

## CHAPTER I

### SOFT SETS AND SOFT SEMIGROUPS

#### Definition : 1.1

Let  $U$  be an initial universe and  $E$  be the set of parameters. Let  $P(U)$  denotes the power set of  $U$ , and let  $A \subseteq E$ . A pair  $(F, A)$  is called a **soft set** over  $U$ , where  $F$  is a mapping given by  $F : A \rightarrow P(U)$ .

In other words, a soft set over  $U$  is a parameterized family of subsets of the universe  $U$ . The soft set  $(F, A)$  is also denoted as  $F_A$ .

#### Definition : 1.2

A soft set  $(F, A)$  over  $U$  is said to be **null soft set** if  $F(x) = \emptyset$  for all  $x \in A$ , and is denoted by  $\phi$ .

#### Definition : 1.3

A soft set  $(F, A)$  over  $U$  is said to be **absolute soft set** if  $F(x) = U$  for all  $x \in A$ , and is denoted by  $\tilde{A}$ .

#### Definition : 1.4

Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ . Then  $(F, A)$  is called a **soft subset** of  $(G, B)$  if,

- i)  $A \subseteq B$
- ii)  $F(a) \subseteq G(a)$  for all  $a \in A$ .

we write,  $(F, A) \subseteq (G, B)$ .

#### Definition : 1.5

Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ . Then  $(F, A)$  is said to be a **soft superset** of  $(G, B)$ , if  $(G, B)$  is a soft subset of  $(F, A)$ .

we write,  $(F, A) \supseteq (G, B)$ .

**Definition : 1.6**

Two soft sets  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$  are said to be **soft equal** if  $(F, A)$  is a soft subset of  $(G, B)$  and  $(G, B)$  is a soft subset of  $(F, A)$ .

we write,  $(F, A) = (G, B)$ .

**Definition : 1.7**

Let  $(F, A)$  and  $(G, B)$  be two soft sets over a semigroup  $S$ , their **cartesian product** is defined as  $(F, A) \times (G, B) = (H, A \times B)$ , where  $H(a, b) = F(a) \times G(b)$  for all  $(a, b) \in A \times B$ .

**Definition : 1.8**

Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ , then the **\*** operation for soft sets is defined as

$$(F, A) * (G, B) = (H, A \times B),$$

where  $H(a, b) = F(a) * G(b)$ ,  $a \in A$ ,  $b \in B$  and  $A \times B$  is the cartesian product of the sets  $A$  and  $B$ .

If there does not arise any ambiguity then we can simply write  $(F, A)(G, B)$  instead of  $(F, A) * (G, B)$  and  $F(a)G(b)$  for  $F(a) * G(b)$ .

**Definition : 1.9**

Let  $(F, A)$  and  $(G, B)$  be any two soft sets over a semigroup  $S$ . Then, the **restricted product** of  $(F, A)$  and  $(G, B)$  is defined as the soft set  $(H, C) = (F, A) \hat{\circ} (G, B)$ , where  $C = A \cap B$  and  $H$  is a set valued function from  $C$  to  $P(S)$  defined as

$$H(e) = F(e)G(e) \text{ for all } e \in C.$$

**Definition : 1.10**

Let  $(F, A)$  and  $(G, B)$  be any two soft sets over a semigroup  $S$ . Then, the **extended product** of  $(F, A)$  and  $(G, B)$  is defined as the soft set  $(H, C) = (F, A) \circ (G, B)$ , where  $C = A \cup B$  and  $H$  is a set valued function from  $C$  to  $P(S)$  defined as,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e)G(e) & \text{if } e \in A \cap B \end{cases} \quad \text{for all } e \in C.$$

**Definition : 1.11**

Let  $(F, A)$  and  $(G, B)$  be any two soft sets over a common universe  $U$ . Then the  $\wedge$  - product (**AND-product**) of  $(F, A)$  and  $(G, B)$  is defined as the soft set  $(H, C) = (F, A) \wedge (G, B)$ , where  $C = A \times B$  and  $H(a, b) = F(a) \cap G(b)$  for all  $(a, b) \in A \times B$ .

**Definition: 1.12**

Let  $(F, A)$  and  $(G, B)$  be any two soft sets over a common universe  $U$ . Then the  $\vee$  - product (**OR-product**) of  $(F, A)$  and  $(G, B)$  is defined as the soft set  $(H, C) = (F, A) \vee (G, B)$ , where  $C = A \times B$  and  $H(a, b) = F(a) \cup G(b)$  for all  $(a, b) \in A \times B$ .

**Definition: 1.13**

Let  $(F, A)$  and  $(G, B)$  be any two soft sets over a common universe  $U$ . Then the **extended union** of  $(F, A)$  and  $(G, B)$  is defined as the soft set  $(H, C) = (F, A) \cup_E (G, B)$ , where  $C = A \cup B$  and for all  $e \in C$ ,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \vee G(e) & \text{if } e \in A \cap B \end{cases}$$

**Definition: 1.14**

Let  $(F, A)$  and  $(G, B)$  be any two soft sets over a common universe  $U$ . Then the **extended intersection** of  $(F, A)$  and  $(G, B)$  is defined as the soft set  $(H, C) = (F, A) \cap_E (G, B)$ , where  $C = A \cup B$  and for all  $e \in C$ ,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \wedge G(e) & \text{if } e \in A \cap B \end{cases}$$

**Definition: 1.15**

Let  $(F, A)$  and  $(G, B)$  be any two soft sets over a common universe  $U$ . Then the **restricted intersection** of  $(F, A)$  and  $(G, B)$  is defined as the soft set  $(H, C) = (F, A) \cap_R (G, B)$ , where  $C = A \cap B$  and  $H(e) = F(e) \cap G(e)$  for all  $e \in C$ .

**Definition: 1.16**

Let  $(F, A)$  and  $(G, B)$  be any two soft sets over a common universe  $U$ . Then the **restricted union** of  $(F, A)$  and  $(G, B)$  is defined as the soft set  $(H, C) = (F, A) \cup_R (G, B)$ , where  $C = A \cap B$  and  $H(e) = F(e) \cup G(e)$  for all  $e \in C$ .

**Definition: 1.17**

A **semigroup** is a set  $S$  together with a binary operation  $\cdot$  (a function  $\cdot : S \times S \rightarrow S$ ) that satisfies the associative property

For all  $a, b, c \in S$ , the equation  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  holds.

**Definition: 1.18**

A non-null and non-empty soft set  $(F, A)$  over a semigroup  $S$  is called a **soft semigroup** if

$$(F, A) \hat{\circ} (F, A) \subseteq (F, A).$$

**Definition: 1.19**

A non-null and non-empty soft set  $(F, A)$  over a semigroup  $S$  is called a **soft left (right) ideal** over  $S$ , if  $\tilde{\mathcal{A}} \hat{\circ} (F, A) \subseteq (F, A) \left( (F, A) \hat{\circ} \tilde{\mathcal{A}} \subseteq (F, A) \right)$  where  $\tilde{\mathcal{A}}$  is the absolute soft set over  $S$ .

A soft set over  $S$  is a **soft ideal** if it is both a soft left and a soft right ideal over  $S$ .

**Theorem: 1.20**

- i) A non-null and non-empty soft set  $(F, A)$  over  $S$  is a soft semigroup if and only if  $F(a) \neq \emptyset$  is a subsemigroup of  $S$ .
- ii) A non-null and non-empty soft set  $(F, A)$  over  $S$ , is a soft ideal over  $S$  if and only if  $F(a) \neq \emptyset$  is an ideal of  $S$ .

**Example: 1.21**

Let  $S = \{a, b, c, d\}$  be a semigroup defined by the following table

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define  $F : S \rightarrow P(S)$  as  $F(x) = \{y \in S : y = xu; \text{ for some } u \in S\}$ . It is clear that  $F(a) = \{a\}, F(b) = \{a, b\}, F(c) = \{a, b, c\}, F(d) = \{a, b, d\}$ ,  $F(x)$  is a subsemigroup of  $S$  for all  $x \in S$ . Hence  $(F, S)$  is a soft semigroup over  $S$ . Furthermore  $(F, S)$  is a soft right ideal over  $S$ .

It is clear that not every soft set over a semigroup  $S$ , is a soft semigroup over  $S$ . For example if  $S = \{a, b, c, d\}$  is a semigroup with cayley's table given above and

$G : S \rightarrow P(S)$  is defined by  $G(x) = \{y \in S : y = x\}$  then  $(G, S)$  is a soft set over  $S$ , but not a soft semigroup over  $S$ , because  $G(b) = \{b\}$  is not a subsemigroup of  $S$ .

**Theorem: 1.22**

Let  $(F, A)$  and  $(G, B)$  be two soft semigroups over  $S$ . Then  $(F, A) \cap_R (G, B)$  also a soft semigroup over  $S$ , whenever  $(F, A) \cap_R (G, B)$  is non-null and non-empty.

**Proof:**

By definition,  $(F, A) \cap_R (G, B) = (H, C)$  where  $C = A \cap B \neq \emptyset$  and  $H(c) = F(c) \cap G(c)$  for all  $c \in C$ . Which is clearly either empty or a subsemigroup of  $S$ . Consequently  $(H, C)$  is a soft semigroup over  $S$ .

**Theorem : 1.23**

Let  $(F, A)$  and  $(G, B)$  be two soft semigroups over  $S$  such that  $A \cap B = \emptyset$ . Then  $(F, A) \cup_E (G, B)$  is also a soft semigroup over  $S$ .

**Proof :**

Since  $(F, A) \cup_E (G, B) = (H, C)$  and  $A \cap B = \emptyset$ , so for all  $c \in C = A \cup B$  either  $c \in A - B$  or  $c \in B - A$ . If  $c \in A - B$  then  $H(c) = F(c)$ , and if  $c \in B - A$  then  $H(c) = G(c)$ , in both cases  $H(c)$  is a subsemigroup of  $S$ . Therefore  $(H, C)$  is a soft semigroup over  $S$ .

**Theorem : 1.24**

Let  $(F, A)$  and  $(G, B)$  be two soft semigroups over  $S$ . Then  $(F, A) \wedge (G, B)$  is also a soft semigroup over  $S$ , whenever  $(F, A) \wedge (G, B)$  is non-null.

**Proof :**

Since  $(F, A) \wedge (G, B) = (H, A \times B)$ , and  $H(a, b) = F(a) \cap G(b)$ . As  $F(a)$  and  $G(b)$  are subsemigroup of  $S$ , therefore either  $F(a) \cap G(b) = \emptyset$  or  $F(a) \cap G(b)$  is a sub-semigroup of  $S$ . Consequently  $(H, A \times B)$  is a soft semigroup of  $S$ .

**Theorem : 1.25**

Let  $(F, A)$  and  $(G, B)$  be any two soft semigroups over a commutative semigroup  $S$ . Then  $(F, A) * (G, B)$  is also a soft semigroup over  $S$ , whenever  $(F, A) * (G, B)$  is non-null.

**Example : 1.26**

Consider the semigroup  $S = \{1, b, c, d\}$

.	1	b	c	d
1	1	b	c	d
b	b	b	b	b
c	c	c	c	c
d	d	c	b	d

we define soft semigroups  $(F, S)$  and  $(G, \{1, b\})$  over  $S$  as

$$F(1) = \{1\}, F(b) = \{b\}, F(c) = \{c\}, F(d) = \{d\}$$

and

$$G(1) = \{1, b\}, G(b) = \{1, c\}$$

Now  $(F, S) * (G, \{1, b\}) = (H, S \times \{1, b\})$  and  $H(s, t) = F(s) * G(t)$ , where  $s \in S$ ,  $t \in \{1, b\}$ . Now  $H(d, 1) = \{d\} * \{1, b\} = \{d, c\}$ , which is not a subsemigroup of  $S$ . Therefore  $(H, S \times \{1, b\})$  is not a soft semigroup over  $S$ .

**Theorem : 1.27**

i) Let  $(F, A)$  and  $(G, B)$  be any two soft ideals over  $S$ . Then  $(F, A) * (G, B)$  is also a soft ideal over  $S$ , whenever  $(F, A) * (G, B)$  is non-null.

ii) Let  $(F, A)$  and  $(G, B)$  be any two soft ideals over  $S$ . Then  $(F, A) \circ (G, B)$  is also a soft ideal over  $S$ .

iii) Let  $(F, A)$  and  $(G, B)$  be any two soft semigroups (ideals) over  $S$  and  $T$  respectively. Then  $(F, A) \times (G, B)$  is also a soft semigroup (ideal) over  $S \times T$ , whenever  $(F, A) \times (G, B)$  is non-null.

**Theorem : 1.28**

Let  $(F, A)$  and  $(G, B)$  be any two soft ideals over  $S$ . Then  $(F, A) \cap_R (G, B)$  is also a soft ideal over  $S$ , contained in both  $(F, A)$  and  $(G, B)$ , whenever  $(F, A) \cap_R (G, B)$  is non-null and non-empty.

**Proof :**

By definition,

$$(F, A) \cap_R (G, B) = (H, C)$$

where  $C = A \cap B \neq \emptyset$  and  $H(c) = F(c) \cap G(c)$ , is either empty or an ideal of  $S$ . Consequently  $(H, C)$  is a soft ideal over  $S$ . It is clear that  $A \cap B \subseteq A$  and  $A \cap B \subseteq B$ , also  $H(c) \subseteq F(c)$  and  $H(c) \subseteq G(c)$ . Therefore  $(H, C) \subseteq (F, A)$  also  $(H, C) \subseteq (G, B)$ .

**Theorem : 1.29**

Let  $(F, A)$  and  $(G, B)$  be two soft ideals over  $S$ . Then  $(F, A) \cup_E (G, B)$  is also a soft ideal over  $S$ , containing both  $(F, A)$  and  $(G, B)$ .

**Proof :**

By definition,

$$(F, A) \cup_E (G, B) = (H, C)$$

where  $C = A \cup B$ . If  $c \in A - B$  then  $H(c) = F(c)$ , if  $c \in B - A$  then  $H(c) = G(c)$  and if  $c \in A \cap B$  then  $H(c) = F(c) \cup G(c)$ , in all the cases  $H(c)$  is an ideal of  $S$ . Hence  $(H, C)$  is a soft ideal over  $S$ . Clearly  $(F, A) \subseteq (H, C)$  and  $(G, B) \subseteq (H, C)$ .

**Theorem : 1.30**

Let  $(F, A)$  and  $(G, B)$  be soft ideals over a semigroup  $S$ . Then  $(F, A) \hat{\delta} (G, B)$  is a soft ideal contained in both  $(F, A)$  and  $(G, B)$ , whenever  $(F, A) \hat{\delta} (G, B)$  is non-null and non-empty.

**Proof :**

Since  $(F, A) \hat{\delta} (G, B) = (H, C)$ , where  $C = A \cap B$  and  $H$  is defined by  $H(c) = F(c) G(c)$ , for all  $c \in C$ . So when  $H(c) \neq \emptyset$ , we have

$$H(c)S = F(c)G(c) S \subseteq F(c) G(c) = H(c), \text{ because } G(c) \text{ is an ideal of } S.$$

Similarly,  $SH(c) \subseteq H(c)$ . Hence,  $H(c)$  is an ideal of  $S$  for all  $c \in C$ . This proves that  $(F, A) \hat{\delta} (G, B)$  is a soft ideal over  $S$ . Since  $C = A \cap B \subseteq A$ ,  $C = A \cap B \subseteq B$  and  $H(c) = F(c) G(c) \subseteq F(c) S \subseteq F(c)$  and  $H(c) = F(c) G(c) \subseteq SG(c) \subseteq G(c)$ . Therefore  $(F, A) \hat{\delta} (G, B) \subseteq (F, A)$  and  $(F, A) \hat{\delta} (G, B) \subseteq (G, B)$ .

**Theorem : 1.31**

Let  $(F, A)$  and  $(G, B)$  be any two soft ideals over  $S$ . Then  $(F, A) \wedge (G, B)$  is also a soft ideal over  $S$ , whenever  $(F, A) \wedge (G, B)$  is non-null.

**Proof :**

Since  $(F, A) \wedge (G, B) = (H, A \times B)$  and  $H(a, b) = F(a) \cap G(b)$ . Then either  $F(a) \cap G(b) = \emptyset$  or  $F(a) \cap G(b)$  is an ideal of  $S$ . Therefore  $(H, A \times B)$  is a soft ideal over  $S$ .

**Theorem : 1.32**

Let  $(F, A)$  and  $(G, B)$  be any two soft ideals over  $S$ . Then  $(F, A) \vee (G, B)$  is also a soft ideal over  $S$ .

**Proof :**

Since  $(F, A) \vee (G, B) = (H, A \times B)$  and  $H(a, b) = F(a) \cup G(b)$ . Since  $F(a)$  and  $G(b)$  are ideals of  $S$ , so  $F(a) \cup G(b)$  is also an ideal of  $S$ . Consequently  $(H, A \times B)$  is a soft ideal over  $S$ .

**Definition : 1.33**

Let  $(G, B)$  be a non-null and non-empty soft subset of a soft semigroup  $(F, A)$  over  $S$ , then  $(G, B)$  is called a **soft subsemigroup** of  $(F, A)$ , if  $G(b)$  is a subsemigroup of  $F(b)$  for all  $b \in B$ .

The soft ideal of a soft semigroup defined above is different from usual soft ideal over a semigroup  $S$ . The following example is to depict this fact.

**Example : 1.34**

Let  $S = \{0, a, b, c\}$  be a semigroup.

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Consider the soft set  $(F, S)$ , where  $F: S \rightarrow P(S)$  is defined as  $F(0) = \{0\}$ ,  $F(a) = \{0, a\}$ ,  $F(b) = \{0, a, b\}$ ,  $F(c) = \{0, a, b, c\}$ . It is clear that  $(F, S)$  is a soft semigroup over  $S$ . Now consider the soft set  $(G, \{b\})$  in which  $G: \{b\} \rightarrow P(S)$  is defined as  $G(b) = \{0, b\}$ . As  $\{b\} \subseteq S$  and  $G(b)$  is an ideal of  $F(b)$ , therefore  $(G, \{b\})$  is a soft ideal of  $(F, S)$ . Clearly  $\{0, b\}$  is not an ideal of  $S$ , so  $(G, \{b\})$  is not a soft ideal over  $S$ .

**Theorem : 1.35**

If  $(F, A')$  and  $(G, B')$  are soft ideals of soft semigroups  $(F, A)$  and  $(G, B)$  over semigroups  $S$  and  $T$  respectively, then  $(F, A') \times (G, B')$  is a soft ideal of soft semigroup  $(F, A) \times (G, B)$  over  $S \times T$ , whenever  $(F, A') \times (G, B')$  is non-null.

**Proof :**

Since  $(F, A')$  and  $(G, B')$  are soft ideals of soft semigroups  $(F, A)$  and  $(G, B)$ , therefore  $A' \subseteq A$  and  $B' \subseteq B$ , clearly  $A' \times B' \subseteq A \times B$ . Also is  $\emptyset \neq F'(a')$  is an ideal of  $F(a')$  for all  $a' \in A'$  and  $\emptyset \neq G'(b')$  is an ideal of  $G(b')$  for all  $b' \in B'$ , therefore  $F'(a') \times G'(b')$  is an ideal of  $F(a) \times G(b)$ . Hence  $(F, A') \times (G, B')$  is a soft ideal of soft semigroup  $(F, A) \times (G, B)$  over  $S \times T$ .

**Theorem : 1.36**

Let  $(F, A)$  be a soft semigroup over  $S$  and  $\{(H_i, B_i) ; i \in I\}$  is a non-empty family of soft subsemigroups of  $(F, A)$  then

i)  $\bigcap_{i \in I}^R (H_i, B_i)$  is a soft subsemigroup of  $(F, A)$ , whenever  $\bigcap_{i \in I}^R (H_i, B_i)$  is non-null and non-empty.

ii)  $\bigwedge_{i \in I} (H_i, B_i)$  is a soft subsemigroup of  $\bigwedge_{i \in I} (F, A)$ , whenever  $\bigwedge_{i \in I} (H_i, B_i)$  is non-null.

iii) If  $B_i \cap B_j = \emptyset$  for all different  $i, j \in I$  then  $\bigcup_{i \in I}^R (H_i, B_i)$  is a soft subsemigroup of  $(F, A)$ .

**Theorem : 1.37**

Let  $(F, A)$  be a soft semigroup over  $S$  and  $\{(H_i, B_i) ; i \in I\}$  is a non-empty family of soft ideals of  $(F, A)$  then

i)  $\bigcap_{i \in I}^R (H_i, B_i)$  is a soft ideal of  $(F, A)$ , whenever  $\bigcap_{i \in I}^R (H_i, B_i)$  is non-null and non-empty.

ii)  $\bigwedge_{i \in I} (H_i, B_i)$  is a soft ideal of  $\bigwedge_{i \in I} (F, A)$ , whenever  $\bigwedge_{i \in I} (H_i, B_i)$  is non-null.

iii)  $\bigcup_{i \in I}^R (H_i, B_i)$  is a soft ideal of  $(F, A)$ .

iv)  $\bigvee_{i \in I} (H_i, B_i)$  is a soft ideal of  $\bigvee_{i \in I} (F, A)$ .

**Notation : 1.38**

Let the parameter set  $A$  is fixed. Denote the collection of all the soft ideals with parameter set  $A$  whose restricted intersection is non-null by  $SI(S)_A$ . Then  $SI(S)_A$  is a partially ordered set under inclusion of soft sets.

**Theorem : 1.39**

$(SI(S)_A, \cup_R, \cap_R)$  forms a distributive lattice.

**Proof :**

As we know from theorem 1.28 and 1.29 if  $(F, A), (G, A) \in SI(S)_A$  then  $(F, A) \cup_R (G, A) \in SI(S)_A$  and  $(F, A) \cap_R (G, A) \in SI(S)_A$ . This shows that  $(SI(S)_A, \cup_R, \cap_R)$  forms a lattice. Now we show that this lattice is distributive. For this let  $(F, A), (G, A), (H, A) \in SI(S)_A$ . Consider,

$$\{(F, A) \cup_R (G, A)\} \cap_R (H, A) = (K, A) \cap_R (H, A),$$

Where  $(F, A) \cup_R (G, A) = (K, A)$  and  $K(a) = F(a) \cup G(a)$ .

Now,

$$(K, A) \cap_R (H, A) = (M, A)$$

$$\begin{aligned} M(a) &= K(a) \cap H(a) \\ &= [F(a) \cup G(a)] \cap H(a) \end{aligned}$$

$$= [F(a) \cap H(a)] \cup [G(a) \cap H(a)] \quad \rightarrow (1)$$

Now,

$$(F, A) \cap_R (H, A) = (N, A) \text{ and}$$

$$N(a) = F(a) \cap H(a) \quad \rightarrow (2)$$

also  $(G, A) \cap_R (H, A) = (P, A)$  so

$$P(a) = G(a) \cap H(a) \quad \rightarrow (3)$$

From (1), (2), (3) we have,

$$M(a) = N(a) \cup P(a)$$

Therefore we have,

$$[(F, A) \cup_R (G, A)] \cap_R (H, A) = [(F, A) \cap_R (H, A)] \cup_R [(G, A) \cap_R (H, A)]$$

This shows that  $(SI(S)_A, \cup_R, \cap_R)$  forms a distributive lattice.

Let  $f: S \rightarrow T$  be a homomorphism from a semigroup  $S$  to a semigroup  $T$  and  $(F, A)$  is a soft set over  $S$ . Then  $(f(F), A)$  is a soft set over  $T$  where  $f(F)$  is defined by  $f(F)(a) = f(F(a))$ .

### **Theorem : 1.40**

Let  $(F, A)$  and  $(G, B)$  be two soft semigroups over  $S$  and  $(G, B)$  a soft subsemigroup of  $(F, A)$ , if  $f$  is a homomorphism from  $S$  to  $T$ , then  $(f(F), A)$  and  $(f(G), B)$  are both soft semigroups over  $T$  and  $(f(G), B)$  is a soft subsemigroup of  $(f(F), A)$ .

### **Proof :**

Since  $f$  is a homomorphism from  $S$  to  $T$ , so  $f(F(a))$  and  $f(G(b))$  are subsemigroups of  $T$ . Hence  $(f(F), A)$  and  $(f(G), B)$  are both soft semigroups over  $T$ . Now as  $G(b)$  is a subsemigroup of  $F(b)$ , therefore  $f(G(b))$  is a subsemigroup of  $f(F(b))$ . Therefore by definition  $(f(G), B)$  is a soft subsemigroup of  $(f(F), A)$ .

**Remark : 1.41**

Let  $f$  be a homomorphism from  $S$  to  $T$ . If  $(G, B)$  is a soft ideal of a soft semigroup  $(F, A)$  over  $S$ , then  $(f(G), B)$  is a soft ideal of  $(f(F), A)$ .

**Definition : 1.42**

Let  $(F, A)$  and  $(G, B)$  be two soft semigroups over  $S$  and  $T$  respectively and  $f : S \rightarrow T$  and  $g : A \rightarrow B$  be two functions then we say that  $(f, g)$  is a **soft homomorphism** and  $(F, A)$  is soft homomorphic to  $(G, B)$  if the following conditions hold

- i)  $f$  is a homomorphism from  $S$  onto  $T$ .
- ii)  $g$  is a mapping from  $A$  onto  $B$ .
- iii)  $f(F(a)) = G(g(a))$  for all  $a \in A$ .

If  $f$  is an isomorphism from  $S$  to  $T$  and  $g$  is one to one mapping from  $A$  onto  $B$  then  $(f, g)$  is called a **soft isomorphism** and  $(F, A)$  is soft isomorphic to  $(G, B)$ .

**Theorem : 1.43**

Let  $(F, A)$  and  $(G, B)$  be two soft semigroups over  $S$  and  $T$  respectively,  $(f, g)$  be a soft homomorphism from  $(F, A)$  to  $(G, B)$  if  $(F, A)$  is a soft ideal over  $S$ , then  $(G, B)$  is a soft ideal over  $T$ .

**Proof :**

Since  $(F, A)$  is a soft ideal over  $S$ , therefore  $F(a)$  is an ideal of  $S$ . It is given  $(f, g)$  is a soft homomorphism. Therefore for each  $b \in B$  there exists  $a \in A$  such that  $g(a) = b$ .  $G(b) = G(g(a)) = f(F(a))$ . As  $F(a)$  is an ideal of  $S$ , therefore

$f(F(a))$  is an ideal of  $H(g(b))$ . Hence for each  $b \in B$ ,  $G(b)$  is an ideal of  $H(g(b))$ . Thus  $(G, B)$  is a soft ideal over  $T$ .

**Theorem : 1.44**

Let  $(G, B)$  is a soft ideal of a soft semigroup  $(F, A)$  over  $S$ . Then  $(f(G), g(B))$  is a soft ideal of  $(H, C)$  where  $(H, C)$  is a soft semigroup over  $T$  and  $(f, g)$  is a soft homomorphism from  $(F, A)$  to  $(H, C)$ .

**Proof :**

Since  $G(b)$  is an ideal of  $F(b)$  for all  $b \in B$ . Therefore  $G(b) \subseteq F(b)$ . This implies that  $f(G(b)) \subseteq f(F(b)) = H(g(b))$ . Now since  $G(b)$  is an ideal of  $F(b)$ , therefore  $f(G(b))$  is an ideal of  $T$ . As  $g$  is a function from  $A$  onto  $C$  therefore  $g(B) \subseteq C$ . Hence  $(f(G), g(B))$  is a soft ideal of  $(H, C)$ .

**Soft Prime ideals in semigroups**

**Definition : 1.45**

A soft (left, right) ideal  $(F, A)$  over a semigroup  $S$  is called a **semiprime** over  $S$  if for any soft (left, right) ideal  $(G, B)$  over  $S$ ,  $(G, B) \hat{\circ} (G, B) \subseteq (F, A)$  implies  $(G, B) \subseteq (F, A)$ .

**Theorem : 1.46**

A soft ideal  $(F, A)$  over a semigroup  $S$  is semiprime if and only if  $F(a) \neq \emptyset$ , for each  $a \in A$  is semiprime.

**Proof :**

Let  $(F, A)$  be a soft semiprime ideal over  $S$  and  $I$  an ideal of  $S$ , such that  $I^2 \subseteq F(a)$ . Then define a soft ideal  $(G, B)$  over  $S$ , where  $B = \{a\}$  and  $G(a) = I$ . Then

$$(G, B) \hat{\delta} (G, B) \subseteq (F, A)$$

Since  $(F, A)$  is a soft semiprime ideal over  $S$ , so  $(G, B) \subseteq (F, A)$ . This implies that  $B \subseteq A$  and  $I = G(a) \subseteq F(a)$ .

Conversely, if  $F(a) \neq \emptyset$  for each  $a \in A$  is semiprime and  $(G, B)$  is a soft ideal over  $S$ , such that

$$(G, B) \hat{\delta} (G, B) \subseteq (F, A)$$

then  $(G, B) \hat{\delta} (G, B) = (H, B)$ , where  $H$  is a set valued function from  $B$  into  $P(S)$ , defined by  $H(a) = G(a)G(a)$ , for all  $a \in B \subseteq A$ . Thus,  $G(a)G(a) \subseteq F(a)$ . This leads to  $G(a) \subseteq F(a)$ , for all  $a \in B \subseteq A$ , and hence,  $(G, B) \subseteq (F, A)$ .

**Note :**

In the above Lemma, we obtain an equivalent definition of soft semiprime ideals over a semigroup  $S$ . This Lemma is also valid for one sided soft ideals.

If  $S$  is a semigroup with identity then we have the following theorem.

**Theorem : 1.47**

If  $(F, A)$  is a soft ideal over  $S$  with identity, then the following conditions are equivalent :

- 1)  $(F, A)$  is a soft semiprime ideal over  $S$

2)  $F(\alpha) \neq \emptyset$  is a semiprime ideal of  $S$ , for each  $\alpha \in A$

3)  $aSa \subseteq F(\alpha)$  implies  $a \in F(\alpha)$

4)  $(aS)(aS) \subseteq F(\alpha)$  implies  $a \in F(\alpha)$ .

**Proof :**

(1)  $\Leftrightarrow$  (2) this follows directly from theorem.1.46.

(2)  $\Rightarrow$  (3) Let  $(F, A)$  be a soft ideal over  $S$  and  $aSa \subseteq F(\alpha)$  for  $a \in A$ . This implies that ,

$$SaSaS \subseteq SF(\alpha)S \subseteq F(\alpha)$$

so that  $(SaS)^2 \subseteq F(\alpha)$  implies  $SaS \subseteq F(\alpha)$ , thus  $a \in F(\alpha)$ .

(3)  $\Rightarrow$  (2) Let  $I$  be any ideal of  $S$  such that  $I^2 \subseteq F(\alpha)$ .

Now for any  $a \in I$ , we have  $aSa \subseteq II \subseteq F(\alpha)$ . This implies that  $a \in F(\alpha)$ , and hence,  $I \subseteq F(\alpha)$ .

(3)  $\Rightarrow$  (4) Let  $(aS)(aS) \subseteq F(\alpha)$ . Then, this implies that  $aSaS \subseteq F(\alpha)$ .

Since  $aSa \subseteq aSaS \subseteq F(\alpha)$ . This implies that  $a \in F(\alpha)$ .

(4)  $\Rightarrow$  (3) Consider  $aSa \subseteq F(\alpha)$ . This implies that,

$$aSaS \subseteq F(\alpha)S \subseteq F(\alpha),$$

Thus, by our hypothesis,  $a \in F(\alpha)$ .

**Definition : 1.48**

A semigroup  $S$  is called a **soft semisimple semigroup** if for each soft ideal  $(F, A)$  over  $S$ ,

$$(F, A) \hat{\circ} (F, A) = (F, A)$$

For semisimple semigroups, we have the following theorem.

**Theorem : 1.49**

A semigroup  $S$  is soft semisimple if and only if it is semisimple.

**Proof :**

Let  $S$  be a soft semisimple semigroup and  $I$  an ideal of  $S$ . Then we define a soft ideal  $(F, A)$  over  $S$  as  $I = F(\alpha)$ , where  $\{\alpha\} = A$ . Since  $(F, A) \hat{\circ} (F, A) = (F, A)$ , we have  $F(\alpha)F(\alpha) = F(\alpha)$ , that is,  $I^2 = I$ .

Conversely, assume that  $S$  is semisimple and  $(F, A)$  a soft ideal over  $S$ .

Then,

$$(F, A) \hat{\circ} (F, A) = (G, A)$$

where  $G(\alpha) = F(\alpha)F(\alpha)$ , for all  $a \in A$ . Since  $\emptyset \neq F(\alpha)$  is an ideal of  $S$ , we can easily see that  $G(\alpha) = F(\alpha)F(\alpha) = F(\alpha)$  for all  $a \in A$ .

Hence  $(F, A) \hat{\circ} (F, A) = (F, A)$ .

The following theorem is a consequence of the above theorem.

**Theorem : 1.50**

The following statements are equivalent for a monoid  $S$ .

- 1)  $S$  is semisimple
- 2)  $S$  is soft semisimple
- 3) Every ideal is semiprime
- 4) Every soft ideal over  $S$  is soft semiprime.

**Definition : 1.51**

Let  $(F, A)$  be a soft ideal over a semigroup  $S$ . Then  $(F, A)$  is said to be an **irreducible soft ideal** over  $S$  if for soft ideals  $(G, B)$  and  $(H, C)$  over  $S$ ,  $(G, B) \cap_R (H, C) = (F, A)$ , implies  $(G, B) = (F, A)$  or  $(H, C) = (F, A)$ , where  $B, C \subseteq A$  and  $(F, A) \cap_R (G, B)$  is non-null and non-empty.

**Theorem : 1.52**

If  $(F, A)$  is a soft irreducible ideal over  $S$ , then  $F(\alpha) \neq \emptyset$  is irreducible for all  $\alpha \in A$ .

**Proof :**

Let  $I, J$  be any ideals of  $S$  such that  $I \cap J = F(\alpha)$ , where  $\alpha \in A$ . Then we define the soft ideals  $(G, B)$  and  $(H, C)$  over  $S$  by  $I = G(\alpha)$  and  $J = H(\alpha)$  and  $G(\beta) = H(\beta) = F(\beta)$  for all  $\alpha \neq \beta \in A$ , where  $B = C = A$ . Now, we have

$$(G, B) \cap_R (H, C) = (F, A)$$

Since  $(F, A)$  is soft irreducible, either  $(G, B) = (F, A)$  or  $(H, C) = (F, A)$ . This implies that either  $G(\alpha) = F(\alpha)$  or  $H(\alpha) = F(\alpha)$ . This means that either  $I = F(\alpha)$  or  $J = F(\alpha)$ .

The following Example shows that the converse of the above theorem is not true.

**Example : 1.53**

Let  $S = \{a, b, c, d\}$  be a semigroup.

.	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

In the above semigroup  $S$ , let  $(F, A)$  be a soft ideal over  $S$  with  $A = \{a, b, c\}$  and  $F(a) = \{a, b, d\}$ ,  $F(b) = \{a, c, d\}$ ,  $F(c) = \{a, b, d\}$ . Then every  $F(\alpha)$  is an irreducible ideal of  $S$ . If we consider the soft ideals  $(G, B)$  and  $(H, C)$ , where  $B = C = \{a, b, c\}$ ,  $G(a) = S$ ,  $G(b) = \{a, c, d\}$ ,  $G(c) = \{a, b, d\}$  and  $H(a) = \{a, b, d\}$ ,  $H(b) = S$ ,  $H(c) = S$ , then

$$(G, B) \cap_R (H, C) = (F, A)$$

but neither  $(G, B) = (F, A)$  nor  $(H, C) = (F, A)$ .

This illustrates that  $(F, A)$  is not an irreducible soft ideal.

**Definition : 1.54**

A soft ideal  $(F, A)$  over a semigroup  $S$  is called a **soft prime ideal** if for soft ideals  $(G, B), (H, C)$  over  $S$ ,  $(G, B) \hat{\cap} (H, C) \subseteq (F, A)$  implies that  $(G, B) \subseteq (F, A)$  or  $(H, C) \subseteq (F, A)$ , where  $B, C \subseteq A$  and  $(F, A) \hat{\cap} (G, B)$  is non-null and non-empty.

**Theorem : 1.55**

A soft ideal  $(F, A)$  over a semigroup  $S$  is soft prime if and only if  $(F, A)$  is soft irreducible and soft semiprime ideal over  $S$ .

**Proof :**

Let  $(F, A)$  be a soft prime ideal over  $S$  and  $(G, B)$  a soft ideal over  $S$  such that

$$(G, B) \hat{\cap} (G, B) \subseteq (F, A)$$

Since  $(F, A)$  is a soft prime ideal over  $S$ , so  $(G, B) \subseteq (F, A)$ . Now suppose that

$$(G, B) \cap_R (H, C) \subseteq (F, A)$$

Then by theorem.1.30, we have,

$$(G, B) \hat{\cap} (H, C) \subseteq (G, B) \text{ and } (G, B) \hat{\cap} (H, C) \subseteq (H, C).$$

Therefore,  $(G, B) \hat{\cap} (H, C) \subseteq (G, B) \cap_R (H, C) = (F, A)$ . Since  $(F, A)$  is a soft prime ideal, so we have,  $(G, B) \subseteq (F, A)$  or  $(H, C) \subseteq (F, A)$

However, since  $(G, B) \cap_R (H, C) = (F, A)$ . Therefore either  $(G, B) \subseteq (F, A)$  or  $(H, C) = (F, A)$ .

Conversely, suppose that  $(F, A)$  is both a soft semiprime ideal and a soft irreducible ideal over  $S$ . Let  $(I, D)$  and  $(J, K)$  be soft ideals over  $S$  such that  $(I, D) \hat{\cap} (J, K) \subseteq (F, A)$  where  $D, K \subseteq A$  so we have

$$((I, D) \cap_R (J, K)) \hat{\cap} ((I, D) \cap_R (J, K)) \subseteq ((I, D) \hat{\cap} (J, K)) \subseteq (F, A)$$

since  $(F, A)$  is a soft semiprime ideal over  $S$ , so

$$(I, D) \hat{\cap} (J, K) \subseteq (F, A)$$

Now  $((I, D) \cap_R (J, K)) \cup_E (F, A) = (F, A)$ ,

$$((I, D) \cap_R (J, K)) \cup_E (F, A) = ((I, D) \cup_E (F, A)) \cap_R ((J, K) \cup_E (F, A)) = (F, A).$$

Now, by irreducibility of  $(F, A)$ , we have

$$(I, D) \cup_E (F, A) = (F, A) \text{ or } (J, K) \cup_E (F, A) = (F, A)$$

This implies that

$$(I, D) \subseteq (F, A) \text{ or } (J, K) \subseteq (F, A)$$

### **Theorem : 1.56**

If a soft ideal  $(F, A)$  over a semigroup  $S$  is a soft prime ideal then  $F(\alpha) \neq \emptyset$  for all  $\alpha \in A$ , is a prime ideal of  $S$ .

### **Proof :**

Let  $I, J$  be ideals of  $S$  such that  $IJ \subseteq F(\alpha)$ , where  $\alpha \in A$ . Define the soft ideals  $(G, A)$  and  $(H, A)$  by,

$$G(\alpha) = I \text{ and } G(\alpha) = F(\alpha) \text{ for all } \alpha \neq \beta \in A$$

$$H(\alpha) = J \text{ and } H(\alpha) = F(\alpha) \text{ for all } \alpha \neq \beta \in A$$

Then  $(G, A) \hat{\cap} (H, A) \subseteq (F, A)$ . Since  $(F, A)$  is a soft prime ideal, we have

$$(G, A) \subseteq (F, A) \text{ or } (H, A) = (F, A)$$

Thus either  $G(\alpha) \subseteq F(\alpha)$  or  $H(\alpha) \subseteq F(\alpha)$ . In other words either  $I \subseteq F(\alpha)$  or  $J \subseteq F(\alpha)$ . This proves that  $F(\alpha) \neq \emptyset$  is a prime ideal of  $S$ .

**Note :**

We call a semigroup  $S$  a soft fully prime semigroup if every soft ideal over  $S$  is prime.

**Remark : 1.57**

If  $S$  is soft fully prime then  $S$  itself is fully prime.

The following example shows that converse of the above theorem is not true.

**Example : 1.58**

Consider the semigroup of example.1.53. Let  $(F, A)$  be a soft set over  $S$ , where  $A = \{\alpha, \beta, \gamma, \delta\}$  defined by  $F(\alpha) = \{a, b, d\}$ ,  $F(\beta) = \{a, c, d\}$ ,  $F(\gamma) = \{a, b, d\}$ ,  $F(\delta) = S$ . Then  $(F, A)$  is a soft ideal over  $S$  such that each  $F(\alpha)$  is a prime ideal of  $S$ .

We now consider two soft ideals  $(G, B)$  and  $(H, C)$  such that,

$$B = C = \{\alpha, \beta, \gamma\}$$

and

$$G(\alpha) = S, G(\beta) = \{a, c, d\}, G(\gamma) = \{a, b, d\}, H(\alpha) = \{a, b, d\}, H(\beta) = S, \quad H(\gamma) = \{a, c, d\}.$$

Then it is easy to see that  $(G, B) \hat{\delta} (H, C) \subseteq (F, A)$  but  $(G, B) \not\subseteq (F, A)$  and also  $(H, C) \not\subseteq (F, A)$ .

**Theorem : 1.59**

Let  $(F, A)$  and  $(G, B)$  be soft prime ideals over a semigroup  $S$ . Then  $(F, A) \cap_R (G, B)$  is a soft semiprime ideal over  $S$ , contained in both  $(F, A)$  and  $(G, B)$ , whenever  $(F, A) \cap_R (G, B)$  is non-null and non-empty.

**Proof :**

Let  $(F, A) \cap_R (G, B) = (H, C)$ , where  $C = A \cap B \neq \emptyset$  and  $H(c) = F(c) \cap G(c)$  for all  $c \in C$ . Then, by theorem.1.28 it is clear that  $(H, C)$  is a soft ideal contained in  $(F, A)$  and  $(G, B)$ . Let  $(K, D)$  be a soft ideal over  $S$  such that  $(K, D) \hat{\delta} (K, D) \subseteq (H, C)$ . Then, this implies that  $K(d)K(d) \subseteq F(d) \cap G(d)$  and  $D \subseteq C$ , that is,  $K(d)K(d) \subseteq F(d)$  and  $K(d)K(d) \subseteq G(d)$ . Now by theorem 1.56, we know that  $F(d)$  and  $K(d)$  are both prime ideals of  $S$ .

Consequently,  $K(d) \subseteq F(d)$  and  $K(d) \subseteq G(d)$ . Therefore  $K(d) \subseteq F(d) \cap G(d)$ . This implies that  $(K, D) \subseteq (F, A) \cap_R (G, B)$ . Hence  $(F, A) \cap_R (G, B)$  is a soft semiprime ideal over  $S$ .