Review of Literature

Graph theory is the theory of graphs dealing with nodes and connections or vertices and edges. The subject has skilled rapid growth, due to large measure of its role as a necessary arrangement underpinning modern applied mathematics. Configurations of nodes and connections have great diversity of applications.

Coloring theory started with the problem of coloring the countries of a map in such a way that no two countries that have a common border receive the same color. If we denote the countries by points in the plane and connect each pair of points that correspond to countries with a common border by a curve, we obtain a planar graph. The celebrated Four Color Problem asks if every planar graph can be colored with 4 colors. It seems to have been mentioned for the first time in writing in 1852 letter from A. De Morgan to W.R. Hamilton. Nobody thought at that time that it was the beginning of a new theory. The first “proof” was given by Kempe (1879).

Besides many erroneous proofs (not only that by Kempe), it generated many new directions in Graph Theory. For example, only one sub-direction of chromatic polynomials introduced by Birkhoff (1912) with the aim to solve the problem by algebraic methods counts more than 500 research papers. The main idea of the final proof is quite simple by induction on the number of vertices; but the number of cases is huge. Though the Kempe’s proof was erroneous, his idea of alternating paths and further re-coloring of the respective sub graphs was used in the final proof. A path on which two colors alternate is called a Kempe chain.

Appel et al., (1977) using 1200 hours of computer time, found 1936 unavoidable configurations and proved that all they are reducible. Historically, it was the first time when a famous mathematical problem was solved by extensive use of computers.

Birkhoff (1912) working at Princeton, published the following paper: “A determinant formula for the number of ways of coloring a map”. He introduced a function denoted by $P(G,\lambda)$ which gives the number of proper colorings of a graph G using the colors from set $\{1,2,3,...,\lambda\}$. Since it is a polynomial, it was called the chromatic polynomial of a graph. Sometimes mathematics, and graph theory in particular, looks like a race for generalizations.

In order to generalize graph coloring, Erdos et al., (1966) have introduced hyper graph colourings, the requirement "adjacent vertices must have different colors was generalized to at least two vertices in hyper edge must have different colors". This idea was extremely fruitful and led to many generalizations of graph colorings and many hyper graph classes have been discovered.

Graph coloring deals with the general and widely applicable concept of partitioning the underlying set of a structure into parts, each of which satisfies a given requirement. Here coloring means a vertex coloring of a graph. This inspired Irving et al., (1999) to consider another procedure, which consists of trying to reduce the number of colours by transferring all vertices from one colour class to other colour class. Consideration of proper colourings that are minimal with respect to ψ gives rise to a new parameter, $j(G)$, which is called as b-chromatic number of G.

A k-coloring of a graph G = (V,E) is a mapping C : V → S where S is a set of *k* colors; thus *k*-coloring is an assignment of k colors to the vertices of G. Usually, the set S of colors is taken to be {1,2,...k}. A coloring C is proper if no two adjacent vertices are assigned the same color. Only loop less graphs admits proper coloring. The minimum *k* for which a graph G is k-colorable is called its chromatic number, and denoted $\chi(G)$.

If $\chi(\mathrm{G}) = \mathrm{k}$, the graph G is said to be k chromatic. The b-chromatic number of G, denoted by $\varphi(\mathrm{G})$, is the maximum k for which G has a b-coloring by k colors. A b-coloring of G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex x_i having neighbours in all the other k-1 color classes. Such a vertex x_i is called a b-dominating vertex, and the set of vertices $\{x_1, x_2 ... x_k\}$ is called a b-dominating system. The b-coloring was introduced by Irving, R.W et al., (1999). They proved that determining φ(G) is NP-hard in general and polynomial for trees. They have also proved the following:

- upper bound of $\phi(G), \phi(G) \leq \Delta(G) + 1$.

Many graph invariants related to colorings have been defined. Most of them try to minimize the number of colors used to color the vertices under some constraints. For some invariants, it is meaningful to try to maximize this number.

The b-chromatic number is one such example. When we try to color the vertices of a graph, a simple trick consists in starting from a colouring and trying to decrease the number of colors by reducing in some way, for example by merging two color classes. This motivated the introduction of the achromatic number.

The achromatic number of a graph G admits a coloring with k-colors for which there is an edge between any two color classes. But the process of merging suggested above is impossible if we have such a coloring. So the achromatic number is a measure of complexity to obtain Guy Korsarza, (2006) coloring with few colors.

The b-chromatic number $\varphi(G)$ is the largest integer k such that G admits a b-coloring with k colors. Since then the b-chromatic number has drawn much attention in scientific area. Effantin, B et al., (2005) studied exact values for the b-chromatic number of a power complete k-ary tree. Jakovac, M et al., (2010) derived the b-chromatic number of cubic graphs, and Kouider, M et al., (2006) derived bounds for the b-chromatic number of some families of graph. Kouider, M et al., (2007) proved the b-chromatic number of the Cartesian product of two graphs. El-Sahili, A et al., (2006) about b-coloring of regular graphs, were created lot of ideas to the researchers.

The color classes as different communities, where every community i has a representative that is able to communicate with all the others communities. Even though the b-chromatic number is a simple concept, it is hard to determine the exact values, even for known families of graphs. This lead to studies of lower and upper bounds of following works,

- Balakrishnan, R et al., “Bounds for the b-chromatic number of the Mycielskian of some families of graphs” and “Bounds for the b-chromatic number of $G-v$”.

- Kouider, M et al., (2002) “Some bounds for the b-chromatic number of a graph”.

Kouider, M et al., (2006) proposed some upper bounds for the b-chromatic number of several classes of graphs in function of other graph parameters (clique number, chromatic number, biclique number).

El-Sahili, A et al., (2006) proved by showing that if G is a d-regular graph with girth 5 and without cycles of length 6, then $\phi(G) = d + 1$. Jakovac, M et al., (2010), proved that the b-chromatic number of cubic graphs is four with the exception of Petersen graph, $K_{3,3}$, prism over K_3, and sporadic with 10 vertices. Kouider, M et al., (2002) gave some lower and upper bounds for the b-chromatic number of the Cartesian product of two graphs. Kratochvil, J. (2002) characterized bipartite graphs for which the lower bound on the b-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a connected bipartite graphs which is dominating proper b-coloring even for with $k = \Delta(G) + 1$.

Corteel, S (2005) proved that the b-chromatic number problem is not approximable within 120/133 -∈ for any ∈> 0, unless P = NP. Origami is a kind of art first invented from Japan. It is a feasible to fold up a lot of good-looking figures in origami. Most miraculously, many amazing pieces of origami are shaped from a particular part of paper, with no cuttings. Just like constructions using directly edge and compass, constructions through paper folding is both mathematically interesting and aesthetic, particularly in origami graph by Lang, R.J (1996-2003).

Flat folding have been extensively studied in research on the mathematics of paper folding. The folding patterns that can fold flat with only a single vertex have been completely characterized, for standard models of origami by Thomas Hull (1994), Husimi, K (1979), Justin, J (1986) for rigid origami in which the paper must continuously move from its unfolded state to its folded state without bending anywhere except at its given creases, and even for single-vertex folding patterns whose paper does not form a single flat sheet.

Gallian, J.A (2018) investigated a natural extension of the cycle, by joining the endpoint of a path to a cycle, also known as tadpole, dragon, or kite graphs. Maybe interestingly, this class of graphs was also implicitly used for the lower bound construction on unit edge weight graphs in Miyazaki, S et al., (2009), Morimoto, N (2014).

The b-chromatic number of some special graph classes has been discussed by Kratochvil, J (2002). Effantin, B (2006) presented in a distributed algorithm to determine a b-coloring of a graph G in $O(n\Delta)$ changes of colors, where Δ is the maximum degree of G. They showed that such a coloring can be used for routing information into a network.

Liu, S et al., (2008) proposed a beautiful example of a dynamic distributed algorithm for a b-coloring of graphs. Distributed algorithms are a very attractive topic for a lot of fields and several graph problems arise naturally in distributed systems. Such algorithms are very interesting and 37 efficient for dynamic networks.

Jakovac, M et al., (2010) proved that the b-chromatic number of cubic graphs is four with the exception of Petersen graph, $K_{3,3}$ prism over K_3, and sporadic (isolated) with 10 vertices. Hoang, C.T et al., (2005) characterized the bipartite graphs and the P4- sparse graphs for which each induced subgraph H of G has $\varphi(H) = \chi(H)$.

Kouider, M et al., (2011) presented some lower bounds for the b-chromatic number of connected bipartite graphs. They also discussed some algorithmic consequences of such lower bounds on some subfamilies of connected bipartite graphs.

Betancur Velasquez, C.I (2011) proved that P4−tidy graphs (a generalization of many classes of graphs with few induced P4s) are b-continuous and b-monotonic and also described a polynomial time algorithm to compute the b-chromatic number for this class of graphs.

Cabello, S et al., (2011) proved that every d-regular graph with at least $2d^3$ vertices has b-chromatic number d + 1, the b-chromatic number of an arbitrary d-regular graph with girth $g = 5$ is at least $(d+1)/2$ and every d-regular graph, $d \geq 6$, with diameter at least d and with no 4-cycles, has b-chromatic number $d+1$.

Shaebani, S (2012) proved that, for each d-regular graph G which contains no 4-cycle $\phi(G) \geq [(d+3)/2]$ and besides, if G has a triangle, then $\phi(G) \geq [(d+4)/2]$. Also, if G is a d-regular graph that contains no 4-cycle and diam (G) ≥ 6, then $\phi(G) = d+1$ and also discussed that for any d-regular graph G which does not contain 4-cycle and has vertex connectivity less than or equal to $(d+1)/2$, $\phi(G) = d+1$. Moreover, when the vertex connectivity is less than

$(3d-3)/4$, introduced a lower bound for the b-chromatic number in terms of the vertex connectivity.

Maffray, F et al., (2012) discussed how to compute in polynomial time the b-chromatic number of an outer planar graph of girth at least 8. Havet, F et al., (2012) proved that it is also NP-hard for a tight chordal graph. They also found that the b-chromatic number of a split graph can 38 be computed in polynomial time.

Galcik, F et al., (2013) have been settled this question in a negative way proving that there is no constant ϙ > 0, for which the problem can be approximated within a factor $n^{1/4-ϙ}$, unless P = NP. Maffray, F et al., (2013) discussed that how to find in polynomial time an optimal b-colouring of the Cartesian product of trees by paths, cycles and stars.

Effantin, B et al., (2006) proposed a distributed method to determine a particular coloring of graphs. Thus, they proposed a partitioning method of a graph based on b-colorings. El-Sahili, A (2006) proved any d-regular graph with girth 6 has a b-chromatic number equal to d + 1.

Marko Jakovac et al., (2009) disproved this conjecture “Every d-regular graph with girth at least 5, different from the Petersen graph, has a b-coloring with d+1 colors” and they proved the following: “Peterson graph is a 3-regular graph with girth 5 and b-chromatic number 3”.

Marko Jakovac (2009) proved the following,

- Let P be the Petersen graph. Then $\varphi(P) = 3$.
- Let G be a connected cubic graph.

Then $\varphi(G) = 4$ unless G is $P, K_3 \,\mathrm{o} K_2, K_{3,3}$ or G_1.

In these cases,

$$\varphi(P) = \varphi(\mathrm{K}_3 \,\mathrm{o} \mathrm{K}_3) = \varphi(G_1) = 3 \qquad and \qquad \varphi(K_{3,3}) = 2.$$

Mostafa Blidia et al., (2009) studied the b-coloring number of regular graphs and also verified the following, Let G be a d-regular graph with girth $g(G) \geq 5$, different from the Petersen graph, and with d ≤ 6. Then $\varphi(G) = d + 1$.

Effantin, B (2003) has investigated b-chromatic number of some power graphs.

The following authors derived the below concepts,

- Balakrishnan, R et al., (2012) "For any graph G, $\varphi(G) \le \Delta(G) + 1$".
- Irving, R.W, et al., (1999) "If graph G admits a b-coloring with m-colors, then G must have at least m vertices with degree at least m−1 (Since each color class has a b-vertex)".
- Clark, J et al., (1991) "For any graph G, $\chi(G) \ge 3$ 3 if and only if G has an odd cycle" and "If a graph G contains K_n as a subgraph, $\chi(G) \le n$".
- Alkhateeb, M (2012) $\chi(G) \le \varphi(G) \le m(G)$ and $\chi(W_2) = \begin{cases} 3, & n \quad is odd \\ 4, & n \quad is even \end{cases}$
- Havet, F et al., (2012) "Let G be a tight graph. Then $\varphi(G) = m(G)$ if and only if $\varphi(G^*) = m(G)$".

Alkhateeb, M (2012) have investigated Bounds on b-chromatic number for bipartite graphs, b-chromatic number of regular graphs, b-chromatic number of trees , b-chromatic number of Halin graphs. Blidia, M et al., (2013) have discussed b-coloring of vertex b-critical trees.

Maffray, F et al (2012) have studied b-coloring of outer planar graphs with large girth. Velasquez et al., (2011) have characterized the b-coloring of P4-tidy graphs. Jakovac, M et al., (2012) have determined the bounds on b-chromatic number of trees. Jan Kratochvil et al., (2002) obtained a necessary and sufficient condition for a non-trivial connected graph having b-chromatic number 2.

The following results are due to Yap et al., (1992).

- Let K_n be the complete graph.

  Then $\chi'(K_n) = \begin{cases} n, & if \quad n \quad is \quad odd \\ n+1, & if \quad n \quad is \quad even \end{cases}$

- Let C_n be the cycle graph.

  Then $\chi'(C_n) = \begin{cases} 3, & if \quad n \equiv 0 \bmod 3 \\ 4, & otherwise. \end{cases}$

Also they have found a relationship between the b-chromatic number and three other coloring parameters, the equitable chromatic number, harmonious chromatic number and the achromatic number.

Vijayalakshmi, D et al., (2012a, 2012b) have studied the b-chromatic number of Transformation graph G^{++-} for Cycle, Path and Star graph are obtained. Also they have discussed the b-chromatic number of corona product of Path graph with Cycle and Path graph with complete graph along with its structural properties.

Also b-chromatic number of corona product and Cartesian product of path, cycle and star graphs and explained the b-chromatic number of corona product of Path, Cycle and Star graph with complete graph, the Strong product of Path with Cycle and Cartesian product of cycles were discussed.

Kalpana, M et al., (2018a) discussed on b-coloring of tadpole graphs and Mycielskian graphs. Vernold, J et al., (2012) were discussed on the b-chromatic number of corona graphs.

Venkatachalam, et al., (2014) have investigated and obtained

- b-coloring of fan graphs.
- b-chromatic number of double star graphs.

The G_1 and G_2 are two graphs, the corona of two graph is $G = G_1 \circ G_2$ 2 formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 and every vertex in the i^{th} copy of G_2 are adjacent. This kind of product was introduced by Harary et al., (1970).

Lisna, P.C et al., (2015) discussed the b-Chromatic Number of Corona of Graphs. Santhosh (2005) has shown that $C_n \circ P_4$ is sequential for all odd $n \geq 3$. Corona product of unsigned graphs has been used as a framework to develop models for complex networks in literature by Sharma, R et al., (2017).

Equitable colourings of the corona multi product of graphs was found by Furmanczyk, H. The concept of the corona product has some applications in chemistry for representing chemical compounds was found by Balakrishnan, R et al., (2010). Other applications of this concept include navigation of robots in networks was explained by Khuller, S (1996).

Ayaida, M et al., (2014) applied joint routing and location-based service in VANETs. Graph-based metrics for insider attack detection in VANET multihop data dissemination protocols was given by Dietzel, S et al., (2012).

A Novel Authentication Framework with Conditional Privacy-Preservation and Non-Repudiation for VANETs was discussed by Li, J et al., (2015). Yang, T et al., (2016) proved a graph coloring resource sharing scheme for full-duplex cellular-VANET heterogeneous networks.