Bulk Service Queue With Accessible and Non-Accessible Batches

BY

S. Vaidehi

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE-641 043, IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

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SYNOPSIS

In chapter I,notations,preliminaries,definitions and review of literature are presented.

There are many queueing models, wherein, services are in bulk and arrivals occur either singly or in bulk. Many authors have discussed queues with number of variants ingeneral bulk service rule. Under this general rule, if the server finds less than 'a' units in the queue, he remains idle until it cumulates to a, where upon he starts servicing the batch of a units. On the other hand when the free server finds ($1 \leqslant a \leqslant i$) customers in the queue, he takes min(i,b) customers for service from the head line. Thus at certain epochs, the maximum capacity of the server need not be met for service to begin. Under such cicumstances also the late arrivals can not join the batch in service and hence to wait for the next batch service.

R.Sivasami [10] in his paper cosiders a general bulk service queue with a provision to admit input into certain batches being served while the service is on. In section 1 of chapter II we discuss a single server Poisson queue with this bulk service rule. The steady state probabilities and the waiting time distribution for the system under study are obtained. The Little's formula $L_q = \lambda W_q$ is verified.

In section (2) of chapter II we consider the above bulk service model with repeated vacation. The steady state probabilities, average queue length and occupation time have been calculated when a=b=1, the results are found to coincide with the corresponding results of M/M/1 model with repeated vacation.

In chapter III a single server Poisson quque is discussed with a slight change in the previous bulk service rule. This rule allows the late entries to join a batch in course of ongoing service as long as the number of units in that batch is less than d ($a \leqslant d \leqslant b$) called maximum accesible limit without affecting the service time. When d=b, this model coincides with the model of section(1) of chapter II. The average queue length, distribution of waiting time and occupation time are derived for this model also.

Chapter I

CHAPTER I

INTRODUCTION

Queues have been under intensive investigation for several years by workers in a variety of fields, particularly mathematics and Operations Research. The study of queues is mainly applied in the fields of business, industries, engineering, transportation and every day life. The first work on waiting - line (queues) situation was "The theory of probabilities and telephone converstions" by A.K. Erlang.

QUEUEING SYSTEMS AND THEIR BAISC CHARACTERISTICS.

A system consisting of a service facility, a process of arrival of customers who wish to be served by the facility and the process of service is called a queueing system.

The following characteristics provide an adequate description of any queueing system.

(1). Arrival pattern of customers

This pattern describes the manner in which units arrive (either in batches or singly) at a service facility. The interval between two consecutive arrivals is called the inter arrival time.

(2). Service pattern of Servers

This mechanism describes the management for serving the customers management for serving the customers with the facilities they seek. Service may be deterministic or probabilistic. Customers may be served singly or in batches. In case of batch service, the service system is termed as 'bulk service system'.

(3). Queue discipline

Queue discipline is the rule according to which customers are selected for service. The most common queue discipline is first-in-first out (FIFO). Other queue disciplines are last-in-first out (LIFO) and priority queue disciplines.

(4). System capacity

The system capacity may be finite or infinite.

(5). Service Channels

Queueing system may have a single server or many number of servers. They are respectively called single server model or multi server model. The service channels may be aranged in paralled or in series.

(6). Notations

Kendall [1] proposed a convenient classification of a queueing system by the notation A/B/C/X/Y, where:

A : indicates the inter-arrival distribution.

B : the service pattern as given by the probability distribution for service time.

C : the number of paralled service channels.

X : the capacity of the system.

Y : the queue discipline.

M/Ma,b/1 denotes a queueing system which has a Poisson arrival process, an exponential service distribution with a general bulk service rule.

SOME BULK SERVICE POLICIES.

There could be a number of policies or rules according to which batches for bulk service may be formed. There are actual situations where one rule seems to be more appropriate than the others. The following are the types of bulk service rules frequently discussed in the literature:-

(1). The units are served in batches of not more than b, the capacity of the server. If a server, on completion of a service finds one waiting, he may still start the service. In this case the server is said to be intermittently available. Bailey [2] and Downton [3] consider such a situation when the server never relaxes. On the other hand, the server may wait, if he finds none in queue on completion of a service, till there is atleast one customer available for service. Le Gall, Runnenberg [4], Bloemena [5], Jaiswal[6], Neuts [7], Chaudhry and Templeton [8] consider such a rule. Jaiswal [6] points out the the distribution of the queue length for the modified rule can be obtained from that of the Usual rule.

(2). A service batch may be of a fixed size k. The server will wait till the number reaches k and if there are more than k, he takes a batch of k in order of arrival or even at random, while others wait. Foster, Fabens, Takacs consider such a rule.

(3). Neuts [7] and Medhi [9] consider a batch containing a minimum of 'a' units and a maximum of 'b' units. If immediately, after completion of service of a batch, the server finds less then 'a' units present, he waits till there are 'a' where upon he takes a batch of 'a' for service; if he finds more than 'a' units present but atmost b, he takes them all in a batch and
if he finds b units waiting, he takes in the batch for service a batch of b units in order of arrival or in random order while others wait in the queue. This rule will be called general bulk service rule as the rules (1) and (2) can be considered as specific cases of this rule.

(4). The size of a batch may be variable, that is customers may be served in batches of variable capacity Y_n, where Y_n is a random variable. Bhat, Teghem, Cohen examine this rule.

(5). Further, bulk service may be with accessible batches. If a batch being served does not utilize its full capacity for service, it may remain accessible for units arriving during the service time of the the batch until its full capacity is attained, the total service time is not altered by inclusion of such joining units in course of ongoing service. Newell considers such queues as models for traffic light queues.

Recently Sivasamy [10] deals with a single server Poisson queue with a new bulk service rule. This rule admits each batch served to have not less than 'a' and not more than 'b' units such that the arriving units can enter service, without affecting the service time, if the size of the batch served is less than some fixed d ($a \leq d \leq b$). The steady state distributions of queue length, waiting time and occupation time are derived for this model. In the paper entitled: "A bulk service queueing system with a provision to admit input into certain batches being served", [11]
he has discussed the model under the particular case when d=b.

TYPES OF VACATION

Queueing system in which server leaves for a vacation have been studied by many researchers. The non- availability of a server at the system may be termed as server's vacation.

(1). Repeated Vacations.

Any server on completion of a service, will start servicing again if the system has atleast the minimum number of customers required to start the service. Upon terminating this vacation period, if the server finds less than the required number of customers, he will immediately take another vacation.

(2). Single Vacation.

The assumptions are same as those of repeated vacations, except that, if the server finds less than the minimum number of customers required for service at the end of the vacation, he stays in the system waiting for the queue size to reach the minimum number.

(3). Exceptional first vacation.

Here, the duration of the first vacation is differently distributed from the subsequent vacations.

(4). Gated vacation.

When the server returns from a vacation, he accepts only those customers who were waiting when the server returned, deferring the service of subsequent arrivals until after the next vacation.

A queueing system with single server who services customers according to a general bulk service rule and leaves the system for vacation has been analysed by Nadarajan and Subramanian . Both repeated vacation and single vacation of server are considered. The steady state probability vectors of the number of customers in the system and stability condition have been obtained using matrix-geometric algorithmic approach. Neuts [7] and Mohana Dhas have analysed multiserver general bulk service queue with vacation. Mohana Dhas has obtained the steady state probability distribution for single server bulk service model under repeated vacation using probability generating function method.

Results.

1. Rouche's Theorem:

If $f(z)$ and $g(z)$ are functions analytic inside and on a closed contour C and if $|g(z)|<|f(z)|$ on C then $f(z)$ and $f(z)+g(z)$ will have the same number of zeros inside C.

2. Some of the distributions used in our work

Distribution	Probability function	Parameter	Mean	Variance
Poisson	$\frac{e^{-a}a^{k}}{k!}$	$a>0$	a	a
Exponential	$\lambda e^{-\lambda x}$	$\lambda>0$	$1/\lambda$	$1/\lambda^2$
Gamma	$\frac{\lambda^{k}x^{k-1}e^{-\lambda x}}{k}$ $\lambda>0$	$\lambda>0$ $k>0$	k/λ	k/λ^2

Chapter II

CHAPTER II

A BULK SERVICE QUEUEING SYSTEM WITH A PROVISION TO ADMIT INPUT INTO CERTAIN BATCHES BEING SERVED.

Here, a single server queueing model with Poisson arrivals, exponential service times and a general bulk service queue discipline with a provision to admit input into certain batches being served while the service is on is considered. The steady-state probabilities, average queue length and waiting time distribution are derived in the first section of this chapter. In the second section by introducing the concept of vacation in the same model, we calculate the steady-state probabilities, queue length and occupation time.

Section 2.1

Description of the model under study

Consider a single server facility into which the customers arrive in accordance with a Poisson process at rate λ. The customers are served in batches of varying size in the order of their arrival according to bulk service rule with a minimum of 'a' units and a maximum of 'b' units.

If a batch is being served and does not utilize its full capacity for service it remains accessible for customers arriving during the service time of this batch, until its full capacity for service is utilized and that the service time of the batch is not altered by joining customers.

Assume that the service times, independent of the batch size, are identically, independently and exponentially distributed variables with mean $1/\mu$. Also assume the number of customers in the system as the state variable. This queueing model may be denoted by $M/M_{a,b}/1$.

STEADY STATE PROBABILITIES

Define,

P_n = the steady state probability that there are 'n' units in the system ; $n \geq 0$.

The steady state equations are

$$\lambda P_o = \mu \sum_{n=a}^{b} P_n \qquad (1)$$

$$\lambda P_n = \lambda P_{n-1} + \mu P_{n+b} \quad , \quad 1 \leq n \leq a-1 \qquad (2)$$

$$(\lambda+\mu) P_n = \lambda P_{n-1} + \mu P_{n+b} \quad , \quad n \geq a \qquad (3)$$

It can be easily shown that

$$P_n = \frac{\theta^a - \theta^{b+n+1}}{\theta^a - \theta^{b+1}} P_o \quad ; \quad 1 \leq n \leq a-1 \qquad (4)$$

$$P_n = \frac{1 - \theta^b}{\theta^a - \theta^{b+1}} \theta^{n+1} P_o \quad ; \quad n \geq a \qquad (5)$$

Where θ is a real root in (0,1) which satisfies,

$$\frac{\lambda}{\mu} = \frac{\theta\,(1-\theta^b)}{(1-\theta)} \quad \text{whenever} \quad \rho = \frac{\lambda}{b\mu} < 1.$$

'P_o' can be obtained by using normalizing condition

$$P_o + \sum_{n=1}^{a-1} P_n + \sum_{n=a}^{\infty} P_n = 1.$$

Then

$$P_o = \frac{(\theta^a - \theta^{b+1})\,(1-\theta)}{(a-1)\,(1-\theta)\,\theta^a + \theta^a - \theta^{b+1}}$$

Then the steady - state probabilities are given by (4), (5) and (6).

Next we shall calculate the average queue length, mean waiting time and check the Little's formula for the above model.

MEAN QUEUE LENGTH (L_q)

The expected number of members in the queue is given by

$$L_q = \sum_{n=0}^{a-1} n\,P_n + \sum_{n=b}^{\infty} (n-b)\,P_n.$$

$$= \sum_{n=0}^{a-1} n \frac{\theta^a - \theta^{b+n+1}}{\theta^a - \theta^{b+1}} P_o + \sum_{n=b}^{\infty} (n-b) \frac{(1-\theta^b)\, \theta^{n+1}}{\theta^a - \theta^{b+1}} P_o$$

$$= \frac{P_o}{\theta^a - \theta^{b+1}} \left[\theta^a \sum_{n=0}^{a-1} n - \theta^{b+2} \sum_{n=0}^{a-1} n\, \theta^{n-1} \right.$$

$$\left. + (1-\theta^b)\, \theta^{b+2} \sum_{n=b}^{\infty} (n-b)\, \theta^{n-b-1} \right]$$

Therefore

$$L_q = \frac{P_o}{\theta^a - \theta^{b+1}} \left[\frac{a(a-1)\, \theta^a}{2} + \frac{[a\theta^{a-1} (1-\theta) + \theta^{a-1}]}{(1-\theta)^2} \theta^{b+2} \right.$$

$$\left. + \frac{\lambda}{\mu} \frac{\theta^{b+1}}{(1-\theta)} \right] \qquad (7)$$

DISTRIBUTION OF THE WAITING TIME.

Let the random variable T denote the waiting time in the queue for an arriving unit and --(t)dt = Prob [$t \leq T \leq t+dt$]. An arriving unit may find the system in one of the following states:

(i)	q	, $a-1 \leq q \leq b-1$	
(ii)	q	, $0 \leq q \leq a-2$	
(iii)	n	, $n = kb+q$	; $a-1 \leq q \leq b-1$
(iv)	n	, $n = kb+q$	; $0 \leq q \leq a-2$

In case of (i) the arriving unit does not wait.

In case of (ii) the arriving unit has to wait, till the arrival of a-1-q units and the time for a-1-q arrivals has a gamma distribution with parameters λ, a-1-q.

In case of (iii), the arriving unit has to wait for k batch service completions.

In case of (iv), the arriving unit has to wait till either the services of k-groups are completed or a-1-q units to arrive, whichever occurs later. This duration which is given by maximum of two gamma variates may be denoted by the random variable Z.

(ie) $Z = \text{Max}\,(Z_1(t), Z_2(t)$ where,

$\text{Prob}\,\{t \le Z_1(t) \le t+dt\} \;=\; f\,(\mu,k;t)dt$;

$\text{Prob}\,\{t \le Z_2(t) \le t+dt\} \;=\; f\,(\lambda,a-1-q;t)dt$ and

$f(a,d;t)$ = The probability density function of a gamma variate with parameters s,d,

$$= \frac{s(st)^{d-1}\, e^{-st}}{(d-1)!} \quad ; \qquad t > 0, \quad d = 1,2,\ldots.$$

Denoting the p.d.f of Z by $h_Z(t)$, we have

$$\gamma(t) = \sum_{q=0}^{a-2} P_q\, f(\lambda,\, a-1-q;\, t)$$

$$+ \sum_{k=1}^{\infty} \sum_{q=a-1}^{b-1} P_{kb+q} \quad f(\mu,k;t)$$

$$+ \sum_{k=1}^{\infty} \sum_{q=1}^{a-2} P_{kb+q} \quad h_Z\,(t) \qquad ; \quad t \ge 0.$$

$$= \frac{P_o}{\theta^a - \theta^{b+1}} \left[\{\theta^a - \theta^b e^{-\mu(1-\theta^b)t}\} e^{-\lambda t} \sum_{k=0}^{a-2} \frac{(\lambda t)^k}{k!} + \theta^b(1-\theta^b)\, e^{-\mu(1-\theta^b)t} \right]$$

The expected waiting time in the queue is given by

$$E(T) = W_q = \int_0^\infty t\, \gamma(t)$$

$$= \frac{P_o}{\lambda\, \theta^a - \theta^{b+1}} \left[\frac{a(a-1)\,\theta^a}{2} + \frac{\theta^{b+2}\{a(1-\theta)\theta^{a-1} + \theta^{a-1}\}}{(1-\theta)^2} + \frac{\lambda\, \theta^{b+1}}{\mu\,(1-\theta)} \right] \quad (8)$$

From (7) & (8) we have $L_q = \lambda\, W_q$

SECTION 2.2

In bulk service models, the required number of customers may not be available for extending service as soon as the server is free. The server may utilize his idle time in a useful and optimal way to perform additional jobs.

In this section we consider the model discussed in section 2.1 with repeated vacation.

Under repeated vacation, if the server completes a service and finds less than 'a' customers in the queue, he leaves for a vacation and returns to the system only when he finds 'a' or more members. The vacation time is exponentially distributed with parameter 'α'.

The mathematical model is formulated in terms of difference-differential equations and the steady-state probabilities, average queue length and occcupation-time are calculated.

The process is in the state (i,o), $i \geq o$ when there are 'i' customers in the system and the server is away on a vacation. The process is in the state $(i,1)$, $i \geq o$ when there are 'i' customers in the system and the server is busy.

Define,

$P_{i\ j}(t) = P_r$ {At time t, the system is in the state (i,j) / $i \geq o,\ j = o,1$}

Difference equations are given by

$$P_{o\ o}(t+\Delta t) = P_{o\ o}(t)\ (1-\lambda\Delta t) + \sum_{n=a}^{b} P_{n\ 1}(t)\ \mu\Delta t$$

$$P_{i\ o}(t+\Delta t) = P_{i\ o}(t)\ (1-\lambda\Delta t) + P_{i+b\ 1}(t)\ \mu\Delta t + P_{i-1\ 0}(t)\ \lambda\Delta t, \qquad i < a$$

$$P_{i\ o}(t+\Delta t) = P_{i\ o}(t)\ (1-\overline{\lambda+\alpha}\ \Delta t) + P_{i-1\ 0}(t)\ \lambda\Delta t, \qquad i \geq a$$

$$P_{i\ 1}(t+\Delta t) = P_{i\ 1}(t)\ (1-\overline{\lambda+\mu}\ \Delta t) + \lambda P_{i-1\ 1}(t) + P_{i+b\ 1}(t)\ \mu\Delta t + P_{i\ o}(t)\ \alpha\Delta t, \qquad i > a$$

$$P_{a\ 1}(t+\Delta t) = P_{a\ 1}(t)\ (1-\overline{\lambda+\mu}\ \Delta t) + P_{a+b\ 1}(t)\ \mu\Delta t + P_{a\ o}(t)\ \alpha\Delta t.$$

At steady state the above set of equations yield.

$$0 = -\lambda P_{o\ o} + \mu \sum_{n=a}^{b} P_{n\ 1} \qquad (1)$$

$$0 = -\lambda P_{i\ o} + \mu P_{i+b\ 1} + \lambda P_{i-1\ o}, \qquad i < a \quad (2)$$

$$0 = -(\lambda+\alpha)\ P_{i\ o} + \lambda P_{i-1\ o}, \qquad i \geq a \quad (3)$$

$$0 = -(\lambda+\mu)\ P_{i\ 1} + \lambda P_{i-1\ 1} + \mu P_{i+b\ 1} + \alpha P_{i\ o}, \quad i > a \qquad (4)$$

$$0 = -(\lambda + \mu)P_{a\ 1} + \mu P_{a+b\ 1} + \alpha P_{a\ o} \qquad (5)$$

From (3), we have

$$P_{i\ o} = \left[\frac{\lambda}{\lambda+\alpha}\right] P_{i-1\ o} \quad , \qquad i \geq a.$$

$$P_{a\ o} = \left[\frac{\lambda}{\lambda+\alpha}\right] P_{a-1\ o}$$

When n = a+1,

$$P_{a+1\ o} = \left[\frac{\lambda}{\lambda+\alpha}\right] P_{a\ o}$$

$$= \left[\frac{\lambda}{\lambda+\alpha}\right]^2 P_{a-1\ o}$$

Generalizing,

$$P_{i\ o} = \left[\frac{\lambda}{\lambda+\alpha}\right]^{i-a+1} P_{a-1\ o} \quad , \qquad i \geq a \quad (6)$$

Let E denote the shifting operator defined by $EP_{1\ n} = P_{1\ n+1}$

The difference equation (4) can be written as

$$h(E)\ \{ P_{i\ 1} \} = \alpha P_{i+1\ o} \quad , \qquad i > a-1.$$

Where $h(z) = [-\mu z^{b+1} + (\lambda + \mu)z - \lambda] = \alpha P_{i+1\ o}$.

Taking

$f(z) = (\lambda+\mu)z.$

$g(z) = -\mu z^{b+1} - \lambda.$

Then for $|z|=1$, we have $|g(z)| < |f(z)|$ and applying Rouche's theorem h(z) will have only one root say 'r_o' inside $|z| = 1$

Hence

$$P_{i\,1} = Ar_o^i + \alpha \frac{[\lambda/\lambda+\alpha]^{i-a+2} P_{a-1\,o}}{(\lambda+\mu)[\lambda/\lambda+\alpha] - \lambda - \mu [\lambda/\lambda+\alpha]^{b+1}}$$

Taking $\frac{\lambda}{\lambda+\alpha} = r_1$, we have

$$P_{i\,1} = Ar_o^i - \frac{\alpha\, r_1^{i-a+2} P_{a-1\,o}}{\mu\, r_1^{b+1} - (\lambda+\mu)r_1 + \lambda} \qquad (7)$$

Summing equation (2) over i from 1 to k, where $o \le k \le a-1$, we get

$$o = -\lambda \sum_{i=1}^{k} P_{i\,o} + \mu \sum_{i=1}^{k} P_{i+b\,1}, + \lambda \sum_{i=1}^{k} P_{i-1\,o}$$

$$1 \le k \le a-1$$

$$P_{k\,o} = \frac{\mu}{\lambda} \sum_{i=1}^{k} P_{i+b\,1} + P_{o\,o}.$$

Substituting the value of $P_{o\,o}$ from (1) we get

$$P_{k\,o} = \frac{\mu}{\lambda} \sum_{i=1}^{k} P_{i+b\,1} + \frac{\mu}{\lambda} \sum_{i=a}^{b} P_{i\,1}$$

$$= \frac{\mu}{\lambda} \sum_{i=a}^{b+k} P_{i\,1} \qquad , o \le k \le a-1$$

Using (7) for the value of $P_{i\,1}$, we have

$$P_{k\,o} = \frac{\mu}{\lambda} \sum_{i=a}^{b+k} \left[A r_o^{\,i} - a\, r_1^{\,i-a+2} \frac{P_{a-1\,o}}{\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda} \right]$$

$$= \frac{\mu}{\lambda} \left[A \left(\frac{r_o^{a} - r_o^{b+k+1}}{1-r_o} \right) - a\, P_{a-1\,o} \frac{(r_1^{2} - r_1^{b+k-a+3})}{(1-r_1)(\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda)} \right]$$

$$o \le K \le a-1.$$

The constant A is calculated by the use of (6) &(7) in (5) as follows.

$$o = -(\lambda+\mu)\, P_{a\,1} + \mu P_{a+b\,1} + a P_{a\,o}.$$

$$o = -(\lambda+\mu) \left[A r_o^{a} - \frac{a\, r_1^{2} P_{a-1\,o}}{\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda} \right]$$

$$+ \mu \left[A r_o^{a+b} - \frac{a\, r_1^{b+2} P_{a-1\,o}}{\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda} \right] + a r_1 P_{a-1\,o}.$$

$$A\,[-(\lambda+\mu)\, r_o^{a} + \mu\, r_o^{a+b}]$$

$$= P_{a-1\,o} \left[\frac{-(\lambda+\mu)\, a r_1^{2} + a\mu r_1^{b+2}}{\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda} - a r_1 \right]$$

$$A = \frac{a \lambda r_1 P_{a-1\,o}}{((\lambda+\mu) r_1 - \mu r_1^{b+1} - \lambda)\,(-(\lambda+\mu)\, r_o^{a} + \mu r_o^{a+b})}$$

Since $(\lambda+\mu) r_o - \lambda - \mu r_o^{b+1} = 0$, we have

$$= \frac{a r_1 \, r_o^{1-a} \, P_{a-1\ o}}{\mu r_1^{b+1} - (\lambda+\mu)\, r_1 + \lambda}$$

Then the steady-state probabilities are given by

$$P_{i\ o} = \left[\frac{\lambda}{\lambda+a}\right]^{i-a+1} P_{a-1\ o} \qquad i \geq a.$$

$$P_{i\ 1} = \frac{a r_1 \, P_{a-1\ o} \, [r_o^{i-a+1} - r_1^{i-a+1}]}{\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda} \qquad i \geq a.$$

$$P_{i\ o} = \frac{\mu \, a r_1 \, P_{a-1\ o}}{\lambda(\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda)} \left[\left(\frac{r_o - r_o^{i+b-a+2}}{1-r_o}\right) - \left(\frac{r_1 - r_1^{i+b-a+2}}{1-r_1}\right)\right]$$

$$o \leq i \leq a-1$$

where $r_1 = \dfrac{\lambda}{\lambda+a}$

AVERAGE QUEUE LENGTH (L_q)

Next we shall calculate the average number of units in the queue.

$$L_q = \sum_{n=o}^{\infty} n \, P_{n\ o} + \sum_{n=o}^{\infty} n P_{n+b\ 1} \, .$$

$$L_q = \sum_{n=o}^{a-1} n P_{n\ o} + \sum_{n=a}^{\infty} n P_{n\ o} + \sum_{n=o}^{\infty} n \, P_{n+b\ 1}$$

Consider the 1st term.

$$\sum_{n=o}^{a-1} n\, P_{n\,o} = \sum_{n=o}^{a-1} n \;\frac{\mu}{\lambda}\; \frac{\alpha\, P_{a-1} \quad r_1}{(\mu r_1{}^{b+1} -(\lambda+\mu) r_1 + \lambda)} \left[\frac{r_o - r_o{}^{b-a+n+2}}{1-r_o} - \frac{r_1 - r_1{}^{b-a+n+2}}{1-r_1} \right]$$

$$= \frac{\mu}{\lambda}\; \frac{\alpha\, P_{a-1\,o} \quad r_1}{(\mu r_1{}^{b+1}-(\lambda+\mu)r_1+\lambda)} \left[\frac{a(a-1)}{2}\; \frac{r_o}{(1-r_o)} - \frac{r_o{}^{b-a+3}}{1-r_o}\; \frac{d}{dr_o}\; \frac{(1-r_o{}^{a})}{(1-r_o)} - \frac{a(a-1)}{2}\; \frac{r_1}{1-r_1} + \frac{r_1{}^{b-a+3}}{1-r_1}\; \frac{d}{dr_1}\; \frac{(1-r_1{}^{a})}{(1-r_1)} \right]$$

$$= \frac{\mu\; \alpha\; P_{a-1\;o} \quad r_1}{\lambda\; (\mu r_1{}^{b+1}-\; (\lambda+\mu) r_1\; + \lambda)} \left[\frac{a(a-1)}{2} \left[\frac{r_o}{1-r_o} - \frac{r_1}{1-r_1} \right] - \frac{r_o{}^{b-a+3}}{1-r_o} \left[\frac{-(1-r_o) a r_o{}^{a-1} + 1 - r_o{}^{a}}{(1-r_o)^2} \right] + \frac{r_1{}^{b-a+3}}{1-r_1} \left[\frac{-(1-r_1) a r_1{}^{a-1} + 1 - r_1{}^{a}}{(1-r_1)^2} \right] \right]$$

Consider the 2nd term

$$\sum_{n=a}^{\infty} nP_{n\;o} = P_{a-1\;o} \sum_{n=a}^{\infty} n \left[\frac{\lambda}{\lambda+\alpha} \right]^{n-a+1}$$

$$= P_{a-1\,o} \sum_{n=a}^{\infty} n\, r_1^{\,n-a+1}.$$

$$= P_{a-1\,o}\, r_1^{\,2-a} \left[\frac{(1-r_1)a r_1^{\,a-1} + r_1^{\,a}}{(1-r_1)^2}\right]$$

Consider the 3rd term

$$\sum_{n=o}^{\infty} n\, P_{n+b\ 1} = \sum_{n=o}^{\infty} n \frac{\alpha r_1\, P_{a-1\ o}}{\mu r_1^{\,b+1} - (\lambda+\mu) r_1 + \lambda} \left[r_o^{\,n+b-a+1} - r_1^{\,n+b-a+1}\right]$$

$$= \frac{\alpha r_1\, P_{a-1\ o}}{\mu r_1^{\,b+1} - (\lambda+\mu) r_1 + \lambda} \left[r_o^{\,b-a+2} \frac{d}{dr_o}\left[\frac{1}{1-r_o}\right] - r_1^{\,b-a+2} \cdot \frac{d}{dr_1}\left[\frac{1}{1-r_1}\right]\right]$$

$$= \frac{\alpha r_1\, P_{a-1 o}}{\mu r_1^{\,b+1} - (\lambda+\mu) + \lambda} \left[\frac{r_o^{\,b-a+2}}{(1-r_o)^2} - \frac{r_1^{\,b-a+2}}{(1-r_1)^2}\right]$$

Therefore

$$L_q = \frac{\mu\alpha P_{a-1\ o}\ r_1}{\lambda(\mu r_1^{\,b+1} - (\lambda+\mu) r_1 + \lambda)} \left[\frac{a(a-1)}{2}\left(\frac{r_o}{1-r_o} - \frac{r_1}{1-r_1}\right)\right.$$

$$- \frac{r_o^{\ b-a+3}}{1-r_o}\left[\frac{-(1-r_o)a r_o^{\,a-1} + 1 - r_o^{\,a}}{(1-r_o)^2}\right]$$

$$\left. - \frac{r_1^{\,b-a+3}}{1-r_1}\left[\frac{-(1-r_1)a r_1^{\,a-1} + 1 - r_1^{\,a}}{(1-r_1)^2}\right]\right] +$$

$$+ \; P_{a-1\,o}\, r_1^{2-a} \left[\frac{(1-r_1) a r_1^{a-1} + r_1^a}{(1-r_1)^2} \right]$$

$$+ \; \frac{a r_1 \, P_{a-1\,o}}{\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda} \left[\frac{r_o^{b-a+2}}{(1-r_o)^2} - \frac{r_1^{b-a+2}}{(1-r_1)^2} \right]$$

It is checked that when a=b=1, the average queue length coincides with the corresponding result of M/M/1-repeated vacation model.

Occupation Time

The server's occupation time, say 't' under the steady-state conditions in queue is non-zero if the system is in one of the following states:-

(i) $(i,1)$, $i=kb+q$, $q=o,1,2,\ldots\ldots a-1$ & $k=1,2..$

(ii) $(i,1)$, $i=kb+q$, $a \le q \le b-1$

In case of (i), his busy time follows a gamma distribution with parameter $\mu, k+1$.

In case of (ii), his busy time follows a gamma distribution with parameter μ, $k+2$.

$$g(t) = \sum_{k=o}^{\infty} \sum_{q=o}^{a-1} P_{kb+q\cdot 1} \frac{\mu e^{-\mu t} (\mu t)^k}{k!}$$

$$+ \sum_{k=o}^{\infty} \sum_{q=a}^{b-1} P_{kb+q\cdot 1} \frac{\mu (\mu t)^{k+1} \cdot e^{-\mu t}}{(k+1)!}$$

Consider the 1st term

$$\sum_{k=o}^{\infty} \sum_{q=o}^{a-1} P_{kb+q\ 1} \frac{\mu e^{-\mu t} (\mu t)^k}{k!}$$

$$= \mu e^{-\mu t} \sum_{k=0}^{\infty} \frac{(\mu t)^k}{k!} \sum_{q=0}^{a-1} \frac{a r_1 P_{a-1\ o} [r_o^{kb+q-a+1} - r_1^{kb+q-a+1}]}{\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda}$$

$$= \frac{\mu e^{-\mu t} a r_1 P_{a-1\ o}}{\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda} \left[r_o^{1-a} \frac{1-r_o^a}{1-r_o} e^{r_o b \mu t} - r_1^{1-a} \frac{1-r_1^a}{1-r_1} e^{r_1 b \mu t} \right]$$

Consider the 2nd term

$$\sum_{k=o}^{\infty} \sum_{q=a}^{b-1} P_{kb+q \cdot 1} \frac{\mu (\mu t)^{k+1} e^{-\mu t}}{(k+1)!}$$

$$= \sum_{k=o}^{\infty} \sum_{q=a}^{b-1} \frac{a r_1 P_{a-1\ o} [r_o^{kb+q-a+1} - r_1^{kb+q-a+1}]}{(\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda)} \times \frac{\mu (\mu t)^{k+1} e^{-\mu t}}{(k+1)!}$$

$$= \frac{a r_1 P_{a-1\ o} \mu e^{-\mu t}}{\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda} \left[r_o^{1-a} \sum_{k=o}^{\infty} \frac{(r_o b \mu t)^{k+1}}{(k+1)!} r_o^{-b} \right.$$

$$\times \sum_{q=a}^{b-1} r_o^{q} - r_1^{1-a} \sum_{k=o}^{\infty} \frac{(r_1^{b}\mu t)^{k+1}}{(k+1)!} \cdot r_1^{-b} \sum_{q=a}^{b-1} r_1^{q} \Bigg]$$

$$= \frac{a\, r_1\, P_{a-1\,o}\, \mu e^{-\mu t}}{\mu\, r_1^{b+1} - (\lambda+\mu) r_1 + \lambda} \left[\frac{r_o^{1-a-b} (r_o^{a} - r_o^{b}) (e^{\mu t r_o^{b}} - 1)}{(1-r_o)} \right.$$

$$\left. - \frac{r_1^{1-a-b} (r_1^{a} - r_1^{b}) (e^{\mu t r_1^{b}} - 1)}{(1-r_1)} \right]$$

Accordingly we have

$$g(t) = \frac{a r_1\, P_{a-1\,o}\, \mu e^{-\mu t}}{\mu r_1^{b+1} - (\lambda+\mu) r_1 + \lambda} \left[\frac{e^{r_o^{b} \mu t}}{(1-r_o)} (r_o^{1-b} - r_o) \right.$$

$$+ \frac{e^{r_1^{b} \mu t}}{(1-r_1)} (r_1^{1-b} - r_1)$$

$$\left. - \frac{(r_o^{1-b} - r_o^{1-a})}{(1-r_o)} + \frac{(r_1^{1-b} - r_1^{1-a})}{(1-r_1)} \right]$$

Chapter III

CHAPTER III

A BULK SERVICE QUEUE WITH ACCESSIBLE AND NON - ACCESSIBLE BATCHES.

In this chapter, we consider a single - server Poisson queue with a new bulk service rule. Units arrive into a service facility according to a Poisson process with rate λ. The units are served by a single server in batches of size $n(a \leq n \leq b)$. The service times, irrespective of the batch size, are assumed to have independently, identically, exponential distribution with parameter μ. The service rule is assumed to operate as follows:-

The server starts service only when a minimum number of 'a' units is reached in the queue and the maximum capacity is 'b' units. Unlike the general bulk service rule introduced by Neuts, the late entries are allowed to join the batch being served as long as the number of units in that batch is less than d. $(a \leq d \leq b)$.

At every departure epoch, before intiating the next service the server may find the queue length n in one of the following 3 categories.

(i) $o \leq n \leq a-1$.

(ii) $a \leq n \leq d-1$.

(iii) $n \geq d$.

In case (1) the server cannot initiate service with the available units. He remains idle until the queue length becomes 'a' and then he starts on the batch with the minimum of 'a' units.

In case (2) the server takes the entire queue for batch service. He also admits the subsequent arrivals in the batch, till the service of the current batch is over or till the accessible limit d is reached whichever occurs first. Such a batch is called an accessible batch (AB).

In case (3) be takes min (n,b) units at the begining for the service and does not allow further arrivals into the batch being served even if the current batch capacity is not 'b'. In this case the batch becomes non-accessible for late arriving units.

STATE SPACE OF THE SYSTEM

The queue is studied as a Markov process on the state space

$$\{(o,n)\ ,\ o \le n \le a-1)\}\ U\ \{(o,n),a \le n \le d\}\ U\ \{(1,n),n \ge o\}$$

The system is said to be in the state

(i) (o,n) , $o \le n \le a-1$ if the server is idle and there are 'n' members in the queue.

(ii) (o,n) , $a \le n \le d$ if the server is busy and 'n' members are in the system.

(iii) (1,n), $n \ge o$ if the server is busy and there are 'n' members in the queue.

Define,

$P_{i\,n}(t)$ = Prob {At time t the system is in state (i,n) / i = o,1 & n = o,1,2,...,}.

DIFFERENCE DIFFERENTIAL EQUATIONS.

The time dependent difference equations for the above model are obtained as follows

$$P_{o\,o}(t+\Delta t) = P_{o\,o}(t)\,(1-\lambda\Delta t) + \sum_{n=a}^{d-1} P_{o\,n}(t)\,\mu\Delta t + P_{1\,o}(t)\mu\Delta t$$

$$P'_{o\,o}(t) = -\lambda P_{o\,o}(t) + \mu\sum_{n=a}^{d-1} P_{o\,n}(t) + \mu P_{1\,o}(t).$$

$$P_{o\,n}(t+\Delta t) = P_{o\,n}(t)\,(1-\lambda\Delta t) + P_{o\,n-1}(t)\,\lambda\Delta t + P_{1\,n}(t)\,\mu\Delta t.$$

$$1 \le n \le a-1.$$

$$P'_{o\,n}(t) = -\lambda P_{o\,n}(t) + \lambda P_{o\,n-1}(t) + \mu P_{1\,n}(t).$$

$$P'_{o\,n}(t) = -(\lambda+\mu)\,P_{o\,n}(t) + \lambda P_{o\,n-1}(t) + \mu P_{1\,n}(t)\quad,\ a \le n \le d-1$$

$$P_{1\,o}(t+\Delta t) = P_{1\,o}(t)\,(1-\overline{\lambda+\mu}\,\Delta t) + \sum_{n=d}^{b} P_{1\,n}(t)\,\mu\Delta t + P_{o\,d-1}(t)\,\lambda\Delta t$$

$$P'_{1\,o}(t) = -(\lambda+\mu)\, P_{1\,o}(t) + \lambda\, P_{o\,d-1}(t) + \mu \sum_{n=d}^{b} P_{1\,n}(t)$$

$$P_{1\,n}(t+\Delta t) = P_{1\,n}(t)\,(1-\overline{\lambda+\mu}\; t) + P_{1\,n-1}(t)\,\lambda\Delta t$$
$$+ P_{1\,n+b}(t)\,\mu\Delta t \;, n \geq 1$$

$$P'_{1\,n}(t) = -(\lambda+\mu)\, P_{1\,n}(t) + \lambda P_{1\,n-1}(t) + \mu P_{1\,n+b}(t)$$

STEADY - STATE PROBABILITIES

Assuming the steady - state exists we get the following steady - state equations.

$$\lambda P_{o\,o} = \mu \left[\sum_{n=a}^{d-1} P_{o\,n} + P_{1\,o} \right] \quad (1)$$

$$\lambda P_{o\,n} = \lambda P_{o\,n-1} + \mu\, P_{1\,n}, \quad 1 \leq n \leq a-1 \quad (2)$$

$$(\lambda+\mu)\, P_{o\,n} = \lambda P_{o\,n-1} + \mu\, P_{1\,n}, \quad a \leq n \leq d-1 \quad (3)$$

$$(\lambda+\mu)\, P_{1\,o} = \lambda\, P_{o\,d-1} + \mu \sum_{n=d}^{b} P_{1\,n} \quad (4)$$

$$(\lambda+\mu)\, P_{1\,n} = \lambda\, P_{1\,n-1} + \mu\, P_{1\,n+b}, \quad n \geq 1 \quad (5)$$

When d=a, (3) can not occur and when a = 1, (2) cannot occur.

Let $D = \sum_{n=a}^{d-1} P_{o\,n}$

From equation (5) we have

$$(\lambda+\mu)\, P_{1\,n+1} - \lambda\, P_{1\,n} - \mu\, P_{1\,n+b+1} = o$$

Let E denote the shifting operator defined by $E\ P_{1\,n} = P_{1\,n+1}$. Then the above equation becomes $h(E)\ \{P_{1\,n}\} = o$ where

$h(z) \equiv \mu z^{b+1} - (\lambda+\mu)\ z + \lambda$.

Suppose that $f(z) = -\ (\lambda+\mu)z$ and that $g(z) = \mu z^{b+1}+\lambda$, then for $|z| = 1$, $|g(z)| < |f(z)|$ and hence by Rouche's theorem, $f(z)$ and $f(z) + g(z)$ will have the same number of zeros inside $|z| = 1$. Since $f(z)$ has only one zero inside $|z| = 1$, there will be only one zero of $f(z)+g(z)=h(z)$ inside $|z| = 1$.

Let θ be the root such that $o < \theta < 1$ and $\theta_1,\ \theta_2,\ldots.\theta_b$ (with $|\theta i| \geq 1$) be the other b roots of $h(z) = o$. Let $\rho = \dfrac{\lambda}{b\mu}$ be the traffic intensity.

Then from (6), we have

$$\frac{\lambda}{\mu} = \frac{\theta(1-\theta^b)}{1-\ \theta} \qquad (7)$$

Now we can write the solution of (5) as:

$$P_{1\,n} = B\ \theta^n + \sum_{i=1}^{b} B_i\ \theta_i{}^n\ , \qquad n = o,1,2,\ldots$$

Since $\sum_{n=o}^{\infty} P_{1\,n} < 1$, we must have $B_i = o$ for $i = 1,2,..b$

Hence $P_{1\,n} = B\ \theta^n$, $\quad n \geq o \qquad (8)$

From (1), we get

$$P_{o\,o} = \frac{\mu}{\lambda}\left[\sum_{n=a}^{d-1} P_{o\,n} + B\right]$$

when n = 1, (2) gives

$$P_{o\,1} = P_{o\,o} + \frac{\mu}{\lambda} P_{1\,o}$$

$$= \frac{\mu}{\lambda}[D+B] + \frac{\mu}{\lambda} B\theta \quad \text{(Substituting for } P_{o\,o}\text{)}$$

$$= \frac{\mu}{\lambda}\left[D + B\sum_{r=o}^{1} \theta^r\right]$$

By taking n = 2, (2) becomes

$$P_{o\,2} = \frac{\mu}{\lambda}\left[D + B\sum_{r=o}^{2} \theta^r\right]$$

In general we have

$$P_{o\,n} = \frac{\mu}{\lambda}\left[D + B\sum_{r=o}^{n} \theta^r\right], \qquad o \le n \le a-1 \quad (9)$$

when n = a , (3) becomes

$$P_{o\,a} = \frac{\lambda}{\lambda+\mu} P_{o\,a-1} + \frac{\mu}{\lambda+\mu} P_{1\,a}$$

Substituting for $P_{o\,a-1}$ and $P_{1\,a}$ from (9),(8),

$$P_{o\,a} = \frac{\lambda}{\lambda+\mu}\left[\frac{\mu}{\lambda}\left[D + B\sum_{r=o}^{a-1} \theta^r\right]\right] + \frac{\mu}{\lambda+\mu} B\,\theta^a$$

$$= \frac{\mu}{\lambda+\mu} \left[\left[D + B \sum_{r=o}^{a-1} \theta^r \right] + B\,\theta^a \right]$$

Similarly

$$P_{o\;a+1} = \frac{\mu}{\lambda+\mu} \left[\frac{\lambda}{\lambda+\mu} \left[D + B \sum_{r=o}^{a-1} \theta^r \right] + B\theta^{a+1} \left[1 + \frac{\lambda}{(\lambda + \mu)\theta} \right] \right].$$

In general we have

$$P_{o\;n} = \frac{\mu}{\lambda+\mu} \left[\left[\frac{\lambda}{\lambda+\mu} \right]^{n-a} \left[D + B \sum_{r=o}^{a-1} \theta^r \right] + B\theta^n \sum_{r=o}^{n-a} \left[\frac{\lambda}{(\lambda + \mu)\theta} \right]^r \right] \quad a \le n \le d-1 \quad (10)$$

Equating the expressions for $P_{o\;d-1}$ obtained from (10) and (4) we get D in terms of B as follows:

From (4) we have

$$P_{o\;d-1} = \frac{\lambda + \mu}{\lambda} B - \frac{\mu}{\lambda} \sum_{n=d}^{b} B\theta^n$$

$$\left[\frac{\mu}{\lambda + \mu} \right] \left[\frac{\lambda}{\lambda + \mu} \right]^{d-a-1} D = B \left[- \frac{\mu}{\lambda+\mu} \left[\frac{1-\theta^a}{1-\theta} \left[\frac{\lambda}{\lambda+\mu} \right]^{d-a-1} - \theta^{d-1} \left[\frac{\mu}{\lambda+\mu} \right] \frac{\left(1 - \left[\frac{\lambda}{(\lambda+\mu)\theta} \right]^{d-a} \right)}{\left(1 - \frac{\lambda}{(\lambda+\mu)\theta} \right)} \right. \right. +$$

$$+ \frac{\lambda + \mu}{\lambda} - \frac{\mu}{\lambda} \frac{(\theta^{d} - \theta^{b+1})}{(1-\theta)} \Bigg]$$

Simiplifying,

$$D = \frac{B}{(1-\theta)\theta^{b}} \left[\theta^{a} - \theta^{b} - \left[\frac{\lambda+\mu}{\lambda} \right]^{d-a} (\theta^{d} - \theta^{b}) \right] \quad (11)$$

The only unknown B involved in the steady - state probabilities can be obtained from the normalizing condition

Then

$$B = \left(\frac{\lambda}{\mu} \right) \theta^{b} \ (1-\theta)\ A, \text{ where}$$

$$A = \frac{1-\theta}{\left[a(1-\theta)\{\theta^{a} - (\theta^{d} - \theta^{b}) \left[\frac{\lambda+\mu}{\lambda} \right]^{d-a} \} + \theta^{a+1} - \theta^{b+1} - (1-\theta^{b})(\theta^{d+1} - \theta^{b+1}) \left[\frac{\lambda+\mu}{\lambda} \right]^{d-a} \right]}$$

We now give the closed expressions for $P_{i\,n}$

From equation (8)

$$P_{1\,n} = B\theta^{n} = \left(\frac{\lambda}{\mu} \right) \theta^{b+n} (1-\theta) \quad , \quad n \geq o.$$

From equation (9) we have

$$P_{o\,n} = \frac{\mu}{\lambda} \left[D + B \sum_{r=o}^{n} \theta^{r} \right] \quad , \quad n = o,1,2,..,a-1.$$

$$= \frac{\mu}{\lambda}\left[\frac{\lambda}{\mu}\frac{\theta^b(1-\theta)A}{\theta^b(1-\theta^b)}\left[\theta^a - \theta^b - (\theta^d - \theta^b)\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a}\right]\right.$$

$$\left. + \frac{\lambda}{\mu}\theta^b(1-\theta)A\frac{(1-\theta^{n+1})}{(1-\theta)}\right]$$

On simiplifying further we get

$$P_{o\,n} = A\left[\theta^a - \theta^{b+n+1} - (\theta^d - \theta^b)\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a}\right]$$

$$n = o,1,2,\ldots.a-1.$$

Using (10) we have

$$P_{o\,n} = \frac{\mu}{\lambda+\mu}\left[\left[\frac{\lambda}{\lambda+\mu}\right]^{n-a}\left[D + B\sum_{r=o}^{a-1}\theta^r\right] + B\,\theta^n\sum_{r=o}^{n-a}\left[\frac{\lambda}{(\lambda+\mu)\theta}\right]^r\right]$$

$$= B.\frac{\mu}{\lambda+\mu}\left[\left[\frac{\lambda}{\lambda+\mu}\right]^{n-a}\left[\frac{\theta^a - \theta^b - (\theta^d - \theta^b)((\lambda+\mu/\lambda))^{d-a}}{\theta^b(1-\theta)}\right.\right.$$

$$\left.\left. + \frac{(1-\theta^a)}{(1-\theta)}\right] + \frac{\theta^n(1-(\lambda/(\lambda+\mu)\theta)^{n-a+1})}{(1-\lambda/(\lambda+\mu)\theta)}\right]$$

Substituting for B and regrouping we get

$$P_{o\,n} = \frac{\lambda}{\mu}\theta^b(1-\theta)\,A.\frac{\mu}{\lambda+\mu}.\frac{1}{(1-\theta)\theta^b}\left[-(\theta^d - \theta^b)\left(\frac{\lambda+\mu}{\lambda}\right)^{d-a}\right.$$

$$+ \frac{\theta^n(1-\theta^{b+1})\;\theta^b(1-\theta)}{\theta^b(1-\theta)} + \left[\frac{\lambda}{\lambda+\mu}\right]^{n-a} x$$

$$\left.\left[\theta^a - \theta^b + \frac{(1-\theta^a)\theta^b(1-\theta)}{(1-\theta)} - \frac{\theta^n(1-\theta^{b+1})\lambda\theta^b(1-\theta)}{\theta^b(1-\theta)(\lambda+\mu)\theta^{n-a+1}}\right]\right]$$

Using (7) and simplifying we get

$$P_{o\,n} = -(\theta^d - \theta^b)\, A \left(\frac{\lambda+\mu}{\lambda}\right)^{d-n-1} +$$

$$+ A \frac{\lambda}{\lambda+\mu}\left[\theta^n (1-\theta^{b+1}) + \left[\frac{\lambda}{\lambda+\mu}\right]^{n-a} \left[\theta^a - \theta^{a+b} - \left(\frac{\lambda}{\lambda+\mu}\right)(1-\theta^{b+1})\,\theta^{a-1}\right]\right]$$

$$= A \left[\theta^{n+1} (1-\theta^b) - (\theta^d - \theta^b) \left[\frac{\lambda+\mu}{\lambda}\right]^{d-n-1}\right]$$

Accordingly we have

$$P_{o\,n} = A \left[\theta^a - \theta^{b+n+1} - (\theta^d - \theta^b) \left[\frac{\lambda+\mu}{\lambda}\right]^{d-a}\right],$$

$$n = 0,1,2 \ldots\ a-1. \qquad (14)$$

$$P_{o\,n} = A\left[\theta^{n+1} (1-\theta^b) - (\theta^d - \theta^b) \left[\frac{\lambda+\mu}{\lambda}\right]^{d-n-1}\right]$$

$$n = a, a+1, \ldots\ d-1 \qquad (14)$$

$$P_{1\,n} = A\left(\frac{\lambda}{\mu}\right)(1-\theta)\,\theta^{b+n}, \qquad n \geq o \qquad (15)$$

AVERAGE QUEUE LENGTH (L_q)

Next we shall calculte the average number of units in the queue.

$$L_q = \sum_{n=o}^{a-1} n P_{o\,n} + \sum_{n=o}^{\infty} n\, P_{1\,n}$$

$$\sum_{n=o}^{a-1} n P_{o\,n} = \sum_{n=o}^{a-1} n A \left[\theta^a - \theta^{b+n+1} - (\theta^d - \theta^b) \left[\frac{\lambda+\mu}{\lambda}\right]^{d-a} \right]$$

$$= A \frac{a(a-1)}{2} \left[\theta^a - (\theta^d - \theta^b) \left[\frac{\lambda+\mu}{\lambda}\right]^{d-a} \right]$$

$$+ \frac{A \theta^{b+2} [(1-\theta)a\theta^{a-1} + \theta^{a-1}]}{(1-\theta)^2}$$

$$\sum_{n=o}^{\infty} n P_{1\,n} = \sum_{n=o}^{\infty} n A \left(\frac{\lambda}{\mu}\right) (1-\theta)\, \theta^{b+n}$$

$$= A \frac{\lambda}{\mu} (1-\theta)\, \theta^{b+1} \frac{1}{(1-\theta)^2}$$

$$= A \frac{\lambda}{\mu} \frac{\theta^{b+1}}{1-\theta}.$$

Thus

$$L_q = \left[A \frac{\lambda}{\mu} \frac{\theta^{b+1}}{(1-\theta)} + A \frac{a(a-1)}{2} \left[\theta^a - (\theta^d - \theta^b) \left[\frac{\lambda+\mu}{\lambda}\right]^{d-a} \right] + \frac{A \theta^{b+2} [a\theta^{a-1} (1-\theta) + \theta^a - 1]}{(1-\theta)^2} \right]$$

DISTRIBUTION OF WAITING TIME AND OCCUPATION TIME.

To obtain the distribution of waiting time we use the following notations:-

Let the random variable T denote the waiting time in the queue for an arriving unit.

Let $\gamma(t)$ be the pdf of T.

Let $f(\alpha,k;t)$ be the pdf of gamma distribution with parameters α, k.

$$\text{(ie)}\ f(\alpha,k;t) = \frac{\alpha^k\, t^{k-1}\, e^{-\alpha t}}{\Gamma(k)}, \qquad t > o \quad k = 1,2...$$

$\Gamma_x(\alpha,k)$ be the incomplete gamma function.

$$\Gamma_x(\alpha,k) = \int_o^x f(\alpha,\, k;t)dt$$

$$= 1-\sum_{s=o}^{k-1} \frac{e^{-\alpha x}\,(\alpha x)^s}{s!}$$

An arriving unit has to wait if it finds the system in any one of the following states:

(i) (o,n) , $o \le n \le a-2$.

(ii) $(1,n)$, $n = kb+q$, $a-1 \le q \le b-1$ & $k = o,1,2,...$

(iii) $(1,n)$, $n = kb+q$, $o \le q \le a-2$ & $k = o,1,2,...$

In case of (i) the arriving unit has to wait for the arrival of (a-1-n) units and the time needed for (a-1-n) arrivals has a gamma distribution with parameters \, a-1-n.

In case of (ii) the arriving unit has to wait for the completion of services of (k+1) batches; the time required for this has a gamma distribution with parameters μ, k+1.

In case of (iii), the arriving unit has to wait till either the services of (k+1) batches are completed or (a-1-q) units arrive, whichever occurs later. This duration which is given by the maximum of two gamma variates may be denoted by the random variable Z (ie)

Z = max {gamma variate with parameters λ, a-1-q; gamma variable with parameters μ, k+1}.

The distribution function $F_Z(t)$ and the pdf $h_Z(t)$ of Z are given by

$$F_Z(t) = \text{Prob}\{Z \leq t\} = \Gamma_t(\lambda, a-1-q)\,\Gamma_t(\mu, k+1)$$

and

$$h_Z(t) = F_Z(t) = f(\lambda, a-1-q; t)\,\Gamma_t(\mu, k+1) + \Gamma_t(\lambda, a-1-q)\, f(\mu, k+1; t).$$

Distribution of waiting time $\gamma(t)$ is given by

$$\gamma(t) = \sum_{n=o}^{a-2} P_{o\,n}\, f(\lambda, a-1-n; t) + \sum_{q=a-1}^{b-1} \sum_{k=o}^{\infty} P_{1\; kb+q}\, f(\mu, k+1; t) + \sum_{k=o}^{\infty} \sum_{q=o}^{a-2} P_{1\; kb+q}\, h_Z(t), \quad o<t<\infty$$

First term of $\gamma(t)$ is

$$\sum_{n=o}^{a-2} P_{o\,n}\, f(\lambda, a-1-n, t)$$

$$= \sum_{n=o}^{a-2} A \left[\theta^{a} - \theta^{b+n+1} - (\theta^{d} - \theta^{b}) \left[\frac{\lambda+\mu}{\lambda}\right]^{d-a} \right]$$

$$\times \frac{\lambda^{a-1-n}\, t^{a-2-n}\, e^{-\lambda t}}{(a-n-2)\,!} - A \sum_{n=o}^{a-2} \frac{\theta^{b+n+1}\, \lambda^{a-1-n}\, t^{a-2-n}\, e^{-\lambda t}}{(a-2-n)!}$$

$$= A\, e^{-\lambda t} \left[\theta^{a} - (\theta^{d} - \theta^{b}) \left[\frac{\lambda+\mu}{\lambda}\right]^{d-a} \right] \cdot \lambda \sum_{r=o}^{a-2} \frac{(\lambda t)^{r}}{r!}$$

$$-A\, \theta^{b+a-1}\, \lambda \quad e^{-\lambda t} \sum_{r=o}^{a-2} \frac{(\lambda t/\theta)^{r}}{r!}$$

Second term gives

$$\sum_{k=o}^{\infty} \sum_{q=a-1}^{b-1} P_{1\;kb+q} \quad f(\mu, k+1; t)$$

$$= \sum_{k=o}^{\infty} \sum_{q=a-1}^{b-1} A.\, \frac{\lambda}{\mu}\, (1-\theta)\, \theta^{b+kb+q} \frac{\mu^{k+1}.\, t^{k}.\, e^{-\mu t}}{k!}$$

$$= \theta^{b}\, A \frac{\lambda}{\mu} (1-\theta) \frac{(\theta^{a-1} - \theta^{b})}{(1-\theta)} \sum_{k=o}^{\infty} \frac{(\theta^{b}\, \mu t)^{k}}{k!} \quad e^{-\mu t}.\, \mu.$$

$$= A\, \lambda \theta^{b}\, (\theta^{a-1} - \theta^{b})\, e^{\mu t(\theta^{b} - 1)}$$

The third term of $\gamma(t)$ is

$$\left[\sum_{k=o}^{\infty} \sum_{q=o}^{a-2} A \frac{\lambda}{\mu} (1-\theta)\, \theta^{b+kb+q} \right] \left[\frac{\lambda^{a-1-q}\, t^{a-2-q}\, e^{-\lambda t}}{(a-2-q)!} \right]$$

$$\times \left[1 - \sum_{s=o}^{k} \frac{e^{-\mu t}\, (\mu t)^{s}}{s!} \right]$$

$$+ \frac{\mu^{k+1} t^k e^{-\mu t}}{k!} \left[1 - \sum_{s=o}^{a-2-q} \frac{e^{-\lambda t} (\lambda t)^s}{s!} \right] \Bigg]$$

Consider

$$\sum_{k=o}^{\infty} \sum_{q=o}^{a-2} A \frac{\lambda}{\mu} (1-\theta) \theta^{b+Kb+q} \frac{\lambda^{a-1-q} t^{a-2-q} e^{-\lambda t}}{(a-2-q)!}$$

Substituting for $\frac{\lambda}{\mu} = \frac{(1-\theta^b)\theta}{(1-\theta)}$ and simplifying we

get

$$A \lambda e^{-\lambda t} \theta^{b+a-1} \sum_{r=o}^{a-2} \frac{(\lambda t/\theta)^r}{r!}$$

Consider

$$\sum_{k=o}^{\infty} \sum_{q=o}^{a-2} A \frac{\lambda}{\mu} (1-\theta) \theta^{b+kb+q} \frac{\mu^{k+1} t^k e^{-\mu t}}{k!}$$

$$= A \theta^b \frac{\lambda}{\mu} (1-\theta) \frac{(1-\theta^{a-1})}{(1-\theta)} \sum_{k=o}^{\infty} \frac{(\theta^b \mu t)^k}{k!} e^{-\mu t} \mu$$

$$= (1-\theta^{a-1}) A \lambda \theta^b e^{\mu t(\theta^b - 1)}$$

Consider

$$- \sum_{k=o}^{\infty} \sum_{q=o}^{a-2} A \frac{\lambda}{\mu} (1-\theta) \theta^{b+kb+q}$$

$$\times \frac{\lambda^{a-1-q} t^{a-2-q} e^{-\lambda t}}{(a-2-q)!} \sum_{s=o}^{k} \frac{e^{-\mu t} (\mu t)^s}{(s)!}$$

$$= -A \frac{\lambda}{\mu} (1-\theta)\theta^{b} \sum_{k=o}^{\infty} (\theta^{b})^{k} \sum_{q=o}^{a-2} (\lambda t/\theta)^{a-2-q} \frac{\lambda \cdot \theta^{a-2} \cdot e^{-\lambda t}}{(a-2-q)\,!}$$

$$\times \sum_{s=o}^{k} \frac{e^{-\mu t} (\mu t)^{s}}{s!}$$

$$= \left[-A\, e^{-\lambda t} \frac{\lambda}{\mu} \theta^{b} (1-\theta)\, \theta^{a-2}\, e^{-\mu t} \cdot \lambda \sum_{r=o}^{a-2} \frac{(\lambda t/\theta)^{r}}{r!} \right]$$

$$\times \left[1+ \theta^{b} \sum_{s=o}^{1} \frac{(\mu t)^{s}}{s!} + \theta^{2b} \sum_{s=o}^{2} \frac{(\mu t)^{s}}{s!} + \ldots \right]$$

$$= \left[- A\; e^{-\lambda t} \frac{\lambda}{\mu} \theta^{b} (1-\theta)\, \theta^{a-2}\; e^{-\mu t} \lambda \sum_{r=o}^{a-2} \frac{(\lambda t/\theta)^{r}}{r!} \right]$$

$$\times \left[\frac{e^{\mu t}\, \theta^{b}}{1-\theta^{b}} \right]$$

$$= -A \lambda\, e^{-\lambda t}\; \theta^{a+b-1}\; e^{-\mu t (1-\theta^{b})} \sum_{r=o}^{a-2} \frac{(\lambda t/\theta)^{r}}{r!}$$

Consider

$$\sum_{k=o}^{\infty} \sum_{q=o}^{a-2} A \frac{\lambda}{\mu} (1-\theta)\theta^{b+kb+q} \frac{\mu^{k+1}\, t^{k}\, e^{-\mu t}}{k!}$$

$$\times \sum_{s=o}^{a-2-q} \frac{e^{-\lambda t} (\lambda t)^{s}}{s!}$$

$$= -A \frac{\lambda}{\mu} e^{-\mu t} (1-\theta)\, \theta^{b} \sum_{k=o}^{\infty} \frac{\theta^{kb}\, \mu^{k+1}\, t^{k}}{k!}$$

$$\sum_{q=o}^{a-2} \theta^{q} \sum_{s=o}^{a-2-q} \frac{e^{-\lambda t} (\lambda t)^{s}}{s!}$$

$$= - A \frac{\lambda}{\mu} e^{-\mu t} (1-\theta)\theta^{b}\, e^{-\lambda t}\, \mu \sum_{k=o}^{\infty} \frac{(\theta^{b} \mu t)^{k}}{k!}$$

$$\times \left[\theta^{a-2} . 1 + \theta^{a-3} (1+\lambda t/1!) + \ldots + \sum_{r=o}^{a-2} \frac{(\lambda t)^{r}}{r!} \right]$$

$$= \left(- A \lambda\, e^{-\mu t(1-\theta^{b})} (1-\theta)\, \theta^{b}\, e^{-\lambda t}\right)$$

$$\times \frac{1}{1-\theta} \left[(1-\theta^{a-1}) + \frac{\lambda t}{1!} (1-\theta^{a-2}) + \ldots + \frac{(\lambda t)^{a-2}}{(a-2)!} (1-\theta) \right]$$

$$= - A \lambda\, e^{-\mu t(1-\theta^{b})}\, \theta^{b} \sum_{r=o}^{a-2} \frac{(\lambda t)^{r}}{r!}$$

$$+ A \lambda\, e^{-\mu t(1-\theta^{b})}\, \theta^{b}\, e^{-\lambda t}\, \theta^{a-1} \sum_{r=o}^{a-2} \frac{(\lambda t/\theta)^{r}}{r!}$$

Substituting in $\gamma(t)$, we have

$$\gamma(t) = A\lambda \left[\theta^{b} (1-\theta^{b})\, e^{-\mu t(1-\theta^{b})} \right.$$

$$+ \left[\theta^{a} - (\theta^{d} - \theta^{b}) (\lambda+\mu/\lambda)^{d-a} - \theta^{b} e^{-\mu t(1-\theta^{b})} \right]$$

$$\left. \times\, e^{-\lambda t} \sum_{r=o}^{a-2} \frac{(\lambda t)^{r}}{r!} \right]$$

The mean waiting time in the queue $w_q = \int t\gamma(t)$ for an arriving unit is found to satisfy the Little's formula $L_q = \lambda w_q$.

$$w_q = \int_0^\infty t\, \gamma(t)$$

$$= \left[\int_0^\infty t\, A\lambda \left[\theta^a (1-\theta^b)\, e^{-\mu t(1-\theta^b)}\right]\right.$$

$$+ A\lambda \int_0^\infty t \left[\theta^a - (\theta^d - \theta^b) \quad (\lambda+\mu/\lambda)^{d-a} \right.$$

$$\left.\left. - \theta^b\, e^{-\mu t(1-\theta^b)} \right] e^{-\lambda t} \sum_{r=0}^{a-2} \frac{(\lambda t)^r}{r!} \right]$$

$$= A\lambda \left[(1-\theta^b)\, \theta^b \left[-\int_0^\infty \frac{e^{-\mu t(1-\theta^b)}}{-\mu(1-\theta^b)}\, dt \right] \right.$$

$$+ \{ \theta^a - (\theta^d - \theta^b\, (\lambda+\mu/\lambda)^{d-a} \} \sum_{r=0}^{a-2} \lambda^r/r! \int_0^\infty t^{r+1} \quad e^{-\lambda t}\, dt$$

$$\left. -\theta^b \sum_{r=0}^{a-2} \lambda^r/r! \int_0^\infty e^{-(\lambda+\mu(1-\theta^b))t} \quad t^{r+1}\, dt \right]$$

$$= A\,\lambda \left[\frac{(1-\theta^b)\theta^b}{[\mu(1-\theta^b)]^2} + \{\theta^a - (\theta^d - \theta^b) \left[\frac{\lambda+\mu}{\lambda}\right]^{d-a} \} \sum_{r=0}^{a-2} \frac{\lambda^r}{r!} \right.$$

$$\left. \times \frac{\Gamma(r+2)}{\lambda^{r+2}} - \theta^b \sum_{r=0}^{a-2} \frac{\lambda^r}{r!} \quad \frac{\Gamma(r+2)}{[\lambda+\mu(1-\theta^b)]^{r+2}} \right]$$

substituting for $\frac{\mu}{\lambda} = \frac{1-\theta}{\theta(1-\theta^b)}$ we get

$$w_q = A\lambda\left[\frac{(1-\theta^b)\theta^b}{\mu^2(1-\theta^b)^2} + \left[\theta^a - (\theta^d-\theta^b)\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a}\right]\sum_{r=o}^{a-2}\frac{\lambda^r}{r!}\right.$$

$$\left. \times \frac{(r+1)!}{\lambda^{r+2}} - \theta^b \sum_{r=o}^{a-2}\frac{\lambda^r}{r!}\;\frac{(r+1)!}{[\lambda+\mu(1-\theta^b)]^{r+2}}\right]$$

$$= \frac{A\theta^{b+1}}{\mu(1-\theta)} + A\left[\theta^a - (\theta^d-\theta^b)\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a}\right]\frac{a(a-1)}{2\lambda}$$

$$- \frac{\theta^b \quad A\lambda}{[\lambda+\mu(1-\theta^b)]^2}\sum_{r=o}^{a-2}\frac{(r+1)}{[\,1+((1-\theta)\,/\,\theta(1-\theta^b))\,(1-\theta^b)\,]^r}$$

Simplifying, we get

$$w_q = \frac{A\theta^{b+1}}{\mu(1-\theta)} + A\left[\theta^a-(\theta^d-\theta^b)\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a}\right]\frac{a(a-1)}{2\lambda}$$

$$+ \frac{A\theta^{b+2}}{\lambda(1-\theta)^2}\quad\{\theta^{a-1} + a\theta^{a-1}\,(1-\theta)\}$$

It is seen that $L_q = \lambda w_q$.

Hence the Little's formula is checked

OCCUPATION TIME

The server's occupation time 't' under the steady state conditions in queue is non-zero if the system is in one of the following three mutually exclusive states.

(i) (o,n), $n = a, a+1, \ldots, d-1$.

(ii) (1,n), $n = kb+q$, $q = o, 1, \ldots, a-1$ and $k = o, 1, 2 \ldots$

(iii) (1,n), $n = kb+q$, $q = a, a+1, \ldots, b-1$ and $k = o, 1, 2, \ldots$

In case of (i) the server will be busy with a batch with probability $P_{o\,n}$ $(a \le n \le d-1)$ and his occupation time in this case is $\mu e^{-\mu t}$.

In case of (ii) his busy time follows a gamma distribution with parameter $(\mu, k+1)$ and occupation time is $\mu e^{-\mu t}(\mu t)^k/k!$ with probability $P_{1\,kb+q}$

In case of (iii) his busy time follows a gamma distribution with parameter $(\mu, k+2)$ and the occupation time is $\mu e^{-\mu t}(\mu t)^{k+1}/(k+1)!$ with prob $P_{1\,kb+q}$

Hence pdf g(t) of the occupation time t is given by

$$g(t) = \sum_{n=a}^{d-1} P_{o\,n}\, \mu e^{-\mu t} + \sum_{k=o}^{\infty} \sum_{q=o}^{a-1} P_{1\,kb+q} \frac{\mu e^{-\mu t}(\mu t)^k}{k!}$$

$$+ \sum_{k=o}^{\infty} \sum_{q=a}^{b-1} P_{1\,kb+q} \frac{\mu(\mu t)^{k+1} e^{-\mu t}}{(k+1)!}$$

Consider the first term.

$$\sum_{n=a}^{d-1} P_{o\,n}\,\mu e^{-\mu t}$$

$$= A \sum_{n=a}^{d-1} \left[\theta^{n+1}(1-\theta^{b}) - (\theta^{d}-\theta^{b}) \left[\frac{\lambda+\mu}{\lambda}\right]^{d-n-1} \right] \mu e^{-\mu t}$$

$$= A\,\mu e^{-\mu t} \left[(1-\theta^{b}) \frac{(\theta^{a+1}-\theta^{d+1})}{(1-\theta)} - \frac{(\theta^{d}-\theta^{b})\,((\lambda/\lambda+\mu)^{a-d+1} - (\lambda/\lambda+\mu))}{1-(\lambda/\lambda+\mu)} \right]$$

$$= A\,\lambda e^{-\mu t} [\, \theta^{a}-\theta^{b}-(\theta^{d}-\theta^{b}) \quad (\lambda / \lambda+\mu)^{a-d} \,]$$

Second term

$$\sum_{k=o}^{\infty} \sum_{q=o}^{a-1} A \frac{\lambda}{\mu} (1-\theta)\theta^{b+kb+q}\,\mu \frac{(\mu t)^{k} e^{-\mu t}}{k!}$$

$$= A \frac{\lambda}{\mu} (1-\theta)\,\theta^{b} e^{-\mu t}\,\mu \sum_{k=o}^{\infty} \frac{(\theta^{b}\mu t)^{k}}{k!} \sum_{q=o}^{a-1} \theta^{q}$$

$$= A\,\lambda e^{-\mu t}(1-\theta^{b})\,\theta^{b}\,(1-\theta^{a})$$

Third term

$$\sum_{k=o}^{\infty} \sum_{q=a}^{b-1} A \frac{\lambda}{\mu} (1-\theta)\theta^{b+kb+q} \frac{\mu(\mu t)^{k+1}\,e^{-\mu t}}{(k+1)!}$$

$$= A\,\lambda e^{-\mu t}(1-\theta^{b}).(\theta^{a}-\theta^{b}) - A\,\lambda e^{-\mu t}(\theta^{a}-\theta^{b})$$

$$g(t) = A\lambda\left[\theta^{a}(1-\theta^{b})e^{-\mu t(1-\theta^{b})t} - (\theta^{d}-\theta^{b})\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a} . e^{-\mu t}\right]$$

Denoting the mean occupation time $\bar{E}(t)$, given $t>0$ by U_q

$$U_q = \int_0^\infty t.\ g(t)$$

$$= \int_0^\infty t\ A\lambda\left[\ \theta^{a}(1-\theta^{b})\ e^{-\mu(1-\theta^{b})t} - (\theta^{d}-\theta^{b})\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a} e^{-\mu t}\right]$$

$$= \frac{A\lambda\theta^{a}(1-\theta^{b})}{[\mu(1-\theta^{b})\ \ ]^{2}}\left[-e^{\mu(1-\theta^{b})t}\right]_0^\infty$$

$$- \frac{A\lambda\ (\theta^{d}-\theta^{b})}{\mu^{2}}\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a}\left[-e^{-\mu t}\right]_0^\infty$$

$$= \frac{A\theta(1-\theta^{b})\theta^{a}}{(1-\theta)\mu(1-\theta^{b})} - \frac{A\lambda(\theta^{d}-\theta^{b})}{\mu^{2}}\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a}$$

$$= \frac{A}{\mu(1-\theta)}\left[\theta^{a+1} - (\theta^{d}-\theta^{b})\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a}\frac{(1-\theta).\theta(1-\theta^{b})}{(1-\theta)}\right]$$

$$= \frac{A\theta}{\mu(1-\theta)}\left[\theta^{a}-(\theta^{d}-\theta^{b})\left[\frac{\lambda+\mu}{\lambda}\right]^{d-a}(1-\theta^{b})\right]$$

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