Chapter VI

CHAPTER VI

ORDERED FUZZY TOPOLOGICAL SPACES

Section: 1

Fuzzy ordered sets

Definition: 6.1.1

A fuzzy set μ , in a preordered set X, is called:

(1) **Increasing** if $x \leq y$ implies $\mu(x) \leq \mu(y)$.

(2) **Decreasing** if $x \leq y$ implies $\mu(x) \geq \mu(y)$.

(3) **Order-convex** if $x \leq z \leq y$ implies $\mu(z) \geq \min\{\mu(x), \mu(x)\}$.

Theorem: 6.1.2

Let $(\mu_i)_{i \in A}$ be a family of fuzzy sets in a preordered set X and let $\mu = \inf_{i \in A}\mu_i$. Then, if each μ_i is increasing (resp. decreasing), then μ is increasing (resp. decreasing). Also if each μ_i is order-convex, the same is true for the fuzzy set μ.

Definition: 6.1.3

Given a fuzzy set μ, in a preorder set X, there exists the smallest increasing fuzzy set ρ in X with $\rho \geq \mu$. This fuzzy set is called the **increasing hull of** μ and we will denote it by $i(\mu)$. The **decreasing hull** $d(\mu)$ and the **order-convex hull** $c(\mu)$ of μ are defined analogously.

Theorem: 6.1.4

Let μ be a fuzzy set in a preordered set X. Then, for each $x \in X$, we have

$$i(\mu)(x) = \sup\{\mu(y) : y \leq x\}$$

$$d(\mu)(x) = \sup\{\mu(y) : y \geq x\}$$

$$c(\mu)(x) = \sup\{\min\{\mu(x_1), \mu(x_2)\} : x_1 \leq x \leq x_2\}$$

Proof

Define μ_0 on X by

$$\mu_0(x) = \sup\{\mu(y) : y \leq x\}$$

Since $x \leq x$, for each $x \in X$, we have $\mu \leq \mu_0$. Also it is clear that $\mu_0(x_1) \leq \mu_0(x_2)$ if $x_1 \leq x_2$. Hence μ_0 is increasing. Finally, let ρ be an increasing fuzzy set X with $\mu \leq \rho$. If $y \leq x$, then $\rho(x) \geq \rho(y) \geq \mu(y)$ and thus $\mu_0(x) \leq \rho(x)$. This proves that $\mu_0 = i(\mu)$. The proof for the $d(\mu)$ is similar.

To prove the result for the order-convex hull of μ, let us consider the fuzzy set μ_1 defined by

$$\mu_1(x) = \sup\{\min\{\mu(x_1), \mu(x_2)\} : x_1 \leq x \leq x_2\}$$

Since $x \leq x \leq x$, we have $\mu(x) \leq \mu_1(x)$ for each x. Also μ_1 is order-convex. In fact, let $x \leq z \leq y$ and let $\theta < \min\{\mu_1(x), \mu_1(y)\}$. Since $\mu_1(x) > \theta$, there are $x_1, x_2 \in X$ with $x_1 \leq x \leq x_2$ and $\min\{\mu(x_1), \mu(x_2)\} > \theta$. Similarly, from the $\mu_1(y) > \theta$ follows that there are y_1, y_2 in X with $y_1 \leq y \leq y_2$ and $\min\{\mu(y_1), \mu(y_2)\} > \theta$..

Now

$$x_1 \leq x \leq z \leq y \leq y_2$$

and thus

$\mu_1(z) \geq \min\{\mu(x_1), \mu(y_2)\} > \theta$.

Since $\mu_1(z) > \theta$ for each $\theta < \min\{\mu_1(x), \mu_1(y)\}$, it follows that

$\mu_1(z) \geq \min\{\mu_1(x), \mu_1(y)\}$

which proves that μ_1 is order-convex. Finally, let ρ be order-convex, $\rho \geq \mu$. If $x_1 \leq x \leq x_2$, then

$\rho(x) \geq \min\{\rho(x_1), \rho(x_2)\} \geq \min\{\mu(x_1), \mu(x_2)\}$.

It follows that $\rho(x) \geq \mu_1(x)$, for each $x \in X$, and this completes the proof.

Lemma: 6.1.5

Let μ_1 be an increasing fuzzy set and μ_2 a decreasing fuzzy set in a preordered set X. Then $\mu = \mu_1 \wedge \mu_2$ is order convex.

Proof

Let $x \leq z \leq y$. Since $\mu_1(z) \geq \mu_1(x) \geq \mu(x)$ and $\mu_2(z) \geq \mu_2(y) \geq \mu(y)$, we have

$\mu(z) = \min\{\mu_1(z), \mu_2(z)\} \geq \min\{\mu(x), \mu(y)\}$.

This proves that μ is order-convex.

Theorem: 6.1.6

Let μ be a fuzzy set in a preordered set X. Then $c(\mu) = i(\mu) \wedge d(\mu)$.

Proof

Let $\rho = i(\mu) \wedge d(\mu)$. By the Lemma 6.1.5 ρ is order-convex. Also it is clear that $\mu \leq \rho$. Therefore $c(\mu) \leq \rho$. Suppose, by way of contradiction, that there exists an x with $c(\mu)(x) \neq \rho(x)$. Then, there exists θ with

$c(\mu)(x) < \theta < \rho(x)$

Since $i(\mu)(x) \geq \rho(x) > \theta$ there exists (by Theorem 6.1.4) $x_1 \leq x$ with $\mu(x_1) > \theta$. Similarly from the $d(\mu)(x) > \theta$ follows that there exists $x_2 \geq x$ with $\mu(x_2) > \theta$. Since $x_1 \leq x \leq x_2$, we have (by Theorem 6.1.4).

$$c(\mu)(x) \geq \min\{\mu(x_1), \mu(x_2)\} > \theta$$

which is a contradiction. This contradiction completes the proof.

Theorem: 6.1.7

Let μ be a fuzzy set in preordered set X. Then μ is order-convex iff there exists on increasing fuzzy set μ_1 and a decreasing fuzzy set μ_2 such that $\mu = \mu_1 \wedge \mu_2$.

Proof

The sufficiency of the condition follows form Lemma 6.1.5. Conversely, if μ is order-convex, then

$$\mu = c(\mu) = i(\mu) \wedge d(\mu)$$

by the Theorem 6.1.6. The result follows.

Definition: 6.1.8

Let X, Y be preordered sets and f a function from X to Y. Then f is called **increasing** (resp. **decreasing)** if $x \leq y$ in X implies $f(x) \leq f(y)$ (resp. $f(y) \leq f(x)$).

Theorem: 6.1.9

Let X, Y be preordered sets, f a function from X to Y and μ a fuzzy set in Y. Then:

1) If f is increasing and μ is increasing (resp. decreasing), then $f^{-1}(\mu)$ is increasing (resp. decreasing).

2) If f is decreasing and μ is increasing (resp. decreasing), then $f^{-1}(\mu)$ is decreasing (resp. decreasing).

3) If μ is order-convex and if f is either increasing or decreasing, then $f^{-1}(\mu)$ is order-convex.

Definition: 6.1.10

Let X be preordered set. The relation ~ on X, defined by $x \sim y$ iff $x \leq y \leq x$, is an equivalence relation. Denote by $\hat{x}$ the equivalence class in which x belongs. Let

$$\hat{X} = \{\hat{x} : x \in X\}$$

The relation $\hat{x} \leq \hat{y}$ iff $x \leq y$ is a well defined partial order on $\hat{X}$. Thus $\hat{X}$ becomes on ordered set. We will refer to $\hat{X}$ to as the **ordered set corresponding to the preordered set X**. We will call the function

$$e : X \rightarrow \hat{X} \qquad e(x) = \hat{x}.$$

the **quotient map** from X onto $\hat{X}$.

Remark: 6.1.11

Clearly $x \leq y$ iff $e(x) \leq e(y)$. If μ is an order-convex fuzzy set in X, then μ is constant on each $\hat{x}$. In fact if $y \in \hat{x}$ then $x \leq y \leq x$ and $y \leq x \leq y$. Hence $\mu(y) \geq \mu(x)$ and $\mu(x) \geq \mu(y)$ and so $\mu(x) = \mu(y)$. Also, $e(\mu)$ is order-convex in $\hat{X}$. In fact $e(\mu)(\hat{x}) = \sup\{\mu(y) : y \in \hat{x}\} = \mu(x)$.

If $\hat{x} \leq \hat{z} \leq \hat{y}$, then $x \leq z \leq y$ and thus

$$e(\mu)(\hat{z}) = \mu(z) \geq \min\{\mu(x), \mu(y)\}$$
$$= \min\{e(\mu)(\hat{x}), e(\mu)(\hat{y})\}.$$

Section: 2

Definition and properties of ordered fuzzy topological space.

Definition: 6.2.1

A preordered (resp. an ordered) set on which there is given a fuzzy topology is called a **preordered (resp. an ordered) fuzzy topological space**.

Definition: 6.2.2

A preordered fuzzy topological space X is called **locally order convex** if, for each $x \in X$, the collection of all order-convex neighbourhoods of x is a base for the system of all neighbourhoods of x.

Definition: 6.2.3

A preordered fuzzy topological space X is called **order-convex** if the family of all open order-convex fuzzy sets is a base for the topology.

Note: 6.2.4

An order-convex fuzzy topological space is locally order-convex.

Theorem: 6.2.6

Let X be a preordered fuzzy topological space and suppose that the family of all open decreasing and all open increasing fuzzy sets in X is a subbase for the topology. Then X is order-convex.

Proof

If $\mu_1, \mu_2, \ldots\ldots, \mu_n$ are open increasing (resp. decreasing) fuzzy sets in X, then $\mu_1 \wedge \mu_2 \wedge \ldots\ldots \wedge \mu_n$ is an open increasing (resp. decreasing) fuzzy set. Thus, the hypothesis implies that the family of all fuzzy sets in X of the form $\mu_1 \wedge \mu_2$, with μ_1 open increasing and μ_2 open decreasing is a base for the topology of X. In view of Lemma 6.1.5 the result follows.

Definition: 6.2.6

A net (x_α) in a fuzzy topological space X is said to **converge** to $x \in X$, and write $x = \lim x_\alpha$ or $x_\alpha \to x$, if for each neighbourhood μ of x there exists an index α_0 such that μ is a neighbourhood of x_α for each $\alpha \geq \alpha_0$.

Theorem: 6.2.7

Let X be a locally order-convex preordered fuzzy topological space. If $x_\alpha \leq y_\alpha \leq z_\alpha$, $x_\alpha \to x$ and $z_\alpha \to x$, then $y_\alpha \to x$.

Definition: 6.2.8

A preorder $\leq$ on a fuzzy topological space X is said to be:

(1) **Semiclosed on the left** if $x_\alpha \to x$ and $x_\alpha \leq y$ imply that $x \leq y$;

(2) **Semiclosed on the right** if $x_\alpha \to x$ and $x_\alpha \geq y$ imply that $x \geq y$;

(3) **Semiclosed** if it is semiclosed both on the left and on the right;

(4) **Closed** if $x_\alpha \leq y_\alpha$, $x_\alpha \to x$ and $y_\alpha \to y$ imply $x \leq y$.

Theorem: 6.2.9

The preorder of a preordered fuzzy topological space X is:

(1) Semiclosed on the left iff the following condition is satisfied: If $x \leq y$ is false, then there exists a neighbourhood μ of x such that $i(\mu)(y) = 0$;

(2) Semiclosed on the right iff the following holds: If $x \leq y$ is false, then there exists a neighbourhood μ of y such that $d(\mu)(x) = 0$;

(3) Closed iff the following condition is satisfied: If $x \leq y$ is false, then there are neighbourhood μ, ρ of x, y, respectively, such that $i(\mu) \wedge d(\mu) = 0$,

Proof

(1) Suppose that the preorder is semiclosed on the left and let $x \leq y$ be false. Assume, by way of contradiction, that for each neighbourhood μ of x we have $i(\mu)(y) > 0$. By theorem 6.1.4 there exists an $x_\mu \leq y$ such that $\mu(x_\mu) > 0$. Let

$$A = \{\mu : \mu \text{ open neighbourhood of } x\}.$$

The set A is directed by the relation

$$\mu_1 \leq \mu_2 \ \text{ iff } \ \mu_2 \leq \mu_1.$$

It is easy to see that the net $(x_\mu)_{\mu \in A}$ converges to x. Since $x_\mu \leq y$, our hypothesis implies that $x \leq y$ which is a contradiction.

Conversely, suppose that the condition is satisfied and let $x_\alpha \rightarrow x$, $x_\alpha \leq y$. We need to show that $x \leq y$. Assume that this is false. Our hypothesis implies that there exists a neighbourhood μ of x such that $i(\mu)(y) = 0$. Let ρ be an open neighbourhood of x such that $\rho \leq \mu$. For each α we have

$$\rho(x_\alpha) \leq \mu(x_\alpha) \leq i(\mu)(y) = 0$$

and thus ρ is not a neighbouhood of x_α. This contradicts our hypothesis that $x_\alpha \rightarrow x$.

(2) The proof is similar to that of (1).

(3) Suppose that the preorder of X is closed and let x, y be such that $x \leq y$ is false. We need to show that there are neighbourhoods μ, ρ of x, y respectively, such that $i(\mu) \wedge d(\rho) = 0$. Assume the contrary and let μ, ρ be open neighbourhoods of x, y respectively. Let $z \in X$ such that

$i(\mu)\,(z) > 0,\ \ d(\rho)\,(z) > 0.$

In view of the Theorem 6.1.4 there are z_1, z_2 in X such that $z_1 \le z \le z_2$. $\mu(z_1) > 0$ and $p(z_2) > 0$. Thus for all pairs (μ,ρ), where μ is an open neighbourhood of x and ρ an open neighbourhood of y, there are $x_{(\mu,\rho)} = z_1$ and $y_{(\mu,\rho)} = z_2$ with $z_1 \le z_2$, $\mu(z_1) > 0$ and $\rho(z_2) > 0$. Let

$B = \{(\mu, \rho) : \mu, \rho$ open neighbourhoods of x, y, respectively$\}$.

The relation

$(\mu_1, \rho_1) \le (\mu_2, \rho_2)$ iff $\mu_1 \ge \mu_2$ and $\rho_1 \ge \rho_2$ makes B into a directed set. In this way we get two nets

$(x_{(\mu,\rho)})_{(\mu,\rho)\in B}$, $(y_{(\mu,\rho)})_{(\mu,\rho)\in B}$

such that $x_{(\mu,\rho)} \le y_{(\mu,\rho)}$. Let μ_0, ρ_0 be open neighbourhoods of x, y respectively. If $(\mu, \rho) \ge (\mu_0, \rho_0)$, then $\mu \le \mu_0$ and $\rho \le \rho_0$.

Hence

$\mu_0(x_{(\mu,\rho)}) \ge \mu(x_{(\mu,\rho)}) > 0$ and

$\rho_0(y_{(\mu,\rho)}) \ge \rho\,(y_{(\mu,\rho)}) > 0$

This proves that $x_{(\mu,\rho)} \to x$ and $y_{(\mu,\rho)} \to y$. By hypothesis $x \le y$ which is a contradiction.

Conversely, assume that the condition is satisfied and let $x_\alpha \le y_\alpha$, $x_\alpha \to x$, $y_\alpha \to x$. Assume that $x \le y$ is false. By hypothesis there are open neighbourhoods of x, y, respectively, such that $i(\mu) \wedge d(\rho) = 0$. Since $x_\alpha \to x$ and $y_\alpha \to x$, there exists some index α such that $\mu(x_\alpha) > 0$ and $\rho(y_\alpha) > 0$. Let $z = y_\alpha$. Since $z \ge x_\alpha$, we have $i(\mu)\,(z) \ge \mu(x_\alpha) > 0$. Also

$d(\rho)(z) \geq \rho(z) > 0$. Thus $i(\mu) \wedge d(\rho) \neq 0$ which is a contradiction. This contradiction completes the proof.

Definition: 6.2.10

Let X be a preordered fuzzy topological space and let $\hat{X}$ be the ordered set which corresponds to the preordered set X. Let e be the quotient mapping from X onto $\hat{X}$. Let τ be the quotient topology on $\hat{X}$ ($\mu \in \tau$ iff $e^{-1}(\mu)$ is open in $\hat{X}$). Then X, with the topology τ, is an ordered fuzzy topological space. We will refer to $(\hat{X}, \tau)$ as the **ordered fuzzy topological space corresponding to the preordered fuzzy topological space X**.

Theorem: 6.2.11

Let X be a preordered fuzzy topological space whose topology is locally order-convex. Let $\hat{X}$ be the corresponding ordered fuzzy topological space and $e : X \rightarrow \hat{X}$ the quotient mapping. Then:

(1) $\mu = e^{-1}(e(\mu))$ for each open set μ in X.

(2) e is an open map.

(3) μ is open in X iff $\mu = e^{-1}(\rho)$ for some open set ρ in $\hat{X}$.

Proof

(1) Let μ be open in X. Clearly $\mu \leq e^{-1}(e(\mu))$,

Suppose, by way of contradiction, that there exists $x \in X$ such that

$$\mu(x) < e^{-1}(e(\mu))\,(x) = e(\mu)\,(\hat{x}).$$

Since $e^{-1}[\hat{x}] = \hat{x}$, there exists $y \in \hat{x}$ such that $\mu(y) > \mu(x)$. Since μ is open, μ is a neighbourhood of y. Our hypothesis and the $\mu(y) > \mu(x)$ imply that there exists an order-convex neighbourhood μ_1 of y such that

$\mu_1 \leq \mu$ and $\mu_1(y) > \mu(x)$. But $\mu_1(y) = \mu_1(x)$ since μ_1 is order-convex and $y \in \hat{x}$. Hence

$$\mu(x) \geq \mu_1(x) = \mu_1(y) > \mu(x).$$

This contradiction shows that $\mu = e^{-1}(e(\mu))$.

(2) It is direct consequence of (1) since ρ is open in $\hat{X}$ iff $e^{-1}(\rho)$ is open in X.

(3) It follows form (1) and from the definition of the topology of $\hat{X}$.

Lemma: 6.2.12

Let X be a preordered fuzzy topological space which is locally order-convex. If μ is an order-convex neigthbourhood of a point x in X, then $e(\mu)$ is an order-convex neighbourhood of $\hat{x}$ in $\hat{X}$.

Proof

By theorem 6.2.11, e is an open map. Let now μ be an order-convex neighbourhood of $x \in X$. Then $e(\mu)$ is order-convex and $e(\mu)(\hat{y}) = \mu(y)$ for each $y \in X$. In particular $e(\mu)(\hat{x}) = \mu(x)$. Since μ is a neighbourhood of x, there exists an open neighbourhood ρ of x with $\rho \leq \mu$ and $\rho(x) = \mu(x)$. Now $e(\rho)$ is an open neighbourhood of $\hat{x}$ and $e(\rho) \leq e(\mu)$. Moreover

$$e(\rho)(\hat{x}) \geq \rho(x) = \mu(x) = e(\mu)(\hat{x}).$$

and hence $e(\rho)(\hat{x}) = e(\mu)(\hat{x})$. This proves that $e(\mu)$ is an order-convex neighbourhood of $\hat{x}$.

Theorem: 6.2.13

If the preordered fuzzy topological space X is locally order-convex, then $\hat{X}$ is locally order-convex.

Proof

Let X be locally order-convex and let μ be a neighbourhood of $\hat{x}$ in $\hat{X}$. Let $0 < \theta < \mu(\hat{x})$. Sin ce e is continuous, $e^{-1}(\mu)$ is a neighbourhood of x in X. Moreover, $e^{-1}(\mu)(x) = \mu(\hat{x}) > \theta$. Hence, there exists an order–convex neighbourhood ρ of x, $\rho \leq e^{-1}(\mu)$, with $\rho(x) > \theta$. By the preceding lemma, $e(\rho)$ is an order-convex neighbourhood of $\hat{x}$. Also, $e(\rho) \leq e(e^{-1}(\mu)) = \mu$. Finally,

$$e(\rho)(\hat{x}) \geq \rho(x) > \theta$$

This shows that the family of order-convex neighbourhoods of $\hat{x}$ is a base for the system of all neighbourhoods of $\hat{x}$. Hence $\hat{X}$ is locally order-convex.

Theorem: 6.2.14

Let X be a preordered fuzzy topological space. Then X is locally order-convex iff there exist an ordered fuzzy topological space Y whose topology is locally order-convex and an increasing function f from X to Y such that a fuzzy set μ in X is open iff $\mu = f^{-1}(\rho)$ for some open fuzzy set ρ in Y.

Proof

If X is locally order-convex, then we may take $Y = \hat{X}$ and f the quotient map $e : X \rightarrow \hat{X}$.

Conversely, suppose that there exist Y and f with the properties mentioned in the statement of the theorem. Let μ be a neighbourhood of x in X. There exists an open neighbourhood μ_1 of x with $\mu_1 \leq \mu$ and $\mu_1(x) = \mu(x)$.

Let ρ be open in Y with $\mu_1 = f^{-1}(\rho)$ and let $0 < \theta < \mu(x) = \mu_1(x) = \rho(f(x))$. Since Y is locally order-convex, there exists an order convex neighbourhood ρ_1 of f(x) with $\rho_1 \leq \rho$ and $\rho_1(f(x)) > \theta$. Let $\mu_2 = f^{-1}(\rho_1)$. Since f is increasing, μ_2 is order-convex (Theorem 6.1.9). Also since f is continuous, μ_2 is a neighbourhood of x. Clearly $\mu_2 \leq \mu_1 \leq \mu$. Also, $\mu_2(x) = \rho_1(f(x)) > \theta$. This proves that the family of all order-convex neighbourhoods of x is a base or the system of all neighbourhoods of x. Hence X is locally order-convex.