CHAPTER - III

3. An $M^{[X]}/G/1$ Retrial G-Queue with Server Breakdown

Batch arrival retrial queue with positive and negative customers is considered. If the server is idle upon the arrival of a batch, one of the customers in the batch receives service immediately and others join the orbit. If the server is busy, all the customers join the orbit. The arrival of negative customer brings the server down and removes the customer in service from the system. The server is subject to random breakdown while it is working. Using supplementary variable technique, expected number of customers in the orbit and expected number of customers in the system are derived. Stochastic decomposition property is established. Some special cases are discussed and numerical results are presented.

3.1 Model Description

Consider a single server retrial queueing system with positive and negative customers. Positive customers arrive in batches according to Poisson process with rates λ^+. Negative customers arrive singly with Poisson arrival rate λ^-. At every arrival epoch, a batch of k customers arrive with probability C_k. The generating function of the sequence $\{C_k\}$ is C(z) with first two moments m_1 and m_2. There is no waiting space in front of the server and therefore if an arriving batch of positive customers finds the server idle, then one of the customers receives the service and others join the retrial queue. If the server is busy, then the arriving batch enters the orbit. The retrial time of the customers is generally distributed with distribution function A(x), density function a(x), Laplace-Stieltjes transform $A^*(s)$ and hazard rate function $\eta(x)=\dfrac{a(x)}{1-A(x)}$. The service time of positive customers is generally distributed with distribution function B(x), density function b(x), Laplace-Stieltjes transform $B^*(s)$ and hazard rate function $\mu(x)=\dfrac{b(x)}{1-B(x)}$. The arrival of a negative customer removes the positive customer in service from the system and causes the server breakdown. The server is subject to unpredictable breakdown while it is working. The life time of the server is exponentially distributed with rate α. The repair time of the failed server is generally distributed with distribution function R(x), density function r(x) and hazard rate function $\beta(x)=\frac{r(x)}{1-R(x)}$. All stochastic processes involved in the system are assumed to be independent of each other. Throughout the rest of the paper, we also denote $\bar{F}(x)=1-F(x)$ the tail of distribution function F(x). We also denote $F^*(s)=\int_0^\infty e^{-sx}dF(x)$, $\tilde{\bar{F}}(s)=\int_0^\infty e^{-sx}\bar{F}(x)dx$.

3.2 Stability Condition

Let $N(t_n^+)$ be the number of customers in the orbit just after the time t_n. Then the sequence of random variables $Y_n = N(t_n^+)$ form a Markov chain, which is the embedded Markov chain for this queueing system.

Theorem 3.1

The embedded Markov chain $\{Y_n, n\in N\}$ is ergodic if and only if

$$(1-B^*(\lambda^- +\alpha))(\lambda^+ m_1(1+\beta_1(\lambda^- +\alpha))+\alpha+m_1(\lambda^- +\alpha)(1-A^*(\lambda^+)))$$
$$<(\lambda^- +\alpha)(1-m_1(1-A^*(\lambda^+))B^*(\lambda^- +\alpha))$$

The theorem can be proved as in Gomez-Corral (1999).

3.3 Steady State Distribution

In this section, by treating elapsed service time and elapsed repair time of the server as supplementary variables, the steady state probability generating functions of the orbit size distribution are derived.

Define the states of the server as

$$C(t)=\begin{cases}0, & \text{if the server is idle at time t,}\\ 1, & \text{if the server is busy at time t,}\\ 2, & \text{if the server is under repair at time t.}\end{cases}$$

For $t\geq 0$, we define the random variable $\xi(t)$ as follows:

(i) If $C(t)=0$, then $\xi(t)$ represents the elapsed retrial time at time t;

(ii) If $C(t)=1$, then $\xi(t)$ represents the elapsed service time at time t;

(iii) If $C(t)=2$, then $\xi(t)$ represents the elapsed repair time at time t.

Then, the process $\{X(t); t \geq 0\} = \{C(t), N(t), \xi(t), t \geq 0\}$ is a Markov process. For the process $\{X(t); t \geq 0\}$, we define the following probability densities

$I_0(t) = P\{C(t) = 0, N(t) = n\}$

$I_n(x, t)\, dx = P\{C(t) = 0, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 1$

$P_n(x, t)\, dx = P\{C(t) = 1, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 0$

$R_n(x, t)\, dx = P\{C(t) = 2, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 0$

The system of equation that governs the model under supplementary variable technique are given below

$$\frac{d}{dt} I_0(t) = -\lambda^+ I_0(t) + \int_0^\infty P_0(x,t)\, \mu(x)dx + \int_0^\infty R_0(x,t)\beta(x)dx, \tag{3.1}$$

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) I_n(x,t) = -(\lambda^+ + \eta(x)) I_n(x,t), \quad n \geq 1 \tag{3.2}$$

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) P_n(x,t) = -(\lambda^+ + \lambda^- + \alpha + \mu(x)) P_n(x,t) + \lambda^+ \sum_{k=1}^{n} C_k P_{n-k}(x,t), \quad n \geq 0 \tag{3.3}$$

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) R_n(x,t) = -(\lambda^+ + \beta(x)) R_n(x,t) + \lambda^+ \sum_{k=1}^{n} C_k R_{n-k}(x,t), \quad n \geq 0 \tag{3.4}$$

with boundary conditions

$$I_n(0,t) = \int_0^\infty P_n(x,t)\mu(x)dx + \int_0^\infty R_n(x,t)\,\beta(x)dx, \quad n \geq 1 \tag{3.5}$$

$$P_0(0,t) = \lambda^+ C_1 I_0(t) + \int_0^\infty I_1(x,t)\,\eta(x)dx, \tag{3.6}$$

$$P_n(0,t) = \lambda^+ C_{n+1} I_0(t) + \int_0^\infty I_{n+1}(x,t)\,\eta(x)dx + \lambda^+ \sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x,t)dx, \quad n \geq 1 \tag{3.7}$$

$$R_0(0,t) = \lambda^- \int_0^\infty P_0(x,t)dx \tag{3.8}$$

$$R_n(0,t) = \lambda^- \int_0^\infty P_n(x,t)dx + \alpha \int_0^\infty P_{n-1}(x,t)dx, \ \ n \geq 1 \tag{3.9}$$

Define the steady state probabilities

$$I_0 = \lim_{t\to\infty} I_0(t);$$

$$I_n(x) = \lim_{t\to\infty} I_n(x,t), \ \ x \geq 0, n \geq 1 \ ;$$

$$P_n(x) = \lim_{t\to\infty} P_n(x,t), \ \ x \geq 0, n \geq 0 \ \text{ and}$$

$$R_n(x) = \lim_{t\to\infty} R_n(x,t), x \geq 0, n \geq 0 \ .$$

Taking limit as $t \to \infty$ on both sides of the equations, we get the following steady state equations.

$$\lambda^+ I_0 = \int_0^\infty P_0(x)\mu(x)dx + \int_0^\infty R_0(x)\beta(x)dx, \tag{3.10}$$

$$\frac{d}{dx} I_n(x) = -(\lambda^+ + \eta(x))I_n(x), \ \ n \geq 1 \tag{3.11}$$

$$\frac{d}{dx} P_n(x) = -(\lambda^+ + \lambda^- + \alpha + \mu(x))P_n(x) + \lambda^+ \sum_{k=1}^{n} C_k P_{n-k}(x), \ \ n \geq 0 \tag{3.12}$$

$$\frac{d}{dx} R_n(x) = -(\lambda^+ + \beta(x))R_n(x) + \lambda^+ \sum_{k=1}^{n} C_k R_{n-k}(x), \ \ n \geq 0 \tag{3.13}$$

with boundary conditions

$$I_n(0) = \int_0^\infty P_n(x)\mu(x)dx + \int_0^\infty R_n(x)\beta(x)dx, \ \ n \geq 1 \tag{3.14}$$

$$P_0(0) = \lambda^+ C_1 I_0 + \int_0^\infty I_1(x)\eta(x)dx, \tag{3.15}$$

$$P_n(0) = \lambda^+ C_{n+1} I_0 + \int_0^\infty I_{n+1}(x)\eta(x)dx + \lambda^+ \sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x)dx, \ \ n \geq 1 \tag{3.16}$$

$$R_0(0) = \lambda^- \int_0^\infty P_0(x)dx \tag{3.17}$$

$$R_n(0) = \lambda^- \int_0^\infty P_n(x)dx + \alpha \int_0^\infty P_{n-1}(x)dx, \quad n \geq 1 \tag{3.18}$$

The normalising condition is

$$I_0 + \sum_{n=1}^\infty \int_0^\infty I_n(x)dx + \sum_{n=0}^\infty \int_0^\infty P_n(x)dx + \sum_{n=0}^\infty \int_0^\infty R_n(x)dx = 1 \tag{3.19}$$

Define the probability generating functions for $|z| \leq 1$:

$$I(x,z) = \sum_{n=1}^\infty I_n(x) z^n; \quad P(x,z) = \sum_{n=0}^\infty P_n(x) z^n; \quad R(x,z) = \sum_{n=0}^\infty R_n(x) z^n$$

Multiplying equation (3.11) by z^n and summing over n we get

$$\left(\frac{d}{dx} + \lambda^+ + \eta(x)\right) I(x, z) = 0 \tag{3.20}$$

Solving the partial differential equation (3.20), we get

$$\begin{aligned} I(x,z) &= Ce^{-\lambda^+ x} e^{-\int \eta(x)dx} \\ &= Ce^{-\lambda^+ x} e^{\log(1-A(x))} \\ &= Ce^{-\lambda^+ x}(1 - A(x)) \end{aligned}$$

Eliminating C by taking x = 0, we get

$$I(x,z) = I(0,z)e^{-\lambda^+ x}(1 - A(x)) \tag{3.21}$$

The partial differential equations in (3.12) and (3.13) yield

$$P(x,z) = P(0,z)e^{-(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))x}(1 - B(x)) \tag{3.22}$$

$$R(x,z) = R(0,z)e^{-(\lambda^+ - \lambda^+ C(z))x}(1 - R(x)) \tag{3.23}$$

Multiplying equations (3.14) to (3.18) by z^n and summing over n, we get

$$I(0,z) = \int_0^\infty P(x,z)\mu(x)dx + \int_0^\infty R(x,z)\beta(x)dx - \lambda^+ I_0 \tag{3.24}$$

$$P(0,z) = \frac{\lambda^+ C(z)}{z} I_0 + \frac{1}{z}\int_0^\infty I(x,z)\eta(x)dx + \frac{\lambda^+ C(z)}{z}\int_0^\infty I(x,z)dx \tag{3.25}$$

$$R(0,z) = (\lambda^- + \alpha z)\int_0^\infty P(x,z)dx \tag{3.26}$$

Using equations (3.22) and (3.23) in equation (3.24), we get

$$I(0,z) = P(0,z)[B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) + (\lambda^- + \alpha z)K(z)R^*(\lambda^+ - \lambda^+ C(z))] - \lambda^+ I_0 \tag{3.27}$$

where $K(z) = \dfrac{1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))}{(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))}$

Substituting the expression in equation (3.21), in equation (3.25) and simplifying we obtain

$$P(0,z) = \frac{\lambda^+ C(z)}{z} I_0 + \frac{I(0,z)}{z}(C(z) + (1 - C(z))A^*(\lambda^+)) \tag{3.28}$$

Solving equations (3.27) and (3.28), we get

$$I(0,z) = \lambda^+ I_0[C(z)B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) - z + C(z)(\lambda^- + \alpha z)K(z)R^*(\lambda^+ - \lambda^+ C(z))]/T(z) \tag{3.29}$$

and

$$P(0,z) = \lambda^+ I_0 A^*(\lambda^+)(C(z) - 1)/T(z) \tag{3.30}$$

where

$$T(z) = z - (C(z) + (1 - C(z))A^*(\lambda^+))B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) - (\lambda^- + \alpha z)K(z)$$
$$R^*(\lambda^+ - \lambda^+ C(z))(C(z) + (1 - C(z))A^*(\lambda^+))$$

Substituting the result in equation (3.22) in equation (3.26), we get

$$R(0,z) = (\lambda^- + \alpha z)P(0,z)K(z) \tag{3.31}$$

Using equation (3.30), equation (3.31) yields

$$R(0,z) = \lambda^+(\lambda^- + \alpha z)I_0 A^*(\lambda^+)(C(z)-1)K(z)/T(z) \tag{3.32}$$

Substituting the expressions of I(0,z), P(0,z) and R(0,z), in equations (3.21), (3.22), and (3.23) respectively, we get

$$I(x,z) = \lambda^+ I_0[C(z)B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) - z + C(z)(\lambda^- + \alpha z)K(z)R^*(\lambda^+ - \lambda^+ C(z))]$$

$$e^{-\lambda^+ x}(1-A(x))/T(z) \tag{3.33}$$

$$P(x,z) = \lambda^+ I_0 A^*(\lambda^+)(C(z)-1)e^{-(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))x}(1-B(x))/T(z) \tag{3.34}$$

$$R(x,z) = \lambda^+(\lambda^- + \alpha z)I_0 A^*(\lambda^+)(C(z)-1)K(z)e^{-(\lambda^+ - \lambda^+ C(z))x}(1-R(x))/T(z) \tag{3.35}$$

The partial probability generating function of the orbit size when the server is idle is

$$I(z) = \int_0^\infty I(x,z)dx$$

$$= I_0(1-A^*(\lambda^+))[(C(z)B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) - z)(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) + (\lambda^- + \alpha z)$$

$$C(z)(1-B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))R^*(\lambda^+ - \lambda^+ C(z))]/T_1(z) \tag{3.36}$$

where

$$T_1(z) = (z - (C(z) + (1-C(z))A^*(\lambda^+))B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))$$

$$-(\lambda^- + \alpha z)(1-B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))R^*(\lambda^+ - \lambda^+ C(z))(C(z) + (1-C(z))A^*(\lambda^+))$$

The partial probability generating function of the orbit size when the server is busy is

$$P(z) = \int_0^\infty P(x,z)dx$$

$$= \lambda^+ I_0 A^*(\lambda^+)(C(z)-1)(1-B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))/T_1(z) \tag{3.37}$$

The partial probability generating function of the orbit size when the server is under repair is

$$R(z) = \int_0^\infty R(x,z)dx$$

$$= (\lambda^- + \alpha z)I_0 A^*(\lambda^+)(1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))(R^*(\lambda^+ - \lambda^+ C(z)) - 1)/T_1(z) \quad (3.38)$$

The partial probability generating function of the orbit size is

$$P_q(z) = I_0 + I(z) + P(z) + R(z)$$

$$= I_0 A^*(\lambda^+)[\lambda^+(z-1)(1 - C(z)) + (z-1)(\lambda^- + \alpha B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))]/T_1(z)$$

(3.39)

The partial probability generating function of the system size is

$$P_s(z) = I_0 + I(z) + zP(z) + R(z)$$

$$= I_0 A^*(\lambda^+)[\lambda^+(z-1)(1 - C(z))B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) + (z-1)$$

$$(\lambda^- + \alpha B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))]/T_1(z) \quad (3.40)$$

3.4 Performance Measures

- The probability that the server is idle during retrial time is given by

$$I = \lim_{z \to 1} I(z)$$

$$= I_0(1 - A^*(\lambda^+))[(1 - B^*(\lambda^- + \alpha))(\lambda^+ m_1(1 + \beta_1(\lambda^- + \alpha)) + \alpha + m_1(\lambda^- + \alpha))$$

$$+ (\lambda^- + \alpha)(m_1 B^*(\lambda^- + \alpha) - 1)]/T_1'(1) \quad (3.41)$$

where

$$T_1'(1) = (B^*(\lambda^- + \alpha) - 1)[\lambda^+ m_1(1 + \beta_1(\lambda^- + \alpha)) + \alpha + m_1(\lambda^- + \alpha)(1 - A^*(\lambda^+))]$$

$$+ (\lambda^- + \alpha)(1 - m_1(1 - A^*(\lambda^+))B^*(\lambda^- + \alpha))$$

- The probability that the server is busy is given by

$$P = \lim_{z\to 1} P(z)$$

$$= \lambda^{+} m_1 I_0 A^{*}(\lambda^{+})(1 - B^{*}(\lambda^{-} + \alpha)) / T_1'(1) \quad (3.42)$$

- The probability that the server is down is given by

$$R = \lim_{z\to 1} R(z)$$

$$= \lambda^{+}(\lambda^{-} + \alpha) m_1 \beta_1 I_0 A^{*}(\lambda^{+})(1 - B^{*}(\lambda^{-} + \alpha)) / T_1'(1) \quad (3.43)$$

The normalizing equation (3.19) is equivalent to

$$I_0 + I + P + R = 1 \quad (3.44)$$

Using equations (3.41) to (3.43), equation (3.44) yields

$$I_0 = (B^{*}(\lambda^{-} + \alpha) - 1)(\lambda^{+} m_1 (1 + \beta_1(\lambda^{-} + \alpha)) + \alpha + m_1(\lambda^{-} + \alpha)(1 - A^{*}(\lambda^{+}))) + (\lambda^{-} + \alpha)$$

$$(1 - m_1(1 - A^{*}(\lambda^{+}))B^{*}(\lambda^{-} + \alpha)) / A^{*}(\lambda^{+})((\lambda^{-} + \alpha) + \alpha(B^{*}(\lambda^{-} + \alpha) - 1)) \quad (3.45)$$

- The mean number of customers in the orbit is given by

$$L_q = \lim_{z\to 1} \frac{d}{dz} P_q(z) \quad (3.46)$$

$$= \frac{Dr'(1)Nr''(1) - Nr'(1)Dr''(1)}{2Dr'(1)^2}$$

where Nr(z) and Dr(z) are the Numerator and Denominator of $P_q(z)$

$$Nr'(1) = I_0 A^{*}(\lambda^{+})((\lambda^{-} + \alpha) - \alpha(1 - B^{*}(\lambda^{-} + \alpha)))$$

$$Nr''(1) = I_0 A^{*}(\lambda^{+})[2\lambda^{+}(\alpha \int_0^{\infty} e^{-(\lambda^{-}+\alpha)x} x b_1(x) dx (m_1) - m_1)]$$

$$Dr'(1) = T_1'(1)$$

$$Dr''(1) = (B^*(\lambda^- + \alpha) - 1)[\lambda^+ m_2 + (\lambda^- + \alpha)\lambda^+ m_2\beta_2 + 2\lambda^+ m_1\beta_1\alpha + 2m_1$$
$$(1 - A^*(\lambda^+))\,((\lambda^- + \alpha)\lambda^+ m_1\beta_1 + \alpha) + (\lambda^- + \alpha)m_2(1 - A^*(\lambda^+))] - 2\lambda^+ m_1$$
$$(1 - m_1(1 - A^*(\lambda^+))B^*(\lambda^- + \alpha)) - (\lambda^- + \alpha)m_2(1 - A^*(\lambda^+))B^*(\lambda^- + \alpha)$$
$$+ 2h_1(\lambda^+ m_1\beta_1(\lambda^- + \alpha) + \alpha + \lambda^+ m_1)$$

where $h_1 = \lambda^+ m_1 \int_0^\infty e^{-(\lambda^- + \alpha)x}\, x\, b_1(x)dx$

- The mean number of customers in the system is given by

$$L_s = \lim_{z \to 1} \frac{d}{dz} P_s(z)$$
$$= L_q + P \tag{3.47}$$

3.5 Stochastic Decomposition

Theorem 3.2

The number of customers in the system (L_s) can be expressed as the sum of two independent random variables, one of which is the mean number of customers in batch arrival G-queue with server breakdown (L) and the other is the mean number of customers in the orbit given that the server is idle (L_I).

Proof

The probability generating function Φ(z) of the number of customers in batch arrival G-queue with server breakdown is given by

$$\Phi(z) = I_0[\lambda^+(z - 1)(1 - C(z))B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) + (z - 1)$$
$$(\lambda^- + \alpha B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))] / T_2(z) \tag{3.48}$$

where

$$T_2(z) = (\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))(z - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))) - (1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))$$
$$(\lambda^- + \alpha z)R^*(\lambda^+ - \lambda^+ C(z))$$

The probability generating function ψ(z) of the number of customers in the orbit given that the server is idle, is given by

$$\Psi(z) = \frac{I_0 + I(z)}{I_0 + I(1)}$$

$$= \frac{\begin{array}{l}[I_0 A^*(\lambda^+)((\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))(z - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))) - (\lambda^- + \alpha z) \\ (1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))R^*(\lambda^+ - \lambda^+ C(z)))][(\lambda^- + \alpha) + \alpha(B^*(\lambda^- + \alpha) - 1)]\end{array}}{T_1(z)[(\lambda^- + \alpha) + (B^*(\lambda^- + \alpha) - 1)(\lambda^+ m_1 + \alpha + \lambda^+ m_1 \beta_1(\lambda^- + \alpha))]}$$

(3.49)

From equations (3.40), (3.48) and (3.49) we get

$$P_s(z) = \Phi(z)\Psi(z)$$

Differentiating $P_s(z)$ with respect to z and taking limit as $z \to 1$, we get $L_s = L + L_I$

3.6 Reliability Analysis

Let A(t) be the system availability at time t, that is, the probability that the server is idle or working for a customer. Then under steady state condition, the availability of the server is given by

$$A = I_0 + \lim_{z \to 1} \left[\int_0^\infty I(x,z)dx + \int_0^\infty P(x,z)dx \right]$$

$$= I_0 + I + P$$

$$= \frac{(\lambda^- + \alpha) + (B^*(\lambda^- + \alpha) - 1)(\alpha + \lambda^+ m_1 \beta_1(\lambda^- + \alpha))}{(\lambda^- + \alpha) + \alpha(B^*(\lambda^- + \alpha) - 1)} \quad (3.50)$$

The steady state failure frequency of the server is given by

$$W_f = \lambda^- P$$

$$= \frac{\lambda^+ \lambda^- m_1 (1 - B^*(\lambda^- + \alpha))}{(\lambda^- + \alpha) + \alpha(B^*(\lambda^- + \alpha) - 1)} \quad (3.51)$$

Theorem 3.3

Let τ be the time to the first failure of the server. Then the Laplace transform of reliability function $\zeta(t) = P(\tau > t)$ of the server is given by

$$\tilde{\zeta}(s) = (1 - A^*(s+\lambda^+)) + (\lambda^+ + sA^*(s+\lambda^+))\tilde{\tilde{B}}(s+\lambda^- + \alpha) + \tilde{I}_0(s)[(\lambda^+ B^*(s+\lambda^- + \alpha) - (s+\lambda^+))(1 - A^*(s+\lambda^+)) - s(s+\lambda^+)A^*(s+\lambda^+)\tilde{\tilde{B}}(s+\lambda^- + \alpha)] / F(1,s) \tag{3.52}$$

where

$$F(1,s) = (s+\lambda^+) - (\lambda^+ + sA^*(s+\lambda^+))B^*(s+\lambda^- + \alpha)$$

Proof

Considering failure states of the server as absorbing states, we obtain a new system with the following governing equations.

$$\frac{d}{dt} I_0(t) = -\lambda^+ I_0(t) + \int_0^\infty P_0(x,t)\,\mu(x)dx, \tag{3.53}$$

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) I_n(x,t) = -(\lambda^+ + \eta(x)) I_n(x,t), \quad n \geq 1 \tag{3.54}$$

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) P_n(x,t) = -(\lambda^+ + \lambda^- + \alpha + \mu(x)) P_n(x,t) + \lambda^+ \sum_{k=1}^{n} C_k P_{n-k}(x,t), \quad n \geq 0 \tag{3.55}$$

$$I_n(0,t) = \int_0^\infty P_n(x,t)\mu(x)dx, \quad n \geq 1 \tag{3.56}$$

$$P_0(0,t) = \lambda^+ C_1 I_0(t) + \int_0^\infty I_1(x,t)\,\eta(x)dx, \tag{3.57}$$

$$P_n(0,t) = \lambda^+ C_{n+1} I_0(t) + \int_0^\infty I_{n+1}(x,t)\,\eta(x)dx + \lambda^+ \sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x,t)dx, \quad n \geq 1 \tag{3.58}$$

Let the initial condition be

$$I_n(0) = \delta_{n,0}, \quad I_n(x,0) = 0, \quad P_n(x,0) = 0.$$

Taking Laplace transforms of equations (3.53) to (3.58), we obtain

$$(s + \lambda^+)\tilde{I}_0(s) - 1 = \int_0^\infty \tilde{P}_0(x,s)\mu(x)dx \tag{3.59}$$

$$\frac{d}{dx}\widetilde{I}_n(x,s) = -(s+\lambda^+ + \eta(x))\widetilde{I}_n(x,s), \quad n \geq 1 \tag{3.60}$$

$$\frac{d}{dx}\widetilde{P}_n(x,s) = -(s+\lambda^+ + \lambda^- + \alpha + \mu(x))\widetilde{P}_n(x,s) + \lambda^+ \sum_{k=1}^{n} C_k \widetilde{P}_{n-k}(x,s), \quad n \geq 0 \tag{3.61}$$

$$\widetilde{I}_n(0,s) = \int_0^\infty \widetilde{P}_n(x,s)\,\mu(x)dx, \quad n \geq 1 \tag{3.62}$$

$$\widetilde{P}_0(0,s) = \lambda^+ C_1\, \widetilde{I}_0(s) + \int_0^\infty \widetilde{I}_1(x,s)\,\eta(x)dx \tag{3.63}$$

$$\widetilde{P}_n(0,s) = \lambda^+ C_{n+1}\, \widetilde{I}_0(s) + \int_0^\infty \widetilde{I}_{n+1}(x,s)\,\eta(x)dx + \lambda^+ \sum_{k=1}^{n} C_k \int_0^\infty \widetilde{I}_{n-k+1}(x,s)dx, \; n \geq 1 \tag{3.64}$$

Define the following probability generating functions for $|z| \leq 1$:

$$\widetilde{I}(z,x,s) = \sum_{n=1}^{\infty} \widetilde{I}_n(x,z)z^n\,; \quad \widetilde{P}(z,x,s) = \sum_{n=0}^{\infty} \widetilde{P}_n(x,s)z^n\,.$$

Then equations (3.60) to (3.64) yield

$$\frac{d}{dx}\widetilde{I}(z,x,s) = -(\lambda^+ + s + \eta(x))\widetilde{I}(z,x,s) \tag{3.65}$$

$$\frac{d}{dx}\widetilde{P}(z,x,s) = -(s+\lambda^+ + \lambda^- + \alpha + \mu(x) - \lambda^+ C(z))\; \widetilde{P}(z,x,s) \tag{3.66}$$

$$\widetilde{I}(z,0,s) = \int_0^\infty \widetilde{P}(z,x,s)\,\mu(x)dx \; - (s+\lambda^+)\,\widetilde{I}_0(s) + 1 \tag{3.67}$$

$$\widetilde{P}(z,0,s) = \frac{\lambda^+ C(z)}{z}\widetilde{I}_0(s) + \frac{1}{z}\int_0^\infty \widetilde{I}(z,x,s)\eta(x)dx + \frac{\lambda^+ C(z)}{z}\int_0^\infty \widetilde{I}(z,x,s)dx \tag{3.68}$$

The solutions of the partial differential equations (3.65) and (3.66) are given by

$$\widetilde{I}(z,x,s) = \widetilde{I}(z,0,s)e^{-(s+\lambda^+)x}(1 - A(x)) \tag{3.69}$$

$$\widetilde{P}(z,x,s) = \widetilde{P}(z,0,s)e^{-(s+\lambda^+(1-C(z))+\lambda^- + \alpha)x}(1 - B(x)) \tag{3.70}$$

Using equation (3.70) in equation (3.67), we get

$$\widetilde{I}(z,0,s) = \widetilde{P}(z,0,s)B^*(s+\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) - (s+\lambda^+)\widetilde{I}_0(s) + 1 \tag{3.71}$$

Using equation (3.69) in equation (3.68) and simplifying we obtain

$$\widetilde{P}(z,0,s) = \frac{\lambda^+ C(z)}{z}\widetilde{I}_0(s) + \frac{\widetilde{I}(z,0,s)}{z(s+\lambda^+)}(C(z)\lambda^+ + (s+\lambda^+(1-C(z)))A^*(s+\lambda^+)) \tag{3.72}$$

Solving equation (3.71) and (3.72), we get

$$\widetilde{I}(z,0,s) = (s+\lambda^+)[z + \widetilde{I}_0(s)(\lambda^+ C(z)B^*(s+\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) - z(s+\lambda^+))]/F(z,s) \tag{3.73}$$

$$\widetilde{P}(z,0,s) = [C(z)\lambda^+ + (s+\lambda^+(1-C(z)))A^*(s+\lambda^+) - \widetilde{I}_0(s)(s+\lambda^+)(s+\lambda^+(1-C(z)))$$

$$A^*(s+\lambda^+)]/F(z,s) \tag{3.74}$$

where

$$F(z,s) = z(s+\lambda^+) - (\lambda^+ C(z) + (s+\lambda^+(1-C(z)))A^*(s+\lambda^+))B^*(s+\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))$$

Substituting the expressions of $\widetilde{I}(z,0,s)$ and $\widetilde{P}(z,0,s)$ in equations (3.69) and (3.70) we get

$$\widetilde{I}(z,x,s) = (s+\lambda^+)[z + \widetilde{I}_0(s)(\lambda^+ C(z)B^*(s+\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) - z(s+\lambda^+))]$$

$$e^{-(s+\lambda^+)x}(1 - A(x))/F(z,s) \tag{3.75}$$

$$\widetilde{P}(z,x,s) = [\lambda^+ C(z) + (s+\lambda^+(1-C(z)))A^*(s+\lambda^+) - \widetilde{I}_0(s)(s+\lambda^+)(s+\lambda^+(1-C(z)))$$

$$A^*(s+\lambda^+)]e^{-(s+\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z))x}(1 - B(x))/F(z,s) \tag{3.76}$$

From equations (3.75) and (3.76) we can obtain

$$\widetilde{I}(z,s) = \int_0^\infty \widetilde{I}(z,x,s)dx$$

$$= [z + \widetilde{I}_0(s)(\lambda^+ C(z)B^*(s+\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) - z(s+\lambda^+))](1 - A^*(s+\lambda^+))/F(z,s) \tag{3.77}$$

$$\tilde{P}(z,s) = \int_0^\infty \tilde{P}(z,x,s)dx$$

$$= [\lambda^+ C(z) + (s+\lambda^+(1-C(z)))A^*(s+\lambda^+) - \tilde{I}_0(s)(s+\lambda^+)(s+\lambda^+(1-C(z)))A^*(s+\lambda^+)]$$

$$\tilde{\tilde{B}}(s+\lambda^+ +\lambda^- +\alpha - \lambda^+ C(z)) / F(z,s) \qquad (3.78)$$

Now $\tilde{\zeta}(s)$ is given by

$$\tilde{\zeta}(s) = \tilde{I}(1,s) + \tilde{P}(1,s)$$

$$\tilde{I}(1,s) = [1 + \tilde{I}_0(s)(\lambda^+ B^*(s+\lambda^- +\alpha) - (s+\lambda^+))](1 - A^*(s+\lambda^+)) / F(1,s) \qquad (3.79)$$

$$\tilde{P}(1,s) = [\lambda^+ + sA^*(s+\lambda^+) - \tilde{I}_0(s)s(s+\lambda^+)A^*(s+\lambda^+)]\tilde{\tilde{B}}(s+\lambda^- +\alpha) / F(1,s) \qquad (3.80)$$

Substituting the expressions of $\tilde{I}(1,s)$ and $\tilde{P}(1,s)$ we get the result in equation (3.52).

Corollary 3.1

The mean time to the first failure (MTTFF) of the server is given by

$$\text{MTTFF} = (1 - A^*(\lambda^+)) + \lambda^+ \tilde{\tilde{B}}(\lambda^- +\alpha) + \tilde{I}_0(0)\lambda^+(B^*(\lambda^- +\alpha) - 1)(1 - A^*(\lambda^+)) / \lambda^+(1 - B^*(\lambda^- +\alpha)) \qquad (3.81)$$

Proof

$$\text{MTTFF} = \int_0^\infty \zeta(t)dt = \lim_{s\to 0} \tilde{\zeta}(s)$$

Taking limit as $s \to 0$ on both sides of equation (3.52) we get the result in equation (3.81).

3.7 Special Cases

Case (i): No negative customers ($\lambda^- \to 0$)

In this case our model reduces to $M^X/G/1$ retrial G-queue with server breakdown having,

$$P_s(z) = I_0 A^*(\lambda^+)[\lambda^+(z-1)(1-C(z))B^*(\lambda^+ +\alpha - \lambda^+ C(z)) + (z-1)\alpha B^*(\lambda^+ +\alpha - \lambda^+ C(z))] / T_3(z)$$

where

$$T_3(z) = (\lambda^+ + \alpha - \lambda^+ C(z))(z - (C(z) + (1 - C(z))A^*(\lambda^+))B^*(\lambda^+ + \alpha - \lambda^+ C(z)))$$

$$- \alpha z(C(z) + (1 - C(z))A^*(\lambda^+))(1 - B^*(\lambda^+ + \alpha - \lambda^+ C(z)))R^*(\lambda^+ - \lambda^+ C(z))$$

$$I_0 = \lambda^+ m_1(1 + \beta_1\alpha)(B^*(\alpha) - 1) + \alpha B^*(\alpha) - \alpha m_1(1 - A^*(\lambda^+))/A^*(\lambda^+)(\alpha + \alpha(B^*(\alpha) - 1))$$

Case (ii): No retrial ($A^*(\lambda^+) \to 1$)

In this case our model reduces to $M^X/G/1$ G-queue with server breakdown and two types of customers with

$$P_s(z) = I_0[\lambda^+(z-1)(1-C(z))B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)) + (\lambda^- + \alpha)(z - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))$$

$$- (\lambda^- + \alpha z)(1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ C(z)))]/T_2(z)$$

and

$$I_0 = [(B^*(\lambda^- + \alpha) - 1)(\lambda^+ m_1(1 + \beta_1(\lambda^- + \alpha)) + \alpha) + (\lambda^- + \alpha)]/((\lambda^- + \alpha) + \alpha(B^*(\lambda^- + \alpha) - 1))$$

3.8 Numerical Results

Numerical results are calculated by assuming the distributions of retrial time, service time and repair time as exponential with rates η,μ and β.

For the parameters $\lambda^+ = 1$, $\lambda^- = 0.5$, $\alpha = 0.7$, $\mu = 6$, $\beta = 3$, $\eta = 40$, $c_1 = 0.5$, $c_2 = 0.5$, the performance measures I_0-the probability that the system is empty, I-the probability that the server is idle in non-empty system, P-the probability that the server is busy, R-the probability that the server is under repair, A-the availability of the server, F-the failure frequency of the server and Ls-the mean number of customers in the system are calculated and presented in tables 3.1 to 3.6 respectively for various rates of λ^+, λ^-, α, η, μ and β.

Table 3.1 reveals that I_0 and A monotonically decrease and I, P, R, F and Ls increase as λ^+ increases. Table 3.2 indicates that increase in λ^- decreases I_0, P, A and Ls and increases other performance measures. From Table 3.3 and 3.6 we observe that the parameters α and β have no effect on the performance measures P and F. For increasing values of α, the values of I, R and Ls increase and I_0 and A decrease. As β increases I, R and Ls decrease I_0 and A increase. From Table 3.4 we observe that the parameter η has no effect on the performance measures P, A and F. For increasing values of η, I_0 increases and I and Ls decrease. Table 3.5 depicts the effect of μ on the performance measures. It is noted that I_0 and A increase and I, P, R, F and Ls decrease for increasing values of μ.

Table 3.1 Performance measures by varying λ^+

λ^+	I_0	I	P	R	A	F	Ls
0.5000	0.8282	0.0103	0.1154	0.0462	0.9538	0.0577	0.2173
1.0000	0.6523	0.0246	0.2308	0.0923	0.9077	0.1154	0.6331
1.5000	0.4724	0.0430	0.3462	0.1385	0.8615	0.1731	1.4160
2.0000	0.2885	0.0654	0.4615	0.1846	0.8154	0.2308	3.1777
2.5000	0.1005	0.0918	0.5769	0.2308	0.7692	0.2885	11.1806

Table 3.2 Performance measures by varying λ^-

λ^-	I_0	I	P	R	A	F	Ls
1.0000	0.6396	0.0246	0.2143	0.1214	0.8786	0.2143	0.6157
2.0000	0.6191	0.0247	0.1875	0.1687	0.8313	0.3750	0.5859
3.0000	0.6031	0.0247	0.1667	0.2056	0.7944	0.5000	0.5614
4.0000	0.5903	0.0248	0.1500	0.2350	0.7650	0.6000	0.5408
5.0000	0.5798	0.0248	0.1364	0.2591	0.7409	0.6818	0.5233

Table 3.3 Performance measures by varying α

α	I_0	I	P	R	A	F	Ls
1.0000	0.6269	0.0269	0.2308	0.1154	0.8846	0.1154	0.7099
2.0000	0.5423	0.0346	0.2308	0.1923	0.8077	0.1154	1.0178
3.0000	0.4577	0.0423	0.2308	0.2692	0.7308	0.1154	1.4397
4.0000	0.3731	0.0500	0.2308	0.3462	0.6538	0.1154	2.0528
5.0000	0.2885	0.0577	0.2308	0.4231	0.5769	0.1154	3.0257

Table 3.4 Performance measures by varying η

η	I_0	I	P	R	A	F	Ls
2.0000	0.1846	0.4923	0.2308	0.0923	0.9077	0.1154	5.4167
4.0000	0.4308	0.2462	0.2308	0.0923	0.9077	0.1154	1.6042
6.0000	0.5128	0.1641	0.2308	0.0923	0.9077	0.1154	1.1467
8.0000	0.5538	0.1231	0.2308	0.0923	0.9077	0.1154	0.9688
10.0000	0.5785	0.0985	0.2308	0.0923	0.9077	0.1154	0.8741

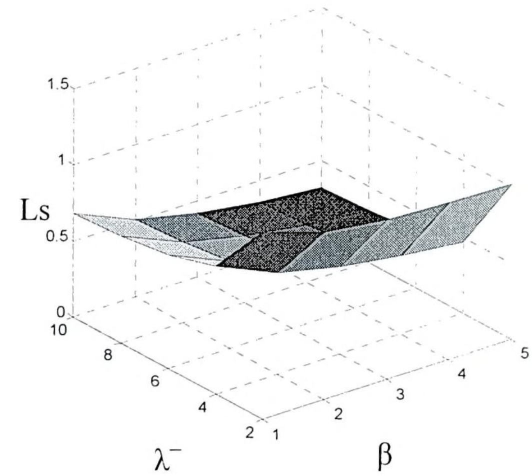
Table 3.5 Performance measures by varying μ

μ	I_0	I	P	R	A	F	Ls
2.0000	0.1160	0.0440	0.6000	0.2400	0.7600	0.3000	9.0833
4.0000	0.5033	0.0300	0.3333	0.1333	0.8667	0.1667	1.1740
6.0000	0.6523	0.0246	0.2308	0.0923	0.9077	0.1154	0.6331
8.0000	0.7312	0.0218	0.1765	0.0706	0.9294	0.0882	0.4359
10.0000	0.7800	0.0200	0.1429	0.0571	0.9429	0.0714	0.3338

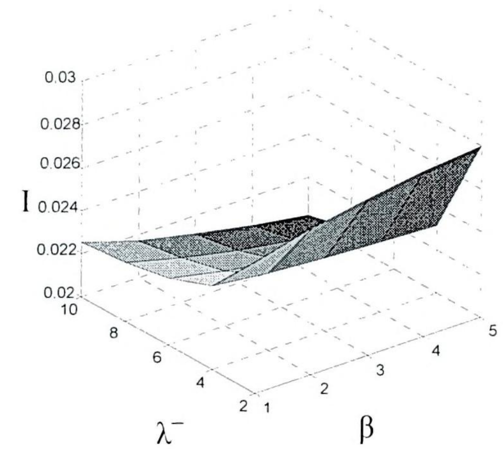
Table 3.6 Performance measures by varying β

β	I_0	I	P	R	A	F	Ls
1.0000	0.4631	0.0292	0.2308	0.2769	0.7231	0.1154	1.5083
2.0000	0.6050	0.0258	0.2308	0.1385	0.8615	0.1154	0.7745
3.0000	0.6523	0.0246	0.2308	0.0923	0.9077	0.1154	0.6331
4.0000	0.6760	0.0240	0.2308	0.0692	0.9308	0.1154	0.5756
5.0000	0.6902	0.0237	0.2308	0.0554	0.9446	0.1154	0.5449

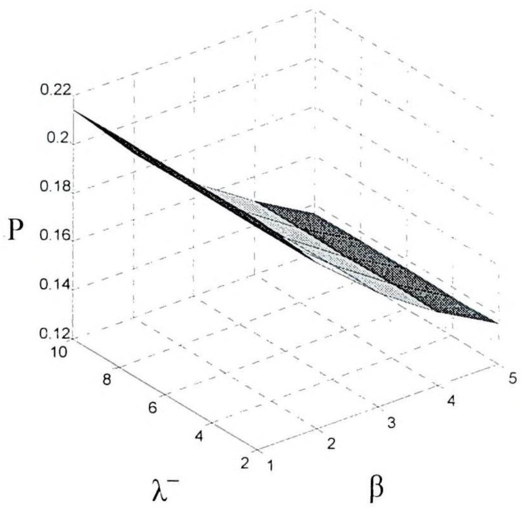
The increasing and decreasing trends of the performance measures by varying the parameters are as expected.



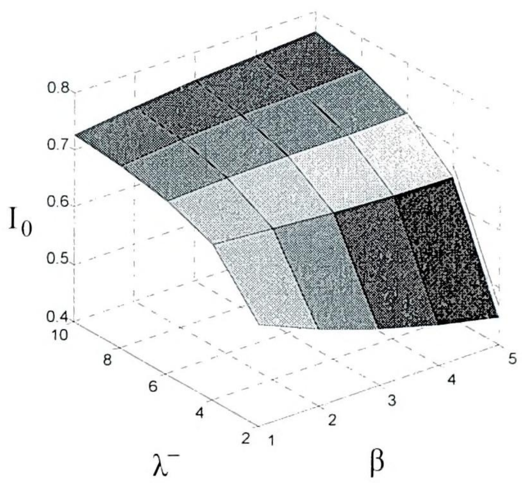
(a) Ls versus (λ^-, β)



(b) I versus (λ^-, β)

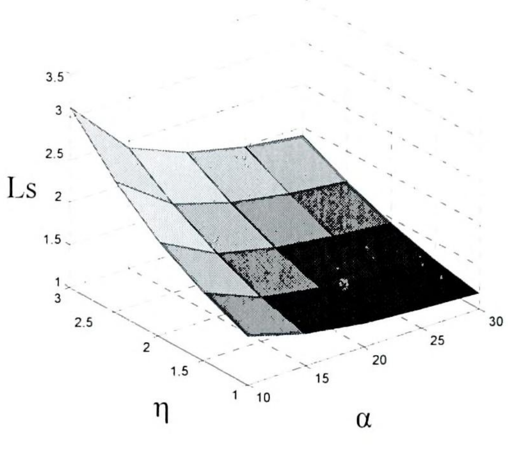


(c) P versus (λ^-, β)

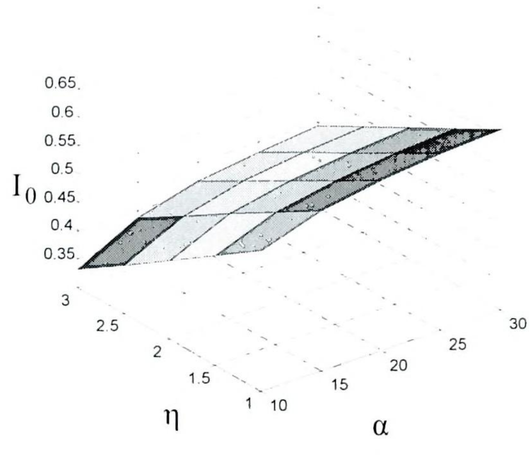


(d) I_0 versus (λ^-, β)

Fig. 3.1 Performance Measures Versus (λ^-, β)

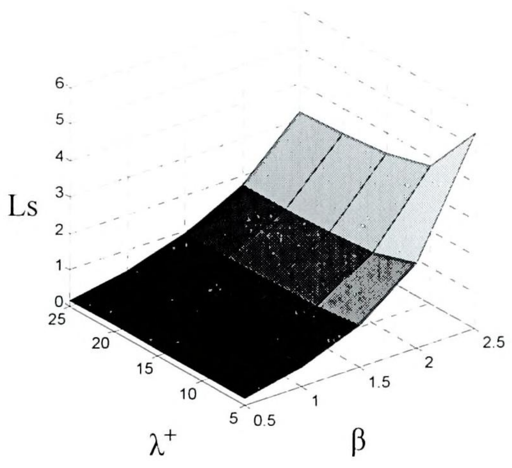


(a) Ls versus (η, α)

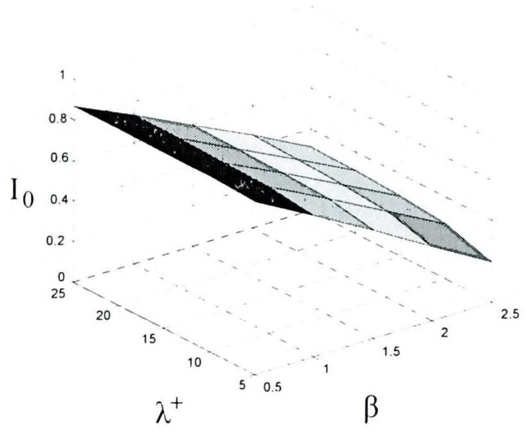


(b) I_0 versus (η, α)

Fig. 3.2 Performance Measures Versus (η, α)

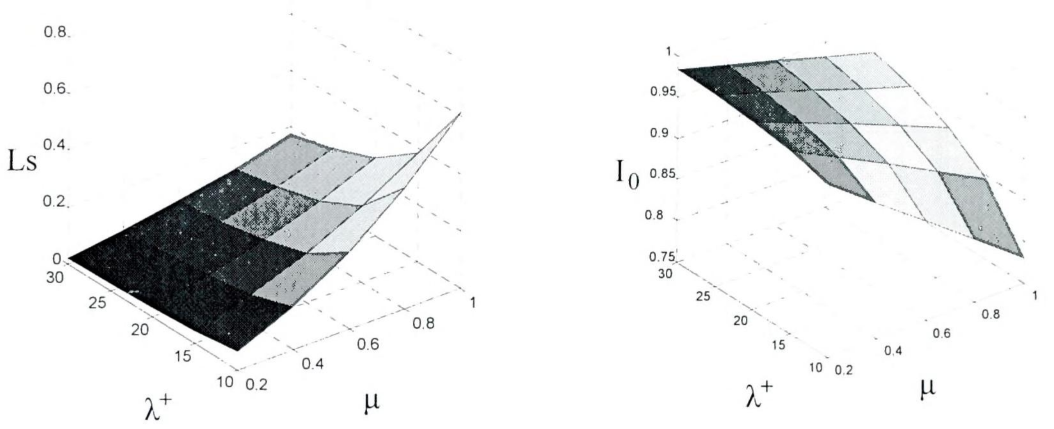


(a) Ls versus (λ^+, β)



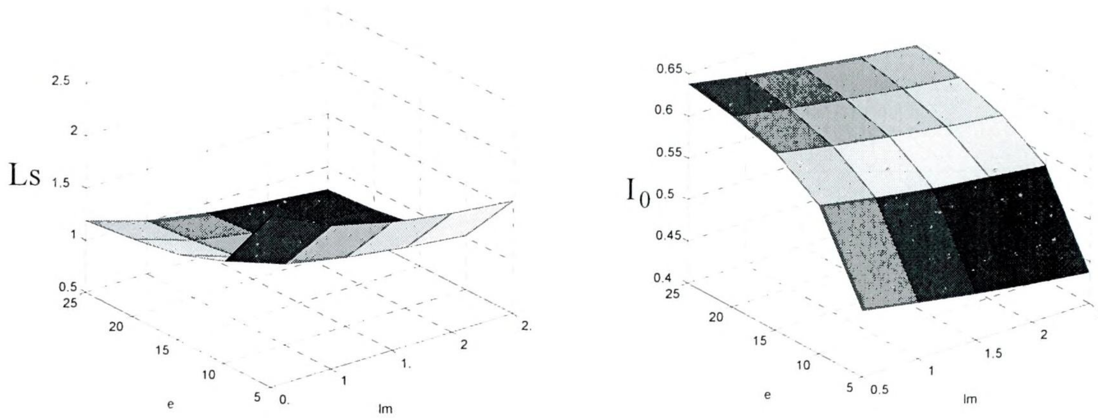
(b) I_0 versus (λ^+, β)

Fig. 3.3 Performance Measures Versus (λ^+, β)



(a) Ls versus (λ^+, μ) **(b) I_0 versus (λ^+, μ)**

Fig. 3.4 Performance Measures Versus (λ^+, μ)



(c) Ls versus (λ^-, β) **(d) I_0 versus (λ^-, β)**

Fig. 3.5 Performance Measures Versus (λ^-, β)