A STUDY OF d-ALGEBRAS AND FUZZY d-ALGEBRAS

By

SHRUTHI GOPAL
(10PM19)

A DISSERTATION SUBMITTED TO THE
AVINASHILINGAM DEEMED UNIVERSITY FOR WOMEN
COIMBATORE – 641 043

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF

MASTER OF SCIENCE IN MATHEMATICS

APRIL 2012

A STUDY OF d-ALGEBRAS AND FUZZY d-ALGEBRAS

BY

SHRUTHI GOPAL
(10PM19)

A DISSERTATION SUBMITTED TO THE
AVINASHILINGAM DEEMED UNIVERSITY FOR WOMEN
COIMBATORE – 641 043

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE
DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

APRIL 2012

CERTIFIED AS BONAFIED RESEARCH WORK

Parvathi

SIGNATURE OF THE
HEAD OF THE DEPARTMENT

P. Jeyalakshmi

SIGNATURE OF
THE GUIDE

ACKNOWLEDGEMENT

ACKNOWLEDGEMENT

First and foremost, the investigator is extremely thankful to the **LORD ALMIGHTY** for his graces and blessings showered on her.

The author takes immense pleasure in thanking **Thiru.T.S.K.MEENAKSHI SUNDARAM**, M.A., M.Phil. Chancellor, Avinashilingam Deemed University for women, Coimbatore, for providing the conducive infrastructure for the conduct of the research study.

The author expresses her sincere thanks to **Thiru.T.K.SHANMUGANANDAM**, B.A., B.L., Former Chancellor, Avinashilingam Deemed University for women, Coimbatore, for giving an opportunity to do research work.

The author would like to thank **Dr.(Mrs.) SHEELA RAMACHANDRAN**, M.Sc., P.G.Dip., Ph.D.(Avinashlingam) Vice Chancellor, Avinashilingam Deemed University for women, Coimbatore, for the support extended by her throughout the study.

The author records her deep sense of gratitude and indebtedness to, **Hony.Col.Dr.(Tmt.)SAROJA PRABHAKARAN,** M.A., Dip.Ed., (Madras), Ph.D.(Mother Teresa), Former Vice Chancellor, Avinashilingam Deemed University for women, Coimbatore, for providing adequate help towards the completion of the study.

The author extends her heart felt thanks to **Dr.(Tmt.) GOWRI RAMAKRISHNAN**, M.Sc. (Madras), M.Phil., Ph.D. (Avinashilingam), Registrar, Avinashilingam Deemed University for women, Coimbatore, for the encouragement given by her.

The author is immensely pleased to express her deep sense of gratitude to **Dr.(Tmt.) R.PARVATHAM**, M.Sc., Dip. Ed., M.Phil. (Madras), Ph.D. (Avinashilingam), Dean Faculty of Science, Avinashilingam Deemed University for women, Coimbatore, for all the encouragement.

The investigator would like to thank **Dr.(Tmt.) A.PARVATHI**, M.Sc., Dip.Ed., M.Phil. (Madras), Ph.D. (Bharathiar), Professor and Head of the Department of Mathematics, Avinashilingam Deemed University for women, Coimbatore, for her excellent advice and valuable guidance.

The author is deeply indebted to her thesis advisor **Dr.(Tmt.) P.JEYALAKSHMI**, MSc., M.Phil. (Annamalai), Ph.D. (Avinashilingam), Professor, Department of Mathematics, Avinashilingam Deemed University for women, Coimbatore, for the invaluable help, guidance, and persistent efforts.

The investigator would like to express her sincere thanks to all the **STAFF MEMBERS OF DEPARTMENT OF MATHEMATICS** who were responsible for the good finish of this dissertation.

Words fail to express her deep indebtedness to her **LOVING PARENTS, FRIENDS** and **ALL WELL WISHERS** for being the motivating forces behind this dissertation and the providing moral support and encouragement in carrying out work.

CONTENTS

CONTENTS PAGE NO

INTRODUCTION

INTRODUCTION

Y. Imai and K. Iseki [13] and K. Iseki [14] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

J. Neggers and H. S. Kim [33] introduced the notion of d-algebras which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK -algebras. J. Neggers, Y. B. Jun and H. S. Kim [34] discussed ideal theory in d-algebras and introduced the notions of d-subalgebra, d-ideal, $d^{\#}$ -ideal and d^{*}-ideal and investigated some relations among them.

L. A. Zadeh [40] introduced the notion of fuzzy sets, and A. Rosenfeld [37] introduced the notion of fuzzy group. Following the idea of fuzzy groups, O. G. Xi [39] introduced the notion of fuzzy BCK-algebras. In [6] M. Akram fuzzified d-algebras.

This thesis is devoted to the study of d-algebras and fuzzy d-algebras.
The following articles are chosen for our discussion:

1) "ON d-ALGEBRAS" by Neggers, J., and Kim, H.S., [34].
2) "ON d-IDEALS IN d-ALGEBRAS" by Neggers, J., Jun, Y.B., and Kim, H.S., [33].
3) "ON d^{*}-SUBALGEBRAS OF d-TRANSITIVE d^{*}-ALGEBRAS" by Lee, Y.C., and Kim, H.S., [26].
4) "COMPANION d-ALGEBRAS" by Allen, P.J., Kim, H.S., and Neggers, J., [7].
5) "SOME CONSTRUCTIONS OF IMPLICATIVE/COMMUTATIVE d-ALGEBRAS" by Ahn, S.S., and Kim, Y.H., [3].
6) "CONSTRUCTION OF MANY d-ALGEBRAS" by Allen, P.J., [8].
7) "ON FUZZY d-ALGEBRAS" by Akram, A., and Dar, K.J., [6].

This thesis is divided into five chapters.

The first chapter deals with the study of d-algebras and d-ideals in d-algebras. In this chapter the notion of d-algebras which is a generalization of BCK-algebras, and several relations between d-algebras and BCK-algebras are discussed. Also relations among d-subalgebras, d-ideals, $d^{\#}$-ideals and d^{*}-ideals in d-algebras are investigated. In this chapter, the following interesting results are discussed.

1) Let $(X;*, 0)$ be a d-transitive edge d-algebra. Then $(X;*, 0)$ is a BCK -algebra.
2) In a d^{*}-algebra, every BCK-ideal is a d-ideal.
3) If $(X; *, 0)$ is a BCK-algebra, then every BCK-ideal of X is a d^{*}-ideal of X.

The chapter 2 deals with Quotient d-algebra and d^{*}-subalgebras of d-transitive d^{*}-algebras. In this chapter the number of d^{*}-subalgebras of order i in a d-transitive d^{*}-algebras which is a generalization of BCK-algebras is estimated.
The following main results are discussed:

1) If $f: X \rightarrow Y$ is a d-morphism from a d-algebra X onto a d-transitive d-algebra Y, then $X/\mathrm{Ker}\, f \cong Y$.
2) Let X be a d-transitive d^{*}-algebra of order n. Then

$$1 \le N(i) \le \binom{n-1}{i-1} \qquad (i=1, 2, 3, \ldots\ldots\ldots, n)$$

where N(i) denotes the number of d^{*}-subalgebras of order i in X.

In chapter 3, a theory of companion d-algebras is developed. In this chapter the following important results are discussed.

1) Let $(X;*, \odot, 0)$ be a companion d-algebra. Assume that $x * 0 = x$ for any $x \in X$.

   (i) X is positive implicative,

   (ii) if $x \leq y$, then $x \odot y = y$,

   (iii) $x \odot x = x$ for any $x, y \in X$.

   Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

2) Let $(X;*, \odot, 0)$ be a companion edge d^*-algebra.

   If $(z * (x \odot y)) * ((z * x) * y) = 0$, then X has a dsu condition:

   $(x * y) * (x \odot y) = 0$.

3) Let $(X;*, \odot, 0)$ be a companion d-algebra. If we define a partial binary relation << by $x << y \Leftrightarrow (x \odot z) * (y \odot z) = 0$ for all $z \in X$, then << is reflexive and anti-symmetric.

In chapter 4, some constructions of implicative/commutative d-algebras which are not BCK-algebras are given. This demonstrates that the notions of implicative/commutative d-algebras are indeed generalizations of the same in BCK-algebras. Also some properties of the constructive function d-algebras on $\mathbb{R}$ determined by constructive function triple (f, g, h) are discussed. In this chapter the following interesting results are discussed.

1) If the φ function d-algebra $(X; *, 0)$ is implicative, then $(x * y) * y = x * y$ for any $x, y \in X$.

2) Let ($\mathbb{C}$; $*$, e) be a constructive function d-algebra on the algebraically closed field $\mathbb{C}$ of complex numbers. If we define $x * y = (x - y)(e - x) + e$, then the solution set of $F(x, y) = x * (x * y) - y * (y * x) = 0$ is

$$\left\{(x, y) \mid y = x \text{ or } \left(x - e + \frac{1}{2}\right)^2 + \left(y - e + \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2\right\}.$$

Chapter 5 deals with the study of fuzzy d-algebras. In this chapter the notion of fuzzy subalgebras and d-ideals in d-algebras are studied and some of their properties are investigated.

1) Any subalgebra of a d-algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X.
2) If $\lambda \times \mu$ is a fuzzy d-ideal of $X \times X$, then λ or μ is a fuzzy d-ideal of X.
3) Let $f : X \rightarrow Y$ be a homomorphism of d-algebras. If μ is a fuzzy d-ideal of Y, then μ^f defined by $\mu^f(x) = \mu(f(x))$ for all $x \in X$, is a fuzzy d-ideal of X.

REVIEW OF

LITERATURE

REVIEW OF LITERATURE

Y. Imai and K. Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([17,15]). BCK-algebras have some connections with other areas: D. Mundici [32] proved that MV -algebras are categorically equivalent to bounded commutative BCK-algebras, and J. Meng [29] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Z. Riecanova [36] showed that extendable commutative BCK-algebras directed upwards are equivalent to generalized MV-effect algebras. G. Georgescu and A. Iorgulescu [11] introduced the notion of pseudo-BCK algebras as an extension of BCK-algebras. X. H. Zhang and W. H. Li [42] established the connections between BCC-algebras, pseudo-BCK algebras, pseudo-BL algebras and weak pseudo-BL algebras (pseudo-MTL algebras). J. Neggers and H. S. Kim introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs [33]. After that some further aspects were studied [26, 35, 34]. J. S. Han et al. [12] defined a variety of special d-algebras, such as strong d-algebras, (weakly) selective d-algebras and others.

The main assertion is that the squared algebra $(X; *, 0)$ of a d-algebra is a d-algebra if and only if the root $(X; *, 0)$ of the squared algebra $(X; *, 0)$ is a strong d-algebra.

L. A. Zadeh [40] introduced the notion of fuzzy sets, and A. Rosenfeld [37] introduced the notion of fuzzy group. Following the idea of fuzzy groups, O. G. Xi [39] introduced the notion of fuzzy BCK-algebras. After that, Y. B. Jun and J. Meng [19] studied fuzzy BCK-algebras. B. Ahmad [1] fuzzified BCI-algebras. In [6] M. Akram fuzzified d-algebras.

Here we present abstracts of some important articles on BCK/BCI/d-algebras and fuzzy d-algebras.

1. An introduction to the theory of BCK-algebras

K.Iseki and S.Tanaka (1978) [15]

In this paper the definition of BCK-algebras and its fundamental properties are studied. Various ideals in BCK-algebras are discussed in detailed manner. Also, the homomorphism properties on BCK-algebras are discussed.

2. Fuzzy commutative ideals in BCK-algebras

Y. B. Jun and E. H. Roh (1994) [20]

In this paper, the authors defined and discussed the fuzzy commutative ideals in BCK-algebras. A fuzzy characteristic commutative ideals is characterized in terms of its level commutative ideals.

3. Fuzzy implicative ideals in BCK – algebras

Mostafa M.Swamy (1997) [31]

The notion of fuzzy implicative ideals of BCK-algebras is discussed. The following result is obtained: for a BCK-algebra X, suppose that μ and γ are fuzzy ideals of X with $\mu < \gamma$ and $\mu(0) = \gamma(0)$, if μ is a fuzzy implicative ideal of X then so is γ.

4. On d-fuzzy functions in d-algebras

J. Neggers, A. Dvurecenskij and H. S. Kim [2000] [35]

In this paper the author introduced the concept of d-fuzzy function which generalized the concept of fuzzy subalgebra to a much larger class of functions in a natural way. In addition they discussed a method of fuzzification of a wide class of algebraic systems onto [0, 1] along with some consequences.

5. On fuzzy topological d-algebras

Y. B. Jun and H. S. Kim (2001) [21]

In this paper the authors introduced the concept of fuzzy topological d-algebras and applied some of Foster's results on homomorphic images and inverse images to fuzzy topological d-algebras.

6. Intutionistic fuzzy d-algebras

Y. B. Jun, H. S. Kim and D. S. Yoo (2006) [22]

The intuitionistic fuzzification of a d-algbera is considered, and related results are investigated. The notion of equivalence relations on the family of all intuitionistic fuzzy d-algebras of a d-algebra is introduced, and then some properties are discussed. The concept of intuitionistic fuzzy topological d-algebras is introduced, and some related results are obtained.

7. Involutive Brouwerian D-Algebras

Mircea Sularia (2006) [30]

The author introduced the structure of Brouwerian D-algebra in order to obtain the algebraic counterpart of a logic of problem solving. A Brouwerian D-algebra is defined by a subdirect product of a couple of algebras associated with a Heyting lattice and a Brouwer lattice. Then in this paper the author introduced the notion of involutive Brouwerian D-algebra having as models direct products between the lattice of open sets and the lattice of closed sets of a topological space. Different properties including basic distributivity equations for complete Brouwerian D-algebras are presented. An extension of the real graded membership space for fuzzy sets is obtained. A connection with the notion of prelinear Heyting MV-algebra is also established.

8. Vague set theory based on d- algebras

K. J. Lee, Y. H. Kim and Y. U. Cho (2008) [27]

The notions of vague d-subalgebras, vague BCK-ideals, vague d-ideals, vague $d^{\#}$-ideals and vague d^{*}-ideals are introduced, and their properties are investigated. Relations between vague d-subalgebras, vague BCK-ideals, vague d-ideals, vague $d^{\#}$-ideals and vague d^{*}-ideals are established.

9. Intutionistic fuzzy quick ideals in d-algebras

S. S. Ahn and G.H. Han (2009) [2]

Quick ideals and fuzzification of quick ideals in d-algebras are considered, and some related properties are investigated. The intuitionistic fuzzification of quick ideals of d-algebra is established, and related results are studied. The notion of equivalence relations on the family of all intuitionistic fuzzy quick ideals of a d-algebra is introduced, and then some properties are discussed.

10. Fuzzy dot BCK/BCI-algebras

Ashram Borumand Saeid (2010) [10]

In this paper the notion of fuzzy dot BCK-subalgebra is introduced. The author have stated and proved some theorem in fuzzy dot BCK-subalgebra and level subalgebras.

11. Rough fuzzy quick ideal in d-algebras

S, S. Ahn and G. H. Han (2010) [4]

The authors introduced the notion of a rough set in d-algebras. Using a quick ideal in d-algebras, they obtained some relations between quick ideals and upper (lower) rough quick ideals in d-algebras. Also they considered the notion of rough fuzzy quick ideals in d-algebras and gave some properties of such ideals.

12. Structures on bipolar fuzzy d-ideals under (T.S) norms

S. V. Manimaran and B. Chellappa (2010) [28]

In this paper, the authors applied the notion of bipolar-valued fuzzy set to groups. They introduced the concept of bipolar fuzzy groups /fuzzy d-ideals of groups under (T.S) norm and investigate several properties. They gave relations between a bipolar fuzzy group and bipolar fuzzy d-ideal. They provided a condition for bipolar fuzzy groups to be a bipolar fuzzy d-ideal. They also gave characterizations of bipolar fuzzy ideal. They considered the concept of strongest bipolar fuzzy relations on bipolar fuzzy d-ideals of a group and discussed some related properties.

13. A class of BCK – algebra

Zahara M. Samaci, Mohammad Ali N. Azadani and Leila N. Ranjbar (2011) [41]

In this article, the authors introduced a BCK-algebra, and showed that this BCK-algebra is commutative, with the relative cancellation property, lower semi lattice and also it's with condition (S) but it's not positive implicative in some cases. Also they gave two examples for this BCK-algebra and introduced a BCK-algebra on fuzzy set, and they showed that this BCK-algebra is bounded, commutative and also it is a lattice but it is not an implicative BCK-algebra.

14. Deformations of d/BCK-algebras

P. J. Allen, H. S. Kim and J. Neggers (2011) [9]

In this paper, the authors studied the effects of a deformation mapping on the resulting deformation d/BCK-algebra obtained via such a deformation mapping. Besides providing a method of constructing d-algebras from BCK-algebras, it also highlights the special properties of the standard BCK-algebras of posets as opposed to the properties of the class of divisible d/BCK-algebras which appear to be of interest and which form a new class of d/BCK-algebras in so far as its not having been identified before.

15. Primary decompositions of fuzzy dot ideals in d-algebras

A. Sobiraju and A. Prasanna (2011) [38]

K. H. Kim (2009) introduced the notation of fuzzy dot subalgebra of d-algebra. In this paper some algebraic properties of fuzzy dot-d-ideals of d-algebra, are discussed. The author also gave characterizations of fuzzy d-ideal. They considered the concept of strongest fuzzy relations on fuzzy d-ideals of a group and discuss some related properties.

16.The theory of falling shadows applied to d-ideals in d-algebras

Y. B. Jun and S. S. Ahn (2011) [24]

On the basis of the theory of a falling shadow, the notion of falling d^*-ideals in d algebras is introduced, and related properties are investigated. Characterizations of a falling d^*-ideal are established. Relations among falling d^*-ideals, falling d-ideals, falling $d^\#$-ideals, falling d-subalgebras and falling BCK-ideals are discussed.

17. Falling d-ideals in d-algebras

Y. B. Jun, S. S. Ahn and K. J. Lee (2011) [23]

Based on the theory of a falling shadow, a theoretical approach of the ideal structure in d-algebras is established. The notions of a falling d-subalgebra, a falling d-ideal, a falling BCK-ideal, and a falling d^*-ideal of a d-algebra are introduced. Some fundamental properties are investigated. Relations among a falling d-subalgebra, a falling d-ideal, a falling BCK-ideal, and a falling d^*-ideal are stated. Characterizations of falling d-ideals and falling d^*-ideals are discussed. A relation between a fuzzy d-subalgebra and a falling d-subalgebra is provided.

18. Ideal theory of d-algebras based on N-structures

S. S. Ahn and G. H. Han [2011] [5]

The notions of N-subalgebra, (positive implicative) N-ideals of d-algebras are introduced, and related properties are investigated. Characterizations of an N-subalgebra and a (positive implicative) N-ideals of d-algebras are given. Relations between an N-subalgebra, an N-ideal and a positive implicative N-ideal of d-algebras are discussed.

CHAPTER I

CHAPTER - 1

ON d-ALGEBRAS AND d-IDEALS IN d-ALGEBRAS

Section 1.1

Preliminary results in d-algebras and edge d-algebras

Definition: 1.1.1 [15]

A BCK-Algebra is a non-empty set X with a constant 0 and a binary operation $*$, denoted by $(X; *, 0)$ satisfying the following conditions:

(BCK 1) $((x * y) * (x * z)) * (z * y) = 0$,

(BCK 2) $(x * (x * y)) * y = 0$,

(BCK 3) $(x * x) = 0$,

(BCK 4) $(x * y) = 0, (y * x) = 0 \implies x = y$,

(BCK 5) $0 * x = 0$

for all $x, y, z \in X$

Definition: 1.1.2 [15]

A non-empty set X with a constant 0 and a binary operation $*$, denoted by $(X; *, 0)$ satisfying the conditions (BCK 1), (BCK 2), (BCK 3) and (BCK 4) is called a BCI-Algebra.

Proposition: 1.1.3

In a BCK-algebra $(X; *, 0)$ the following hold:

1) $(x * y) * x = 0$,

2) $((x * z) * (y * z)) * (x * y) = 0$

for arbitrary $x, y, z \in X$.

Proof

Obvious.

Definition: 1.1.4 [33]

A non-empty set X with a constant 0 and a binary operation $*$, denoted by (X; $*$, 0) is called a d-algebra, if it satisfies the following axioms:

(d1) $x * x = 0$,

(d2) $0 * x = 0$,

(d3) $x * y = 0$ and $y * x = 0 \implies x = y$

for all x, y $\in$ X.

Note:

In BCK/BCI/d-algebras we can define a binary relation "$\le$" by $x \le y$ if and only if $x * y = 0$.

Proposition: 1.1.5

Let X be a d-algebra. If $x \neq y$ and $x * y = 0$, then $y * x \neq 0$.

Proof

Let X be a d-algebra. Let $x \neq y$ and $x * y = 0$.

Claim: $y * x \neq 0$

Suppose $y * x = 0$. Then by (d3) x = y which is a contradiction.

Therefore $y * x \neq 0$.

Example: 1.1.6

i. Every BCK-algebra is a d-algebra.

ii. Let X = {0, 1, 2} be a set with the following Table.

*	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Then (X; $*$, 0) is a d-algebra, but not a BCK-algebra, since $(2 * (2 * 2)) * 2 = (2 * 0) * 2 = 1 * 2 = 2 \neq 0$.

iii. Let $\mathbb{R}$ be the set of all real numbers and define $x * y = x \bullet (x - y)$, $x, y \in \mathbb{R}$, where $\bullet$ and $-$ are the ordinary product and subtraction of real numbers.

Then $x * x = 0$, $0 * x = 0$ and $x * 0 = x \bullet (x - 0) = x \bullet x = x^2$

If $x * y = y * x = 0$, then

$x \bullet (x - y) = 0$ and $y \bullet (y - x) = 0$

$\Rightarrow x^2 - xy = 0$ and $y^2 - yx = 0$

$\Rightarrow x^2 = xy$ and $y^2 = yx$

If $x = 0$, then $y^2 = 0 \Rightarrow y = 0$

If $y = 0$, then $x^2 = 0 \Rightarrow x = 0$

If $xy \neq 0$ then $x^2 = y^2 \Rightarrow x = y$

Hence in any case $x = y$.

Hence $(\mathbb{R}; *, 0)$ is a d-algebra.

Definition: 1.1.7

Let $(X; *, 0)$ be a d-algebra and $x \in X$. Define $x * X = \{x * a \mid a \in X\}$. X is said to be edge if for any x in X, $x * X = \{x, 0\}$.

Lemma: 1.1.8

Let $(X; *, 0)$ be an edge d-algebra. Then $x * 0 = x$ for any $x \in X$.

Proof

Since $(X; *, 0)$ is an edge d-algebra, either $x * 0 = x$ or $x * 0 = 0$ for any $x \in X$.

If $x \neq 0$ and $x * 0 = 0$, then by (d3) $x = 0$, a contradiction.

$\therefore x * 0 = x$ for any $x \in X$.

Proposition: 1.1.9

If $(X; *, 0)$ is an edge d-algebra, then the condition $(x * (x * y)) * y = 0$ $\forall x, y \in X$ holds.

Proof

If $x = 0$, then $(x * (x * y)) * y = 0$ by (d2).

Let $x \neq 0$.

Assume $(x * (x * y)) * y \neq 0$ for some $y \in X$.

Let $\alpha = (x * (x * y))$.

Then $\alpha * y \neq 0$ and $\alpha \neq 0$.

This means that $x \neq x * y$, but, $x * y \in x * X = \{x, 0\}$ and hence $x * y = 0$.

It follows that, by Lemma 1.1.8, $(x * (x * y)) * y = (x * 0) * y = (x * y) = 0$, a contradiction.

$\therefore (x * (x * y)) * y = 0 \ \forall \ x, y \in X$.

Hence proved.

Definition: 1.1.10

A d-algebra $(X; *, 0)$ is said to be d-transitive if $x * z = 0$ and $z * y = 0$ imply $x * y = 0$.

Theorem 1.1.11

Let $(X; *, 0)$ be a d-transitive edge d-algebra. Then $(X; *, 0)$ is a BCK -algebra.

Proof

Let $(X; *, 0)$ be a d-transitive edge d-algebra.

To prove: Then $(X; *, 0)$ is a BCK -algebra.

So it is enough to show that condition $((x * y) * (x * z)) * (z * y) = 0$ holds. Assume that $((x * y) * (x * z)) * (z * y) \neq 0$ for some $x, y, z \in X$.

Since $(x * y) * (x * z) \in (x * y) * X = \{(x * y), 0\}$,

$$(x * y) * (x * z) = (x * y) \tag{1}$$

If $(x * y) = 0$, then $0 \neq ((x * y) * (x * z)) * (z * y)$

$= (0 * (x * z)) * (z * y)$

$= 0 * (z * y) = 0$, a contradiction.

It follows that $x * y = x$. (2)

Hence $x = x * y$ [by (2)]

$= (x * y) * (x * z)$ [by (1)]

$= x * (x * z)$ [by (2)]

That is, $x = x * (x * z)$. (3)

Claim 1: $x * z = 0$.

Assume $x * z \neq 0$, then $x * z = x$, since X is an edge d-algebra.

By applying **(**d3**),** $x = x * (x * z) = x * x = 0$.

This means that $0 \neq ((x * y) * (x * z)) * (z * y)$

$= (x * x) * (z * y)$ [by (2) and $x * z = x$]

$= 0 * (z * y)$

$= 0$, a contradiction.

Thus, $x * z = 0$.

Claim 2: $z * y = 0$.

Assume $z * y \neq 0$, then $z * y = z$, since X is an edge d-algebra.

This means that $0 \neq ((x * y) * (x * z)) * (z * y)$

$= (x * y) * 0) * z$

$= (x * y) * z$

$= x * z$

$= 0$, a contradiction.

Thus, $z * y = 0$.

Hence by claim (1) and claim (2), we obtain that $x * z = 0$ and $z * y = 0$.

Since X is a d-transitive, $x * y = 0$.

Hence $0 \neq ((x * y) * (x * z)) * (z * y) = 0$, a contradiction.

Hence the theorem.

Note:

Both conditions, i.e., to have the d-transitive and edge properties, are necessary for a d-algebra of this type to be a BCK-algebra. Thus, arbitrary BCK-algebras do not always have the edge property.

Example: 1.1.12

Consider the following d-algebra X with the Table.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Then $1 * 2 = 0$, $2 * 3 = 0$, but $1 * 3 = 1$, and hence $(X; *, 0)$ is non-d-transitive edge d-algebra.

Since $((1 * 3) * (1 * 2)) * (2 * 3) = 1 \neq 0$, $(X; *, 0)$ is not a BCK -algebra.

Example: 1.1.13

Let $X = \{0,1,2,...\}$ and let the binary operation $*$ be defined as follows:

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases}$$

Then $x * z = 0$, $z * y = 0$ implies $x \leq z$, $z \leq y$ and in particular $x \leq y$, i.e., $x * y = 0$ also.

Furthermore, $x * x = 0$, $0 * x = 0$ and $x * y = y * x = 0$ if $x \leq y$, $y \leq x$, whence $x = y$.

Thus, the algebra $(X; *, 0)$ is a d-transitive non-edge d-algebra.

Also, $(2 * (2 * 0)) * 0 = (2 * 1) * 0 = 1 * 0 = 1$, so that $(X; *, 0)$ is not a BCK-algebra.

Theorem: 1.1.14

Given a d-algebra $(X; *, 0)$ we can construct an edge d-algebra $(X; \oplus, 0)$, called the extended edge d-algebra.

Proof

Suppose that $(X; *, 0)$ is an arbitrary d-algebra.

Assume that $(X; *, 0)$ is not an edge d-algebra.

Define a binary operation $\oplus : X \times X \to X$ by

$$x \oplus y = \begin{cases} x & \text{if} \quad x * y \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $(X; \oplus, 0)$ is a d-algebra.

Suppose now that $x \oplus X = \{0\}$.

Then $x * y = 0$ for all $y \in X$.

In particular, $x * 0 = 0 = 0 * x$, so that $x = 0$.

Hence, if $x \neq 0$, then $x \oplus X = \{x, 0\}$.

Hence $(X; \oplus, 0)$ is an edge d-algebra.

Hence the proof.

Proposition: 1.1.15

A d-algebra $(X; *, 0)$ is d-transitive if and only if its extended edge d-algebra $(X; \oplus, 0)$ is d-transitive.

Proof

Assume $(X; *, 0)$ is d-transitive.

Then $x \oplus z = 0$ and $z \oplus y = 0$ imply $x * z = 0 = z * y$,

$\Rightarrow x * y = 0$

$\Rightarrow x \oplus y = 0$.

Hence $(X; \oplus, 0)$ is d-transitive.

Conversely, assume that $(X; \oplus, 0)$ is d-transitive.

Then $x * z = 0$ and $z * y = 0$

$\Rightarrow x \oplus z = 0 = z \oplus x$.

$\Rightarrow x \oplus y = 0$.

Hence $x * y = 0$.

(i.e) $(X; *, 0)$ is d-transitive.

Hence the proof.

Example 1.1.16

There are 27 d-algebras as follows:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	u
b	b	v	0	0
c	c	0	w	0

where $u, v, w \in \{a,b,c\}$.

All of these algebras have the same unique edge d-algebra as follows:

⊕	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	0	c	0

This d-algebra is not d-transitive, since $a \oplus b = b \oplus c = 0$, while $a \oplus c = a \neq 0$.

It also has the following d-chain property: $x \oplus y \neq 0$ implies $y \oplus x = 0$.

Section 1.2

On d-ideals in d-algebras

Definition: 1.2.1

A non empty subset I of a BCK-algebra X is called a BCK-ideal of BCK-algebra X if

i. $0 \in I$,

ii. $x \in I$ and $y * x \in I$, imply $y \in I$ for all $x, y \in X$.

Definition: 1.2.2 [34]

Let S be a non empty subset of a d-algebra $(X; *, 0)$, then S is called a subalgebra of d-algebra X (d-subalgebra of X) if $x * y \in S$ for all $x, y \in S$.

Definition: 1.2.3 [34]

Let $(X; *, 0)$be a d-algebra and I be a non empty subset of X, then I is called a d-ideal of d- algebra X (or d- ideal of X) if it satisfies the following conditions:

(Id1) $0 \in I$,

(Id2) $x * y \in I$ and $y \in I \Rightarrow x \in I$,

(Id3) $x \in I$ and $y \in X \Rightarrow x * y \in I$, i.e., $I \times X \subseteq I$.

Note:

In a d-algebra X, a non empty subset I is called a BCKd-ideal of X if it satisfies (Id1) and (Id2).

Example: 1.2.4

Let X = {0, a, b, c, d} be a d-algebra

∗	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	c	0
c	c	c	b	0	c
d	c	c	a	a	0

Then I = {0, a} is a d-ideal of X.

Example: 1.2.5

Let X = {0, a, b, c} be a d-algebra

∗	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Then I = {0, a, c} satisfies (Id3), but not (Id2) since $b * c = 0 \in I$ and $c \in I$, but $b \in I$ i.e., I is a d-subalgebra, but not a BCKd-ideal of X.

In a d-algebra a BCKd-ideal need not be a d-subalgebra, and also d-subalgebra need not be a BCKd-ideal as shown in the following example.

Example: 1.2.6

Let X = {0, a, b, c} be a d-algebra

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	c	0	0
c	c	c	c	0

Then I = {0, a, b} is a BCKd-ideal which is not a d-subalgebra of X, while J = {0, c} is a d-subalgebra which is not a BCKd-ideal of X.

Note:

Clearly, {0} is a d-subalgebra of every d-algebra X and every d-ideal of X is a d-subalgebra.

Example: 1.2.6

Let X = {0, a, b, c} be a d-algebra

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Then I = {0, a} is a d-subalgebra of X, but not a d-ideal of X, since $a * c = b \notin I$

Lemma: 1.2.8

Let I be a non-empty subset of a d-algebra X, which satisfies (Id2) and (Id3). Then $0 \in I$.

Proof

Since $I \neq \phi$, there exists x in I and hence $0 = x * x \in I$ by (Id3).

Note:

By above Lemma I is called a d-ideal of a d-algebra X, it is enough to satisfy (Id2) and (Id3).

Proposition: 1.2.9

Let I be a d-ideal of a d-algebra X. If $x \in I$ and $y * x = 0$, then $y \in I$.

Proof

Assume that $x \in I$ and $y * x = 0$.

By Lemma 1.2.8 $0 \in I$ (1)

Then $y * x \in I$

$\Rightarrow y \in I$ [by (Id2)]

Hence the proof.

Definition: 1.2.10

Let X be a d-algebra. A d-ideal I of X is called a $d^{\#}$ -ideal of X if, for arbitrary $x, y, z \in X$,

(Id4) $x * z \in I$ whenever $x * y \in I$ and $y * z \in I$.

Example: 1.2.11

Let X be a d-algebra as in Example 1.2.7. Then $K = \{0, a, b\}$ is a $d^{\#}$ -ideal of X.

Note:

Obviously, every $d^{\#}$ -ideal is a d-ideal, but the converse need not be true.

Example: 1.2.12

Let X be a d-algebra as in Example 1.2.7. Then L = {0, a} is a d-ideal. Since $b * d = 0 \in L$, $d * c = a \in L$, but $b * c = c \notin L$.

Then L is not a $d^{\#}$ -ideal of X.

Note:

$d^{\#}$-ideal $\subset$ d-ideal $\subset$ d-subalgebra and

$d^{\#}$-ideal $\subset$ d-ideal $\subset$ BCKd-ideal in d-algebras.

Note:

In a d-algebra X, the identity $(x * y) * x = 0$ does not hold in general. For instance, in Example 1.2.7, we know that $(a * c) * a = b * a = b \neq 0$.

Definition: 1.2.13

A d-algebra X is called a d^*-algebra if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Note:

Clearly, a BCK-algebra is a d^*-algebra, but the converse need not be true.

Example 1.2.14

Let X = { 0 , 1 , 2 , . . . } and let the binary operation $*$ be defined as follows: $x * y = \begin{cases} 0 & \text{if} \quad x \leq y, \\ 1 & \text{otherwise.} \end{cases}$

Then $(X; * , 0)$ is a d-algebra as well as d^*-algebra but not a BCK-algebra.

Theorem: 1.2.15

In a d^*-algebra $(X; *, 0)$, every BCKd-ideal is a d-ideal.

Proof

Let I be a BCKd-ideal of a d^*-algebra X and let $x \in I$, $y \in X$.

Since $(x * y) * x = 0$ for all $x, y \in X$, it follows from Proposition 1.2.9 that $x * y \in I$. Hence I is a d-ideal of X.

Corollary: 1.2.16

In a d^*-algebra, every BCKd-ideal is a d-subalgebra.

Proof

Obvious.

Definition: 1.2.17

If a $d^\#$ -ideal I of a d-algebra X satisfies (Id5): $x * y \in I$ and $y * x \in I$ imply $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$ for all $x, y, z \in X$, then we say that I is a d^*-ideal of X.

Note:

In Example 1.2.5 the set I = {0, a} is a d^*-ideal of X. Obviously, every d^*-ideal in a d-algebra is a $d^\#$ -ideal, but the converse does not hold in general.

Example: 1.2.18

Let X = {0, a, b, c} be a set with the following Cayley table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	c	b	0	c
c	c	b	b	0

Then $(X; *, 0)$ is a d-algebra.

Let I = {0, a}.Then I is a $d^\#$-ideal.

Since $0 * a = 0 \in I$ and $a * 0 = a \in I$, but $(c * 0) * (c * a) = c * b = b \notin I$, I is not d^*-ideal.

Lemma: 1.2.19

Let I be a BCK-ideal of a BCK-algebra X. If $x \in I$ and $y * x = 0$ then $y \in I$.

Proof

Obvious.

Theorem 1.2.20

If $(X; *, 0)$ is a BCK-algebra, then every BCK-ideal of X is a d^*-ideal of X.

Proof

Let I be a BCK-ideal of X and let $x \in I$ and $y \in X$.

Since $(X; * , 0)$ is a BCK-algebra $(x * y) * x = 0$.

Hence $x \in I, y \in X \Rightarrow x * y \in I$. (1)

Assume that $x * y \in I$ and $y * z \in I$ for all $x, y, z \in I$.

Since $(X; *, 0)$ is a BCK-algebra $((x * z) * (y * z)) * (x * y) = 0$,

and hence $(x * z) * (y * z) \in I$.

Since $y * z \in I$ by (Id1) we have $x * z \in I$.

Hence $x * y \in I$ and $y * z \in I$

$\Rightarrow x * z \in I$. (2)

Let $x * y, y * x \in I$ for all $x, y \in X$.
Then, we have

$((z * x) * (z * y)) * (y * x) = 0$ and $((x * z) * (y * z)) * (x * y) = 0$,

$\Rightarrow (z * x) * (z * y) \in I$ and $(x * z) * (y * z) \in I$. (3)

By (1), (2) and (3) I is a d^*-ideal of X.

CHAPTER II

CHAPTER - 2

ON QUOTIENT d-ALGEBRAS AND d^*-SUBALGEBRAS OF d-TRANSITIVE d^*-ALGEBRAS

Section 2.1

On quotient d-algebras

Definition: 2.1.1 [34]

Let $(X;*, 0_X)$ and $(Y;*, 0_Y)$ be two d-algebras. A mapping $f: X \rightarrow Y$ is called a d-morphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Note:

i. $f(0_X) = 0_Y$.

ii. $\text{Ker } f = \{ x \in X \mid f(x) = 0_Y \}$.

Proposition: 2.1.2

Let $f: X \rightarrow Y$ be a d-morphism from a d-algebra X into a d-transitive d-algebra Y. Then Ker f is a d^*-ideal of X.

Proof

The properties (Id1), (Id2) and (Id3) are simple.

If $x * y, y * z \in$ Ker f, then $f(x) * f(y) = 0_Y = f(y) * f(z)$.

Since Y is d-transitive, we obtain $f(x) * f(z) = 0$.

Hence $x * z \in$ Ker f, which proves (Id4).

Let $x * y, y * x \in$ Ker f.

Then $f(x) * f(y) = 0_Y = f(y) * f(x)$.

$\Rightarrow f(x) = f(y)$.

It follows that

$f((x * z)*(y * z)) = f(x * z) * f(y * z) = (f(x) * f(z)) * (f(y) * f(z)) = 0_Y$

Hence $(x * z) * (y * z) \in$ Ker f.

Similarly, $(z * x) * (z * y) \in$ Ker f, which proves (Id5).

Hence Ker f is a d^*-ideal of X.

Example: 2.1.3

Let X be a d-algebra as in Example 1.2.5, and let Y be a d-transitive d-algebra as in Example 1.2.4.

Define a map f: $X \rightarrow Y$ by f(0) = f(a) = 0, f(b) = f(c) = a. Then f is a d-morphism. Obviously, Ker f = {0, a} is a d^*-ideal of X.

Proposition: 2.1.4

Let I be a d^*-ideal of d-algebra $(X; *, 0_X)$. For any x, y in X, define x ~ y if and only if $x * y \in I$ and $y * x \in I$. Then ~ is an equivalence relation on X.

Proof

i. Since $0 \in I$, we have $x * x = 0 \in I$

$\Rightarrow$ x ~ x for any x.

Hence ~ is reflexive.

ii. If x ~ y and y ~ z then $x * y, y * x \in I$ and $y * z, z * y \in I$.

By (Id4) $x * z, z * x \in I$

$\Rightarrow$ x ~ z.

This proves ~ is transitive.

iii. The symmetry of ~ is trivial.

Note:

By (Id5) we can easily see that ~ is a congruence relation on X.

Definition: 2.1.5

The congruence class containing x is defined as $[x]_I = \{ y \in X \mid x \sim y \}$.

Note:

$x \sim y$ if and only if $[x]_I = [y]_I$.

Definition: 2.1.6

The set of all equivalence classes of X is defined as $\frac{X}{I} = \{[x]_I \mid x \in X \}$.

Lemma: 2.1.7:

Let I be a d^*-ideal of a d-algebra $(X;*, 0_X)$. Then $I = [0]_I$.

Proof

If $x \in I$, then $x * 0 \in I * X \subseteq I$ and hence $x \in [0]_I$ i.e., $I \subseteq [0]_I$.

Since $[0]_I = \{ x \in X \mid x \sim 0\}$

$= \{ x \in X \mid x * 0, 0 * x \in I \}$

$= \{ x \in X \mid x * 0 \in I \}$

$\subseteq I$.

It follows that $I = [0]_I$.

Theorem 2.1.8:

Let $(X; *, 0)$ be a d-algebra and I be a d^*-ideal of X. If we define

$[x]_I * [y]_I = [x * y]_I$ for all $x, y \in X$, then $(X/I; *, 0)$ is a d-algebra, called the quotient d-algebra.

Proof

Since ~ is a congruence relation on X, $x * y \sim x' * y'$ for any $x \sim x'$, $y \sim y'$.

This means that $[x]_I * [y]_I = [x * y]_I$ is well-defined.

Let $[x]_I * [y]_I \in X/I$ with $[x]_I * [y]_I = [0]_I = [y]_I * [y]_I$.

Then $[x * y]_I = [0]_I = [y * x]_I$ and $x * y, y * x \in I$.

Thus $x \sim y$ and $[x]_I = [y]_I$.

The rest is trivial.

Hence the proof.

Proposition: 2.1.9:

Let I be a d^*-ideal of the d-algebra X. Then the mapping $\pi: X \to X/I$ defined by $\pi(x) = [x]_I$ is a d-morphism of X onto the quotient d-algebra X/I and the kernel of π is precisely the set I.

Proof

Since $[x * y]_I = [x]_I * [y]_I$, π is a d-morphism.

By Lemma 2.1.7 we know that,

$\text{Ker } \pi = \{x \in X \text{I } \pi(x) = [0]_I\}$

$= \{x \in X \mid [x]_I = [0]_I \}$

$= \{x \in x \mid x \sim 0\}$

$= [0]_I$

$= I.$

Theorem 2.1.10

If $f: X \to Y$ is a d-morphism from a d-algebra X onto a d-transitive d-algebra Y, then $X/\text{Ker } f \cong Y$.

Proof

Assume $\mu: X/\text{Ker } f \to Y$ such that $\mu([x]_{\text{Ker } f}) = f(x)$.

If $[x]_{\text{Ker } f} = [y]_{\text{Ker } f}$ then $x * y$, $y * x \in \text{Ker } f$, and so $f(x) * f(y) = 0 = f(y) * f(x)$.

$\Rightarrow f(x) = f(y)$, i.e., $\mu([x]_{Ker\ f}) = \mu([y]_{Ker\ f})$.

This means that μ is well-defined.

For any $y \in Y$, there is an $x \in X$ such that $y = f(x)$, since f is onto.

Hence $\mu([x]_{Ker\ f}) = f(x) = y$, which means that μ is onto.

If $\mu([x]_{Ker\ f}) \neq \mu([y]_{Ker\ f})$ then either $x * y \notin Ker\ f$ or $y * x \notin Ker\ f$.

Without loss of generality, we may assume $x * y \notin Ker\ f$.

It follows that $f(x) * f(y) = f(x * y) \neq 0$.

Hence $f(x) \neq f(y)$.

This means that μ is one-one.

$$\begin{aligned}\text{Since } \mu([x]_{Ker\ f} * [y]_{Ker\ f}) &= \mu([x * y]_{Ker\ f}) \\ &= f(x * y) \\ &= f(x) * f(y) \\ &= \mu([x]_{Ker\ f}) * \mu([y]_{Ker\ f})\end{aligned}$$

$\Rightarrow$ μ is a d-morphism.

Thus we have $X/Ker\ f \cong Y$.

Hence the proof.

Section 2.2

On d^*-subalgebras of d-transitive d^*-algebras

Definition: 2.2.1 [26]

Let $(X;*, 0)$ be a d^*-algebra.

i. A non empty subset I of X is called a d^*-subalgebra of X if $x * y \in I$ whenever $x \in I$ and $y \in I$.

ii. X is said to be d-transitive d^*-algebra if $x * z = 0$ and $z * y = 0$ imply $x * y = 0$.

Proposition: 2.2.2

Let $(X;*, 0)$ be a d-(d^*-) algebra and let S be a d-(d^*-) subalgebra of X. Then we have:

(a) $0 \in X$.

(b) $(S; *, 0)$ is also a d-(d^*-) algebra of X .

(c) X is a d-(d^*-) subalgebra of X.

(d) $\{0\}$ is a d-(d^*-) subalgebra of X.

Proof

Obvious.

Note:

If $(X; *, 0)$ is a BCK- algebra and $0 \neq x_0 \in X$, then $(\{0, x_0\}; *, 0)$ is a d-subalgebra of X, but this does not hold in the case of d- (d^*-) algebra.

Example: 2.2.3

Let X = {0, 1, 2} be a d-algebra with the table.

$*$	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Here ({0, 1}; $*$, 0) is not a d-subalgebra of X.

Lemma: 2.2.4:

Let (X;$*$, 0) be a d-algebra. If $x \neq y$ and $x * y = 0$, then $y * x \neq 0$.

Proof

Obvious.

Lemma: 2.2.5

Let (X; $*$, 0) be a d^*-algebra. If $x * y = z$, then $z * x = 0$.

Proof

Let $z = x * y$.

Then $z * x = (x * y) * x = 0$, since X is a d^*-algebra.

Hence the proof.

Remark: 2.2.6

In the above Lemma 2.2.5, the d^*-algebra condition is necessary. Consider the Example 1.1.6(ii). We see that $1* 2 = 2$, but $2 * 1 = 1 \neq 0$, and hence Lemma 2.2.5 does not hold.

Definition: 2.2.7

An ordered n-tuple $a_1,a_2,\ldots,a_n$ of elements in a d-algebra X is called an n-sequence.

Definition: 2.2.8

Given an n-sequence $a_1, a_2, \ldots, a_n$ of a d-algebra X, we construct a $(n-1) \times n$ matrix A as follows:

$$A = \begin{pmatrix} a_1 * a_2 & a_2 * a_1 & \ldots & a_n * a_1 \\ a_1 * a_3 & a_2 * a_3 & \ldots & a_n * a_2 \\ \ldots & \ldots & \ldots & \ldots \\ a_1 * a_n & a_2 * a_n & \ldots & a_n * a_{n-1} \end{pmatrix}$$

A is called the adjoint matrix relative to the n-sequence $a_1, a_2, \ldots, a_n$.

Proposition: 2.2.9

Given a distinct n-sequence $a_1, a_2, .., a_n$ $(n \geq 2)$ of elements of a d-transitive d-algebra X, let A be the adjoint matrix relative to this sequence. Then there exists a column in A which is composed of non-zero elements.

Proof

The proof is by induction on n.

When n = 2, let a_1, a_2 be a 2-sequence, where $a_1 \neq a_2$, then its adjoint matrix is

$$A = (a_1 * a_2 \quad a_2 * a_1).$$

If $a_1 * a_2 = a_2 * a_1 = 0$, then by (d3) $a_1 = a_2$ which is a contradiction.

So the proposition is true for the case n = 2.

Now assume that the proposition is true for n - 1.

Let $a_l, a_2, \ldots, a_n$ be a distinct n-sequence.

Then the adjoint matrix relative to this n-sequence is

$$A_n = \begin{pmatrix} a_1 * a_2 & a_2 * a_1 & \ldots & a_{n-1} * a_1 & a_n * a_1 \\ a_1 * a_3 & a_2 * a_3 & \ldots & a_{n-1} * a_2 & a_n * a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 * a_{n-1} & a_2 * a_{n-1} & \ldots & a_{n-1} * a_{n-2} & a_n * a_{n-2} \\ a_1 * a_n & a_2 * a_n & \ldots & a_{n-1} * a_n & a_n * a_{n-1} \end{pmatrix}$$

Set

$$A_{n-1} = \begin{pmatrix} a_1 * a_2 & a_2 * a_1 & \dots & a_{n-1} * a_1 \\ a_1 * a_3 & a_2 * a_3 & \dots & a_{n-1} * a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 * a_{n-1} & a_2 * a_{n-1} & \dots & a_{n-1} * a_{n-2} \end{pmatrix}$$

It is obvious that A_{n-1} is the adjoint matrix relative to the (n-1) sequence a_1, a_2, …, a_{n-1}. For this (n -1) sequence we certainly have $a_i \neq a_j$ whenever $i \neq j$. Then, by the induction hypothesis, we know that there exists in A_{n-1} a column which is composed of non-zero elements.

Without loss of generality, we can assume that the first column of A_{n-1} is composed of non-zero elements, i.e.,

$$\begin{cases} a_1 * a_2 \neq 0, \\ a_1 * a_3 \neq 0, \\ \vdots \\ a_1 * a_{n-1} \neq 0, \end{cases} \qquad (1)$$

Now, if $a_1 * a_n \neq 0$, then the elements in the first column of A_n are all non-zero, so we are done.

If $a_1 * a_n = 0$, then since $a_1 \neq a_n$, by Lemma 2.2.4,

we have $a_n * a_1 \neq 0$. (2)

For $2 \leq i \leq n-1$, we shall show that

we also have $a_n * a_i \neq 0$. (3)

In fact, if $a_n * a_i = 0$, then since $a_1 * a_n = 0$,

we have $a_1 * a_i = 0$ $(2 \leq i \leq n-1)$. (4)

But (4) contradicts (1).

By (2) and (3) we know that the n^{th} column of A_n is composed of non-zero elements.

Therefore the conclusion is also true for n.

Hence the proof.

Proposition: 2.2.10

Every d-transitive d^*-algebra X of order n + 1 contains a d^*-algebra of order n ($n \geq 1$).

Proof

Let $X = \{0, a_1, a_2, \ldots, a_n\}$ be a d-transitive d^*-algebra of order n + 1, where $a_1, a_2, \ldots a_n$ are distinct non-zero elements of X. We construct the adjoint matrix A_n relative to $a_1, a_2, \ldots, a_n$ as follows:

$$A_n = \begin{pmatrix} a_1 * a_2 & a_2 * a_1 & \cdots & a_{n-1} * a_1 & a_n * a_1 \\ a_1 * a_3 & a_2 * a_3 & \cdots & a_{n-1} * a_2 & a_n * a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 * a_{n-1} & a_2 * a_{n-1} & \cdots & a_{n-1} * a_{n-2} & a_n * a_{n-2} \\ a_1 * a_n & a_2 * a_n & \cdots & a_{n-1} * a_n & a_n * a_{n-1} \end{pmatrix}$$

By Proposition 2.2.9 there exists in A_n a column which is composed of nonzero elements.

Without loss of generality, we can assume that the elements in the n^{th} column of A_n are all non-zero, i.e., $a_n * a_i \neq 0$, $i = 1, 2, \ldots, n-1$ (1)

Now we shall show that $T = \{0, a_1, a_2, \ldots, a_{n-1}\}$ is a subalgebra of order n in X. In fact, if T is not a subalgebra of X, then there exist i, j ($1 \leq i, j \leq n-1$) such that $i \neq j$ and $a_i * a_j = a_n$.

Since X is a d^*-algebra, by Lemma 2.2.5, we have $a_n * a_i = 0$.
which contradicts (1).
Hence the proof.

Note:

As a consequence of proposition 2.2.10 we may estimate the number of d^*-algebras of order i in a d-transitive d^*-algebra.

Theorem 2.2.11:

Let X be a d-transitive d^*-algebra of order n. Then

$$1 \leq N(i) \leq \binom{n-1}{i-1} \qquad (i=1, 2, 3, \ldots\ldots\ldots\ldots, n)$$

where N(i) denotes the number of d^* -subalgebras of order i in X.

Proof

This is a direct consequence of Proposition: 2.2.10.

CHAPTER III

CHAPTER - 3

ON COMPANION d-ALGEBRAS

Section 3.1

Interesting results on companion d-algebras

Definition: 3.1.1 [16]

A BCK-algebra $(X;*,0)$ is said to have a condition(S) if $A(a, b) = \{x \in X : x * a \leq b\}$ has a greatest element for any $a, b \in X$.

Definition: 3.1.2 [7]

Let $(X;*,0)$ be a d-algebra.

i. Define a binary operation $\odot$ on X by $((x \odot y) * x) * y = 0$ for any $x, y \in X$, which is called a subcompanion operation of X.
ii. A subcompanion operation $\odot$ is said to be a companion operation of X if $(z * x) * y = 0$, then $z * (x \odot y) = 0$ for any $x, y, z \in X$.

Example 3.1.3

Let X={0, 1, 2,3} be a set with the following tables:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	2	2	2	0

⊙	0	1	2	3
0	0	1	3	3
1	1	1	3	3
2	2	2	3	3
3	3	3	3	3

Then $(X;*,0)$ is a d-algebra, which is not a BCK/BCI-algebra, and the binary operation $\odot$ defined above is a companion operation on X.

Definition: 3.1.4 [7]

A d-algebra X is said to be companion d-algebra if it has a companion operation.

Proposition: 3.1.5

Let $(X; *, 0)$ be a d-algebra. If X has a companion operation $\odot$, then it is unique.

Proof

Assume the binary operations $\odot_1$ and $\odot_2$ are companion operations on X.

Then $((x \odot_i y) * x) * y = 0$ for any $x, y \in X$ $(i = 1, 2)$.

We obtain $(x \odot_1 y) * (x \odot_2 y) = 0$ (1)

Interchange $\odot_1$ with $\odot_2$.

Then $(x \odot_2 y) * (x \odot_1 y) = 0$ (2)

$\Rightarrow (x \odot_1 y) = (x \odot_2 y)$

$\Rightarrow \odot_1 = \odot_2$

Hence the operation $\odot$ is unique.

Example: 3.1.6

Every BCK-algebra with condition (S) is a companion d-algebra.

Note:

Example 3.1.3 is a companion d-algebra which is not a BCK/BCI-algebra. This means that a companion d-algebra is a generalization of a BCK/BCI algebra with condition (S).

Proposition: 3.1.7

Let $(X; *, \odot, 0)$ be a companion d-algebra. Then for any $x, y, z \in X$. We have

(i) if $x * z = 0$, then $x * (z \odot y) = 0$.

(ii) $x * (x \odot y) = 0$.

(iii) $x \odot 0 = x$.

Proof

Let $(X; *, \odot, 0)$ be a companion d-algebra.

To prove(i): Assume $x * z = 0$

Then $(x * z) * y = 0$

$\Rightarrow 0 * y = 0$

$\Rightarrow x * (z \odot y) = 0$

To prove(ii): We know $x * x = 0$

Put $z = x$ in (i)

$\Rightarrow x * (x \odot y) = 0$

To prove(iii): Assume $x * 0 = 0$

Since $0 * x = 0$, we have $x = 0$ by (d3).

Since X is a companion d-algebra,

$((x \odot 0) * x) * 0 = 0$

$\Rightarrow (x \odot 0) * x = 0$ (1)

If we put $y = 0$ in (ii), then $x * (x \odot 0) = 0$ (2)

From (1) and (2) we get $(x \odot 0) = x$.

Theorem: 3.1.8

Let (X; $*$, $\odot$, 0) be a companion d-algebra. Let $\Diamond$ be a binary operation on X such that $(x * y) * z = x * (y \Diamond z)$. Then X is a companion d-algebra and $\Diamond$ is exactly the operation $\odot$.

Proof

Let $\Diamond$ be a binary operation on X such that $(x * y) * z = x * (y \Diamond z)$. (1)

Then $((x \Diamond y) * x) * y = (x \Diamond y) * (x \Diamond y)$ by (1)

$= 0$, by (d1) (2)

Let $z \in X$ with $(z * x) * y = 0$.

Then by (1) $z * (x \Diamond y) = (z * x) * y = 0$

Hence $\Diamond$ is a companion operation.

By proposition 3.1.5 $\Diamond$ is unique.

Proposition: 3.1.9

Let (X; $*$, $\odot$, 0) be a bounded companion d-algebra. That is, there is an element $1 \in X$ such that $x * 1 = 0$ for any $x \in X$, then $x \odot 1 = 1$ for any $x \in X$.

Proof

Since $u * x \le 1$ for any $u \in X$, $(u * x) * 1 = 0$.

But X is a companion d-algebra. Therefore, we have $u \le x \odot 1$, for any $u \in X$, which implies $1 = x \odot 1$.

Definition: 3.1.10

A d/BCK-algebra (X; $*$, 0) is said to be positive implicative if $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

Proposition: 3.1.11

Let $(X; *, \odot, 0)$ be a companion d-algebra. Then

i. $0 \leq x \odot y$, $x \leq x \odot y$, for any $x, y \in X$.

ii. If X is positive implicative, then $y \leq x \odot y$ for any $x, y \in X$.

Proof

Let $(X; *, \odot, 0)$ be a companion d-algebra.

To prove (i): Since $(0 * x) * y = 0$, $0 \leq x \odot y$.

From $(x * x) * y = 0 * y = 0$, we obtain $x \leq x \odot y$.

To prove (ii): Since X is positive implicative,

$(y * x) * y = (y * y) * (x * y) = 0 * (x * y) = 0$.

Hence $y \leq x \odot y$.

Theorem: 3.1.12

Let $(X; *, \odot, 0)$ be a companion d-algebra. Assume that $x * 0 = x$ for any $x \in X$.

i. X is positive implicative.

ii. If $x \leq y$, then $x \odot y = y$.

iii. $x \odot x = x$

for any $x, y \in X$.

Then i $\Rightarrow$ ii $\Rightarrow$ iii.

Proof

Let $(X; *, \odot, 0)$ be a companion d-algebra and $x * 0 = x$ for any $x \in X$.

To prove: i $\Rightarrow$ ii

If $x \leq y$, then

$0 = ((x \odot y) * x) * y)$

$= ((x \odot y) * y) (x * y)$ [$\because$ X is positive implicative]

$= ((x \odot y) * y) * 0$ [$\because$ $x * y = 0$]

$= (x \odot y) * y,$ $[\because x * 0 = x]$

which means that $x \odot y \leq y$.

By applying proposition 3.1.11-(ii), we have $x \odot y = y$.

To prove: ii $\Rightarrow$ iii

Let $y = x$ in (ii)

Then $x \odot x = x$ for any $x, y \in X$.

Hence the proof.

Definition: 3.1.13

Let $(X; *, \odot, 0)$ be a companion d-algebra and $\phi \neq I \subseteq X$. I is called a $\odot$-subalgebra if $x \odot y \in I$ for any $x, y \in I$.

Example: 3.1.14

In Example 3.1.3, the set $I_1 = \{0,1\}$ is a $\odot$-subalgebra of X. The set $I_2 = \{0,1, 2\}$ is not a $\odot$-subalgebra of X.

Theorem: 3.1.15

Let $(X; *, \odot, 0)$ be a companion d-algebra. If I is a BCKd-ideal of X, then I is a $\odot$ -subalgebra of X.

Proof

Given X is a companion d-algebra

$\Rightarrow ((x \odot y) * x) * y = 0 \in I$ for any $x, y \in I$.

Since I is a BCKd-ideal of X and $y \in I$, we have $(x \odot y) * x \in I$.

Moreover, since $x \in I$, we obtain $x \odot y \in I$.

Hence I is a $\odot$-subalgebra of X.

Note :

The converse of Theorem 3.1.15 need not be true in general.

Example: 3.1.16

In example 3.1.3, J = {0,1, 2, 3} is a $\odot$ -subalgebra of X, but not a BCKd-ideal of X, since $2 * 3 = 0 \in J$, $3 \in J$, but $2 \notin J$.

Proposition 3.1.17

Let (X; $*$, $\odot$, 0) be a companion d-algebra and let I be a BCKd -ideal of X. If $x \odot y \in I$, then $x \in I$ where $x, y \in X$.

Proof

By Proposition 3.1.7-(ii), $x * (x \odot y) = 0 \in I$.
Since $x \odot y \in I$ and I is a BCKd-ideal of X, we have $x \in I$.

Corollary: 3.1.18

Let (X; $*$, $\odot$, 0) be a companion d-algebra and let I be a BCKd -ideal of X . If $x \odot y = y \odot x \in I$, then $x, y \in I$ where $x, y \in X$.

Proof

Obvious.

Corollary 3.1.19

Let (X; $*$, $\odot$, 0) be a companion d-algebra and let I be a BCKd -ideal of X . Then $x \in I \Leftrightarrow x \odot x \in I$.

Proof

It follows immediately from Theorem 3.1.15 and Proposition 3.1.17.

Section 3.2

On complete companion d-algebras

Definition: 3.2.1 [7]

Let $(X; *, \odot, 0)$ be a companion d-algebra. X is said to have a dsu condition if $(x * y) * (x \odot y) = 0$ for any $x, y \in X$.

Proposition: 3.2.2

Let $(X; *, \odot, 0)$ be a companion d-algebra having the dsu condition. If I is a BCKd-ideal of X, then it is a d-subalgebra of X.

Proof

By Theorem 3.1.15, $x \odot y \in I$ for any $x, y \in I$.

Since X has the dsu condition, $(x * y) * (x \odot y) = 0 \in I$ and I is a BCKd-ideal of X,

$\Rightarrow x * y \in I$

Hence the proof.

Theorem: 3.2.3

Let $(X;*, \odot, 0)$ be a companion edge d^*-algebra.

If $(z * (x \odot y)) * ((z * x) * y) = 0$, then X has a dsu condition.

Proof

Given X be a companion edge d^*-algebra and

consider $(z * (x \odot y)) * ((z * x) * y) = 0$ (1)

Put $z = x * y$ in (1). Then

$((x * y) * (x \odot y)) * (((x * y) * x) * y) = 0$

$\Rightarrow ((x * y) * (x \odot y)) * (0 * y) = 0$ [X: d^*-algebra]

$\Rightarrow ((x * y) * (x \odot y)) * 0 = 0$

$\Rightarrow ((x * y) * (x \odot y)) = 0$ [X: edge]

Hence X has a dsu condition.

Hence the proof.

Proposition: 3.2.4

Let $(X; *, \odot, 0)$ be a companion edge d-algebra.

If $(z * (x \odot y)) * ((x * z) * y) = 0$, then X has a dsu condition.

Proof

Given X be a companion edge d-algebra

To prove: X has a dsu condition.

Consider $(z * (x \odot y)) * ((x * z) * y) = 0$, (1)

Put $z = x * y$ in (1). Then

$$((x * y) * (x \odot y)) * ((x * (x * y)) * y) = 0$$

$\Rightarrow ((x * y) * (x \odot y)) * 0 = 0$

$\Rightarrow ((x * y) * (x \odot y)) = 0$

Hence X has a dsu condition.

Definition: 3.2.5

A companion d-algebra $(X; *, \odot, 0)$ is said to be complete if for any $x \in X$, there exists an x^* in X such that $x \odot x^* = x$.

Note:

x^* in the above definition need not be unique.

For example, in Example 3.1.3,

we find $2 \odot 0 = 2 \odot 1 = 2$, and $3 \odot 1 = 3 \odot 2 = 3$

Here x^* is not unique.

Proposition: 3.2.6

Let $(X; *, \odot, 0)$be a companion d-algebra. If we define a partial binary relation << by $x << y \Leftrightarrow (x \odot z) * (y \odot z) = 0$ for all $z \in X$, then << is reflexive and anti-symmetric.

Proof

Clearly, << is reflexive.

If $x << y$, $y << x$, then $(x \odot z) * (y \odot z) = 0 = (y \odot z) * (x \odot z)$ for any $z \in X$. By applying (d3) we have

$x \odot z = y \odot z$ for any $z \in X$ (1)

Since X is complete, there exist $x^*, y^* \in X$ such that $x = x \odot x^*$, $y = y \odot y^*$.

Put $z = x^*$ and $z = y^*$ in (1), respectively, then $x = x \odot x^* = y \odot x^*$,

$y = y \odot y^* = x \odot y^*$.

Thus by Proposition 3.1.7-(ii),

$x * y = x * (x \odot y^*) = 0$ and $y * x = y * (y \odot x^*) = 0$

$\Rightarrow x = y$.

Hence the proof.

Note:

For any BCK/BCI-algebras the following transitivity condition holds:

If $x * y = 0$ and $y * z = 0$, then $x * z = 0$.

This condition does not hold in d-algebra in general.

Note:

If a d-algebra satisfies the transitivity condition, then the natural order $\leq$ given by $x \leq y$ if and only if $x * y = 0$ is a partial order.

Proposition: 3.2.7

Let $(X; *, \odot, 0)$be a complete companion d-algebra. If X satisfies the transitivity condition, then $(X; <<)$ is a poset.

Proof

Obvious.

Proposition: 3.2.8

Let $(X; *,\odot, 0)$be a complete companion d-algebra. If $x << y$, then $x \leq y$ in X.

Proof

If $x << y$, then $(x \odot \alpha) * (y \odot \alpha) = 0$ for any $\alpha \in X$.

$\Rightarrow (x \odot 0) * (y \odot 0) = 0$

$\Rightarrow x * y = 0$

$\Rightarrow x \leq y$.

Note:

The converse of above Proposition 3.2.8 need not be true in general.

Example: 3.2.9

Let X = {0, a, b, c, d, 1} be a set with the following table:

*	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	0	a	0	0
b	b	a	0	b	a	0
c	c	c	b	0	0	0
d	d	c	b	a	0	0
1	1	d	b	a	a	0

⊙	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	1	b	b	1
c	c	1	1	c	1	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

Then $(X; *, \odot, 0)$ is a companion d-algebra, which is not a BCK/BCI-algebra, since $(c * b) * d = a \neq 0 = (c * b) * b$. We know that $a \leq b$, but $a \odot c = d$ and $b \odot c = b$, and d and b are incomparable. Hence $a << b$ does not hold.

CHAPTER IV

CHAPTER – 4

SOME CONSTRUCTIONS OF IMPLICATIVE/ COMMUTATIVE d-ALGEBRAS AND CONSTRUCTIVE FUNCTION d-ALGEBRAS

Section 4.1

Some constructions of implicative/commutative d-algebras

Definition: 4.1.1 [3]

A field (X, +, ·) is called $\sqrt{3}$-exponential if there is a function φ: X $\rightarrow$ X such that

(E1) $\varphi(\varphi(x)) = x^3$,

(E2) $\varphi(xy) = \varphi(x)\,\varphi(y)$,

(E3) If $x \neq 0$, then $\varphi(x) \neq 0$,

(E4) $\varphi(0) = 0$

for any x, y $\in$ X.

Example: 4.1.2

Let X = $\mathbb{R}$ be the set of all real numbers. If we define a map φ: X $\rightarrow$ X

$$\varphi(x) = \begin{cases} x^{\sqrt{3}} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -y^{\sqrt{3}} & \text{if} \quad x = -y < 0 \end{cases}$$

Then ($\mathbb{R}$; +, ·) is $\sqrt{3}$-exponential.

Proposition: 4.1.3

Let (X; +, ·) be a $\sqrt{3}$-exponential field. If we define a new binary operation Δ on X by $x \,\Delta\, y = x^2\varphi(y)y$ for any x, y $\in$ X, then $x \,\Delta\, (x \,\Delta\, y) = y \,\Delta\, (y \,\Delta\, x)$ for any x, y $\in$ X.

Proof

Given x, y ∈ X, we have

$$x \,\Delta\, (x \,\Delta\, y) = x^2\, \varphi(x \,\Delta\, y)\, (x \,\Delta\, y)$$
$$= x^2\, \varphi(x^2\varphi(y)y)\, (x^2\varphi(y)y)$$
$$= x^4\, y^4\, \varphi(x)^2\, \varphi(y)^2. \qquad (1)$$

Similarly,

$$y \,\Delta\, (y \,\Delta\, x) = y^4\, x^4\, \varphi(y)^2\, \varphi(x)^2 \qquad (2)$$

From (1) and (2)

$x \,\Delta\, (x \,\Delta\, y) = y \,\Delta\, (y \,\Delta\, x)$ for any $x, y \in X$.

Hence the proof.

Definition: 4.1.4

i. A d/BCK-algebra $(X; *, 0)$ is said to be a commutative d/BCK-algebra if $x * (x * y) = y * (y * x)$ $\forall$ $x, y \in X$.

ii. A d/BCK-algebra $(X; *, 0)$ is said to be an implicative d/BCK-algebra if $x = x * (y * x)$ for any $x, y \in X$.

Theorem: 4.1.5

Let X be a set with $0 \in X$. if we define a binary operation $*$ on X by

$$x * y = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

then $(X; *, 0)$ is an implicative BCK-algebra.

Theorem: 4.1.6

Let $(X; +, \cdot)$ be a $\sqrt{3}$-exponential field and let $x \,\Delta\, y = x^2\varphi(y)y$ for any $x, y \in X$. If we define a binary operation "$*$" on X by

$$x * y = \begin{cases} 0 & \text{if} \quad x = 0 \text{ or } x = y, \\ x & \text{if} \quad y = 0, \\ x\Delta y & \text{otherwise.} \end{cases}$$

then $(X; *, 0)$ is an commutative d-algebra.

Proof

Let $x * y = y * x = 0$

If $x = 0$ or $y = 0$, then it is easy to see that $x = y$.

Assume that $xy \neq 0$ and $x \neq y$, then $x^2\varphi(y)y = y^2\varphi(x)x = 0$,

$\Rightarrow \varphi(x) = \varphi(y) = 0$.

By (E3) we obtain $x = y$, a contradiction to our assumption.

Hence $(X; *, 0)$ is a d-algebra.

Claim: $(X; *, 0)$ is commutative.

If $xy \neq 0$ and $x \neq y$, then

$x * (x * y) = x \Delta (x \Delta y) = y \Delta (y \Delta x) = y * (y * x)$ by Proposition 4.1.3.

The other cases are trivial.

Hence the proof.

Note:

The commutative d-algebra $(X; *, 0)$ described in Theorem 4.1.6 need not be a BCK-algebra.

Proposition: 4.1.7

Let X be a field and let $x, y \in X$. If we define $x * y = x(x - y)\,\varphi(x, y)$ where $\varphi : X \times X \rightarrow X$ is a function with $\varphi(x, y) \neq 0$ for any $x, y \in X$. Then $(X; *, 0)$ is a d-algebra.

Proof

Assume $x * y = y * x = 0$.

Then $x(x - y)\varphi(x, y) = 0$ and $y(y - x)\varphi(y, x) = 0$

$\Rightarrow x(x - y) = 0 = y(y - x)$.

This leads to $x = y$, since $x \neq y = 0$ implies $x = 0, y = 0$,

i,e., $x = y$, a contradiction.

Hence $(X; *, 0)$ is a d-algebra.

Note:

A d-algebra (X; $\ast$, 0) described in proposition 4.1.6 is called a φ-function d-algebra.

Proposition: 4.1.8

If (X; $\ast$, 0) is an implicative d-algebra, then $x \ast 0 = 0$ for any $x \in X$.

Proof

If X is implicative, then $x = x \ast (y \ast x)$ for any $x, y \in X$.

Put $y = x$, then $x = x \ast (x \ast x) = x = x \ast 0$.

Hence the proof.

Proposition: 4.1.9

Let (X; $\ast$, 0) be a φ-function d-algebra. Then (X; $\ast$, 0) is implicative if and only if φ satiesfies the condition:

$$\varphi\,(x,\, y \ast x) = \begin{cases} \dfrac{1}{x - y \ast x} & \text{if} \quad x \neq 0, \\ a & \text{otherwise.} \end{cases}$$

where a is an arbitrary element of X.

Proof

Obvious.

Note:

If $x \neq 0$, then $x \neq y \ast x$ in proposition 4.1.9.

Definition: 4.1.10

A d/BCK-algebra (X; $\ast$, 0) is said to be a positive implicative if $(x \ast y) \ast z = (x \ast z) \ast (y \ast x)$ for any $x, y, z \in X$.

Proposition: 4.1.11

There are no positive implicative φ-function d-algebras which are not BCK-algebras.

Proof

Assume that the implicative φ-function d-algebra $(X; *, 0)$ which is not a BCK-algebra is positive implicative.

Then $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

Let $z = x$, then $(x * y) * x = (x * x) * (y * x)$

$$= 0 * (y * x)$$

$$(x * y) * x = 0.$$

Since $(X; *, 0)$ is a φ-function d-algebra,

we have $0 = (x * y)[(x * y) - x]\varphi(x * y, x)$.

Since $\varphi(x, y) \neq 0, \ \forall x, y \in X$,

we obtain $0 = (x * y)[x * y - x]$.

Therefore, either $x * y = 0$ or $x * y = x$,

i.e., $x * y \in \{0, x\}, \ \forall x, y \in X$.

Assume that there are $x, y \in X$ such that $x \neq 0$, $x \neq y$ and $x * y = 0$.

Then $0 = x * y = x(x - y)\ \varphi\ (x, y) \neq 0$, a contradiction.

Hence we have $x * y = 0$ if $x = y$ and $x * y = x$ if $x \neq y$,

i.e., $(X; *, 0)$ is an implicative BCK-algebra by Theorem 4.1.5, a contradiction.

Hence there are no positive implicative φ-function d-algebras which are not BCK-algebras.

Theorem: 4.1.12

A BCK-algebra X is positive implicative if and only if

$(x * y) * y = x * y$ for any $x, y \in X$.

Proof

Obvious.

Theorem: 4.1.13

If the φ -function d-algebra $(X; *, 0)$ is implicative, then

$(x * y) * y = x * y$ for any $x, y \in X$.

Proof

Let the φ -function d-algebra $(X; *, 0)$ be implicative.

Then we have $x = x (x - y * x) \varphi(x, y * x)$ for any $x, y \in X$.
Assume that $x \neq 0$,

since $x = 0$ implies $(0 * y) * z = 0 = (0 * z) * (y * z)$.

Then we have $1 = (x - y * x) \varphi(x, y * x)$.

Hence, $\varphi(x, y * x) = \dfrac{1}{(x - y*x)}$

Also, $y * x \neq 0$ and $y * x \neq x * (y * x)$.

$$\text{Then } \varphi(y * x, x) = \varphi(y * x, x * (y * x)) = \frac{1}{(y*x - x*(y*x))}$$

$$= \frac{1}{(y*x - x)}$$

$$= \frac{-1}{(x - y*x)}$$

$$= -\varphi(x, y * x)$$

Hence $\varphi(y* x, x) = -\varphi(x, y * x)$.
Given $x, y \in X$, we have

$$(y * x) * x = (y * x) (y * x - x) \varphi(y * x, x)$$

$$= (y * x) (y * x - x) [-\varphi(x, y * x)]$$

$$= (y * x) (x - y * x) \varphi(x, y * x).$$

Since $x = x * (y * x) = x(x - y * x) \varphi(x, y * x)$, we have

$$x - (y * x) * x = (y * x - x)^2 \varphi(x, y * x)$$

$$= (y * x - x)^2 \frac{1}{(x - y*x)}$$

$$= x - y * x$$

$\Rightarrow (y * x) * x = y * x.$

Hence the proof.

Note:

In BCK-algebras, the condition $(x * y) * (x * z) = (x * y) * x$ is equivalent to the condition $(x * y) * y = x * y$, but it is not equivalent in d-algebras in general.

Example: 4.1.14

If we define a map $\varphi: X \to X$ by

$$\varphi (x, y) = \begin{cases} \frac{1}{x - y} & \text{if } x(x-y) \neq 0 \\ a & \text{if } x = y \\ b & \text{if } x = 0 \end{cases}$$

then the function φ satisfies the conditions of Proposition 4.1.3, and so it defines a φ-function d-algebra $(X, *, 0)$ where

$$x * y = \begin{cases} x & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

which is an implicative BCK-algebra as described in Theorem 4.1.5.

Example: 4.1.15

If we defined a map φ on X by

$$\varphi\,(x, y) = \begin{cases} \dfrac{-y}{x(y-x)} & \text{if } y(y-x) \neq 0 \\ a & \text{otherwise} \end{cases}$$

for an arbitrary element a in X, then

$$x * y = \begin{cases} -y & \text{if} \quad y(y-x) \neq 0 \\ 0 & \text{if} \quad x = 0 \quad \text{or} \quad x = y \\ x & \text{if} \quad y = 0 \end{cases}$$

leads to a d-algebra.

If y (y - x) ≠ 0, then x ∗ (y ∗ x) = x ∗ (-x) = x for any x, y ∈ X, showing that

(X, ∗, 0) is an implicative d-algebra. Indeed, it is not a BCK-algebra,

since ((3 ∗ 4) ∗ (3 ∗ 5)) ∗ (5 ∗ 4) = 4 ≠ 0.

Example: 4.1.16

If we apply Example 4.1.15 to the finite field $\mathbb{Z}_5$, then we obtain the following table:

∗	0	1	2	3	4
0	0	0	0	0	0
1	1	0	3	2	1
2	2	4	0	2	1
3	3	4	3	0	1
4	4	4	3	2	0

Then it is an implicative d-algebra, which is not a BCK-algebra,

since ((3 ∗ 4) ∗ (3 ∗ 2)) ∗ (2 ∗ 4) = 4 ≠ 0.

Moreover, it is not positive implicative, since

(3 ∗ 4) ∗ 5 = -4 ∗ 5 = -5 and (3 ∗ 5) ∗ (4 ∗ 5) = -5 ∗ -5 = 5.

Remark: 4.1.17

In BCK-algebras, X is an implicative BCK-algebra if and only if it is both a positive implicative and a commutative BCK-algebra. But this does not hold in d-algebras.

Section 4.2

Construction of many d-algebras

Definition: 4.2.1 [3]

Let f, g : $\mathbb{R} \to \mathbb{R}$ be real valued functions such that f(t) = 0 if and only if t = 0 and g(t) = 0 if and only if t = 0. Furthermore, let h : $\mathbb{R}^2 \to \mathbb{R}$ be a real valued function such that h(u, t) $\neq$ 0 when u $\neq$ t. We say a triple (f, g, h) described above is called a constructive function triple on $\mathbb{R}$.

Example: 4.2.2

f(t) = g(t) = t, h(u, t) = 1 is a constructive function triple on $\mathbb{R}$.

Theorem: 4.2.3

Let (f, g, h) be a constructive function triple on $\mathbb{R}$ and e $\in \mathbb{R}$. If we define $x * y = f(x - y)\, g(e - x)\, h(x, y) + e$, where x, y $\in \mathbb{R}$. Then ($\mathbb{R}$; $*$, e) is a d-algebra.

Proof

For any x $\in \mathbb{R}$, $x * x = f(0)\, g(e - x)\, h(x, x) + e = e$

and $e * x = f(e - x)\, g(0)\, h(e, x) + e = e$.

If $x * y = y * x = e$, then $f(x - y)\, g(e - x)\, h(x, y) = 0 = f(y - x)\, g(e - y)\, h(y, x)$.

Assume x $\neq$ y.

Then h(x, y) $\neq$ 0 $\neq$ h(y, x) and f(y - x) g(e - y) = 0.

This means either x - y = 0 or e - x = 0; either y - x = 0 or e - y = 0.

Since x $\neq$ y, we obtain e - x = 0, e - y = 0, i.e., x = e = y, a contradiction.

Hence ($\mathbb{R}$; $*$, e) is a d-algebra.

Example: 4.2.4

The functions $f(t) = e^t - 1$, $g(t) = t^3$ and $h(u, t) = (u - t)^2$ will yield a d-algebra on the reals.

Definition: 4.2.5

The d-algebra ($\mathbb{R}$; $*$, e) described in the above theorem is called a constructive function d-algebra on $\mathbb{R}$ determined by (f, g, h).

Example: 4.2.6

Let $\mathbb{K}$ be any subring of the real numbers $\mathbb{R}$ and let (f, g, h) be a constructive function triple on $\mathbb{K}$. If we define $x * y$ on $\mathbb{K}$ as $x * y = f(x - y)\, g(e - x)\, h(x, y) + e$, where $e \in \mathbb{K}$.

Then ($\mathbb{K}$; $*$, e) is a d-algebra.

Example: 4.2.7

Let $\mathbb{D}$ be any (not necessarily commutative) integral domain and let (f, g, h) be a constructive function triple on $\mathbb{D}$. If we define $x * y$ on $\mathbb{D}$ as $x * y = f(x - y)\, g(e - x)\, h(x, y) + e$, where $e \in \mathbb{D}$, then ($\mathbb{D}$; $*$, e) is a d-algebra.

Proposition: 4.2.8

Let ($\mathbb{R}$; $*$, e) be a constructive function d-algebra determined by (f, g, h) satisfying the condition: $x * e = x$ for all $x \in \mathbb{R}$.

Then $f(t)\, g(-t)\, h(t + e, e) = t$ for any t in $\mathbb{R}$.

Proof

Since $x * e = x$, we have $x = x * e = f(x - e)\, g(e - x)\, h(x, e) + e$.

Thus $f(x - e)\, g(e - x)\, h(x, e) = x - e$.

If $x - e = t$, then $f(t)\, g(-t)\, h(t + e, e) = t$.

Example: 4.2.9

If $f(t) = g(t) = \sqrt[3]{t}$, then $h(t + e, e) = \sqrt[3]{t}$, where $e \in \mathbb{R}$.

If we take t = x-e, then $h(x, e) = \sqrt[3]{x - e}$.

Hence $x * y = \sqrt[3]{x - e}\,\sqrt[3]{x - e}\,\sqrt[3]{x - y} + e$ satisfies $x * e = x$ for all $x \in \mathbb{R}$.

Theorem: 4.2.10

Let $(\mathbb{R}; *, e)$ be a constructive function d-algebra determined by (f, g, h). If it satisfies the condition $(x * (x * y)) * y = e$ for any $x, y \in \mathbb{R}$, then it also satisfies $x * e = x$ for all $x \in \mathbb{R}$.

Proof

Assume $(x * (x * y)) * y = e$ for any $x, y \in \mathbb{R}$.

Let $u = x * (x * y)$.

Then $e = u * y = f(u - y)\, g(e - u)\, h(u, y) + e$

$\Rightarrow f(u - y)g(e - u)h(u, y) = 0$.

If $u \neq y$, then $h(u, y) \neq 0$

$\Rightarrow$either $f(u - y) = 0$ or $g(e - u) = 0$.

Hence $e = u = x * (x * y)$ for any $x, y \in \mathbb{R}$.

If we take y = e, then $e = x * (x * e) = f(x - x * e)\, g(e - x)\, h(x, x * e) + e$.

Thus $f(x - x * e)g(e - x)h(x, x * e) = 0$.

Since it is a constructive function d-algebra, we obtain either $x = x * e$ or $e - x = 0$.

i.e., in any case $x = x * e$ since $e = e * e$ as well.

If u = y, then $x * (x * y) = u = y$.

If we take y = e, then $x * (x * e) = x$,

$\Rightarrow x = x * e$ for any $x \in \mathbb{R}$.

Hence the proof

Theorem: 4.2.11

Let ($\mathbb{C}$; $*$, e) be a constructive function d-algebra on the algebraically closed field $\mathbb{C}$ of complex numbers. If we define $x * y = (x - y)(e - x)+e$, then the solution set of $F(x, y) = x * (x * y) - y * (y * x) = 0$ is

$$\left\{(x, y) \mid y = x \text{ or } \left(x - e + \frac{1}{2}\right)^2 + \left(y - e + \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2\right\}$$

Proof

For any $x, y \in X$,

$x * (x * y) = x * ((x - y)(e - x) + e)$

$= (x - ((x - y)(e - x) + e)) (e - x) + e$ [since $(x * y) = (x-y)(e-x) + e$]

$= (x - (x - y)(e - x) - e) (e - x) + e$

$= ((x - e) - (x - y)(e - x)) (e - x) + e$

$= -((e - x) + (x - y)(e - x)) (e - x) + e$

$x * (x * y) = -(1 + x - y)(e - x)^2 + e.$

Similarly, $y * (y * x) = -(1 + y - x)(e - y)^2 + e.$

Hence $F(x, y) = [x * (x * y)] – [y * (y * x)]$

$= [-(1 + x - y)(e - x)^2 + e] – [-(1 + y - x)(e - y)^2 + e]$

$= -(1 + x - y)(e - x)^2 + e + (1 + y - x)(e - y)^2 - e$

$= -(e - x)^2 - (x - y)(e - x)^2 + (e - y)^2 - (x - y)(e - y)^2$

$= (e - y)^2 -(e - x)^2 + (x - y)[-(e - x)^2 - (e - y)^2]$

$= (e - y + e - x) [(e - y) - (e - x)] + (x - y)[-(e - x)^2 - (e - y)^2]$

$= (x - y) [(e - y) - (e - x)] + (x - y)[-(e - x)^2 - (e - y)^2]$

$= (x - y)[(e - x) + (e -y) - (e - x)^2 - (e - y)^2]$

$F(x, y) = 0$

$\Rightarrow (x - y)[(e - x) + (e - y) - (e - x)^2 - (e - y)^2] = 0$

$\Rightarrow (x - y) = 0$ or $(e - x) + (e - y) - (e - x)^2 - (e - y)^2 = 0$

$\Rightarrow x = y$ or $(e - x) + (e - y) - (e - x)^2 - (e - y)^2 = 0$

$\Rightarrow x = y$ or $(x - e)^2 + (y - e)^2 + (x - e) + (y - e) = 0$

$\Rightarrow x = y$ or $\left(x - e + \frac{1}{2}\right)^2 + \left(y - e + \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$

Hence the solution of F(x,y) is :

$x - y = 0$

or

$$\left(x - e + \frac{1}{2}\right)^2 + \left(y - e + \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2.$$

This is a description of the commutativity set of the d-algebra. This is an algebraic set of the union of two algebraic geometry varieties, viz., the line x = y and the complex circle.

Note:

Consider the equation $E(x, y) = (x * (x * y)) * y - e = 0$. This set is referred as the implicativity set of the d-algebra.

Since $x * y = (x - y)(e - x) + e$, and $x * (x * y) = (y - x - 1)(e - x)^2 + e$,

$$\begin{aligned}(x * (x * y)) * y &= [(y - x - 1)(e - x)^2 + e - y]\,[e - ((y - x - 1)(e - x)^2 + e)] + e \\ &= [(y - x - 1)(e - x)^2 + (e - y)]\,[e - (y - x - 1)(e - x)^2 - e] + e \\ &= [(y - x - 1)(e - x)^2 + (e - y)]\,[- (y - x - 1)(e - x)^2] + e\end{aligned}$$

$(x * (x * y)) * y - e = [(y - x - 1)(e - x)^2 + (e - y)]\,[- (y - x - 1)(e - x)^2]$

$E(x, y) = (y - x - 1)(e - x)^2[y - e - (y - x - 1)(e - x)^2] = 0.$

$E(x, y) = 0$

$\Rightarrow (y - x - 1)(e - x)^2[y - e - (y - x - 1)(e - x)^2] = 0.$

$\Rightarrow x = e$ or $y = x + 1$, or $y(1 - (e - x)^2) = e - (x + 1)(e - x)^2$.

If $(e - x)^2 = 1$, then $x = e \pm 1$,

while otherwise, $y = \dfrac{e-(x+1)(e-x)^2}{1-(e-x)^2}$

If $e = 0$, then $y = x + 1 + \dfrac{1}{x-1}$ with asymptote $y = x + 1$, which is also on the implicativite set.

CHAPTER V

Chapter 5

ON FUZZY d-ALGEBRAS

Section 5.1

Preliminary Definitions and results in fuzzy sets

Definition: 5.1.1 [40]

Let X be any arbitrary set. Let I = [0,1] be the unit interval. Any function μ: X $\rightarrow$ [0,1] is called a fuzzy set on X. The collection of all fuzzy sets defined on X is defined by I^X.

Definition: 5.1.2

Let μ and γ be fuzzy sets on X, then

i. $\mu = \gamma \Leftrightarrow \mu(x) = \gamma(x), \ \forall x \in X.$

ii. $\mu \leq \gamma \Leftrightarrow \mu(x) \leq \gamma(x), \ \forall x \in X.$

iii. $(\mu \vee \gamma)(x) \Leftrightarrow \max\{\mu(x), \gamma(x) / x \in X\}.$

iv. $(\mu \wedge \gamma)(x) \Leftrightarrow \min\{\mu(x), \gamma(x) / x \in X\}.$

v. $\mu^c(x) \Leftrightarrow 1 - \mu(x), \ x \in X.$

Definition: 5.1.3

For a family of fuzzy sets $\{\mu_\lambda\}_{\lambda \in \wedge}$, the union $\bigvee_{\lambda \in \wedge}$ and the intersection $\bigwedge_{\lambda \in \wedge}$ are defined by

$$(\bigvee_{\lambda \in \wedge} \mu_\lambda)(x) = \sup_{\lambda \in \wedge}\{\mu_\lambda(x) / x \in X\}$$

$$(\bigwedge_{\lambda \in \wedge} \mu_\lambda)(x) = \inf_{\lambda \in \wedge}\{\mu_\lambda(x) / x \in X\}.$$

Definition: 5.1.4

The symbol 0_x or ϕ will be used to denote an empty fuzzy set 0_x is defined as $0_x(x) = 0$ for all $x \in X$ and 1_x or X denotes the whole fuzzy set where 1_x is defined as $1_x(x) = 1 \ \forall x \in X$.

Definition: 5.1.5

The constant fuzzy set, denoted by α, is defined as, $\alpha(x) = \alpha, \ \forall x \in X$.

Definition: 5.1.6

Any subset μ of X can be identified with a fuzzy set χ_μ, the characteristic function of μ. The function $\chi_\mu : X \to [0,1]$ is defined by

$$\chi_\mu(x) = 1 \text{ if } x \in \mu$$

$$= 0 \text{ if } x \notin \mu$$

Definition: 5.1.7

Let μ be a fuzzy set on X. The set $\{x \in \mu \mid \mu(x) > 0\}$ is called the support of μ and is denoted by supp μ. If μ takes only the values 0, 1 then μ is called a crisp set in X.

Definition: 5.1.8

Let μ be a fuzzy set of a set X. For a fixed $s \in [0, 1]$, the set $\mu_s = U(\mu;s) = \{x \in \mu \mid \mu(x) \geq s\}$ is called an upper level subset of μ.

Section 5.2

On fuzzy subalgebras

Definition: 5.2.1 [6]

A fuzzy set μ in d-algebra X is called a fuzzy subalgebra of X if it satisfies $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Example: 5.2.2

Consider a d-algebra $(X; *, 0)$ as in Example 1.1.6-(ii).

Define a fuzzy set $\mu: X \to [0,1]$ by $\mu(0) = 0.7$, $\mu(x) = 0.02$, where for all $x \neq 0$.

Then μ is a fuzzy subalgebra of X.

Example 5.2.3

Let $X = \{0, 1, 2, ...\}$ be a set and the operation $*$ be defined as follows:

$$x * y = \begin{cases} 0 & \text{if } x \leq y \\ x - y & \text{if } y < x \end{cases}$$

Then $(X; *, 0)$ is an infinite d-algebra.

Define a fuzzy set $\mu: X \to [0,1]$ by $\mu(0) = t_1$, $\mu(x) = t_2$, where for all $x \neq 0$, where $t_1 > t_2$.

Then μ is a fuzzy subalgebra of X.

Proposition: 5.2.4

A fuzzy set μ of a d-algebra X is a fuzzy subalgebra if and only if for every $t \in [0, 1]$ the upper level subset μ_t is either empty or a subalgebra of X.

Proof

Suppose that μ is a fuzzy subalgebra of a d-algebra X and $\mu t \neq 0$, then for any $x, y \in \mu_t$, we have $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \geq t$.

This implies $x * y \in \mu_t$.

Hence μ_t is a subalgebra of X.

Conversely,

Take $t = \min \{\mu(x), \mu(y)\}$, for any $x, y \in X$.

Then by assumption, μ_t is a subalgebra of X.

This implies $x * y \in \mu_t$.

Therefore $\mu(x * y) \geq t = \min \{\mu(x), \mu(y)\}$.

Hence μ is a fuzzy subalgebra of X.

Proposition: 5.2.5

Any subalgebra of a d-algebra X can be realized as an upper level subalgebra of some fuzzy subalgebra of X.

Proof

Let A be a subalgebra of a d-algebra X and μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t, & \text{if } \quad x \in A \\ 0 & \text{otherwise} \end{cases} \qquad \text{where } t \in (0,1).$$

It is clear that $\mu_t = A$.

Let $x, y \in X$.

If $x, y \in A$, then $x * y \in A$.

So $\mu(x) = \mu(y) = \mu(x * y) = t$ and $\mu(x * y) \geq \min \{\mu(x), \mu(y)\}$.

If $x, y \notin A$, then $\mu(x) = \mu(y) = 0$.

Thus $\mu(x * y) \geq \min \{\mu(x), \mu(y)\} = 0$.

If at most one of $x, y \in A$, then at least one of $\mu(x)$ and $\mu(y)$ is equal to 0.

Therefore, $\min\{\mu(x), \mu(y)\} = 0$ and $\mu(x * y) \geq 0$ which completes the proof.

Corollary: 5.2.6

Let A be a subset of X. Then the characteristic function χ_A is a fuzzy subalgebra of X if and only if A is a subalgebra of X.

Proof

Obvious.

Lemma: 5.2.7

Let μ be a fuzzy subalgebra of a d-algebra X with finite image. If $\mu_s = \mu_t$ for some s, t $\in$ Im(μ), then s = t.

Proof

Obvious.

Lemma: 5.2.8

Let μ and λ be two fuzzy subalgebras of a d-algebra X with identical family of level subalgebras. If Im(μ) = $\{t_1, t_2, \ldots, t_n\}$ and Im(λ) = $\{s_1, s_2, \ldots, s_m\}$, where $t_1 \geq t_2 \geq \ldots \geq t_n$ and $s_1 \geq s_2 \geq \ldots \geq s_m$. Then

(1) m = n.

(2) $\mu_{t_i} = \lambda_{s_i}$, for i= 1, 2, ... , n.

(3) If $\mu(x) = s_i$, then ¸$(x) = s_i$, for all x $\in$ X and i=1, 2, …, n.

Proof

Obvious.

Proposition: 5.2.9

Let μ and λ be two fuzzy subalgebras of a d-algebra X with identical family of level subalgebras. Then $\mu = \lambda \Rightarrow \mathrm{Im}(\mu) = \mathrm{Im}(\lambda)$.

Proof

Let Im(μ) = Im(λ) = $\{s_1, \ldots, s_n\}$ and s1 > ... > s_n.

By lemma 5.2.8, for any x $\in$ X, there exists s_i such that $\mu(x) = s_i = \lambda(x)$.

Thus $\mu(x) = \lambda(x)$, $\forall$ x $\in$ X.

Hence proved.

Proposition: 5.2.10

Let X be a d-algebra.Two level subalgebras μ_s and μ_t, $(s < t)$ of a fuzzy subalgebra μ are equal if and only if there is no $x \in X$ such that $s \leq \mu(x) < t$.

Proof

Suppose that $\mu_s = \mu_t$ for some $s < t$.

If there exists $x \in X$ such that $s \leq \mu(x) < t$, then μ_t is a proper subset of μ_s, which is contradicting the hypothesis.

Conversely,

suppose that there is no $x \in X$ such that $s \leq \mu(x) < t$.

If $x \in \mu_s$, then $\mu(x) \geq s$ and so $\mu(x) \geq t$, since $\mu(x)$ does not lie between s and t.

Thus $x \in \mu_t$, which gives $\mu_s \subseteq \mu_t$.

The converse inclusion, $\mu_t \subseteq \mu_s$ is obvious since $s < t$.

Therefore, $\mu_s = \mu_t$.

Hence the proof.

Section 5.3

On fuzzy d-ideals in d-algebras

Definition: 5.3.1 [6]

A fuzzy set μ in a d-algebra X is called fuzzy BCKd-ideal of X if it satisfies the following inequalities:

(1) $\mu(0) \geq \mu(x)$,

(2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

for all x, y $\in$ X.

Definition: 5.3.2 [6]

A fuzzy set μ in a d-algebra X is called fuzzy d-ideal of X if it satisfies the following inequalities:

(Fd1) $\mu(0) \geq \mu(x)$,

(Fd2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$,

(Fd3) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

for all x, y $\in$ X.

Example: 5.3.3 [37]

Let X = {0, 1, 2, 3} be a d-algebra with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Define fuzzy set μ in X by $\mu(0) = 0.8$ and $\mu(x) = 0.01$ for all $x \neq 0$ in X. This shows that μ is a d-ideal of X.

Note:

(1) In a d-algebra X, every fuzzy d-ideal is a fuzzy BCKd-ideal, and every fuzzy BCKd-ideal is a fuzzy subalgebra of X.

(2) Every fuzzy d-ideal of a d-algebra X is a fuzzy subalgebra of X.

Theorem: 5.3.4

Let X be a d-algebra.If each non-empty level subset $U(\mu; t)$ of μ is a fuzzy ideal of X then μ is a fuzzy d-ideal of X, where $t \in [0, 1]$.

Proof

Assume that each non-empty level subset $U(\mu; s)$ of μ is a d-ideal.

Then μ satisfies (Fd1) and (Fd2).

Assume that $\mu(x * y) < \min\{\mu(x), \mu(y)\}$ for some $x, y \in X$.

Take $t_0 = 1/2\{\mu(x * y) + \min(\mu(x), \mu(y))\}$, then $x, y \in U(\mu; t_0)$.

Since μ is a d-ideal of X, $x * y \in U(\mu; t_0)$.

Therefore, $\mu(x * y) \geq t_0$, a contradiction.

Hence assumption is wrong.

Hence the proof.

Definition: 5.3.5

Let λ and μ be the fuzzy sets in a set X. The cartesian product $\lambda \times \mu : X \times X \to [0, 1]$ is defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(x)\}$, $\forall x, y \in X$.

Theorem: 5.3.6

If λ and μ are fuzzy d-ideals of a d-algebra X. Then $\lambda \times \mu$ is a fuzzy d-ideal of $X \times X$.

Proof

For any $(x, y) \in X \times X$, we have

$(\lambda \times \mu)(0, 0) = \min\{\lambda(0), \mu(0)\} \geq \min\{\lambda(x), \mu(y)\} = (\lambda \times \mu)(x, y)$

Let (x_1, x_2) and $(y_1, y_2) \in X \times X$.

Then $(\lambda \times \mu)((x_1, x_2)) = \min\{\lambda(x_1), \mu(x_2)\}$

$\geq \min\{\min\{\lambda(x_1 * y_1), \lambda(y_1)\}, \min\{\mu(x_2 * y_2), \mu(y_2)\}\}$

$= \min\{\min\{\lambda(x_1 * y_1), \mu(x_2 * y_2)\}, \min\{\lambda(y_1), \mu(y_2)\}\}$

$= \min\{(\lambda \times \mu)((x_1 * y_1, x_2 * y_2)), (\lambda \times \mu)((y_1, y_2))\}$

$= \min\{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)((y_1, y_2))\}$

and $(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)) = (\lambda \times \mu)((x_1 * y_1, x_2 * y_2))$

$= \min\{\lambda(x_1 * y_1), \mu(x_2 * y_2)\}$

$\geq \min\{\min\{\lambda(x_1),$ ¸$(y_1)\}, \min\{\mu(x_2), \mu(y_2)\}\}$

$= \min\{\min\{\lambda(x_1), \mu(x_2)\}, \min\{\lambda(y_1), \mu(y_2)\}\}$

$= \min\{(\lambda \times \mu)((x_1, x_2)), (\lambda \times \mu)((y_1, y_2))\}$.

Hence $\lambda \times \mu$ is a fuzzy d-ideal of $X \times X$.

Theorem: 5.3.7

Let λ and μ be fuzzy sets in a d-algebra X such that $\lambda \times \mu$ is a fuzzy d-ideal of $X \times X$. Then

(i) either $\lambda(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$, $\forall x \in X$.

(ii) If $\lambda(0) \geq \lambda(x)$, $\forall x \in X$, then either $\mu(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$.

(iii) If $\mu(0) \geq \mu(x)$, $\forall x \in X$, then either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$.

Proof

(i) Assume $\lambda(x) > \lambda(0)$ and $\mu(y) > \mu(0)$, for some $x, y \in X$.

Then $(\lambda \times \mu)(x, y) = \min \{\lambda(x), \mu(y)\} > \min \{\lambda(0), \mu(0)\} = (\lambda \times \mu)(0, 0)$

$\Rightarrow (\lambda \times \mu)(x, y) > (\lambda \times \mu)(0, 0), \ \forall \ x, y \in X$

which is a contradiction.

Hence (i) is proved.

(ii) Assume $\mu(0) < \lambda(x)$ and $\mu(0) < \mu(y), \ \forall \ x, y \in X$.

Then $(\lambda \times \mu)(0, 0) = \min \{\lambda(0), \mu(0)\} = \mu(0)$

and $(\lambda \times \mu)(x, y) = \min \{\lambda(x), \mu(y)\} > \mu(0) = (\lambda \times \mu)(0, 0)$

$\Rightarrow (\lambda \times \mu)(x, y) > (\lambda \times \mu)(0, 0)$

which is a contradiction.

Hence (ii) is proved.

Similarly (iii) can be proved.

Theorem: 5.3.8

If $\lambda \times \mu$ is a fuzzy d-ideal of $X \times X$, then λ or μ is a fuzzy d-ideal of X.

Proof

By Theorem 5.3.7(i), without loss of generality we assume that $\mu(0) \geq \mu(x), \ \forall x \in X$.

It follows from Theorem 5.3.7(iii) that either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$.

If $\lambda(0) \geq \mu(x), \ \forall \ x \in X$.

Then $(\lambda \times \mu)(0, x) = \min \{\lambda(0), \mu(x)\} = \mu(x)$ (1)

Since $\lambda \times \mu$ is a fuzzy d-ideal of $X \times X$.

Therefore, $(\lambda \times \mu)(x_1, x_2) \geq \min \{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\}$

and $(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)) \geq \min \{(\lambda \times \mu)((x_1, x_2), (\lambda \times \mu)(y_1, y_2)\}$

$\Rightarrow (\lambda \times \mu)(x_1, x_2) \geq \min \{(\lambda \times \mu)((x_1 * y_1, x_2 * y_2)), (\lambda \times \mu)(y_1, y_2)\}$

and $(\lambda \times \mu)((x_1 * y_1, x_2 * y_2)) \geq \min \{(\lambda \times \mu)((x_1, x_2), (\lambda \times \mu)(y_1, y_2)\}$.

Putting $x_1 = y_1 = 0$,

we have $(\lambda \times \mu)(0, x_2) \geq \min\{(\lambda \times \mu)(0, x_2 * y_2), (\lambda \times \mu)(0, y_2)\}$

and $(\lambda \times \mu)(0, x_2 * y_2) \geq \min\{(\lambda \times \mu)(0, x_2), (\lambda \times \mu)(0, y_2)\}$.

Using equation (1),

we have $\mu(x_2) \geq \min\{\mu(x_2 * y_2), \mu(y_2)\}$ and $\mu(x_2 * y_2) \geq \min\{\mu(x_2), \mu(y_2)\}$.

$\Rightarrow \mu$ is a fuzzy d-ideal of X.

The second part is similar.

Hence the proof.

Definition: 5.3.9

Let A be a fuzzy set in a set S, the strongest fuzzy relation on S that is fuzzy relation on A is μ_A given by $\mu_A(x, y) = \min\{A(x), A(y)\}$, for all $x, y \in S$.

Theorem: 5.3.10

Let A be a fuzzy set in a d-algebra X and μ_A be the strongest fuzzy relation on X. Then A is a fuzzy d-ideal of X if and only if μ_A is a fuzzy d-ideal of $X \times X$.

Proof

Suppose that A is a fuzzy d-ideal of X.

Then $\mu_A(0,0) = \min\{A(0),A(0)\} \geq \min\{A(x),A(y)\} = \mu_A(x,y)$, $\forall (x, y) \in X \times X$.

For any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X \times X$,

we have $\mu_A(x) = \mu_A(x_1, x_2)$

$$= \min\{A(x_1), A(x_2)\}$$

$$\geq \min\{\min\{A(x_1 * y_1), A(y_1)\}, \min\{A(x_2 * y_2), A(y_2)\}\}$$

$$= \min\{\min\{A(x_1 * y_1), A(x_2 * y_2)\}, \min\{A(y_1), A(y_2)\}\}$$

$$= \min\{\mu_A(x_1 * y_1, x_2 * y_2), \mu_A(y_1, y_2)\}$$

$$= \min\{\mu_A((x_1, x_2) * (y_1 * y_2)), \mu_A(y_1, y_2)\}$$

$$= \min\{\mu_A(x * y), \mu_A(y)\}$$

and $\mu_A(x * y) = \mu_A((x_1, x_2) * (y_1, y_2))$

$$= \mu_A((x_1 * y_1, x_2 * y_2))$$

$$= \min \{A(x_1 * y_1), A(x_2 * y_2)\}$$

$$\geq \min \{\min \{A(x_1), A(y_1)\}, \min \{A(x_2), A(y_2)\}\}$$

$$= \min \{ \min \{A(x_1), A(x_2)\}, \min \{A(y_1), A(y_2)\}\}$$

$$= \min \{\mu_A((x_1, x_2), \mu_A(y_1, y_2)\}$$

$$= \min \{\mu_A(x), \mu_A(y)\}.$$

Hence μ_A is a fuzzy d-ideal of $X \times X$.

Conversely,

suppose that μ_A is a fuzzy d-ideal of $X \times X$.

Then $\min\{A(0),A(0)\} = \mu_A(0,0) \geq \mu_A(x,y) = \min \{A(x),A(y)\}$, $\forall$ $(x, y) \in X \times X$.

It follows that $A(x) \leq A(0)$, $\forall x \in X$.

For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in X \times X$, we have

$$\min\{A(x_1),A(x_2)\} = \mu_A(x_1, x_2)$$

$$\geq \min \{\mu_A((x_1, x_2) * (y_1, y_2)), \mu_A(y_1, y_2)\}$$

$$= \min \{\mu_A(x_1 * y_1, x_2 * y_2), \mu_A(y_1, y_2)\}$$

$$= \min\{\min\{A(x_1 * y_1),A(x_2 * y_2)\}, \min\{A(y_1),A(y_2)\}\}$$

$$= \min\{\min\{A(x_1 * y_1),A(y_1)\},\min\{A(x_2 * y_2),A(y_2)\}\}.$$

Putting $x_2 = y_2 = 0$,

we have $\mu_A(x_1) \geq \min \{\mu_A(x_1 * y_1), \mu_A(y_1)\}$.

Likewise, $\mu_A(x_1 * y_1) \geq \min \{\mu_A(x_1), \mu_A(y_1)\}$.

Hence A is fuzzy d-ideal of X.

Definition: 5.3.11

Let $f : X \to Y$ be a mapping of d-algebras and μ be a fuzzy set of Y. The map μ^f is the pre-image of μ under f, if $\mu^f(x) = \mu(f(x)),\ \forall\ x \in X$.

Theorem: 5.3.12

Let $f : X \to Y$ be a homomorphism of d-algebras. If μ is a fuzzy d-ideal of Y, then μ^f is a fuzzy d-ideal of X.

Proof

For any $x \in X$, we have

$\mu^f(x) = \mu(f(x)) \leq \mu(\acute{}0) = \mu(f(0)) = \mu^f(0)$

Let $x, y \in X$.

Then $\min\{\mu^f(x * y), \mu^f(y)\} = \min\{\mu(f(x * y)), \mu(f(y))\}$

$= \min\{\mu(f(x) * f(y)), \mu(f(y))\}$

$\leq \mu(f(x))$

$= \mu^f(x)$.

and $\min\{\mu^f(x), \mu^f(y)\} = \min\{\mu(f(x)), \mu(f(y))\}$

$= \min\{\mu(f(x)), \mu(f(y))\}$

$\leq \mu(f(x) * f(y))$

$= \mu(f(x * y))$

$= \mu^f(x * y)$.

Hence μ^f is a fuzzy d- ideal of X.

Theorem: 5.3.13

Let $f : X \to Y$ be an epimorphism of d-algebras. If μ^f is a fuzzy d-ideal of X, then μ is a fuzzy d-ideal of Y.

Proof

Let $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Then $\mu(y) = \mu(f(x)) = \mu^f(x) \leq \mu^f(0) = \mu(f(0)) = \mu(\acute{}0)$

Let $x, y \in Y$.

Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$.

$$\begin{aligned}
\Rightarrow \mu(x) = \mu(f(a)) &= \mu^f(a) \\
&\geq \min \{\mu^f(a * b), \mu^f(b)\} \\
&= \min \{\mu(f(a * b)), \mu(f(b))\} \\
&= \min \{\mu(f(a) * f(b)), \mu(f(b))\} \\
&= \min \{\mu(x * y), \mu(y)\}
\end{aligned}$$

$$\begin{aligned}
\text{and } \mu(x * y) = \mu(f(a) * f(b)) &= \mu^f(a * b) \\
&\geq \min \{\mu^f(a), \mu^f(b)\} \\
&= \min \{\mu(f(a)), \mu(f(b))\} \\
&= \min \{\mu(x), \mu(y)\}.
\end{aligned}$$

Hence μ is a fuzzy d- ideal of Y.

SUMMARY AND CONCLUSION

SUMMARY AND CONCLUSION

Y. Imai and K. Iseki [13, 14] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the notion of BCI-algebras is a generalization of BCK-algebras. J. Neggers and H. S. Kim [33] introduced the class of d-algebras which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK-algebras.

L. A. Zadeh [40] introduced the notion of fuzzy sets and A. Rosenfeld [37] introduced the notion of fuzzy group. Following the idea of fuzzy groups, O. G. Xi [39] introduced the notion of fuzzy BCK-algebras. In [6] M. Akram fuzzified d-algebras.

In this thesis we have made an attempt to study the properties of d-algebras and fuzzy d-algebras.

In chapter 1, the notion of d-algebras which is another generalization of BCK-algebras and ideal theory in d-algebras are investigated due to J. Neggers [33, 34].

In chapter 2, the properties of quotient d-algebras are given. Also the number of d^*-subalgebras of order i in a d-transitive d^*-algebra is estimated [34, 26].

In chapter 3, a theory of companion d-algebras is studied in a detailed manner due to P. J. Allen, H. S. Kim and J. Neggers [7].

In chapter 4, some constructions of implicative/commutative d-algebras which are not BCK-algebras are given [3]. Also some properties of the constructive function d-algebras on $\mathbb{R}$ determined by constructive function triple (f, g, h) are discussed [8].

In chapter 5, the fuzzification of d-algebras and d-ideals in d-algebras are studied due to M. Akram [6].

A deep study of d-algebras and fuzzy d-algebras can be extended to intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets. So it provides a lot of scope for further research.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Ahmad, B., **Fuzzy BCI- algebras**, J. Fuzzy Math 2(1993), 445-452.
2. Ahn, S.S., and Han, G.H., **Intutionistic fuzzy quick ideals in d-algebras**, Honam Mathematical Journal. 31(2009), No 3, 351-368.
3. Ahn, S.S., and Kim, H.S., **Some construction of implicative commutative d-algebras**, Bull. Korean Math Soc. 46(2009), No.1, 147-153.
4. Ahn, S.S., and Han, G.H., **Rough fuzzy quick ideal in d-algebras**, Commun. Korean Math. Soc. 25(2010), No.4, 511-522.
5. Ahn, S.S., and Han, G.H., **Ideal theory of d-algebras based on N-structures**, J. Appl. Math. And Informatics, Vol. 29(2011), No. 5-6, 1489-1500.
6. Akram, M., and Dar, K.H., **On fuzzy d-algebras**, Journal of Mathematics, Vol. 37(2005), 61-76.
7. Allen, P.J., Kim, H.S., and Neggers, J.,**On companion d-algebras**, Math, Slovaca 57(2007), 93-106.
8. Allen, P.J., **Construction of many d-algebras**, Commun. Korean Math, Soc. 24(2009), No. 3, 361-366
9. Allen, P.J., Kim, H.S., and Neggers, J., **Deformations of d/BCK-algebras**, Bull. Korean Math. Soc. 48(2011), No.2, 315-324.
10. Ashram Borumand Saeid, **Fuzzy dot BCK/BCI-algebras**, International Journal of Algebras, Vol. 4(2010), No.7, 341-352.
11. Georgescu, G., and Iorgulescu, A., **Pseudo- BCK-algebras an extention of BCK-algebras**, Combinatorics, Computability and logic (Constanta, 2011), 97-114, Springer ser. Discrete Math. Theor. Comput. Sci., Springer, London, 2011.
12. Han, J.S., Kim, H.S., and Neggers, J., **Strong and ordinary d-algebras**, J.Mutiple-Valued logic and Soft computing, (to appear).
13. Imai, Y., and Iseki, K., **On axiom system of propositional calculi XIV**, Proc. Tapan Acad. Ser. Appl. Math. Sci. 42(1996), 19-22.

14. Iseki, K., **An algebra related with a propositional calculus**, Proc. Japan. Acad. Ser.Appl. Math, Sci. 42(1996), 26-29
15. Iseki, K., and Tanaka, S., **An introduction to the theory of BCK-algebra**, Math. Japonica, 23(1978), 935-942.
16. Iseki, K., **BCK-algebras with condition**(s), Math. Japonica, 24, No. 1(1979), 107-117.
17. Iseki, K., **On BCI-algebras**, Math. Semin. Notes, Kobe Univ. 8(1980), 125-130.
18. Ivan Chajda and Jan, K.H., **Algebriac structures derived from BCK-algebras**, Miskok Mathematical Notes, Vol. 8(2009), No. 1, 11-21.
19. Jun, Y.B., and Meng, J., **Characterization of fuzzy sub-algebras by their level sub-algebras**, selected papers on BCK and BCI- algebras. 1(1992), 60-65.
20. Jun. Y.B., and Roh, E.H., **Fuzzy commutative ideals in BCK-algebras**, Fuzzy sets and systems, 64(1994), 401-405.
21. Jun. Y.B., and Kim, H.S., **On fuzzy topological d-algebras**, Math. Slovaca, 51(2001), No.2, 167-173.
22. Jun. Y.B., Kim, H.S., and Yoo, D.S., **Intutionistic fuzzy d-algebras**, Scientiae Mathematicae Japonicae, (2007), 117-125.
23. Jun. Y.B., Ahn, S.S., and Lee, K.J., **Falling d-ideals in d-algebras**, Hindawi Publishing Corporation, Vol.2011(2011), 1-14.
24. Jun, Y.B., and Ahn, S.S., **The theory of falling shadows a lied to d-ideals in d-algebras**, Hindawi Publishing Corporation, Vol. 2012(2011), 1-9.
25. Kim, H.S., **On fuzzy dot sub-algebras of d-algebras**, International Mathematical Forum, 4(2009), No.13, 654-651.
26. Lee, Y.C., and Kim, H.S., **On d^* sub-algebras of d-transitive d^*- algebras**, Math. Slovaca, 49(1999), No.1, 27-33.

27. Lee, K.J., Kim, Y.H., and Cho, Y.U., **Vague set theory based on d-algebras**, J.Appl, Math and informatics, Vol 26 (2008), No 5-6, 1221-1232.
28. Manimaran, S.V., and Chella a, B., **Structures on bipolar fuzzy d-ideals under (T.S) norms**, International Journal of computer Application, Vol 9 (2010), No.12, 1-4.
29. Meng, J., **Implicative commutative semi groups are equivalent to a class of BCK – algebras**, semi group forum 50 (1995), 89-96.
30. Mircca Subria, **Involutive Brouwerian D- algebras**, Proceeding of the 6th WSEAS Int. Conf. On systems theory and scientific computation, (2006), 81-89.
31. Mostafa M.Swamy, **Fuzzy implicative ideals in BCK - algebras**, Fuzzy sets and system, Vol – 87, Issue 3 (1997), 361-368.
32. Mundici, D., **MV – algebras are categorically equivalent to bounded commutative BCK - algebras**, Math. Japan. St (1986), 889-894.
33. Neggers, J., and Kim, H.S., **On d- algebras**, Math. Slovaca 49 (1999), No.1, 19-26.
34. Neggers, J., Jun, Y.B., and Kim, H.S., **On d-ideals in d- algebras**, Math. Slovaca, 49 (1999), No 3, 243-251.
35. Neggers, J., Dvurecens, A., and Kim, H.S., **On d-fuzzy functions in d-algebras**, Foundation of Physics Vol.80, No.10, (2000), 1807-1816.
36. Riecanova, Z., **On embeddings of generalised effect algebras into complete effect algebras**, soft computing 10 (2006), 476-482.
37. Rosenseld, A., **Fuzzy groups**, J. Math Anal. Appl. 35 (1971), 512-517.
38. Sobiraju, A., and Prasanna, A., **Primary decompositions of fuzzy dot ideals in d - algebras**, International Journal of Mathematical Sciences and A licaions, Vol. 1, No. 2 (2011), 671-677.
39. Xi, O.G., **Fuzzy BCK - algebras**, Math – Japan, 36 (1991), 935-942.
40. Zadeh, L.H., **Fuzzy sets**, Information control, 8 (1965), 338-353.

41. Zahara, M. Samaci, Mohammad Ali N. Azadani and Leila N. Ranjbar, **A class of BCK – algebra**, International Journal of algebra, Vol 5, (2011), No 28, 1379-1385.

42. Zhang, X.H., and Li, W.H., **On pseudo – BL algebras and BCC - algebras**, soft computing 10 (2006), 941-952.