Chapter III

CHAPTER III

SEPARATION AXIOMS IN TOPOLOGICAL ORDERED SPACES

Definition: 3.1

A topological ordered space $(X, \tau, \leq)$ is said to be **upper strongly T_1-ordered** if and only if for each pair of elements $a \not\leq b$ in X, there exists a decreasing τ-open neighbourhood W of b such that $a \notin W$.

Definition: 3.2

A topological ordered space $(X, \tau, \leq)$ is said to be **lower strongly T_1-ordered** if and only if for each pair of elements $a \not\leq b$ in X, there exists an increasing τ-open neighbourhood W of a such that $b \notin W$.

Definition: 3.3

$(X, \tau, \leq)$ is said to be **strongly T_1-ordered** if and only if is both lower and upper strongly T_1-ordered.

Theorem: 3.4

Let $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be a bijective I-open map as well as a poset isomorphism (i.e., $x \leq y$ if and only if $f(x) \leq^* f(y)$ for all $x,y \in X$). If $(X, \tau, \leq)$ is a lower strongly T_1-ordered space, then so is $(X^*, \tau^*, \leq^*)$.

Proof

Let, $a, b \in X^*$ such that $a \not\leq^* b$. Then $f^{-1}(a) \not\leq^* f^{-1}(b)$. Since $(X, \tau, \leq)$ is a lower strongly T_1-ordered space, then there exists an increasing open neighbourhood U of $f^{-1}(a)$ such that $f^{-1}(b) \notin U$. Thus f(U) is an increasing open neighbourhood of $f(f^{-1}(a)) = a$ such that $b = f(f^{-1}(b)) \notin f(U)$. Therefore $(X^*, \tau^*, \leq^*)$ is a lower strongly T_1-ordered space.

Theorem: 3.5

Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a bijective D-open map as well as a poset isomorphism. If $(X, \tau, \leq)$ is an upper strongly T_1-ordered space, then so is $(X^*, \tau^*, \leq^*)$.

Proof

Similar to the Theorem 3.4.

Theorem: 3.6

Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a bijective B-open map. If f is a poset isomorphism and $(X, \tau, \leq)$ is a strongly T_1-ordered space, then so is $(X^*, \tau^*, \leq^*)$.

Proof

Follows from the above two Theorems 3.4 and 3.5 the fact that every B-open map is both I-open and D-open.

Definition: 3.7

A topological ordered space $(X, \tau, \leq)$ is called **strongly T_2-ordered (or strongly Hausdorff – ordered or strongly H-closed)** if and only if for each pair of elements $a \not\leq b$ in X, there exists disjoint τ-open neighbourhoods U and V of a and b respectively such that U is an increasing set and V is a decreasing set.

Theorem: 3.8

Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a bijective B-open map. If (X, τ) is a Hausdorff space, then $(X^*, \tau^*, \leq^*)$ is strongly Hausdorff-ordered.

Proof

Let $a, b \in X^*$ such that $a \not\leq^* b$. Then $f^{-1}(a) \neq f^{-1}(b)$. Since X is Haudorff, then there exist disjoint τ-open neighbourhoods U and V of $f^{-1}(a)$ and $f^{-1}(b)$ respectively. Since f is B-open, then f(U) and f(V) are two disjoint τ^*-open neighbourhoods of a and b respectively such that f(U) is an increasing set and f(V) is a decreasing set. Therefore $(X^*, \tau^*, \leq^*)$ is a strongly Hausdorff-ordered space.

Definition: 3.9

A topological ordered space $(X, \tau, \leq)$ is said to be a **lower strongly regular ordered space** if and only if for each decreasing τ-closed set F and each element $a \notin F$, there exist τ-open neighbourhoods U of a and V of F such that U is increasing and V is decreasing set in X and $U \cap V = \phi$.

Definition: 3.10

A topological ordered space $(X, \tau, \leq)$ is said to be an **upper strongly regular ordered space** if and only if for each increasing τ-closed set F and each element $a \notin F$, there exist τ-open neighbourhoods U of a and V of F such that U is decreasing and V is increasing set in X and $U \cap V = \phi$.

Theorem: 3.11

Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a bijective I-continuous, B-open map. If (X, τ) is a regular space, then $(X^*, \tau^*, \leq^*)$ is a lower strongly regular-ordered space.

Proof

Let F be a decreasing closed subset of X^* and $a \in X^*$ such that $a \notin F$. Since f is I-continuous, immediately it follows that $f^{-1}(F)$ is a decreasing closed set in X. [By the Theorem 2.12(4)]. Moreover $f^{-1}(a) \notin f^{-1}(F)$ and $f^{-1}(a) \in X$. Since f is a regular space, there exist two disjoint open neighbourhoods U of $f^{-1}(a)$ and V of $f^{-1}(F)$ in X. Since f is B-open, then clearly f(U) is an increasing open set and f(V) is a decreasing open set in X^*. Also $a \in f(U)$ and $F \subseteq f(U)$. Therefore $(X^*, \tau^*, \leq^*)$ is a lower strongly regular-ordered space.

Theorem: 3.12

Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a bijective D-continuous, B-open map. If (X, τ) is a regular space, then $(X^*, \tau^*, \leq^*)$ is an upper strongly regular-ordered space.

Proof

Similar to the Theorem 3.11.

Theorem: 3.13

Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a B-homeomorphism. If (X, τ) is a regular space, then $(X^*, \tau^*, \leq^*)$ is a strongly regular-ordered space.

Proof

Follows from the Theorem 3.11 and 3.12.

Definition: 3.14

A topological ordered space $(X, \tau, \leq)$ is said to be **strongly normally ordered space** if and only if for each pair of disjoint τ-closed sets F_1 and F_2, where F_1 is increasing and F_2 is decreasing, there exists two disjoint τ-open neighbourhoods U_1 of F_1 and U_2 of F_2 such that U_1 is increasing and U_2 is decreasing in X.

Definition: 3.15

A topological ordered space $(X, \tau, \leq)$ is said to be **strongly T_3 ordered** if and only if it is both strongly T_1 ordered and strongly regular ordered.

Definition: 3.16

A topological ordered space $(X, \tau, \leq)$ is said to be **strongly T_4 ordered** if and only if it is both strongly T_1 ordered and strongly normally ordered.

Using the above definitions the following theorems can be proved.

Theorem: 3.17

Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a B-homeomorphism. Then

(1) If (X, τ) is normal, then $(X^*, \tau^*, \leq^*)$ is a strongly normally-ordered space.

(2) If f is a poset isomorphism and (X, τ) is T_3, then $(X^*, \tau^*, \leq^*)$ is a strongly T_3 ordered space.

(3) If f is a poset isomorphism and (X, τ) is T_4, then $(X^*, \tau^*, \leq^*)$ is a strongly T_4 ordered space.

Theorem: 3.18

Let $f : (X, \tau, \leq_1) \rightarrow (X, \sigma, \leq_2)$ and $g : (Y, \sigma, \leq_2) \rightarrow (Z, \eta, \leq_3)$ be any two mappings. Then

(1) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-homeomorphism if f is a homeomorphism and g is x-homeomorphism for x = I, D, B.

(2) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-homeomorphism if both f and g are x-homeomorphism for x = I, D, B.

(3) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-homeomorphism if f is y-homeomorphism and g is x-homeomorphism for all x, y$\in$\{ I, D, B\}.