Interval-Valued Fuzzy Structures in Z-Algebras

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In 1975, Zadeh [74] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [73] in which the values of the membership degrees are intervals of numbers instead of the numbers. This chapter is divided into two sections. In the first section, we discuss the notion of interval-valued fuzzy Z-Subalgebras of Z-algebras. In the second section, we discuss the concept of interval-valued fuzzy Z-ideals of Z-algebras and prove some interesting results.

7.1 Interval valued fuzzy Z-Subalgebras in Z-algebras

In this section, we define the notion of an interval-valued fuzzy Z-Subalgebra of a Z-algebra. Also we have proved some interesting results.

Definition 7.1.1: Let $(X,*,0)$ be a Z-algebra. An interval-valued fuzzy set $A=\{(x,\tilde{\mu}_A(x)) \mid x\in X\}$ in X is said to be an **interval –valued fuzzy Z-Subalgebra** of a Z-algebra X if: $\tilde{\mu}_A(x*y)\geq r\min\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$ for all $x,y\in X$.

Example 7.1.2: Consider a Z-algebra X= {0,1,2,3} with the following Cayley table :

*	0	1	2	3
0	0	1	2	3
1	0	1	1	2
2	0	1	2	3
3	0	2	3	3

Define an interval-valued fuzzy set $A=\{(x,\tilde{\mu}_A(x)) \mid x\in X\}$ by

$$\tilde{\mu}_A(x)=\begin{cases}[0.4,0.8] & \text{if } x\in\{0,2\}\\ [0.2,0.5] & \text{otherwise}\end{cases}$$

Then A is an interval-valued fuzzy Z-Subalgebra of a Z-algebra sX.

Theorem 7.1.3: An interval-valued fuzzy set $A = \{(x, \widetilde{\mu}_A(x)) | x \in X\}$ in a Z-algebra X where $\widetilde{\mu}_A = [\mu_A^L, \mu_A^U]$ is an interval-valued fuzzy Z-Subalgebra of X if and only if $\mu_A^L : X \to [0,1]$ and $\mu_A^U : X \to [0,1]$ are fuzzy Z-Subalgebras of X.

Proof: Let $A = \{(x, \widetilde{\mu}_A(x)) | x \in X\}$ where $\widetilde{\mu}_A = [\mu_A^L, \mu_A^U]$ be an interval-valued fuzzy set in a Z-algebra X. Let $\mu_A^L : X \to [0,1]$ and $\mu_A^U : X \to [0,1]$ are fuzzy Z-Subalgebras of X and $x, y \in X$.

Consider $\widetilde{\mu}_A(x * y) = [\mu_A^L(x * y), \mu_A^U(x * y)] \geq [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}]$

$$= r\min\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\}$$

$$= r\min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$$

Thus $A = \{(x, \widetilde{\mu}_A(x)) | x \in X\}$ is an interval-valued fuzzy Z-Subalgebra of a Z-algebra X.

Conversely, for any $x, y \in X$,

$$[\mu_A^L(x * y), \mu_A^U(x * y)] = \widetilde{\mu}_A(x * y) \geq r\min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$$

$$= r\min\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\}$$

$$= [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}]$$

Hence, $\mu_A^L(x * y) \geq \min\{\mu_A^L(x), \mu_A^L(y)\}$ and $\mu_A^U(x * y) \geq \min\{\mu_A^U(x), \mu_A^U(y)\}$.

Thus, μ_A^L and μ_A^U are fuzzy Z-Subalgebras of a Z-algebra X.

Theorem 7.1.4: Let $A_1 = \{(x, \widetilde{\mu}_{A_1}(x)) | x \in X\}$ and $A_2 = \{(x, \widetilde{\mu}_{A_2}(x)) | x \in X\}$ are interval-valued fuzzy Z-Subalgebras of a Z-algebra X. Then $A_1 \cap A_2 = \{(x, \widetilde{\mu}_{A_1 \cap A_2}(x)) | x \in X\}$ is an interval-valued fuzzy Z-Subalgebra of X.

Proof: Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2.

$$\widetilde{\mu}_{A_1 \cap A_2}(x * y) = r\min\{\widetilde{\mu}_{A_1}(x * y), \widetilde{\mu}_{A_2}(x * y)\}$$

$$= r\min\{[\mu_{A_1}^L(x * y), \mu_{A_1}^U(x * y)], [\mu_{A_2}^L(x * y), \mu_{A_2}^U(x * y)]\}$$

$$= [\min\{\mu_{A_1}^L(x * y), \mu_{A_2}^L(x * y)\}, \min\{\mu_{A_1}^U(x * y), \mu_{A_2}^U(x * y)\}]$$

$$\geq [\min(\min\{\mu_{A_1}^L(x), \mu_{A_1}^L(y)\}, \min\{\mu_{A_2}^L(x), \mu_{A_2}^L(y)\}), \min(\min\{\mu_{A_1}^U(x), \mu_{A_1}^U(y)\}, \min\{\mu_{A_2}^U(x), \mu_{A_2}^U(y)\})]$$

$= [\min(\min\{\mu^L_{A_1}(x), \mu^L_{A_2}(x)\}, \min\{\mu^L_{A_1}(y), \mu^L_{A_2}(y)\}), \min(\min\{\mu^U_{A_1}(x), \mu^U_{A_2}(x)\}, \min\{\mu^U_{A_1}(y), \mu^U_{A_2}(y)\})]$

$= [\min\{\mu^L_{A_1\cap A_2}(x), \mu^L_{A_1\cap A_2}(y)\}, \min\{\mu^U_{A_1\cap A_2}(x), \mu^U_{A_1\cap A_2}(y)\}]$

$= r\min\{[\mu^L_{A_1\cap A_2}(x), \mu^U_{A_1\cap A_2}(x)], [\mu^L_{A_1\cap A_2}(y), \mu^U_{A_1\cap A_2}(y)]\}$

$= r\min\{\tilde{\mu}_{A_1\cap A_2}(x), \tilde{\mu}_{A_1\cap A_2}(y)\}$

Therefore, $A_1 \cap A_2$ is an interval-valued fuzzy Z-Subalgebra of a Z-algebra X.

Corollary 7.1.5: Let $\{A_i = \{(x, \tilde{\mu}_{A_i}(x)) \mid x \in X\} \mid i \in \Omega\}$ be a family of interval-valued fuzzy Z-Subalgebras of a Z-algebra X. Then $\bigcap_{i\in\Omega} A_i$ is also an interval-valued fuzzy Z-Subalgebra of X.

Theorem 7.1.6: Let $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ be an interval-valued fuzzy set in a Z-algebra X. Then A is an interval-valued fuzzy Z-Subalgebra of X if and only if the nonempty set $U(\tilde{\mu}_A;[s_1,s_2]) = \{x \in X \mid \tilde{\mu}_A(x) \geq [s_1,s_2]\}$ is a Z-Subalgebra of X for every $[s_1,s_2] \in D[0,1]$.

Proof: Assume that $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ is an interval-valued fuzzy Z-Subalgebra of a Z-algebra X and let $[s_1,s_2] \in D[0,1]$ be such that $x, y \in U(\tilde{\mu}_A;[s_1,s_2])$.

This implies $\tilde{\mu}_A(x) \geq [s_1,s_2]$ and $\tilde{\mu}_A(y) \geq [s_1,s_2]$.

Then $\tilde{\mu}_A(x * y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \geq r\min\{[s_1,s_2],[s_1,s_2]\} = [\min\{s_1,s_1\}, \min\{s_2,s_2\}]$

$= [s_1,s_2]$

and so $x * y \in U(\tilde{\mu}_A;[s_1,s_2])$.

Thus $U(\tilde{\mu}_A;[s_1,s_2])$ is a Z-Subalgebra of a Z-algebra X.

Conversely, assume that $U(\tilde{\mu}_A;[s_1,s_2]) \neq \phi$ is a Z-Subalgebra of a Z-algebra X for every $[s_1,s_2] \in D[0,1]$.

Suppose there exists $x_0, y_0 \in X$ such that

$$\tilde{\mu}_A(x_0 * y_0) < r\min\{\tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0)\}. \quad (1)$$

Let $\tilde{\mu}_A(x_0) = [\gamma_1,\gamma_2]$, $\tilde{\mu}_A(y_0) = [\gamma_3,\gamma_4]$ and $\tilde{\mu}_A(x_0 * y_0) = [s_1,s_2]$.

Then (1) becomes, $[s_1,s_2] < r\min\{[\gamma_1,\gamma_2],[\gamma_3,\gamma_4]\} = [\min\{\gamma_1,\gamma_3\}, \min\{\gamma_2,\gamma_4\}]$

so $s_1 < \min\{\gamma_1,\gamma_3\}$ and $s_2 < \min\{\gamma_2,\gamma_4\}$.

If we take $[\lambda_1,\lambda_2]=\frac{1}{2}[\tilde{\mu}_A(x_0*y_0)+r\min\{\tilde{\mu}_A(x_0),\tilde{\mu}_A(y_0)\}]$

then $[\lambda_1,\lambda_2]=\frac{1}{2}[[s_1,s_2]+r\min\{[\gamma_1,\gamma_2],[\gamma_3,\gamma_4]\}]=\frac{1}{2}[[s_1,s_2]+[\min\{\gamma_1,\gamma_3\},\min\{\gamma_2,\gamma_4\}]]$

$$=\left[\frac{1}{2}(s_1+\min\{\gamma_1,\gamma_3\}),\frac{1}{2}(s_2+\min\{\gamma_2,\gamma_4\})\right]$$

Therefore, $\min\{\gamma_1,\gamma_3\}>\lambda_1=\frac{1}{2}(s_1+\min\{\gamma_1,\gamma_3\})>s_1$

and $\min\{\gamma_2,\gamma_4\}>\lambda_2=\frac{1}{2}(s_2+\min\{\gamma_2,\gamma_4\})>s_2$

Hence, $[\min\{\gamma_1,\gamma_3\},\min\{\gamma_2,\gamma_4\}]>[\lambda_1,\lambda_2]>[s_1,s_2]=\tilde{\mu}_A(x_0*y_0)$

Therefore $x_0*y_0\notin U(\tilde{\mu}_A;[\lambda_1,\lambda_2])$. But,

$\tilde{\mu}_A(x_0)=[\gamma_1,\gamma_2]\geq[\min\{\gamma_1,\gamma_3\},\min\{\gamma_2,\gamma_4\}]>[\lambda_1,\lambda_2]$

$\tilde{\mu}_A(y_0)=[\gamma_3,\gamma_4]\geq[\min\{\gamma_1,\gamma_3\},\min\{\gamma_2,\gamma_4\}]>[\lambda_1,\lambda_2]$

$\Rightarrow\ x_0,y_0\in U(\tilde{\mu}_A;[\lambda_1,\lambda_2])$.

It contradicts that $U(\tilde{\mu}_A;[\lambda_1,\lambda_2])$ is a Z-Subalgebra of a Z-algebra X.

Hence $\tilde{\mu}_A(x*y)\geq r\min\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$, for all $x,y\in X$.

Note: We call $U(\tilde{\mu}_A;[s_1,s_2])$ as the interval-valued $[s_1,s_2]$-level Z-Subalgebras of A.

Theorem 7.1.7: Every Z-Subalgebra of a Z-algebra X can be realized as an interval-valued $[s_1,s_2]$-level Z-Subalgebra of an interval-valued fuzzy Z-Subalgebra of X.

Proof: Let Y be a Z-Subalgebra of a Z-algebra X and $A=\{(x,\tilde{\mu}_A(x))|x\in X\}$ be an interval-valued fuzzy set on X defined by

$$\tilde{\mu}_A(x)=\begin{cases}[s_1,s_2] & \text{if } x\in Y\\ [0,0] & \text{otherwise}\end{cases}$$

where $s_1,s_2\in(0,1]$ with $s_1<s_2$ .It is clear that $U(\tilde{\mu}_A;[s_1,s_2])=Y$.

Let $x,y\in X$. We consider the following cases.

Case (1): If $x,y\in Y$ then $x*y\in Y$.

Therefore, $\tilde{\mu}_A(x * y) = [s_1, s_2] = r\min\{[s_1,s_2],[s_1,s_2]\} = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

Case (2): If $x, y \notin Y$ then $\tilde{\mu}_A(x) = [0,0] = \tilde{\mu}_A(y)$ and so

$\tilde{\mu}_A(x * y) \geq [0,0] = r\min\{[0,0],[0,0]\} = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

Case (3): If $x \in Y$ and $y \notin Y$ then $\tilde{\mu}_A(x) = [s_1, s_2]$ and $\tilde{\mu}_A(y) = [0,0]$.

Thus, $\tilde{\mu}_A(x * y) \geq [0,0] = r\min\{[s_1,s_2],[0,0]\} = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

Case (4): If $y \in Y$ and $x \notin Y$ then $\tilde{\mu}_A(x) = [0,0]$ and $\tilde{\mu}_A(y) = [s_1, s_2]$.

Thus, $\tilde{\mu}_A(x * y) \geq [0,0] = r\min\{[0,0],[s_1,s_2]\} = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

Therefore $\tilde{\mu}_A$ is an interval-valued fuzzy Z-Subalgebra of a Z-algebra X.

Theorem 7.1.8: Let Y be a subset of a Z-algebra X and $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ be an interval-valued fuzzy set on X which is given in the proof of Theorem 7.1.7. If A is an interval-valued fuzzy Z-Subalgebra of a Z-algebra X then Y is a Z-Subalgebra of a Z-algebra X.

Proof: Let $x, y \in Y$. Then, $\tilde{\mu}_A(x) = [s_1, s_2] = \tilde{\mu}_A(y)$.

$\tilde{\mu}_A(x * y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = r\min\{[s_1,s_2],[s_1,s_2]\} = [\min\{s_1,s_1\},\{s_2,s_2\}] = [s_1, s_2]$

Thus $x * y \in Y$.

Hence Y is a Z-Subalgebra of a Z-algebra X.

Theorem 7.1.9: Let h be a Z-homomorphism from a Z-algebra $(X,*,0)$ onto a Z-algebra $(Y,*',0')$. If $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ is an interval-valued fuzzy Z-Subalgebra of X with the rsup property, then the image h(A) is an interval-valued fuzzy Z-Subalgebra of Y.

Proof: Assume that $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ where $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ is an interval-valued fuzzy Z-Subalgebra of a Z-algebra X, then by Theorem 7.1.3, μ_A^L and μ_A^U are fuzzy Z-Subalgebras of X. By Theorem 2.1.21, $\mu_{h(A)}^L$ and $\mu_{h(A)}^U$ are fuzzy Z-Subalgebras of Y. By lemma 1.1.20 and Theorem 7.1.3, we can conclude that $h(A) = [\mu_{h(A)}^L, \mu_{h(A)}^U]$ is an interval-valued fuzzy Z-Subalgebra of Y.

Proposition 7.1.10: Let h be a Z-homomorphism from a Z-algebra $(X,*,0)$ into a Z-algebra $(Y,*',0')$ and $B = \{(y, \tilde{\mu}_B(y)) | y \in Y\}$ be an interval-valued fuzzy Z-Subalgebra of Y. Then the inverse image $h^{-1}(B)$ of B is an interval-valued fuzzy Z-Subalgebra of X.

Proof: Since $B=[\mu_B^L,\mu_B^U]$ is an interval-valued fuzzy Z-Subalgebra of a Z-algebra Y, by Theorem 7.1.3, we get that μ_B^L and μ_B^U are fuzzy Z-Subalgebras of Y.

By Theorem 2.1.22, $\mu^L_{h^{-1}(B)}$ and $\mu^U_{h^{-1}(B)}$ are fuzzy Z-Subalgebras of a Z-algebra X. Now, by lemma 1.1.20 and Theorem 7.1.3 we can conclude that $h^{-1}(B)=[\mu^L_{h^{-1}(B)},\mu^U_{h^{-1}(B)}]$ is an interval-valued fuzzy Z-Subalgebra of a Z-algebra Y.

Theorem 7.1.11: If $A=\{(x,\tilde{\mu}_A(x))|x\in X\}$ and $B=\{(x,\tilde{\mu}_B(x))|x\in X\}$ are interval-valued fuzzy Z-Subalgebras of a Z-algebra X then $A\times B=\{((x_1,x_2),\tilde{\mu}_{A\times B}(x_1,x_2))|(x_1,x_2)\in X\times X\}$ is also an interval-valued fuzzy Z-Subalgebra of $X\times X$.

Proof: For any (x_1,x_2), $(y_1,y_2)\in X\times X$, we have

$$
\begin{aligned}
\tilde{\mu}_{A\times B}((x_1,x_2)*(y_1,y_2)) &= \tilde{\mu}_{A\times B}(x_1*y_1,\ x_2*y_2)\\
&= \text{rmin}\{\tilde{\mu}_A(x_1*y_1),\ \tilde{\mu}_B(x_2*y_2)\}\\
&\geq \text{rmin}\{\text{rmin}\{\tilde{\mu}_A(x_1),\ \tilde{\mu}_A(y_1)\},\ \text{rmin}\{\tilde{\mu}_B(x_2),\ \tilde{\mu}_B(y_2)\}\}\\
&= \text{rmin}\{\text{rmin}\{\tilde{\mu}_A(x_1),\ \tilde{\mu}_B(x_2)\},\ \text{rmin}\{\tilde{\mu}_A(y_1),\ \tilde{\mu}_B(y_2)\}\}\\
&= \text{rmin}\{\tilde{\mu}_{A\times B}(x_1,x_2),\ \tilde{\mu}_{A\times B}(y_1,y_2)\}
\end{aligned}
$$

Hence $A\times B$ is also an interval-valued fuzzy Z-Subalgebra of $X\times X$.

7.2 Interval-Valued Fuzzy Z-Ideals in Z-algebras

In this section, we introduce the notion of an interval-valued fuzzy Z-ideals of Z-algebras. Also we have proved some interesting results.

Definition 7.2.1: Let $(X,*,0)$ be a Z-algebra. An interval-valued fuzzy set $A=\{(x,\tilde{\mu}_A(x))|x\in X\}$ in X is said to be an **interval-valued fuzzy Z-ideal** of a Z-algebra X if:

(i) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$

(ii) $\tilde{\mu}_A(x) \geq \text{rmin}\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$, for all $x,y\in X$.

Example 7.2.2: Consider a Z-algebra X={0,1,2,3} with the following Cayley table :

*	0	1	2	3
0	0	1	2	3
1	0	1	1	1
2	0	1	2	3
3	0	1	3	3

Define an interval-valued fuzzy set A by $\tilde{\mu}_A(x)=[0.6,0.8]$ for all $x\in X$. Then, A is an interval-valued fuzzy Z-ideal of a Z-algebra X.

Theorem 7.2.3: Let $A=\{(x,\tilde{\mu}_A(x))|x\in X\}$ be an interval-valued fuzzy set in a Z-algebra X. Then A is an interval-valued fuzzy Z-ideal of a Z-algebra X if and only if the nonempty set $U(\tilde{\mu}_A;[\gamma_1,\gamma_2])$ is an Z-ideal of a Z-algebra X for all $[\gamma_1,\gamma_2]\in D[0,1]$.

Proof: Assume that $A=\{(x,\tilde{\mu}_A(x))|x\in X\}$ is an interval-valued fuzzy Z-ideal of a Z-algebra X and let $[\gamma_1,\gamma_2]\in D[0,1]$ be such that $x\in U(\tilde{\mu}_A;[\gamma_1,\gamma_2])$. Then $\tilde{\mu}_A(0)\geq\tilde{\mu}_A(x)\geq[\gamma_1,\gamma_2]$. That is, $0\in U(\tilde{\mu}_A;[\gamma_1,\gamma_2])$.

Let $x,y\in X$ be such that $x*y\in U(\tilde{\mu}_A;[\gamma_1,\gamma_2])$ and $y\in U(\tilde{\mu}_A;[\gamma_1,\gamma_2])$.

Then $\tilde{\mu}_A(x*y)\geq[\gamma_1,\gamma_2]$ and $\tilde{\mu}_A(y)\geq[\gamma_1,\gamma_2]$.

Now, $\tilde{\mu}_A(x)\geq \text{r}\min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}\geq \text{r}\min\{[\gamma_1,\gamma_2],[\gamma_1,\gamma_2]\}=[\gamma_1,\gamma_2]$

so that $x \in U(\tilde{\mu}_A;[\gamma_1,\gamma_2])$.

Hence $U(\tilde{\mu}_A;[\gamma_1,\gamma_2])$ is an Z-ideal of a Z-algebra X.

Conversely, suppose that $U(\tilde{\mu}_A;[\gamma_1,\gamma_2]) \neq \phi$ is an Z-ideal of a Z-algebra X for all $[\gamma_1,\gamma_2] \in D[0,1]$.

Suppose that there exists $a \in X$ such that $\tilde{\mu}_A(0) < \tilde{\mu}_A(a)$. That is,

$[\mu_A^L(0), \mu_A^U(0)] < [\mu_A^L(a), \mu_A^U(a)]$. Then $\mu_A^L(0) < \mu_A^L(a)$ and $\mu_A^U(0) < \mu_A^U(a)$.

If we take $[\delta_1,\delta_2] = \frac{1}{2}(\tilde{\mu}_A(0) + \tilde{\mu}_A(a))$, then $[\delta_1,\delta_2] = \left[\frac{1}{2}(\mu_A^L(0)+\mu_A^L(a)), \frac{1}{2}(\mu_A^U(0)+\mu_A^U(a))\right]$.

Hence $\mu_A^L(0) < \delta_1 < \mu_A^L(a)$ and $\mu_A^U(0) < \delta_2 < \mu_A^U(a)$, which imply that

$\tilde{\mu}_A(0) = [\mu_A^L(0), \mu_A^U(0)] < [\delta_1,\delta_2] < [\mu_A^L(a), \mu_A^U(a)] = \tilde{\mu}_A(a)$.

This shows that $0 \notin U(\tilde{\mu}_A;[\delta_1,\delta_2])$, which leads to a contradiction.

Therefore $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, for all $x \in X$.

Suppose there exists $a, b \in X$ such that $\tilde{\mu}_A(a) < r\min\{\tilde{\mu}_A(a*b), \tilde{\mu}_A(b)\}$

where $\tilde{\mu}_A(a) = [\mu_A^L(a), \mu_A^U(a)]$, $\tilde{\mu}_A(a*b) = [\mu_A^L(a*b), \mu_A^U(a*b)]$ and $\tilde{\mu}_A(b) = [\mu_A^L(b), \mu_A^U(b)]$.

Put $[\varphi_1,\varphi_2] = \frac{1}{2}(\tilde{\mu}_A(a) + r\min\{\tilde{\mu}_A(a*b), \tilde{\mu}_A(b)\})$.

Then, $[\varphi_1,\varphi_2] = \left[\frac{1}{2}\left(\mu_A^L(a) + \min\{\mu_A^L(a*b), \mu_A^L(b)\}\right), \frac{1}{2}\left(\mu_A^U(a) + \min\{\mu_A^U(a*b), \mu_A^U(b)\}\right)\right]$

and so $\mu_A^L(a) < \varphi_1 < \min\{\mu_A^L(a*b), \mu_A^L(b)\}$ and $\mu_A^U(a) < \varphi_2 < \min\{\mu_A^U(a*b), \mu_A^U(b)\}$.

It follows that $\tilde{\mu}_A(a) = [\mu_A^L(a), \mu_A^U(a)] < [\varphi_1,\varphi_2] < [\min\{\mu_A^L(a*b), \mu_A^L(b)\}, \min\{\mu_A^U(a*b), \mu_A^U(b)\}]$

so that $a \notin U(\tilde{\mu}_A;[\varphi_1,\varphi_2])$.

But $\tilde{\mu}_A(a*b) = [\mu_A^L(a*b), \mu_A^U(a*b)] > [\varphi_1,\varphi_2]$

and $\tilde{\mu}_A(b) = [\mu_A^L(b), \mu_A^U(b)] > [\varphi_1,\varphi_2]$.

That is, $a*b \in U(\tilde{\mu}_A;[\varphi_1,\varphi_2])$ and $b \in U(\tilde{\mu}_A;[\varphi_1,\varphi_2])$.

This leads to a contradiction.

Consequently, $\tilde{\mu}_A(x) \geq r\min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$, for all $x, y \in X$.

Theorem 7.2.4: An interval-valued fuzzy set $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ where $\tilde{\mu}_A = [\mu_A^L, \mu_A^U]$ in a Z-algebra X is an interval-valued fuzzy Z-ideal of X if and only if μ_A^L and μ_A^U are fuzzy Z-ideals of X.

Proof: Assume that $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ where $\tilde{\mu}_A = [\mu_A^L, \mu_A^U]$ is an interval-valued fuzzy Z-ideal of a Z-algebra X. For any $x \in X$, we have

$[\mu_A^L(0), \mu_A^U(0)] = \tilde{\mu}_A(0) \geq \tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$.

It follows that $\mu_A^L(0) \geq \mu_A^L(x)$ and $\mu_A^U(0) \geq \mu_A^U(x)$.

Let $x, y \in X$. Then, $[\mu_A^L(x), \mu_A^U(x)] = \tilde{\mu}_A(x) \geq r\min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$

$$= r\min\{[\mu_A^L(x * y), \mu_A^U(x * y)], [\mu_A^L(y), \mu_A^U(y)]\}$$

$$= [\min\{\mu_A^L(x * y), \mu_A^L(y)\}, \min\{\mu_A^U(x * y), \mu_A^U(y)\}]$$

and so $\mu_A^L(x) \geq \min\{\mu_A^L(x * y), \mu_A^L(y)\}$ and $\mu_A^U(x) \geq \min\{\mu_A^U(x * y), \mu_A^U(y)\}$.

Hence μ_A^L and μ_A^U are fuzzy Z-ideals of a Z-algebra X.

Conversely, suppose that μ_A^L and μ_A^U are fuzzy Z-ideals of a Z-algebra X. Then the nonempty upper level subsets $U(\mu_A^L; \alpha_1)$ and $U(\mu_A^U; \alpha_2)$ are Z-ideals of a Z-algebra X where $\alpha_1, \alpha_2 \in [0,1]$ and $\alpha_1 \leq \alpha_2$. Noticing that $U(\tilde{\mu}_A; [\alpha_1, \alpha_2]) = U(\mu_A^L; \alpha_1) \cap U(\mu_A^U; \alpha_2)$ which is an Z-ideal of a Z-algebra X, and applying Theorem 7.2.3 we know that $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ is an interval-valued fuzzy Z-ideal of a Z-algebra X.

Theorem 7.2.5: Let h be a Z-homomorphism from a Z-algebra $(X, *, 0)$ onto a Z-algebra $(Y, *', 0')$. If A is an interval-valued fuzzy Z-ideal of X with rsup property, then the image of A denoted by h(A) is an interval-valued fuzzy Z-ideal of Y.

Proof: Given $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ where $\tilde{\mu}_A = [\mu_A^L, \mu_A^U]$ is an interval-valued fuzzy Z-ideal of a Z-algebra X. Then μ_A^L and μ_A^U are fuzzy Z-ideals of X by Theorem 7.2.4.

By Theorem 2.2.8, $\mu_{h(A)}^L$ and $\mu_{h(A)}^U$ are fuzzy Z-ideals of a Z-algebra Y. By Lemma 1.1.20 and Theorem 7.2.4 $h(A) = [\mu_{h(A)}^L, \mu_{h(A)}^U]$ is an interval-valued fuzzy Z-ideal of a Z-algebra Y.

Theorem 7.2.6: Let $h:(X,*,0)\to(Y,*',0')$ be a Z-homomorphism of Z-algebras. If $B=\{(y,\tilde{\mu}_B(y))|y\in Y\}$ is an interval-valued fuzzy Z-ideal of Y, then $h^{-1}(B)$ is an interval-valued fuzzy Z-ideal of X.

Proof: Since $B=[\mu_B^L,\mu_B^U]$ is an interval-valued fuzzy Z-ideal of Y, it follows from Theorem 7.2.4, that is μ_B^L and μ_B^U are fuzzy Z-ideals of Y. By Theorem 2.2.9, $\mu^L_{h^{-1}(B)}$ and $\mu^U_{h^{-1}(B)}$ are fuzzy Z-ideals of X. By Lemma 1.1.20 and Theorem 7.2.4, we conclude that $h^{-1}(B)=[\mu^L_{h^{-1}(B)},\mu^U_{h^{-1}(B)}]$ is an interval-valued fuzzy Z-ideal of Y.

Theorem 7.2.7: Let $h:(X,*,0)\to(Y,*',0')$ be an Z-epimorphism of Z-algebras. Let $B=\{(y,\tilde{\mu}_B(y))|y\in Y\}$ be an interval-valued fuzzy set of Y. If $h^{-1}(B)$ is an interval-valued fuzzy Z-ideal of X then B is an interval-valued fuzzy Z-ideal of Y.

Proof: Let y $\in$ Y, there exists x $\in$ X such that h (x) = y. Then

$$\tilde{\mu}_B(y)=\tilde{\mu}_B(h(x))=\tilde{\mu}_{h^{-1}(B)}(x)\le\tilde{\mu}_{h^{-1}(B)}(0)=\tilde{\mu}_B(h(0))=\tilde{\mu}_B(0')$$

Let x, y $\in$ Y. Then there exists a, b $\in$ X such that h(a) = x and h (b) = y. It follows that

$$\begin{aligned}\tilde{\mu}_B(x)=\tilde{\mu}_B(h(a))=\tilde{\mu}_{h^{-1}(B)}(a)&\ge r\min\{\tilde{\mu}_{h^{-1}(B)}(a*b),\tilde{\mu}_{h^{-1}(B)}(b)\}\\&=r\min\{\tilde{\mu}_B(h(a*b)),\tilde{\mu}_B(h(b))\}\\&=r\min\{\tilde{\mu}_B(h(a)*'h(b)),\tilde{\mu}_B(h(b))\}\\&=r\min\{\tilde{\mu}_B(x*'y),\tilde{\mu}_B(y)\}\end{aligned}$$

Hence B is an interval-valued fuzzy Z-ideal of a Z-algebra Y.

Theorem 7.2.8: If $A=\{(x,\tilde{\mu}_A(x))|x\in X\}$ and $B=\{(x,\tilde{\mu}_B(x))|x\in X\}$ are interval-valued fuzzy Z-ideals in a Z-algebra X then $A\times B$ with membership function $\tilde{\mu}_{A\times B}$ is an interval-valued fuzzy Z-ideal in $X\times X$.

Proof: Let $(x_1,x_2)\in X\times X$,

$$\tilde{\mu}_{A\times B}(0,0)=r\min\{\tilde{\mu}_A(0),\tilde{\mu}_B(0)\}\ge r\min\{\tilde{\mu}_A(x_1),\tilde{\mu}_B(x_2)\}=\tilde{\mu}_{A\times B}(x_1,x_2)$$

Let $(x_1,x_2),(y_1,y_2)\in X\times X$. Then,

$$\begin{aligned}\tilde{\mu}_{A\times B}(x_1,x_2)&=r\min\{\tilde{\mu}_A(x_1),\tilde{\mu}_B(x_2)\}\\&\ge r\min\{r\min\{\tilde{\mu}_A(x_1*y_1),\tilde{\mu}_A(y_1)\},r\min\{\tilde{\mu}_B(x_2*y_2),\tilde{\mu}_B(y_2)\}\}\end{aligned}$$

$= r\min\{r\min\{\tilde{\mu}_A(x_1 * y_1), \tilde{\mu}_B(x_2 * y_2)\}, r\min\{\tilde{\mu}_A(y_1), \tilde{\mu}_B(y_2)\}\}$

$= r\min\{\tilde{\mu}_{A\times B}((x_1 * y_1), (x_2 * y_2)), \tilde{\mu}_{A\times B}(y_1, y_2)\}$

$= r\min\{\tilde{\mu}_{A\times B}((x_1, x_2) * (y_1, y_2)), \tilde{\mu}_{A\times B}(y_1, y_2)\}$

Hence $A \times B$ is an interval-valued fuzzy Z-ideal in $X \times X$.

Theorem 7.2.9: Let $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ and $B = \{(x, \tilde{\mu}_B(x)) | x \in X\}$ be an interval-valued fuzzy sets in a Z-algebra X such that $A \times B$ is an interval-valued fuzzy Z-ideal of $X \times X$. Then,

(i) Either $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ (or) $\tilde{\mu}_B(0) \geq \tilde{\mu}_B(x)$ for all $x \in X$.

(ii) If $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ for all $x \in X$, then either $\tilde{\mu}_B(0) \geq \tilde{\mu}_A(x)$ (or) $\tilde{\mu}_B(0) \geq \tilde{\mu}_B(x)$

(iii) If $\tilde{\mu}_B(0) \geq \tilde{\mu}_B(x)$ for all $x \in X$ then either $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ (or) $\tilde{\mu}_A(0) \geq \tilde{\mu}_B(x)$

Proof: (i) If $\tilde{\mu}_A(0) < \tilde{\mu}_A(x_1)$ and $\tilde{\mu}_B(0) < \tilde{\mu}_B(x_2)$ for some $x \in X$.

Then, $\tilde{\mu}_{A\times B}(x_1, x_2) = r\min\{\tilde{\mu}_A(x_1), \tilde{\mu}_B(x_2)\} > r\min\{\tilde{\mu}_A(0), \tilde{\mu}_B(0)\} = \tilde{\mu}_{A\times B}(0,0)$,

which is a contradiction.

Hence, either $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ (or) $\tilde{\mu}_B(0) \geq \tilde{\mu}_B(x)$ for all $x \in X$.

(ii) Let $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ for all $x \in X$.

Assume that there exists $x_1, x_2 \in X$ such that $\tilde{\mu}_B(0) < \tilde{\mu}_A(x_1)$ and $\tilde{\mu}_B(0) < \tilde{\mu}_B(x_2)$.

Then, $\tilde{\mu}_{A\times B}(0,0) = r\min\{\tilde{\mu}_A(0), \tilde{\mu}_B(0)\} = \tilde{\mu}_B(0)$

$\tilde{\mu}_{A\times B}(x_1, x_2) = r\min\{\tilde{\mu}_A(x_1), \tilde{\mu}_B(x_2)\} > \tilde{\mu}_B(0) = \tilde{\mu}_{A\times B}(0,0)$

$\Rightarrow \tilde{\mu}_{A\times B}(x_1, x_2) > \tilde{\mu}_{A\times B}(0,0)$, which is a contradiction.

Hence either $\tilde{\mu}_B(0) \geq \tilde{\mu}_A(x)$ (or) $\tilde{\mu}_B(0) \geq \tilde{\mu}_B(x)$

(iii) By interchanging the roles of A and B in part (ii) we obtain (iii), for all $x \in X$.

Hence the proof.

Theorem 7.2.10: Let $A = \{(x, \tilde{\mu}_A(x)) | x \in X\}$ and $B = \{(x, \tilde{\mu}_B(x)) | x \in X\}$ be an interval-valued fuzzy sets in a Z-algebra X and $A \times B$ is an interval-valued fuzzy Z-ideal of $X \times X$ then either A or B is an interval-valued fuzzy Z-ideal of X.

Proof : By Theorem 7.2.9(i), we can assume that $\tilde{\mu}_B(0) \geq \tilde{\mu}_B(x)$ for all $x \in X$. Then, by Theorem 7.2.9(iii), either $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ (or) $\tilde{\mu}_A(0) \geq \tilde{\mu}_B(x)$.

Let $\widetilde{\mu}_A(0) \geq \widetilde{\mu}_B(x)$ for any $x \in X$, then

$$\widetilde{\mu}_B(x) = r\min\{\widetilde{\mu}_A(0), \widetilde{\mu}_B(x)\} = \widetilde{\mu}_{A\times B}(0, x)$$

$$\geq r\min\{\widetilde{\mu}_{A\times B}((0, x) * (0, y)), \widetilde{\mu}_{A\times B}(0, y)\}$$

$$= r\min\{\widetilde{\mu}_{A\times B}((0 * 0), (x * y)), \widetilde{\mu}_{A\times B}(0, y)\}$$

$$= r\min\{\widetilde{\mu}_{A\times B}(0, x * y), \widetilde{\mu}_{A\times B}(0, y)\}$$

$$= r\min\{r\min\{\widetilde{\mu}_A(0), \widetilde{\mu}_B(x * y)\}, r\min\{\widetilde{\mu}_A(0), \widetilde{\mu}_B(y)\}\}$$

$$= r\min\{\widetilde{\mu}_B(x * y), \widetilde{\mu}_B(y)\}$$

Therefore, $\widetilde{\mu}_B(x) \geq r\min\{\widetilde{\mu}_B(x * y), \widetilde{\mu}_B(y)\}$, for all $x, y \in X$.

Hence B is an interval-valued fuzzy Z-ideal of a Z-algebra X.

By Theorem 7.2.9 (i), assume that $\widetilde{\mu}_A(0) \geq \widetilde{\mu}_A(x)$, for all $x \in X$.

By Theorem 7.2.9 (ii), assume that $\widetilde{\mu}_B(0) \geq \widetilde{\mu}_A(x)$, for any $x \in X$.

Then A is an interval-valued fuzzy Z-ideal of a Z-algebra X.

Therefore, either A or B is an interval-valued fuzzy Z-ideal of a Z-algebra X.

This completes the proof.