h_2}^{t-\tau(t)}y(s)ds\Big]^T R_2\Big[\int_{t-h_2}^{t-\tau(t)}y(s)ds\Big]. \tag{2.3.17}$$

It is obvious that

$$-2K_1f^T(x(t))f(x(t))+2f^T(x(t))K_1Lx(t) \ge 0, \tag{2.3.18}$$

$$-2K_2f^T(x(t-\tau(t)))f(x(t-\tau(t)))+2f^T(x(t-\tau(t)))K_2Lx(t-\tau(t)) \ge 0. \tag{2.3.19}$$

Substituting from (2.3.8) to (2.3.19) into (2.3.7) we get

$$dV(x_t,t) \le \zeta^T(t)\Psi\zeta(t)+\xi(dw(t)),$$

where

$$\Psi = \begin{bmatrix} \bar{\Omega} & -N & -M & -S & UB_2(t)+VD_3(t) \\ * & -\frac{1}{h_2}R_1 & 0 & 0 & 0 \\ * & * & -\frac{1}{h_2-h_1}(R_1+R_2) & 0 & 0 \\ * & * & * & -\frac{1}{h_2-h_1}R_2 & 0 \\ * & * & * & * & -\frac{1}{\bar{r}}Y \end{bmatrix}$$

with

$$
\begin{aligned}
\bar{\Omega} &= (\varphi_{ij})_{8\times 8} + \Delta\Omega_1 + \Delta\Omega_2,\\
\Delta\Omega_1 &= UHF(t)\bar{T}_1 + \bar{T}_1^T F^T(t)H^T U^T,\\
\Delta\Omega_2 &= VHF(t)\bar{T}_2 + \bar{T}_2^T F^T(t)H^T V^T,\\
\zeta^T(t) &= \Big[\xi^T(t), \int_{t-\tau(t)}^{t} y^T(s)ds, \int_{t-h_2}^{t-\tau(t)} y^T(s)ds, \int_{t-\tau(t)}^{t-h_1} y^T(s)ds, \int_{t-r(t)}^{t} f^T(x(s))ds\Big],\\
\xi(dw(t)) &= -2\xi^T(t)N\int_{t-\tau(t)}^{t} g(s)dw(s)\\
&\quad -2\xi^T(t)M\int_{t-\tau(t)}^{t-h_1} g(s)dw(s) - 2\xi^T(t)S\int_{t-h_2}^{t-\tau(t)} g(s)dw(s) + 2x^T(t)Pg(t)dw(t).
\end{aligned}
$$

According to $\Pi_1 < 0$ and there exist a scalar $\alpha > 0$ such that

$$\Pi_1 + diag\{\alpha I_n, 0, 0, 0, 0, 0, 0, 0, 0\} < 0.$$

Hence we have

$$\frac{\mathbb{E}dV(x_t, t)}{dt} \leq \mathbb{E}(\xi^T(t)\Xi\xi(t)) \leq \alpha\mathbb{E}|x(t)|^2.$$

Thus, if $\Pi_1 < 0$ the equilibrium point of the stochastic neural networks (2.2.4) is robustly asymptotically stochastically stable in the mean square. The proof is completed. □

Remark 2.3.1. *In the following Theorem, the new stability results of the stochastic neural networks are discussed (2.2.4) by using the same Lyapunov-Krasovkii functional as defined in Theorem* 2.3.1

Theorem 2.3.2. *For given scalars $h_2 > h_1 \geq 0$ and μ, the equilibrium solution of stochastic neural networks (2.2.4) is globally asymptotically stable in the mean square if there exist matrices $P_1 > 0$, $Y > 0$, $Q_l = Q_l^T \geq 0$, $l = 1, 2, \cdots, 4$, $R_i = R_i^T > 0$, $i = 1, 2$, diagonal matrices $K_1 > 0, K_2 > 0$, $K = diag\{k_1, k_2, \cdots, k_n\} \geq 0$,* *and positive scalars* $\epsilon_i > 0$ $i = 1, 2, \cdots, 4$ *such that*

$$\Pi_2 = \begin{bmatrix} \Pi_{1,1} & -h_2 N & \Pi_{1,3} & \Pi_{1,4} & \Pi_{1,5} & UH & VH & UH & VH \\ * & -h_2 R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Pi_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{4,4} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Pi_{5,5} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\epsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\epsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & * & -\epsilon_3 I & 0 \\ * & * & * & * & * & * & * & * & -\epsilon_4 I \end{bmatrix} < 0, \quad (2.3.20)$$

where

$$\begin{aligned} \Pi_{1,1} &= (\varphi_{i,j})_{8\times 8} + \epsilon_1 \bar{T}_1^T \bar{T}_1 + \epsilon_2 \bar{T}_2^T \bar{T}_2, \quad \Pi_{1,3} = -(h_2 - h_1)M, \quad \Pi_{1,4} = -(h_2 - h_1)S, \\ \Pi_{1,5} &= UB_2 + VD_3, \quad \Pi_{3,3} = -(h_2 - h_1)(R_1 + R_2), \quad \Pi_{4,4} = -(h_2 - h_1)R_2, \\ \Pi_{5,5} &= -\frac{1}{\bar{r}}Y + \epsilon_3 T_4^T T_4 + \epsilon_4 T_9^T T_9. \end{aligned}$$

The remaining terms are defined in Theorem 2.3.1.

Proof. Consider the same Lyapunov-Krasovskii functional as defined in Theorem

2.3.1 and utilizing proof of the same Theorem, we have

$$
\begin{aligned}
& -2h_2\xi^T(t)N\Big[\int_{t-\tau(t)}^{t} y(s)ds + \int_{t-\tau(t)}^{t} g(s)dw(s)\Big] - h_2\int_{t-\tau(t)}^{t} y^T(s)R_1y(s)ds \\
& -h_2\int_{t-\tau(t)}^{t} g^T(s)R_1g(s)ds \leq -h_2\int_{t-\tau(t)}^{t}\Big[\xi^T(t)N + \Big(\int_{t-\tau(t)}^{t} y(\theta)d\theta + \int_{t-\tau(t)}^{t} g(\theta)dw(\theta)\Big)R_1\Big] \\
& \times R_1^{-1}\Big[N^T\xi(t) + R_1\Big(\int_{t-\tau(t)}^{t} y(\theta)d\theta + \int_{t-\tau(t)}^{t} g(\theta)dw(\theta)\Big)\Big]ds + h_2^2\xi^T(t)NR_1^{-1}N\xi(t) \\
\leq\ & h_2^2\xi^T(t)NR_1^{-1}N\xi(t), \qquad (2.3.21)
\end{aligned}
$$

$$
\begin{aligned}
& -2(h_2-h_1)\xi^T(t)M\Big[\int_{t-h_2}^{t-\tau(t)} y(s)ds + \int_{t-h_2}^{t-\tau(t)} g(s)dw(s)\Big] - (h_2-h_1) \\
& \int_{t-h_2}^{t-\tau(t)} y^T(s)(R_1+R_2)y(s)ds - (h_2-h_1)\int_{t-h_2}^{t-\tau(t)} g^T(s)(R_1+R_2)g(s)ds \\
\leq\ & -(h_2-h_1)\int_{t-h_2}^{t-\tau(t)}\Big[\xi^T(t)M + \Big(\int_{t-h_2}^{t-\tau(t)} y(\theta)d\theta + \int_{t-h_2}^{t-\tau(t)} g(\theta)dw(\theta)\Big)(R_1+R_2)\Big](R_1+R_2)^{-1} \\
& \Big[M^T\xi(t) + (R_1+R_2)\Big(\int_{t-h_2}^{t-\tau(t)} y(\theta)d\theta + \int_{t-h_2}^{t-\tau(t)} g(\theta)dw(\theta)\Big)\Big]ds \\
& +(h_2-h_1)^2\xi^T(t)M(R_1+R_2)^{-1}M\xi(t) \\
\leq\ & (h_2-h_1)^2\xi^T(t)M(R_1+R_2)^{-1}M\xi(t), \qquad (2.3.22)
\end{aligned}
$$

$$
\begin{aligned}
& -2(h_2-h_1)\xi^T(t)S\Big[\int_{t-\tau(t)}^{t-h_1} y(s)ds + \int_{t-\tau(t)}^{t-h_1} g(s)dw(s)\Big] - (h_2-h_1)\int_{t-\tau(t)}^{t-h_1} y^T(s)R_2y(s)ds \\
& -(h_2-h_1)\int_{t-\tau(t)}^{t-h_1} g^T(s)R_2g(s)ds \\
\leq\ & -(h_2-h_1)\int_{t-\tau(t)}^{t-h_1}\Big[\xi^T(t)S + \Big(\int_{t-\tau(t)}^{t-h_1} y(\theta)d\theta + \int_{t-\tau(t)}^{t-h_1} g(\theta)dw(\theta)\Big)R_2\Big] \\
& \times R_2^{-1}\Big[S^T\xi(t) + R_2\Big(\int_{t-\tau(t)}^{t} y(\theta)d\theta + \int_{t-\tau(t)}^{t-h_1} g(\theta)dw(\theta)\Big)\Big]ds + (h_2-h_1)^2\xi^T(t)SR_2^{-1}S\xi(t) \\
\leq\ & (h_2-h_1)^2\xi^T(t)SR_2^{-1}S\xi(t). \qquad (2.3.23)
\end{aligned}
$$

The remaining part of the proof follows from Theorem 2.3.1. Then we have

$$dV(x_t,t) \leq \zeta^T(t)\Psi\zeta(t) + \xi(dw(t)),$$

where

$$\Psi = \begin{bmatrix} \bar{\Omega} & UB_2(t) + VD_3(t) \\ \\ * & -\frac{1}{\bar{r}}Y \end{bmatrix}$$

with

$$\begin{aligned} \bar{\Omega} &= (\varphi_{ij})_{8\times 8} + \Delta\Omega_1 + \Delta\Omega_2 + h_2 N R_1^{-1} N + (h_2 - h_1) M (R_1 + R_2)^{-1} M^T \\ &\quad + (h_2 - h_1) S R_2^{-1} S^T, \\ \Delta\Omega_1 &= UHF(t)\bar{T}_1 + \bar{T}_1^T F^T(t) H^T U^T, \quad \Delta\Omega_2 = VHF(t)\bar{T}_2 + \bar{T}_2^T F^T(t) H^T V^T, \\ \zeta^T(t) &= \Big[\xi^T(t), \int_{t-r(t)}^{t} f^T(x(s))ds\Big], \quad \xi(dw(t)) = 2x^T(t)Pg(t)dw(t). \end{aligned}$$

According to (2.2.6) and Lemma 2.2.3, $\Delta\Omega_1$ and $\Delta\Omega_2$ satisfy the following inequality

$$\Delta\Omega_1 \le \epsilon_1^{-1} UHH^T U^T + \epsilon_1 \bar{T}_1^T \bar{T}_1, \quad \Delta\Omega_2 \le \epsilon_2^{-1} VHH^T V^T + \epsilon_2 \bar{T}_2^T \bar{T}_2.$$

According to $\Pi_1 < 0$ and there exist a scalar $\alpha > 0$ such that

$$\Pi_1 + diag\{\alpha I_n, 0, 0, 0, 0, 0, 0, 0, 0\} < 0.$$

Hence we have

$$\frac{d\mathbb{E}V(x_t, t)}{dt} \le \mathbb{E}(\xi^T(t)\Xi\xi(t)) \le \alpha\mathbb{E}|x(t)|^2.$$

Thus, if $\Pi_2 < 0$ the equilibrium point of the stochastic neural networks (2.2.4) is robustly asymptotically stochastically stable in the mean square. The proof is completed. □

Remark 2.3.2. *In the following Theorem, the new stability results of the stochastic neural networks are discussed (2.2.4) by using the same Lyapunov-Krasovkii functional as defined in Theorem* 2.3.1.

Theorem 2.3.3. *For given scalars $h_2 > h_1 \ge 0$ and μ, the equilibrium solution of stochastic neural networks (2.2.4) is globally asymptotically stable in the mean square if there exist matrices $P_1 > 0$, $Y > 0$, $Q_l = Q_l^T \ge 0$, $l = 1, 2, \cdots, 4$, $R_i = R_i^T >$*

$0,\ i=1,2,$ *diagonal matrices* $K_1>0, K_2>0,\ K=diag\{k_1,k_2,\cdots,k_n\}\geq 0,$ *and positive scalars* $\epsilon_i>0\ i=1,2,\cdots,4$ *such that the linear matrix inequalities (LMIs) are feasible*

$$\Pi_3=\begin{bmatrix} \bar{\Pi}_{1,1} & UB_2+VD_3 & UH & VH & UH & VH \\ * & \bar{\Pi}_{2,2} & 0 & 0 & 0 & 0 \\ * & * & -\epsilon_1 I & 0 & 0 & 0 \\ * & * & * & -\epsilon_2 I & 0 & 0 \\ * & * & * & * & -\epsilon_3 I & 0 \\ * & * & * & * & * & -\epsilon_4 I \end{bmatrix}<0, \tag{2.3.24}$$

$$\Psi_1=\begin{bmatrix} X & N \\ * & R_1 \end{bmatrix}\geq 0, \tag{2.3.25}$$

$$\Psi_2=\begin{bmatrix} Y & S \\ * & R_2 \end{bmatrix}\geq 0, \tag{2.3.26}$$

$$\Psi_3=\begin{bmatrix} X+Y & M \\ * & R_1+R_2 \end{bmatrix}\geq 0, \tag{2.3.27}$$

where

$$\bar{\Pi}_{1,1} = (\Pi_{i,j})_{8\times 8} + \epsilon_1 \bar{T}_1^T \bar{T}_1 + \epsilon_2 \bar{T}_2^T \bar{T}_2, \quad \bar{\Pi}_{2,2} = -\frac{1}{\bar{r}} Y + \epsilon_3 T_4^T T_4 + \epsilon_4 T_9^T T_9,$$

with

$$\begin{aligned}
\varphi_{1,1} &= Q_1 + Q_2 + Q_3 + P_2 + P_2^T + h_2 X_{11} + (h_2 - h_1) Y_{11}, \varphi_{1,2} = -P_2^T + P_3^T - P_4^T \\
&\quad -P_5 A + P_9 C + h_2 X_{12} + (h_2 - h_1) Y_{12}, \varphi_{1,3} = P_4^T - P_6 A + P_{10} C + h_2 X_{13} \\
&\quad +(h_2 - h_1) Y_{13}, \varphi_{1,4} = -P_3^T - P_7 A + P_{11} C + h_2 X_{14} + (h_2 - h_1) Y_{14},
\end{aligned}$$

$$\begin{aligned}
\varphi_{1,5} &= P_1 - A^T P_8 + C^T P_{12} + h_2 X_{15} + (h_2 - h_1) Y_{15}, \\
\varphi_{1,6} &= -A^T P_{16} + C^T P_{17} + P_{13}^T + h_2 X_{16} + (h_2 - h_1) Y_{16}, \\
\varphi_{1,7} &= h_2 X_{17} + (h_2 - h_1) Y_{17} + L^T K_1, \varphi_{1,8} = h_2 X_{18} + (h_2 - h_1) Y_{18}, \varphi_{2,2} = -(1-\mu) Q_1 + P_9 D_0 \\
&\quad +D_0^T P_9^T + b_2 L^T L + h_2 X_{22} + (h_2 - h_1) Y_{22}, \varphi_{2,3} = D_0^T P_{10}^T + h_2 X_{23} + (h_2 - h_1) Y_{23}, \\
\varphi_{2,4} &= D_0^T P_{11}^T + h_2 X_{24} + (h_2 - h_1) Y_{24}, \quad \varphi_{2,5} = -P_5 + D_0^T P_{12}^T + h_2 X_{25} + (h_2 - h_1) Y_{25}, \\
\varphi_{2,6} &= -P_{13}^T + P_{14}^T - P_{15}^T + D_0^T P_{17} + h_2 X_{26} + (h_2 - h_1) Y_{26} \\
\varphi_{2,7} &= -P_9 + P_5 B_0 + P_9 D_1 + h_2 X_{27} + (h_2 - h_1) Y_{27}, \\
\varphi_{2,8} &= P_5 B_1 + P_9 D_2 + h_2 X_{28} + (h_2 - h_1) Y_{28} + L^T K_2, \quad \varphi_{3,3} = -Q_2 + h_2 X_{33} + (h_2 - h_1) Y_{33}, \\
\varphi_{3,4} &= h_2 X_{34} + (h_2 - h_1) Y_{34}, \quad \varphi_{3,5} = -P_6 + h_2 X_{35} + (h_2 - h_1) Y_{35}, \\
\varphi_{3,6} &= P_{15}^T - P_{10} + h_2 X_{36} + (h_2 - h_1) Y_{36}, \quad \varphi_{3,7} = P_6 B_0 + P_{10} D_1 + h_2 X_{37} + (h_2 - h_1) Y_{37}, \\
\varphi_{3,8} &= P_6 B_1 + P_{10} D_2 + h_2 X_{38} + (h_2 - h_1) Y_{38}, \quad \varphi_{4,4} = -Q_3 + h_2 X_{44} + (h_2 - h_1) Y_{44}, \\
\varphi_{4,5} &= -P_7 + h_2 X_{45} + (h_2 - h_1) Y_{45}, \quad \varphi_{4,6} = -P_{14}^T - P_{11} + h_2 X_{46} + (h_2 - h_1) Y_{46}, \\
\varphi_{4,7} &= P_7 B_0 + P_{11} D_1 + h_2 X_{47} + (h_2 - h_1) Y_{47}, \quad \varphi_{4,8} = P_7 B_1 + P_{11} D_2 + h_2 X_{48} + (h_2 - h_1) Y_{48},
\end{aligned}$$

$$\begin{aligned}
\varphi_{5,5} &= -2P_8 + h_1R_1 + (h_2-h_1)R_2 + h_2X_{55} + (h_2-h_1)Y_{55}, \varphi_{5,6} = -P_{16}^T - P_{12} + h_2X_{56} \\
&\quad +(h_2-h_1)Y_{56}, \quad \varphi_{5,7} = K^T + P_8B_0 + P_{12}D_1 + h_2X_{57} + (h_2-h_1)Y_{57}, \\
\varphi_{5,8} &= P_8B_1 + P_{12}D_2 + h_2X_{58} + (h_2-h_1)Y_{58}, \\
\varphi_{6,6} &= P_1 + \bar{k}lI + h_1R_1 + (h_2-h_1)R_2 - 2P_{17} + h_2X_{66} + (h_2-h_1)Y_{66}, \\
\varphi_{6,7} &= P_{16}B_0 + P_{17}D_1 + h_2X_{67} + (h_2-h_1)Y_{67}, \\
\varphi_{6,8} &= P_{16}B_1 + P_{17}D_2 + h_2X_{68} + (h_2-h_1)Y_{68}, \\
\varphi_{7,7} &= Q_4 + \bar{r}Y - 2K_1 + h_2X_{77} + (h_2-h_1)Y_{77}, \\
\varphi_{7,8} &= h_2X_{78} + (h_2-h_1)Y_{78}, \quad \varphi_{8,8} = -(1-\mu)Q_4 - 2K_2 + h_2X_{88} + (h_2-h_1)Y_{88}, \\
\bar{T}_1 &= [-T_1, 0, 0, 0, 0, 0, T_2, T_3], \quad \bar{T}_2 = [T_5, T_6, 0, 0, 0, 0, T_7, T_8] \\
M^T &= [P_2, 0, 0, 0, 0, P_{13}, 0, 0], \quad U = [0, P_5^T, P_6^T, P_7^T, P_8^T, P_{16}^T, 0, 0], \\
N^T &= [P_3, 0, 0, 0, 0, P_{14}, 0, 0], \quad S^T = [P_4, 0, 0, 0, 0, P_{15}, 0, 0], \\
V &= [0, P_9^T, P_{10}^T, P_{11}^T, P_{12}^T, P_{17}^T, 0, 0].
\end{aligned}$$

Proof. From Theorem 2.3.1, it follows that

$$\begin{aligned}
dV(x_t,t) &\le \zeta^T(t)\Psi\zeta(t) - \int_{t-\tau(t)}^{t} \xi^T(t,s)\Psi_1\xi(t,s)ds - \int_{t-\tau(t)}^{t-h_1} \xi^T(t,s)\Psi_2\xi(t,s)ds \\
&\quad - \int_{t-h_2}^{t-\tau(t)} \xi^T(t,s)\Psi_3\xi(t,s)ds + \xi(dw(t)),
\end{aligned}$$

where

$$\Psi = \begin{bmatrix} \bar{\Pi}_{1,1} & UB_2(t) + VD_3(t) \\ * & -\frac{1}{\bar{r}}Y \end{bmatrix},$$

with

$$\begin{aligned}
\bar{\Pi}_{1,1} &= (\Pi_{i,j})_{8\times 8} + \Delta\Omega_1 + \Delta\Omega_2, \quad \zeta^T(t) = \left[\xi^T(t), \int_{t-r(t)}^{t} f^T(x(s))ds\right], \\
\Delta\Omega_1 &= UHF(t)\bar{T}_1 + \bar{T}_1^T F^T(t)H^TU^T, \quad \Delta\Omega_2 = VHF(t)\bar{T}_2 + \bar{T}_2^T F^T(t)H^TV^T,
\end{aligned}$$

$$\begin{aligned}\xi(dw(t)) &= -2\xi^T(t)N\int_{t-\tau(t)}^{t} g(s)dw(s) - 2\xi^T(t)M\int_{t-\tau(t)}^{t-h_1} g(s)dw(s) \\ &\quad -2\xi^T(t)S\int_{t-h_2}^{t-\tau(t)} g(s)dw(s) + 2x^T(t)Pg(t)dw(t), \\ \xi^T(t,s) &= \Big[x^T(t), x^T(t-\tau(t)), x^T(t-h_1), x^T(t-h_2), y^T(t), g^T(t), f^T(x(t)), \\ &\quad f^T(x(t-\tau(t))), y(s)\Big].\end{aligned}$$

According to (2.2.6) and Lemma 2.2.3, $\varDelta\varOmega_1$ and $\varDelta\varOmega_2$ satisfy the following inequality:

$$\varDelta\varOmega_1 \le \epsilon_1^{-1}UHH^TU^T + \epsilon_1\bar{T}_1^T\bar{T}_1, \quad \varDelta\varOmega_2 \le \epsilon_2^{-1}VHH^TV^T + \epsilon_2\bar{T}_2^T\bar{T}_2.$$

According to $\varPi_1 < 0$ and there exist a scalar $\alpha > 0$ such that

$$\varPi_1 + diag\{\alpha I_n, 0, 0, 0, 0, 0, 0, 0, 0\} < 0.$$

Hence we have

$$\frac{d\mathbb{E}V(x_t,t)}{dt} \le \mathbb{E}(\xi^T(t)\varXi\xi(t)) \le \alpha\mathbb{E}|x(t)|^2.$$

Thus, if $\varPi_1 < 0$ and $\varPsi_i \ge 0, i = 1, 2, 3$, the equilibrium point of the stochastic neural networks (2.2.4) is robustly asymptotically stochastically stable in mean square. The proof is completed. □

Remark 2.3.3. *When the derivative of $\tau(t)$ is unknown, and the delay $\tau(t)$ satisfies $0 \le h_1 \le \tau(t) \le h_2$, by setting $Q_i = 0, i = 1, 4$ in (2.3.9), we can know that the system (2.2.4) is delay/interval dependent and rate-independent robustly asymptotically stable in mean square for delays $0 \le h_1 \le \tau(t) \le h_2$ and $0 \le r(t) \le \bar{r}$.*

The stability results for the following uncertain stochastic neural networks with time-varying delays are stated in the following three Corollaries.

$$\begin{aligned}dx(t) &= \Big[-A(t)x(t) + B_0(t)f(x(t)) + B_1(t)f(x(t-\tau(t)))\Big]dt + \Big[C(t)x(t) \\ &\quad + D_0(t)x(t-\tau(t)) + D_1(t)f(x(t)) + D_2(t)f(x(t-\tau(t)))\Big]dw(t), \qquad (2.3.28)\end{aligned}$$

where the time-delay $\tau(t)$ satisfies $0 \le h_1 \le \tau(t) \le h_2$, $\dot{\tau}(t) \le \mu$. Then, we have the following results.

Corollary 2.3.1. *For given scalars $h_2 > h_1 \geq 0$ and μ, the equilibrium solution of stochastic neural networks (2.3.28) is globally asymptotically stable in the mean square if there exist matrices $P_1 > 0$,$Q_l = Q_l^T \geq 0$, $l = 1, 2, \cdots, 4$, $R_i = R_i^T > 0$, $i = 1, 2$, diagonal matrices $K_1 > 0, K_2 > 0$, $K = diag\{k_1, k_2, \cdots, k_n\} \geq 0$, and positive scalars $\epsilon_i > 0$ $i = 1, 2, \cdots, 4$ such that*

$$\Pi_4 = \begin{bmatrix} \Pi_{1,1} & -N & -M & -S & UH & VH \\ * & \Pi_{2,2} & 0 & 0 & 0 & 0 \\ * & * & \Pi_{3,3} & 0 & 0 & 0 \\ * & * & * & \Pi_{4,4} & 0 & 0 \\ * & * & * & * & -\epsilon_1 I & 0 \\ * & * & * & * & * & -\epsilon_2 I \end{bmatrix} < 0, \qquad (2.3.29)$$

the terms $\Pi_{1,1}, \Pi_{2,2}, \Pi_{3,3}, \Pi_{4,4}$ is defined in Theorem 2.3.1.

Corollary 2.3.2. *For given scalars $h_2 > h_1 \geq 0$ and μ, the equilibrium solution of stochastic neural networks (2.3.28) is globally asymptotically stable in the mean square if there exist matrices $P_1 > 0$,$Q_l = Q_l^T \geq 0$, $l = 1, 2, \cdots, 4$, $R_i = R_i^T > 0$, $i = 1, 2$, diagonal matrices $K_1 > 0, K_2 > 0$, $K = diag\{k_1, k_2, \cdots, k_n\} \geq 0$, and* *positive scalars* $\epsilon_i > 0$ $i = 1, 2, \cdots, 4$ *such that*

$$\Pi_5 = \begin{bmatrix} \Pi_{1,1} & -h_2N & \Pi_{1,3} & \Pi_{1,4} & UH & VH \\ * & -h_2R_1 & 0 & 0 & 0 & 0 \\ * & * & \Pi_{3,3} & 0 & 0 & 0 \\ * & * & * & \Pi_{4,4} & 0 & 0 \\ * & * & * & * & -\epsilon_1 I & 0 \\ * & * & * & * & * & -\epsilon_2 I \end{bmatrix} < 0, \qquad (2.3.30)$$

the terms $\Pi_{1,1}, \Pi_{2,2}, \Pi_{3,3}, \Pi_{4,4}$ *is defined in Theorem* 2.3.2.

Corollary 2.3.3. *For given scalars* $h_2 > h_1 \geq 0$ *and* μ*, the equilibrium solution of stochastic neural networks (2.3.28) is globally asymptotically stable in the mean square if there exist matrices* $P = P^T > 0, Q_l = Q_l^T \geq 0$*,* $l = 1, 2, \cdots, 4$*,* $Z_i = Z_i^T > 0$*,* $i = 1, 2$*,* $K = diag\{k_1, k_2, \cdots, k_n\} \geq 0$*,* $X = (X_{ij})_{8\times 8} \geq 0$*,* $Y = (Y_{ij})_{8\times 8} \geq 0$ *diagonal matrices* $K_1 > 0, K_2 > 0$ *and positive scalars* $\epsilon_1 > 0$*,* $\epsilon_2 > 0$*,* $\epsilon_3 > 0$ *and* $\epsilon_4 > 0$ *such that the linear matrix inequalities (LMIs) are feasible*

$$\Pi_6 = \begin{bmatrix} \hat{\Pi}_{1,1} & UH & VH \\ * & -\epsilon_1 I & 0 \\ * & * & -\epsilon_2 I \end{bmatrix} < 0, \qquad (2.3.31)$$

$$\Psi_1 = \begin{bmatrix} X & N \\ \\ * & R_1 \end{bmatrix} \geq 0, \tag{2.3.32}$$

$$\Psi_2 = \begin{bmatrix} Y & S \\ \\ * & R_2 \end{bmatrix} \geq 0, \tag{2.3.33}$$

$$\Psi_3 = \begin{bmatrix} X+Y & M \\ \\ * & R_1+R_2 \end{bmatrix} \geq 0, \tag{2.3.34}$$

where

$$\hat{\Pi}_{1,1} = (\Pi_{i,j})_{8\times 8} + \epsilon_1 \bar{T}_1^T \bar{T}_1 + \epsilon_2 \bar{T}_2^T \bar{T}_2.$$

The remaining terms are defined in Theorem 2.3.3.

2.4 Numerical Examples

In this section, two examples are provided showing the effectiveness of the conditions given here.

Example 2.4.1

Consider the uncertain stochastic neural networks

$$\begin{aligned} dx(t) \;\; = \;\; & \Big[-A(t)x(t) + B_0(t)f(x(t)) + B_1(t)f(x(t-\tau(t))) + B_2(t)\int_{t-r(t)}^{t} f(x(s))ds \Big] dt \\ & + \Big[C(t)x(t) + D_0(t)x(t-\tau(t)) + D_1(t)f(x(t)) + D_2(t)f(x(t-\tau(t))) \\ & + D_3(t)\int_{t-r(t)}^{t} f(x(s))ds \Big] dw(t) \end{aligned}$$

where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.2 & -4 \\ 0.1 & 0.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.7 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.2 & 0.6 \\ 0.5 & 0.1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.3 & 0.6 \\ 0.2 & -0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0.5 & -0.1 \\ -0.5 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$

$$B_2 = D_3 = I, L = 0.2I, \quad \text{and} \quad T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_7 = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

In order to show the significant contributions of this chapter, the comparisons between the obtained results of Theorem 2.3.1, Theorem 2.3.2 and Theorem 2.3.3 are summarised and presented.Table 2.1 gives the comparison results on the maximum allowable upper bounds h_2 and $\bar{r}$. It is found that the equilibrium solution of uncertain stochastic neural networks (2.2.4) is asymptotically stable in mean square.

when $\mu = 0$, $h_1 = 0$	Theorem 2.3.1	Theorem 2.3.2	Theorem 2.3.3
$h_2 = r$	0.4936	0.4806	0.4103

Table 2.1: Comparison of our result

Example 2.4.2

Consider the uncertain stochastic neural networks

$$\begin{aligned} dx(t) &= \Big[- A(t)x(t) + B_0(t)f(x(t)) + B_1(t)f(x(t-\tau(t))) \Big] dt \\ &\quad + \Big[C(t)x(t) + D_0(t)x(t-\tau(t)) + D_1(t)f(x(t)) + D_2(t)f(x(t-\tau(t))) \Big] dw(t) \end{aligned}$$

where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.4 & -0.7 \\ 0.1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.2 & 0.6 \\ 0.5 & -0.1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix},$$

$$L = 0.5I, T_1 = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix}, T_2 = \begin{bmatrix} 0.2 & -0.3 \end{bmatrix}, T_3 = \begin{bmatrix} -0.2 & -0.3 \end{bmatrix} \text{ and}$$

$$T_4 = T_5 = T_6 = T_7 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}.$$

when $\mu = 0$, $h_1 = 0$	Corollary 2.3.1	Corollary 2.3.2	Corollary 2.3.3
h_2	0.8998	0.8998	0.6436

Table 2.2: Comparison of our result

For the above system, applying Theorem 2.3.1 in Huang and Feng (2007) and Theorem 2.3.2 in Yu et al.(2009), it is found that the equilibrium solution of stochastic neural network is robustly exponentially stable in mean square for any delay $\tau(t)$ satisfying $0 < \tau(t) \leq 0.4109$ and $0 < \tau(t) \leq 0.6196$ respectively. However, by our Corollary 2.3.1, Corollary 2.3.2 and Corollary 2.3.3 in this chapter it is concluded that the system (2.3.28) is robustly asymptotically stable in mean square for maximum allowable upper bounds h_2 is given in Table 2.2. Hence the proposed method is finer than the previous works based on the upper bound techniques.

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