CHAPTER 4

Chapter 4

Quasi generalized π-closed function

4.1 Introduction

Functions and of course open functions stand among the most important notions in the whole of mathematical science. Many different forms of open functions have been introduced over the years. Various interesting problems arise when one considers openness. Its importance is significant in various areas of mathematics and related sciences. The notion of a quasi normal space was introduced by Zaitsev (1968). Saeid Jafari and Noiri (2001) introduced α-quasi-irresolute functions and studied the relationships between α-quasi irresolute functions and graphs. Rajesh and Ekici (2008) introduced quasi $\tilde{g}$ s-open and quasi $\tilde{g}$ s-closed functions in topological spaces. Vadivel and Vairamanickam (2010) studied quasi rw-open and quasi rw-closed functions in topological spaces.

In this chapter, we have introduced the concept of quasi gπ-closed and quasi gπ-open functions, gπ-quasi irresolute functions and gπ-quasi closed graphs. The properties of gπ-Hausdorff spaces and gπ-normal spaces are discussed.

4.2 Quasi gπ-open Functions

Definition 4.2.1 A function f : X → Y is said to be quasi gπ-open if the image of every gπ-open set in X is open in Y.

Example 4.2.1(a) Let X = Y = {a, b, c}, τ = {X, φ, {a, b}} and σ = {Y, φ, {a}, {b}, {a, b}}. Define a function f : (X, τ) → (Y, σ) by f (b) = a, f (a) = b, f(c)=c. Then f is quasi open.

It is evident that, the concepts of quasi gπ-openness and gπ-continuity coincide if the function is a bijection

Theorem 4.2.2 A function $f : X \to Y$ is quasi gπ-open iff for every subset U of X, $f(g\pi int(U)) \subset int(f(U))$.

Proof: Let f be a quasi gπ-open function. $g\pi int(U)$ is a gπ-open set and $g\pi int(U) \subset U$, $f(g\pi int(U)) \subset f(U)$. As $f(g\pi int(U))$ is open, $f(g\pi int(U)) \subset int(f(U))$.

Conversely, assume that U is a gπ-open set in X. Then $f(U) = f(g\pi int(U)) \subset int(f(U))$ but $int(f(U)) \subset f(U)$. Consequently $f(U) = int(f(U))$ and hence f is quasi gπ-open.

Lemma 4.2.3 If a function $f : X \to Y$ is quasi gπ-open, then $g\pi int(f^{-1}(G)) \subset f^{-1}(int(G))$ for every subset G of Y .

Proof: Let G be any arbitrary subset of Y. Then, $g\pi int(f^{-1}(G))$ is a gπ-open set in X and f is quasi gπ-open, then $f(g\pi int(f^{-1}(G))) \subset int(f(f^{-1}(G))) \subset int(G)$. Thus $g\pi int(f^{-1}(G)) \subset f^{-1}(int(G)))$.

Theorem 4.2.4 For a function $f : X \to Y$, the following are equivalent.

(i) f is quasi gπ-open.

(ii)For each subset U of X, $f(g\pi int(U)) \subset int(f(U))$.

(iii) For each $x \in X$ and each gπ-neighbourhood U of x in X, there exists a neighbourhood V of f (x) in Y such that $V \subset f(U)$.

Proof: (i) ⇒ (ii) : It follows from Theorem 4.2.2.

(ii) ⇒ (iii) : Let $x \in X$ and U be an arbitrary gπ-neighbourhood of x in X. Then there exist a gπ-open set V in X such that $x \in V \subset U$. Then by (ii), $f(V) = f(g\pi int(V)) \subset int(f(V))$ and hence $f(V) = int(f(V))$. Therefore, it follows that f (V) is open in Y such that $f(x) \in f(V) \subset f(U)$.

(iii) ⇒ (i) : Let U be an arbitrary gπ-open set in X. Then for each $y \in f(U)$, by (iii) there exist a neighbourhood V_y of y in Y such that $V_y \subset f(U)$. As V_y is a neighbourhood of y, there exist an open set W_y in Y such that $y \in W_y \subset V_y$. Thus $f(U) = \cup\{W_y : y \in f(U)\}$ which is an open set in Y. This implies that f is a quasi gπ-open function.

Theorem 4.2.5. A function $f : X \to Y$ is quasi gπ-open if and only if for any subset B of Y and for any gπ-closed set F of X, containing $f^{-1}(B)$, there exist a closed set G of Y containing B such that $f^{-1}(G) \subset F$.

Proof: Assume $f : X \to Y$ is quasi gπ-open. Let $B \subset Y$ and F be a gπ-closed set of X containing $f^{-1}(B)$. Let $G = Y - f(X - F)$. Then $f^{-1}(B) \subset F$ implies $B \subset G$. Since f is quasi gπ-open. G is a closed set of Y and $f^{-1}(G) \subset F$.

Conversely, let U be a gπ-open set of X and put $B = Y - f(U)$. Then $X - U$ is a gπ-closed set in X containing $f^{-1}(B)$. By hypothesis, there exists a closed set F of Y such that $B \subset F$ and $f^{-1}(F) \subset X - U$. Hence, $f(U) \subset Y - F$. Since $B \subset F$, $Y - F \subset Y - B = f(U)$. Thus, $f(U) = Y - F$ which is open and hence f is a quasi gπ-open function.

Theorem 4.2.6. A function $f: X \to Y$ is quasi gπ-open if and only if $f^{-1}(cl(B)) \subset g\pi Cl(f^{-1}(B))$ for every subset B of Y.

Proof: Suppose that f is quasi gπ-open. For any subset B of Y, $f^{-1}(B) \subset g\pi Cl(f^{-1}(B))$. Therefore by Theorem 4.2.5., there exists a closed set F in Y such that $B \subset F$ and $f^{-1}(F) \subset g\pi Cl(f^{-1}(B))$. Therefore, we obtain $f^{-1}(cl(B)) \subset f^{-1}(F) \subset g\pi Cl(f^{-1}(B))$.

Conversely, let $B \subset Y$ and F be a gπ-closed set of X containing $f^{-1}(B)$. Put $W = cl_Y(B)$, then $B \subset W$ and W is a closed set and $f^{-1}(W) \subset g\pi Cl(f^{-1}(B)) \subset F$. Then by Theorem 4.2.5., f is quasi gπ-open.

Lemma 4.2.7 Let $f : X \to Y$ and $g : Y \to Z$ be two functions and $g \circ f : X \to Z$ is quasi gπ-open. If g is continuous injective, then f is quasi gπ-open.

Proof: Let U be a gπ-open set in X, then (g ∘ f) (U) is open in Z. Since g ∘ f is quasi gπ-open. Again g is an injective continuous function, $f(U) = g^{-1}$ (g ∘ f(U)) is open in Y . This shows that f is quasi gπ-open.

4.3 Quasi gπ-closed Functions

Definition 4.3.1 A function f : X → Y is said to quasi gπ-closed if the image of each gπ-closed set in X is closed in Y .

Clearly, every quasi gπ-closed function is closed as well as gπ-closed.

Remark 4.3.2 Every gπ-closed (resp. closed) function need not be quasi gπ-closed as shown by the following example.

Example 4.3.3 Let X = Y = {a, b, c, d}, τ = {X, φ, {b}} and σ = {Y, φ, {b}, {c}, {b, c}}. Define a function f : (X, τ) → (Y, σ) by f (a) = a, f (b) = b, f(c)=c and f (d) = d. Then clearly f is gπ-closed as well as closed but not quasi gπ-closed.

Lemma 4.3.4 If a function f : X → Y is quasi gπ-closed, then f^{-1}(int (B)) ⊂ gπint (f^{-1}(B)) for every subset B of Y .

Proof: This proof is similar to the proof of Lemma 4.2.3

Theorem 4.3.5 A function f : X → Y is quasi gπ-closed if and only if for any subset B of Y and for any gπ-open set G of X containing f^{-1}(B), there exists an open set U of Y containing B such that f^{-1} (U) ⊂ G.

Theorem 4.3.6 If f : X → Y and g : Y → Z are two quasi gπ-closed functions, then g ∘ f is a quasi gπ-closed function.

Theorem 4.3.7 Let f : X → Y and g : Y → Z be any two functions. Then

(i) If f is gπ-closed and g is quasi gπ-closed, then g ∘f is closed.

(ii) If f is quasi gπ-closed and g is gπ-closed, then g ∘ f is $g\pi^*$-closed.

(iii) If f is $g\pi^*$-closed and g is quasi gπ-closed, then $g \circ f$ is quasi gπ-closed.

Theorem 4.3.8 Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$ is quasi gπ-closed.

(i) If f is gπ-irresolute surjective, then g is closed.

(ii) If g is gπ-continuous injective, then f is $g\pi^*$-closed.

Proof: (i) Suppose that F is an arbitrary closed set in Y. As f is gπ-irresolute, $f^{-1}(F)$ is gπ-closed in X. Since $g \circ f$ is quasi gπ-closed and f is surjective, $g \circ f (f^{-1}(F)) = g (F)$ which is closed in Z. This implies g is a closed function.

(ii) Suppose F is any gπ-closed set in X. Since $g \circ f$ is quasi gπ-closed, $(g \circ f) (F)$ is closed in Z. Again g is a gπ-continuous injective function, $g^{-1}(g \circ f (F)) = f (F)$, which is gπ-closed in Y . Hence f is $g\pi^*$-closed.

Theorem 4.3.9 Let X and Y be topological spaces. Then the function $g : X \to Y$ is quasi gπ-closed if and only if $g (X)$ is closed in Y and $g (V) \setminus g (X \setminus V)$ is open in $g (X)$ whenever V is gπ-open in X.

Proof: Necessity: Suppose $g : X \to Y$ is a quasi gπ-closed function. Since X is a gπ-closed, $g (X)$ is closed in Y and $g (V) \setminus g (X \setminus V) = g (V) \cap g (X) \setminus g (X \setminus V)$ is open in $g (X)$ when V is gπ-open in X.

Sufficiency: Suppose $g (X)$ is closed in Y, $g (V) \setminus g (X \setminus V)$ is open in $g (X)$ when V is gπ-open in X, and let C be closed in X. Then $g (C) = g (X) \setminus (g (X \setminus C) \setminus g (C))$ is closed in $g (X)$ and hence, closed in Y.

Corollary 4.3.10 Let X and Y be topological spaces. Then a surjective function $g : X \to Y$ is quasi gπ-closed if and only if $g (V) \setminus g (X \setminus V)$ is open in Y whenever U is gπ-open in X.

Corollary 4.3.11 Let X and Y be topological spaces and let $g : X \to Y$ be a gπ-continuous and quasi gπ-closed surjective function. Then the topology on Y is $\{g(V) \setminus g(X \setminus V) : V \text{ is } g\pi\text{-open in } X\}$.

Proof: Let W be open in Y. Then $g_C(W)$ is gπ-open in X, and $g(g^{-1}(W)) \setminus g(X \setminus g^{-1}(W)) = W$. Hence, all open sets in Y are of the form $g(V) \setminus g(X \setminus V)$, V is gπ-open in X. On the other hand, all sets of the form $g(V) \setminus g(X \setminus V)$, V is gπ-open in X, are open in Y from Corollary 4.3.10.

Definition 4.3.12 A topological space (X, τ) is said to be gπ-normal, if for any pair of disjoint gπ-closed subsets F_1 and F_2 of X, there exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Theorem 4.3.13 Let X and Y be topological spaces such that X is gπ-normal. If $g : X \to Y$ is a gπ-continuous and quasi gπ-closed surjective function. Then Y is normal.

Proof: Let K and M be disjoint closed subsets of Y. Then $g^{-1}(K)$, $g^{-1}(M)$ are disjoint gπ-closed subsets of X. Since X is gπ-normal, there exists disjoint open sets V and W such that $g^{-1}(K) \subset V$ and $g^{-1}(M) \subset W$. Then $K \subset g(V) \setminus g(X \setminus V)$ and $M \subset g(W) \setminus g(X \setminus W)$. Further by Corollary 4.3.10, $g(V) \setminus g(X \setminus V)$ and $g(W) \setminus g(X \setminus W)$ are open sets in Y and clearly $(g(V) \setminus g(X \setminus V)) \cap g(W) \setminus g(X \setminus W)) = \varphi$. This shows that Y is normal.

4.4 Gπ-Quasi-irresolute functions

Definition 4.4.1 A function $f : X \to Y$ is called gπ-quasi -irresolute if for each $x \in X$ and each $V \in SO(Y, f(x))$ there exists a gπ-open set U in X containing x such that $f(U) \subset cl(V)$.

Theorem 4.4.2. Suppose that gπO(X) is closed under arbitrary union, then the following are equivalent for a function $f : X \to Y$

1. f is gπ- quasi-irresolute.

2. $f^{-1}(V) \subset g\pi\ int(f^{-1}(cl(V)))$ for every $V \in SO(Y)$.

3. The inverse image of a regular closed set of Y is gπ-open.

4. The inverse image of a regular open set of Y is gπ-closed.

5. The inverse image of a θ-semi open set of Y is gπ-open.

6. $f^{-1}(int(cl(G)))$ is gπ-closed. for every open subset G of Y.

7. $f^{-1}(cl(int(F)))$ is gπ-open for every closed subset F of Y.

Proof: (1) $\Rightarrow$ (2) : Let $V \in SO(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is gπ-quasi irresolute, there exist a gπ-open set U in X containing x such that $f(U) \subset cl(V)$. It follows that $x \in U \subset f^{-1}(cl(V))$.Hence $x \in g\pi$- $int(f^{-1}(cl(V))$. Therefore $f^{-1}(V) \subset g\pi\ int(f^{-1}(cl(V))$.

(2) $\Rightarrow$ (3) : Let F be any regular closed set of Y. Since $F \in SO(Y)$, then by (2), $f^{-1}(F) \subset g\pi\ int(f^{-1}(F))$. This shows that $f^{-1}(F)$ is gπ-open.

(3) $\Rightarrow$ (4) : is obvious.

(4) $\Rightarrow$(5) : This follows from our assumption and the fact that any θ-semi open set is a union of regular closed sets.

(5) $\Rightarrow$(1). Let $x \in X$ and $V \in SO(Y)$. Since cl (V) is θ-semi open in Y by (5) there exist a gπ-open set U in X containing x such that $x \in U \subset (f^{-1}(cl(V))$. Hence $f(U) \subset cl(V)$. Hence f is gπ-quasi irresolute.

(4) $\Rightarrow$ (6) let G be a open subset of Y. Since int(cl (G)) is regular open , then by (4), $f^{-1}(int(cl(G)))$ is gπ- closed

(6) $\Rightarrow$(4) is obvious.

(3) $\Rightarrow$(7) is similar as (4) $\Rightarrow$ (6) .

Theorem 4.4.3 If $f : X \to Y$ is a gπ-quasi-irresolute function and A is any open subset of X , then the restriction $f|_A: A \to Y$ is gπ-quasi irresolute function.

Proof: Let $F \in RC(Y)$. Then by theorem 4.4.2, $f^{-1}(F) \in g\pi O(X)$.Since A is any open set in X, $(f|_A)^{-1}(F) = f^{-1}(F) \cap A \in g\pi O(A)$. Therefore $f|_A$ is gπ-quasi irresolute function.

Theorem 4.4.4. If $f : X \to Y$ is gπ-quasi irresolute injection and Y is S-Urysohn, then X is gπ-Hausdorff.

Proof: Since f is injective, it follows that $f(x) \neq f(y)$ for any distinct points x and y in X. Since Y is S-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, f(y))$ such that $cl(V) \cap cl(W) = \varphi$. Since f is a gπ-quasi irresolute function, there exist gπ-open sets U and G in X containing x and y respectively, such that $f(U) \subset cl(V)$ and $f(G) \subset cl(W)$ and $U \cap G = \varphi$. Hence X is gπ-Hausdorff.

Theorem 4.4.5. If f is an almost gπ-quasi irresolute injection and Y is weakly Hausdorff, then X is gπ-T_1.

Proof: Suppose that Y is weakly Hausdorff. For any two distinct points x and y in X, there exist regular closed sets V and W in Y such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$, and $f(y) \in W$. Since f is gπ-quasi irresolute injection, then $f^{-1}(V)$ and $f^{-1}(W)$ are gπ-open subsets of X such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$, and $y \in f^{-1}(W)$. This show that X is gπ-T_1.

Theorem 4.4.6 If $f, g : X \to Y$ are gπ-quasi irresolute functions and Y is S-Urysohn, then $E = \{x \in X / f(x) = g(x)\}$ is gπ -closed in X.

Proof: Let $x \in X - E$. Then $f(x) \neq g(x)$.Since Y is S-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, g(x))$ such that $cl(V) \cap cl(W) = \varphi$. Since f and g are gπ-quasi irresolute, there exist $U \in g\pi O(X,x)$ and $G \in g\pi O(X,x)$, such that $f(U) \subset cl(V)$ and $g(G) \subset cl(W)$. Set $D = U \cap G$. Then D is gπ -open in X.

Therefore $D \cap E = \varphi$ and $D \cap g\pi Cl(E) = \varphi$. It follows that $g\pi Cl(E) \subset E$. This shows that E is gπ -closed in X.

Theorem 4.4.7 If $f, g : X \rightarrow Y$ are gπ-quasi irresolute functions and Y is S-Urysohn, then $E = \{ (x, y) / f(x) = f(y) \}$ is gπ -closed in $X \times X$.

Proof: Suppose that $(x,y) \notin E$. It follows that $f(x) \neq g(x)$. Since Y is S-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, f(y))$ such that $cl(V) \cap cl(W) = \varphi$. Since f is gπ-quasi irresolute, there exist $U \in g\pi O(X, x)$ and $G \in g\pi O(X, x)$, such that $f(U) \subset cl(V)$ and $g(G) \subset cl(W)$. Let $D = U \times G$. It follows that $(x, y) \in D \in g\pi O(X \times X)$ and $D \cap E = \varphi$. This means that $g\pi Cl(E) \subset E$ and therefore E is gπ -closed in $X \times X$.

Theorem 4.4.8 Let $f : X \rightarrow Y$ be a function and $g : X \rightarrow X \times Y$ the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is gπ-quasi irresolute , then f is gπ-quasi irresolute .

Proof: Let $F \in RC(Y)$, then $X \times F = X \times cl(int(F)) = cl(int(X) \times cl(int(F)) = cl(int(X \times F))$. Therefore $X \times F \in RC(X \times Y)$. It follows from theorem 4.4.2, that $f^{-1}(F) = g^{-1}(X \times F)$ is gπ-open in X. Thus f is gπ-quasi-irresolute.

Theorem 4.4.9 Let $f : X \rightarrow Y$ and $g : X \rightarrow Y$ be functions. If f is gπ-irresolute and g is gπ-quasi irresolute then $g \circ f : X \rightarrow Z$ is gπ-quasi irresolute.

Proof: Let $x \in V$ and W be a semi-open set in Z containing $(g \circ f)(x)$. since g is gπ-quasi irresolute, there exist a gπ-open set V in Y containing $f(x)$ such that $g(V) \subset cl(W)$. Since f is gπ- irresolute, there exist gπ-open set U in X such that $f(U) \subset V$.

Theorem 4.4.10 If $f: X \rightarrow Y$ is a gπ-open surjective function and $g : Y \rightarrow Z$ is a function such that $g \circ f : X \rightarrow Z$ is gπ-quasi irresolute, then g is gπ-quasi irresolute.

Proof: Suppose that x and y are in X and Y respectively such that $f(x) = y$. Let W be a semi-open set in Z containing $g \circ f(x)$. Then there exist

$U \in g\pi O(X, x)$ such that $g(f(U)) \subset cl(W)$. Since f is gπ-open , then $f(U) \in g\pi O(Y, y)$ such that $g(f(U)) \subset cl(W)$. This implies that g is gπ-quasi irresolute.

4.5 Gπ-quasi-closed graph

Definition 4.5.1. The graph G(f) of a function $f : X \to Y$ is said to be gπ-quasi closed if for each $(x, y) \in X \times Y - G(f)$, there exist $U \in g\pi O(X, x)$ and $V \in SO(Y, y)$ such that $(U \times cl(V)) \cap G(f) = \varphi$

Lemma 4.5.2. The following properties are equivalent for a graph G(f) of a function $f : X \to Y$

1. The graph G (f) is gπ-quasi-closed in $X \times Y$

2. For each point $(x, y) \in X \times Y - G(f)$, there exist $U \in g\pi O(X, x)$ and $V \in SO(Y, y)$ such that $f(U) \cap cl(V) = \varphi$.

3. For each point $(x, y) \in X \times Y - G(f)$, there exist $U \in g\pi O(X, x)$ and $F \in RC(Y, y)$ such that $f(U) \cap F = \varphi$.

Proof: (1) $\Rightarrow$ (2) follows from the definition and the fact that for any subset $A \subset X$, $B \subset Y$, $(A \times B) \cap G(f) = \varphi$ iff $f(A) \cap B = \varphi$.

(2) $\Rightarrow$(3) follows from the fact that $cl(V) \in RC(Y)$ for any $V \in SO (Y)$.

(3)$\Rightarrow$(1). It is obvious since every regular closed set is semi-open and closed.

Theorem 4.5.3. If $f: X \to Y$ is gπ-quasi-irresolute and Y is S-Urysohn, then G(f) is gπ-quasi-closed in $X \times Y$.

Proof: Let $(x, y) \in X \times Y - G(f)$. It follows that $f(x) \neq y$. Since Y is S-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, y)$ such that $cl(V) \cap cl(W) = \varphi$. Since f is gπ-quasi-irresolute, there exist gπ-open set U(X, x) such that $f(U) \subset cl(V)$. Therefore, $f(U) \cap cl(W) \subset cl(V) \cap cl(W) = \varphi$ and G(f) is gπ-quasi closed in $X \times Y$.

Theorem 4.5.4. If $f : X \to Y$ is surjective and $G(f)$ is gπ-quasi-closed then Y is weakly Hausdorff.

Proof: Let y_1 and y_2 be any distinct points of Y. Since f is surjective $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in X \times Y - G(f)$. By lemma 4.5.2, there exist $U \in g\pi O(X, x)$ and $F \in RC(Y, y_2)$ such that $f(U) \cap F = \varphi$. Hence $y_1 \notin F$. This implies Y is weakly hausdorff.

Theorem 4.5.5. If $f : X \to Y$ is gπ-quasi irresolute and gπ-quasi-closed graph then X is gπ- Hausdorff.

Proof: Let x, y be any two distinct points of X. Since f is injective, $f(x) \neq f(y)$ and thus $(x, f(y)) \in X \times Y - G(f)$. Since $G(f)$ is gπ-quasi-closed , there exist $U \in g\pi O(X, x)$ and $V \in SO(Y, f(y))$ such that $f(U) \cap cl(V) = \varphi$. Since f is gπ quasi irresolute there exist $G \in g\pi O(X, y)$ such that $f(G) \subset cl(V)$. Therefore, $f(U) \cap f(G) = \varphi$ and hence $U \cap G = \varphi$. This shows that X is gπ-Hausdorff.