A NEW APPROACH TO THE STUDY OF FUZZY TOPOLOGICAL SPACES

BY

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CERTIFICATE

CERTIFICATE

This is to certify that the thesis entitled "**A NEW APPROACH TO THE STUDY OF FUZZY TOPOLOGICAL SPACES**" submitted to the Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for the award of the Degree of **Doctor of Philosophy in Mathematics**, is a record of original research work done by **Mrs. P. Jeyalakshmi** as a part time Research Scholar, during the period of her study October 1994 - March 1999 in the Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University , Coimbatore, under my supervision and guidance.

I further certify that this research work has not previously formed the basis for the award of any other Degree/Diploma/Associateship/Fellowship or other similar title to any candidate of this or any other University.

K. N. Meenakshi
Signature of the Guide
19.3.99

DECLARATION

DECLARATION

I do hereby declare that the thesis entitled "**A NEW APPROACH TO THE STUDY OF FUZZY TOPOLOGICAL SPACES**" submitted to the Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for the award of the Degree of **Doctor of Philosophy in Mathematics**, is a record of original and independent research work done by me during 1994-1999 under the supervision and guidance of **Dr.K.N.MEENAKSHI**, Professor and Head of the Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore and it has not previously formed the basis for the award of any Degree / Diploma / Associateship / Fellowship or other similar title to any candidate of any other University.

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INTRODUCTION

INTRODUCTION

The concept of fuzzy topology was first introduced by Chang in 1968[10]. Since then many topological concepts like compactness, separation axioms, connectedness etc., have been extended to fuzzy topological spaces. In this thesis, an entirely new approach is followed in the study of fuzzy topological spaces. This approach is new in the sense that it has no counterpart in classical topology.

A fuzzy set which assumes only a finite set of values is called a finite fuzzy set. A fuzzy topology in which all open sets are finite is called a finite fuzzy topology. A finite fuzzy topology δ is said to be a strong finite fuzzy topology if all the members of δ have a common codomain, denoted by $\mathbf{J}(\delta)$. In the literature available so far, many important examples are finite fuzzy topologies.

The first problem discussed in this thesis is to associate a sequence of finite fuzzy topologies with a given fuzzy topology and to study how best this sequence of finite fuzzy topologies reflects the properties of the original fuzzy topology.

The second problem deals with extending the concept of fuzzy topology to intuitionistic fuzzy sets[102].

An **intuitionistic fuzzy set** on a set X is a pair (f, g) of fuzzy sets defined on X such that $0 \leq f(x) + g(x) \leq 1$ for all $x \in X$.

A new concept called $\mathscr{F}$-structure is introduced as follows :

An $\mathscr{F}$ - **structure $\mathcal{T}$ on a set X** is a set $\mathcal{T}$ of intuitionistic fuzzy sets satisfying the following conditions :

$\mathscr{F}_1$. $(1, 0) \in \mathcal{T}$, $(0, 1) \in \mathcal{T}$.

$\mathscr{F}_2$. $(f_\lambda, g_\lambda) \in \mathcal{T}$, $\lambda \in \Lambda \Rightarrow (\vee f_\lambda, \wedge g_\lambda) \in \mathcal{T}$

$\mathscr{F}_3$. $(f_j, g_j) \in \mathcal{T}$, for $j = 1, 2, \ldots, k \Rightarrow (\wedge f_j, \vee g_j) \in \mathcal{T}$

Every $\mathscr{F}$-structure $\mathcal{T}$ induces two fuzzy topologies denoted by $(\mathcal{T}(\delta))^1$ and $(\mathcal{T}(\delta))^2$. Every fuzzy topology $\delta = \{ f_\lambda \mid \lambda \in \Lambda \}$ induces an $\mathscr{F}$-structure $\mathcal{T}_\delta = \{ (f_\lambda, 1 - f_\lambda) \mid \lambda \in \Lambda \}$.

The concept of continuity , separation axioms and compactness are introduced and studied in the case of $\mathscr{F}$-structured spaces.

In the third problem, two sequences of finite valued gradations ${}^n\mathcal{G}$ and ${}_n\mathcal{G}$ are associated with a given gradation of openness and their properties are studied.

The first problem is studied in Chapter I. For every positive integer n , a finite fuzzy topology ${}^n\delta$ is associated with a given fuzzy topology δ.

Given a fuzzy set f on a set X, the **nth upper approximation** ^{n}f is defined as follows :

$$^{n}f(x) = 0 \text{ if } f(x) = 0 .$$

$$^{n}f(x) = \frac{i+1}{n} \text{ if } \frac{i}{n} < f(x) \leq \frac{i+1}{n} \text{ for } i = 0,1,2,3, \ldots, n-1.$$

If δ is a fuzzy topology on X, then $^{n}\delta = \{^{n}f \mid f \in \delta\}$ is a fuzzy topology on X.

Some important properties of the sequence $\{^{n}\delta\}$ can be stated as follows :

(i). sup $\{ i(^{n}\delta) \mid n = 1,2. \ldots .\} = i(\delta)$ where $i(\delta)$ is the topology generated by $\{ f^{-1}(\varepsilon,1] \mid \varepsilon > 0, f \in \delta \}$.

(ii). If $\theta : (X, \delta) \rightarrow (Y, \sigma)$ is fuzzy continuous (open), then $\theta : (X, {}^{n}\delta) \rightarrow (Y, {}^{n}\sigma)$ is fuzzy continuous (open) for all n.

(iii). Let (Y, σ) be a fuzzy topological space and $\theta : X \rightarrow Y$ be a map. If δ is the smallest fuzzy topology on X which makes $\theta : (X,\delta) \rightarrow (Y, \sigma)$ fuzzy continuous, then $^{n}\delta$ is the smallest fuzzy topology on X which makes $\theta : (X, {}^{n}\delta) \rightarrow (Y, {}^{n}\sigma)$ fuzzy continuous.

(iv). Let (X, δ) and (Y, σ) be two fuzzy topological spaces. Let $\delta \times \sigma$ be the product fuzzy topology on $X \times Y$. Then $^{n}(\delta \times \sigma) = {}^{n}\delta \times {}^{n}\sigma$.

(v). Let δ, σ be two fuzzy topologies on a set X. Then $^{n}(\delta \vee \sigma) = {}^{n}\delta \vee {}^{n}\sigma$.

With every fuzzy topology δ, a **fuzzy topology** $\delta^{\Diamond}$ is associated such that $\delta^{\Diamond}$ is the largest fuzzy topology containing δ and $^{n}\delta = {}^{n}(\delta^{\Diamond})$ for every n.

$$\delta^{\Diamond} = \{ f \mid {}^{n}f \in {}^{n}\delta \text{ for every n} \}$$

Some important results on $\delta^{\lozenge}$ are as follows :

(i). $i(\delta) = i(\delta^{\lozenge})$.

(ii). If δ is topologically generated, then $\delta = \delta^{\lozenge}$.

(iii). $\delta \subseteq \sigma \Rightarrow \delta^{\lozenge} \subseteq \sigma^{\lozenge}$.

(iv). $(\delta \vee \sigma)^{\lozenge} = \delta^{\lozenge} \vee \sigma^{\lozenge}$.

(v). $(\delta \times \sigma)^{\lozenge} = \delta^{\lozenge} \times \sigma^{\lozenge}$.

(vi). If $\theta : (X, {}^{n}\delta) \rightarrow (Y, {}^{n}\sigma)$ is fuzzy continuous for every n, then $\theta : (X, \delta^{\lozenge}) \rightarrow (Y, \sigma^{\lozenge})$ is fuzzy continuous.

(vii). If $\theta : (X, \delta) \rightarrow (Y, \sigma)$ is fuzzy continuous, then $\theta : (X, \delta^{\lozenge}) \rightarrow (Y, \sigma^{\lozenge})$ is fuzzy continuous .

In the final section of the first chapter, with every fuzzy topology δ on a set X and with every positive integer n, two new fuzzy topologies $\boxed{\delta(n)}$ and $\enclose{circle}{\delta(n)}$ are associated having the following interesting properties :

(i). ${}^{n}(\boxed{\delta(n)}) = \delta \wedge {}^{n}\delta$

(ii). ${}^{n}\delta \subseteq \delta \Rightarrow \delta \subseteq \boxed{\delta(n)}$

(iii). ${}^{n}\delta \subseteq \delta \Rightarrow {}^{n}\delta = {}^{n}(\boxed{\delta(n)})$

(iv). If $\theta : (X,\delta) \rightarrow (Y,\sigma)$ is fuzzy continuous, then $\theta : (X, \boxed{\delta(n)}) \rightarrow (Y, \boxed{\sigma(n)})$ is fuzzy continuous.

(v) . Let δ be a fuzzy topology defined on X. Then $\enclose{circle}{\delta(n)}$ is the largest fuzzy topology on X containing δ such that ${}^{n}\delta = {}^{n}(\enclose{circle}{\delta(n)})$.

(vi). If $\theta : (X,\delta) \rightarrow (Y,\sigma)$ is fuzzy continuous, then $\theta : (X, \enclose{circle}{\delta(n)}) \rightarrow (Y, \enclose{circle}{\sigma(n)})$ is fuzzy continuous.

Chapter II deals with the separation axioms and compactness properties of the sequence of fuzzy topologies ${}^{n}\delta$.

The following two results are established on compactness in the first section.

(i). $(X, {}^{n}\delta)$ is Lowen fuzzy compact for every n $\Rightarrow (X, \delta)$ is Lowen fuzzy compact.

(ii). (X,δ) is Lowen fuzzy compact $\Leftrightarrow (X, \delta^{\Diamond})$ is Lowen fuzzy compact.

The concepts of separation axioms T_0, T_1, T_2 are extended to fuzzy situation in a number of ways by Ali, D.M., Fora, A.A., Gantner, T.E., Katsaras, A.K., Lal, S.N., Sarkar, M., Srivastava, A. K., Srivastava, R., Steinlage, R.C., and Warren, R.H.

These concepts are investigated with regard to the nth degree approximation ${}^{n}\delta$. The results obtained in this connection are of the following types :

(i). (X,δ) has the property P $\Rightarrow (X, {}^{n}\delta)$ has the property P for every n.

(ii). $(X, {}^{n}\delta)$ has the property P for every n $\Rightarrow (X,\delta)$ has the property P.

(iii). $(X, {}^{n}\delta)$ has the property P for some n $\Rightarrow (X,\delta)$ has the property P.

(iv). (X,δ) has the property P $\Rightarrow (X, \delta^{\Diamond})$ has property P.

Here P stands for suitable separation axioms.

The final section of Chapter II is devoted to the study of fuzzy covering uniformity introduced by Chandrika and Meenakshi [9]. With every fuzzy covering uniformity Γ, and with every positive integer n, a fuzzy covering uniformity ${}^n\Gamma$ is associated having the following properties :

(i). ${}^n\Gamma$ has a basis consisting of finite valued functions with codomain

$$I_n = \{ 0, \frac{1}{n}, . . ,1 \}.$$

(ii). ${}^n\Gamma$ and Γ induce the same fuzzy topology on the base set X.

(iii). ${}^n\Gamma \subseteq \Gamma$.

Some interesting subcategories of the category of fuzzy topological spaces are studied in chapter III. For this purpose the following notations are used :

(i). **$\mathscr{C}$FTOP** : Category of fuzzy topological spaces with fuzzy continuous functions as morphisms.

(ii). **$\mathscr{C}_n$ FTOP** : Subcategory of $\mathscr{C}$FTOP of fuzzy topological spaces of the form (X,δ) where δ is a strong finite fuzzy topology with **J(δ)** $= \left\{0, \frac{1}{n}, ..., 1\right\}$.

(iii). **$\mathscr{C}$ FFTOP** : Category of finite fuzzy topological spaces with fuzzy continuous functions as morphisms.

(iv). **$\mathscr{C}$SFTOP** : Category of strong finite fuzzy topological spaces with fuzzy continuous functions as morphisms.

(v). **$\mathscr{C}_{w_n}$FTOP** : Category of fuzzy topological spaces with weak -n-continuous functions as morphisms.

(vi). **$\mathscr{C}^{\Diamond}$FTOP** : Sub category of $\mathscr{C}$FTOP with objects (X,δ) where δ is a saturated fuzzy topology on X.

The fuzzy topology δ is said to be **saturated** if $\delta = \delta^{\Diamond}$.

(vii). **$\mathscr{C}_c$ FTOP** : Category of fuzzy topological spaces with continuous functions as morphisms.

(viii). **$\mathscr{C}^{\Diamond}_{c}$ FTOP** : Category of saturated fuzzy topological spaces with continuous functions as morphisms.

(ix). **$\mathscr{C}_L$FTOP**: Sub category of $\mathscr{C}$FTOP with topologically generated fuzzy topological spaces as objects.

The following functorial properties are established :

(i). The correspondence $(X,\delta) \rightarrow (X, {}^{n}\delta)$ defines a functor ζ_n from $\mathscr{C}$FTOP to $\mathscr{C}_n$FTOP. The functor ζ_n preserves product.

(ii). $\mathscr{C}_n$FTOP is a reflective subcategory of $\mathscr{C}_{w_n}$FTOP.

(iii). The correspondence $(X,\delta) \rightarrow (X,\delta^{\Diamond})$ defines a coreflective functor $\zeta^{\Diamond}$: $\mathscr{C}$ FTOP $\rightarrow$ $\mathscr{C}^{\Diamond}$ FTOP and $\zeta^{\Diamond}$ preserves product.

(iv). The map $(X,\delta) \rightarrow (X,\delta^{\Diamond})$ defines a reflective functor $\zeta^{\Diamond}_{c}$: $\mathscr{C}_c$FTOP $\rightarrow$ $\mathscr{C}^{\Diamond}_{c}$FTOP.

The results on fuzzy Sierpinski space obtained by Srivastava, A.K.[81] for the category $\mathscr{C}$FTOP have been extended to $\mathscr{C}_n$FTOP and $\mathscr{C}^{\Diamond}$ FTOP.

Chapter IV is devoted to the study of $\mathscr{F}$-structured spaces. With every $\mathscr{F}$ -structure $\mathcal{T}$ on a set X, a sequence of finite approximations ${}^{n}\mathcal{T}$ is associated.

Some important results on $\mathscr{F}$-structured spaces are listed below :

(i). $\theta : (X, \mathscr{T}_\delta) \to (Y, \mathscr{T}_\sigma)$ is $\mathscr{F}$-continuous iff
$\theta : (X,\delta) \to (Y,\sigma)$ is fuzzy continuous .

(ii). $(X, \mathscr{T})$ has the property P $\Rightarrow$ $(X, {}^n\mathscr{T})$ has the property P, where P stands for a suitable separation axiom defined for $\mathscr{F}$-structured spaces.

(iii). (X,δ) is Chang fuzzy compact $\Leftrightarrow$ $(X, \mathscr{T}_\delta)$ is Chang $\mathscr{F}$-compact.

(iv). $(X, (\mathscr{T}(\delta))^1)$ is Chang fuzzy compact $\Rightarrow$ $(X, \mathscr{T})$ is Chang $\mathscr{F}$-compact.

(v). $(X, (\mathscr{T}(\delta))^2)$ is Chang fuzzy compact $\Rightarrow$ $(X, \mathscr{T})$is Chang $\mathscr{F}$-compact.

(vi). $(X, {}^n\mathscr{T})$ is Lowen $\mathscr{F}$- compact for every n $\Rightarrow$$(X, \mathscr{T})$ is Lowen $\mathscr{F}$-compact.

The category $\mathscr{C}\mathscr{F}$STR of $\mathscr{F}$-structured spaces with $\mathscr{F}$-continuous functions as morphisms has two interesting subcatagories , $\mathscr{C}$ T$\mathscr{F}$STR and $\mathscr{C}_n\mathscr{F}$STR .

Objects of $\mathscr{C}$ T$\mathscr{F}$STR are $\mathscr{F}$-structured spaces of the form $(X, \mathscr{T}_\delta)$.

Objects of $\mathscr{C}_n\mathscr{F}$STR are $\mathscr{F}$-structured spaces of the form $(X, {}^n\mathscr{T})$.

The correspondence $(X,\delta) \to (X, \mathscr{T}_\delta)$ defines an embedding ζ_e of $\mathscr{C}$FTOP into a full subcategory of $\mathscr{C}\mathscr{F}$STR.

The correspondence $(X, \mathscr{T}) \to (X, {}^n\mathscr{T})$ defines a functor

$\zeta^n_{\mathscr{T}}$: $\mathscr{C}\mathscr{F}\text{STR} \to \mathscr{C}_n\mathscr{F}\text{STR}$.

In 1992, Hazra, Samanta and Chattopadhyay[28] introduced the concept of gradation of openness and gave a new definition of fuzzy topology.

In chapter V , two interesting methods of associating finite valued gradations ${}^{n}\mathcal{G}$ and ${}_{n}\mathcal{G}$ with a given gradation $\mathcal{G}$ are studied. The third method associates with each gradation $\mathcal{G}$, a gradation $({}^{n}\mathcal{G})^{*}$ which depends only on finite valued functions. The following properties are established :

(i). $\mathcal{G}$ and ${}^{n}\mathcal{G}$ induce the same fuzzy topology on X.

(ii). The identity map $(X, {}^{n}\mathcal{G}) \rightarrow (X, \mathcal{G})$ is a gradation preserving map.

(iii). A map $\theta : (X, \mathcal{G}) \rightarrow (Y, \mathcal{G}')$ is a gradation preserving map iff $\theta : (X, {}^{n}\mathcal{G}) \rightarrow (Y, {}^{n}(\mathcal{G}'))$ is a gradation preserving map for all n.

(iv). Let $\theta : X \rightarrow Y$ be a map. Let $\mathcal{G}$ be a gradation of openness on X and $\mathcal{G}'$ be the largest gradation of openness on Y such that $\theta : (X, \mathcal{G}) \rightarrow (Y, \mathcal{G}')$ is a gradation preserving map. Then ${}^{n}(\mathcal{G}')$ is the largest gradation of openness on Y such that $\theta : (X, {}^{n}\mathcal{G}) \rightarrow (Y, {}^{n}(\mathcal{G}'))$ is a gradation preserving map.

In the fifth section of this chapter, given a gradation $\mathcal{G}$ which induces the fuzzy topology δ , a new gradation $({}^{n}\mathcal{G})^{*}$ is constructed such that $({}^{n}\mathcal{G})^{*}$ induces the fuzzy topology ${}^{n}\delta$.

A brief survey of some important developments in the study of fuzzy separation axioms, fuzzy uniform structure and fuzzy compactness.

The concept of fuzzy sets was introduced by Zadeh [101] in 1965. In 1968, Chang C.L. developed the theory of fuzzy topological spaces based on Zadeh's concept. In recent years fuzzy topology has developed considerably. A number of articles have been published on the following topics :

1. Fuzzy neighbourhood spaces .
2. Fuzzy separation axioms.
3. Fuzzy uniformity.
4. Fuzzy convergence.
5. Fuzzy compactness.
6. Fuzzy connectedness.
7. Fuzzy proximity.

In this thesis the three important concepts (1) Fuzzy compactness (2) Fuzzy separation axioms and (3) Fuzzy uniformity are studied with reference to finite approximations.

The first article on fuzzy normality was published by Hutton in 1975 [31]. His definition is a direct generalization of the corresponding topological concept. Here, he has extended the Uryshon lemma to fuzzy topological spaces. He has introduced fuzzy perfectly normal spaces and obtained the following chatracterization which extends the corresponding topological result.

A fuzzy topological space is perfectly normal iff it is normal and every closed set is a countable intersection of open sets.

In 1980, Rodabaugh, S.E. [71] introduced the concepts of α-Hausdorff spaces. It is proved that the concept of α-Hausdorff fuzzy topological spaces is compatible with α-compactness and fuzzy continuity. In 1983, Rodabaugh[74] gave a detailed study of fuzzy real line with special reference to fuzzy separation axioms. It is proved here that the fuzzy real line satisfies the higher order separation axioms of Hutton, B. [31] and Sarkar, M. [77].

In 1981, Sarkar, M.extended the concepts T_0, T_1,T_2,T_3,T_4 to fuzzy situation. She has obtained the following results :

1. $FT_4 \Rightarrow FT_3 \Rightarrow FT_2 \Rightarrow FT_1$.
2. A properly compact set in an FT_2 space is closed.
3. Fuzzy Lindelof sets in a Hausdorff fuzzy P-space are closed.
4. Properly compact sets in an FT_1 P-space have finite supports.
5. FT_1 is a hereditary property.

Srivastava, R., Lal, S.N., and Srivastava, A.K. have published a number of interesting articles on "Fuzzy separation axioms." Their definition of separation axioms are different from the definitions of Sarkar, M. [77] . Here, we give some important results from their articles entitled
"Fuzzy Hausdorff Topological spaces." [86] and
'Fuzzy T_1-Topological Spaces"[87] .

1. Let (X,τ) be a topological space. Then (X,τ) is Hausdorff $\Leftrightarrow$ $(X, \omega(\tau))$ is fuzzy Hausdorff.
2. If fuzzy topological space (X,δ) is fuzzy Hausdorff, then $(X,i(\delta))$ is Hausdorff.
3. A fuzzy subspace (A, τ_A) of a fuzzy Hausdorff topological space is fuzzy Hausdorff .

4. If $\{ X_i, \tau_i) \mid i \in I \}$ is a family of fuzzy Hausdorff topological spaces, their product (X, τ) is also fuzzy Hausdorff.

5. Let (X,τ) be a fuzzy topological space. Consider the following statements :

   (i). Δ_X, the diagonal of X is fuzzy closed in $(X \times X, \tau \times \delta)$ where δ is discrete fuzzy topology on X (That is, δ consists of all fuzzy sets in X).

   (ii). $\{x\}$, for every $x \in X$, is fuzzy closed in (X, τ).

   (iii). (X, τ) is a fuzzy T_1-topological space.

Then in general , (i) ⇒ (ii) ⇒ (iii).

For topologically generated fuzzy topological spaces, the three statements are equivalent .

6 (i) (X, τ) is T_1 ⇔ $(X, \omega(\tau))$ is fuzzy T_1.

  (ii) (X, δ) is fuzzy T_1 ⇒ $(X, i(\delta))$ is T_1.

7. (i) Fuzzy T_1-ness is productive.

   (ii) Fuzzy T_1-ness is hereditary.

Separation axioms for fuzzy neighbourhood spaces and fuzzy uniform spaces are studied in detail by Wuyts and Lowen [100]. They have introduced a number of separation axioms. We collect below the different definitions and results.

Definitions :

1. Given a fuzzy topological space (X, δ), separation properties are defined as follows :

$(WT_0))$: For every x , y in X, $x \neq y$ $\overline{1}_x(y) \wedge \overline{1}_y(x) < 1$

(T_0''') : For every x,y in X, $x \neq y$ for every $\alpha \in I_0$, $\overline{\alpha 1}_x(y) \wedge \overline{\alpha 1}_y(x) < \alpha$.

(T_0'') : For every x, y in X, $x \neq y$, for every $(\alpha, \beta) \in I_0 \times I_0$, $\overline{\alpha 1}_x(y) < \alpha$ or $\overline{\beta 1}_y(x) < \beta$.

(T_0') : For every x,y in $x \neq y$, for every $(\alpha,\beta) \in I_0 \times I_0$, $\overline{\alpha 1}_x(y) \wedge \overline{\beta 1}_y(x) < \alpha \wedge \beta$.

(T_0) : For every x,y in X, $x \neq y$, for every $(\alpha,\beta) \in I_0 \times I_0$, $\overline{\alpha 1}_x(y) \wedge \overline{\beta 1}_y(x) = 0$.

(WT_1) : For every x , y in X , $x \neq y$, $\overline{1_x}(y) < 1$

(T_1') : For every x , y in X , $x \neq y$, for every $\alpha \in I_0$, $\alpha \in I_0$, $\overline{\alpha 1}_x(y) < \alpha$.

(T_1) : For every x in X and α in I , $\overline{\alpha 1_x} = \alpha 1_x$

(WT_2) : For every $\mathcal{F} \in F(X)$, $c(\mathcal{F}) = 1$; $\lim \mathcal{F}(x) = 1$ in atmost one point x.

(T_2') : For every $\mathcal{F} \in F(X)$, $C(\mathcal{F}) > 0$; $\lim \mathcal{F}(x) = c(\mathcal{F})$ in atmost one point x.

(T_2'') : For every $\mathcal{F} \in F(X)$; $\lim \mathcal{F}(x)$ contains a strictly positive maximum in atmost one point x.

(T_2''') : For every $\mathcal{F} \in F(X)$; $\lim \mathcal{F}(x)$ attains a strictly positive maximum in atmost one point x.

(T_2) : For every $\mathcal{F} \in F(x)$; $\lim \mathcal{F}(x) \neq 0$ in atmost one point x.

Results :

1. The implications contained in the following diagram are valid for general fuzzy topological spaces :

$$\begin{tikzcd}
(T_2) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (T_2''') \arrow[r, Rightarrow] & (T_2'') \arrow[r, Rightarrow] & (T_2') \arrow[rrr, Rightarrow] \arrow[d, Rightarrow] & & & (WT_2) \arrow[d, Rightarrow] \\
(T_1) \arrow[rrr, Rightarrow] \arrow[d, Rightarrow] & & & (T_1') \arrow[rrr, Rightarrow] \arrow[d, Rightarrow] & & & (WT_1) \arrow[d, Rightarrow] \\
(T_0) \arrow[rrr, Rightarrow] & & & (T_0') \arrow[r, Rightarrow] & (T_0'') \arrow[r, Rightarrow] & (T_0''') \arrow[r, Rightarrow] & (WT_0)
\end{tikzcd}$$

2. a. (T_0), (T_0'), (T_0'') (T_0''') and (WT_0) are good extensions of the topological (T_0) property.
   b. (T_1), (T_1') and $(W(T_1))$ are good extensions of the topological (T_1) property.
   c. (T_2), (T_2'), (T_2'') , (T_2''') and (WT_2) are good extensions of the topological (T_2) property.

In the above definitions $C(\mathcal{F})$ stands for the characteristic value of a prefilter $\mathcal{F}$ given by

$$C(\mathcal{F}) = \inf_{\lambda \in \mathcal{F}} \sup_{x \in X} \lambda(x)$$

These separation axioms are characterized in the cases of fuzzy neighbourhood spaces and fuzzy uniform spaces.

A new approach to the study of separation axioms is followed by Ghanim, M.H, Kerre, E.E. and Mashhour, A.S, [26]. For example, their definition of FT_2 is as follows :

A fuzzy topological space is said to be FT_2(F-Hausdorff) iff for every pair of fuzzy singletons p_1 and p_2 with different supports, there exist open fuzzy sets O_1, O_2 such that $p_1 \subseteq O_1 \subseteq \text{co } p_2$, $p_2 \subseteq O_2 \subseteq \text{co } p_1$ and $O_1 \subseteq \text{co } O_2$ Or equivalently, iff for every pair of fuzzy singletons p_1 and p_2 with different supports, there exists an open fuzzy set O such that $p_1 \subseteq O \subseteq \text{Cl } O \subseteq \text{co } p_2$. Here, co f stands for the complement (1- f).

They introduced similar definitions for FT_0, FT_1, FT_2, FT, $FT_{2\frac{1}{2}}$ etc. They have extablished the following results :

1. A fuzzy topological space is FT_1 iff every crisp singleton is closed.
2. For every closed fuzzy set F in a FR fuzzy topological space and any fuzzy singleton $p \subseteq \text{co } F$, there exist open fuzzy sets U and V such that $p \subseteq U$, $F \subseteq V$ and $\text{Cl } U \subseteq \text{co}(\text{Cl } V)$.
3. Let X be an FR-Fuzzy topological space which is also FT_0. Then X is $FT_{2\frac{1}{2}}$.
4. For every $j \in \{ 0, 1, 2, 2\frac{1}{2}, 3 \}$ the correspondig FT_j property is hereditary.
5. Fuzzy normality is hereditary with respect to closed sub spaces.
6. For every $j \in \{ 0, 1, 2, 2\frac{1}{2} \}$ the corresponding FT_j property is productive.

7. For every $j \in \{ 0, 1, 2, 2½, 3, 4 \}$ the corresponding FT_j property is additive.

α-Hausdorff property of Rodabaugh [71] is extended to the α-T_i, $i = 0, 3, 4$ and α-T_i', $i = 0, 1, 2, 3, 4$ concepts. The interrelations between these concepts are studied. These concepts are used to study α-almost compactness, α-nearly compactness and α-continuous mappings.

In 1989, Fora, A.A. [18] has introduced the concepts of fuzzy completely regular and fuzzy functionally Hausdorff spaces. Here the problems connecting the hereditary and productive nature of these concepts are investigated.

The concept of quasi-coincidence is used in the definition of separation axioms by Ganguly, S. and Saha, S. [21]. Natural extensions of topological theorems are obtained for fuzzy topological spaces.

The concept of strong separation axioms are introduced by Macho Stadler, M. and De Prada Vicente, M.A. [58].

Let (X,δ) be a fuzzy topological space. For each $t \in [0,1)$, the level topology $i_t(\delta)$ is given by $i_t(\delta) = \{\mu^{-1}(t,1] \mid \mu \in \delta \}$.

1. (X,δ) is said to be a t-T_1 fuzzy topological space iff $(X, i_t(\delta))$ is a T_1-space.
2. (X, δ) is said to be a strong T_1 fuzzy topological space if it is t-T_1 for every $t \in [0,1)$.

3. (X, δ) is said to be an FT_1 fuzzy topological space iff for each $t \in (0,1]$ and $x \in X$, the fuzzy point x_t is a fuzzy closed set.

In a similar manner the concepts regularity, normality, ultra separation axioms, strong separation axioms are defined and a detailed theory connecting these concepts is developed.

There are four important essentially different approaches to the study of fuzzy uniform spaces.

I. The first definition of fuzzy uniform structure is due to Hutton [31].

Let X be any set and $\mathfrak{S}$ be the set of maps $D: L^X \to L^X$ which satisfy the following conditions :

(A1) $f \leq D(f)$ for every f in L^X.

(A2) $D(\vee_{\lambda \in \Lambda} f_\lambda) = V_{\lambda \in \Lambda} D(f_\lambda)$ for f_λ in I^X.

A fuzzy quasi-uniformity on a set X is a subset $\mathfrak{D}$ of $\mathfrak{S}$ such that

(Q1) $\mathfrak{D} \neq \phi$.

(Q2) $D \in \mathfrak{D}$ and $D \subset E \in \mathfrak{S}$ implies $E \in \mathfrak{D}$.

(Q3) $D \in \mathfrak{D}$ and $E \in \mathfrak{D}$ implies $D \wedge E \in \mathfrak{D}$.

(Q4) $D \in \mathfrak{D}$ implies there exists $E \in \mathfrak{D}$ such that $E \circ E \subset \mathbf{D}$, where $E \circ E$ denotes the composition of mappings E and E.

(Q5) A fuzzy quasi uniformity $\mathfrak{D}$ is a fuzzy uniformity if it also satisfies $D \in \mathfrak{D}$ implies $D^{-1} \in \mathfrak{D}$ where $D^{-1}(g) = \wedge \{ f \mid D(f^c) \leq g^c \}$.

Using this definition of fuzzy uniformity, theorems corresponding to many of the theorems in the classical theory of uniform spaces are established. Further it is shown that every fuzzy topological space is fuzzy quasi uniformizable and fuzzy uniformizability is characterized in terms of complete regularity. Moreover, a natural fuzzy uniformity on the fuzzy unit interval is constructed.

II. In 1981 [50], Lowen introduced the concept of fuzzy uniformity on a set X by considering fuzzy sets on X x X. This definition is an extension of the entourage approach [Kelley,40]. A uniformity $i_u(\mathcal{U})$ is associated with a fuzzy uniformity $\mathcal{U}$. A fuzzy uniformity $w_u(\mu)$ is associated with a uniformity μ on X.

The results connecting these concepts are as follows :

1. $i_u(w_u(\mu)) = \mu$, μ uniformity on X.
2. $w_u(i_u(\mathcal{U}))$ is the coarsest fuzzy uniformity generated by a uniformity and which is finer than $\mathcal{U}$.
3. If (X, μ) and (Y,μ') are uniform spaces, then F: X→ Y is uniformly continuous, iff it is fuzzy uniformly continuous when considered as a map between the fuzzy uniform spaces (X, $w_u(\mu)$) and (Y, $w_u(\mu')$).
4. If (X, $\mathcal{U}$) and (Y, $\mathcal{U}'$) are fuzzy uniform spaces and F : X→Y is fuzzy uniformly continuous, then it is also uniformly continuous when considered as function between the uniform spaces (X, $i_u(\mathcal{U})$) and (Y, $i_u(\mathcal{U}')$).

III. The pesudo metric approach is extended to fuzzy situation by Hohle, U. [29] in his paper on probabilistic metrization of fuzzy uniformities published in 1982. Here the relation between this uniformity and Hutton uniformity is studied. A necessary and sufficient condition for Hutton uniformity to be probabilistic metrizable is obtained.

IV. The Fourth important approach is a covering approach due to Chandrika, G.K., and Meenakshi, K.N [9]. The definition of uniformity in terms of a collection of uniform coverings [Isbell,34] is extended to fuzzy situation. With every fuzzy uniformity a fuzzy topology is associated. Many of the results connecting uniformity and topology are extended to fuzzy uniformity and fuzzy topological spaces.

A unified theory of fuzzy topologies, fuzzy proximities and fuzzy uniformities is developed by Katsaras, A.K., and Petalas, C.G. [38].

The relation between quasi-proximities and fuzzy quasi-uniformities were studied by Katsaras in 1988 [37] .

An interesting article on " A Theory of Fuzzy Uniformities with Applications to the Fuzzy Real Lines" is published by Rodabaugh [75] in 1988.

The inter-relationship between fuzzy unifomities and fuzzy proximities are studied by Artico, G. and Moresco, R.[4] in their article entitled " Fuzzy Uniformities Induced by Fuzzy Proximities".

Chang, C. L. in his first paper on fuzzy topological spaces gave a definition of compactness which is a direct extension of the compactness concept to fuzzy topological space. Chang's definition of compactness was criticized by Lowen who constructed an example in which (X, τ) is compact, but $(X, w(\tau))$ is not compact (Chang). Lowen[47] gave a new definition of fuzzy compactness. With this definition he proved the following results :

1. The fuzzy topological space $(E, w(\tau))$ is fuzzy compact iff the space (E, τ) is compact.
2. If $f : (E, \delta) \rightarrow (F, \gamma)$ is fuzzy continuous and ν is a fuzzy compact fuzzy set in (E,δ),then $f(\nu)$ is fuzzy compact in (F, γ).
3. If (E, δ) is fuzzy compact and f a fuzzy continuous map from (E,δ) onto (F,γ), then (F,γ) is fuzzy compact.
4. If (E,δ) is a topologically generated fuzzy compact space, then every closed fuzzy set is fuzzy compact.
5. (E,δ) is fuzzy compact iff for any subbasis σ for δ, for any $\beta \subseteq \sigma$ and for any $\alpha > \varepsilon > 0$ such that $\sup \{ \mu \mid \mu \in \beta \} \geq \alpha$ there exists a finite subset $\beta_0 \subseteq \beta$ such that $\sup \{\mu \mid \mu \in \beta_0 \} \geq \alpha - \varepsilon$.

Gantner, T.E., Steinlage, R.C. and Warren, R.H. [24] have introduced degree of compactness in L-fuzzy topological spaces where L is a completely distributive lattice with a smallest element and a largest element and equipped with an order reversing involution $\alpha \rightarrow \alpha'$.

(X,δ) is a L-fuzzy space and $\alpha \in L$. A collection $u \subseteq \delta$ is called an α-shading (α^* - shading) of X if for each $x \in X$, there exists a $f \in u$ such that $f(x) > \alpha$ ($f(x) \geq \alpha$) .

(X,δ) is α-compact if each α-shading of X has a finite α-subshading. It is proved here that product of α-compact spaces is α-compact, the fuzzy unit interval is α-compact and the fuzzy real line is not α-compact. A one point α-compactification is also constructed.

It is easily proved that (X,δ) is α-compact (resp α^*-compact) iff $(X, i_\alpha(\delta))$ (resp $(X, i_{\alpha^*}(\delta))$) is compact.

A space (X,δ) is strong fuzzy compact if it is α-compact for each $\alpha \in [0,1)$ and ultra fuzzy compact if $(X,i(\delta))$ is compact. α-compact, strong fuzzy compact and ultra fuzzy compact satisfy the Tychonoff product theorem.

Lowen, R. and Wuyts, P. (1982[55], 1983[56]) have studied completeness, compactness, precompactness in uniform spaces. Here the authors use Lowen's definition of fuzzy uniformity in terms of fuzzy entourages. It is proved here that every compact space has a unique uniform structure. Further in fuzzy uniform spaces compactness is equivalent to pre compactness plus completeness.

CHAPTER I

FINITE FUZZY TOPOLOGICAL APPROXIMATIONS

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FINITE FUZZY TOPOLOGICAL APPROXIMATIONS

The first section deals with some preliminary definitions and results. This includes the definition of continuity, product and the two functors $i(\delta)$ and $\omega(\tau)$ which are introduced by Lowen [47].

In section 2, the n^{th} order upper approximation (n^{th} order lower approximation) of a fuzzy set is defined and its properties are developed.

In section 3, given a fuzzy topology δ, a sequence of strong finite fuzzy topology ${}^{n}\delta$ is associated. Some interesting properties of ${}^{n}\delta$ are obtained .

In section 4, given a fuzzy topology δ, a fuzzy topology $\delta^{\Diamond}$ is constructed such that $\delta^{\Diamond}$ is the largest fuzzy topology containing δ and satisfying ${}^{n}\delta = {}^{n}(\delta^{\Diamond})$ for every n.

In the last section, two new types of finite approximations $\boxed{\delta(n)}$, $\textcircled{$\delta(n)$}$ are introduced and some interesting properties are established .

Section : 1.1

Preliminary Definitions and Results

Definition : 1.1.1

Let X be a non empty set. A function f defined on X with values in the closed unit interval [0,1] is called a **fuzzy set** on X.

If f takes only the values 0 and 1, then f is called a **crisp set**.

Support of any fuzzy set f is defined by supp $f = \{ x \in X \mid f(x) > 0\}$.

Given a fuzzy set f on X, the fuzzy set $(1 - f)$ is called the **complement** of f.

Notation :

(i). The constant fuzzy set $\alpha(x) = \alpha$ is denoted by the symbol α .

(ii). Throughout this thesis X,Y, ... will denote non empty sets.

In general fuzzy sets will be denoted by $f, g, h, \ldots$

Definition : 1.1.2

Given a collection $\{ f_\lambda \mid \lambda \in \Lambda \}$ of fuzzy sets defined on a set X, the **union** $\vee_\lambda f_\lambda$ and **intersection** $\wedge_\lambda f_\lambda$ are defined as follows :

$$(\vee_\lambda f_\lambda)(x) = \sup\{ f_\lambda(x) \mid \lambda \in \Lambda \}.$$

$$(\wedge_\lambda f_\lambda)(x) = \inf\{ f_\lambda(x) \mid \lambda \in \Lambda \}.$$

Definition : 1.1.3

A **fuzzy point** x_t in a set X is a fuzzy set in X defined by

$$x_t(x) = t \text{ where } 0 < t \leq 1,$$

$$x_t(y) = 0 \text{ if } y \neq x.$$

'x' is called the support of x_t and 't' is called the value of the fuzzy point x_t.

Definition : 1.1.4

The fuzzy point x_t is said to belong to f, denoted by $x_t \in f$, if $t < f(x)$.

Note : 1.1.5

Every fuzzy set f can be expressed as the union of all the fuzzy points which belong to f.

Definition : 1.1.6

Let $\theta : X \to Y$ be a function. For a fuzzy set f in Y, define the fuzzy set $\theta^{-1}(f)$ in X by $(\theta^{-1}(f))(x) = f(\theta(x))$, for all $x \in X$.

$\theta^{-1}(f)$ is called the **inverse of** f.

Definition : 1.1.7

Let $\theta : X \to Y$ be a function and f be a fuzzy set in X. The **image of** f, written as $\theta(f)$ is a fuzzy set in Y defined by

$$(\theta(f))(y) = \sup\{f(x) \mid x \in \theta^{-1}(y)\} \text{ if } \theta^{-1}(y) \text{ is not empty.}$$

$$= 0 \quad \text{otherwise}$$

where $\theta^{-1}(y) = \{x \mid \theta(x) = y\}$.

Definition [Chang,10] : 1.1.8

Let X be a set. A collection δ of fuzzy sets on X is called a **fuzzy topology** for X if it satisfies the following three conditions :

FT1. The constant functions 0 and 1 belong to δ.

FT2. $f, g \in \delta \Rightarrow f \wedge g \in \delta$.

FT3. $f_\lambda \in \delta$ for each $\lambda \in \Lambda \Rightarrow \sup\{f_\lambda \mid \lambda \in \Lambda\} \in \delta$.

The pair (X, δ) is called a fuzzy topological space.

Members of δ are called open sets of (X, δ).

Complements of open sets are called closed sets of (X, δ).

Lowen modified the above definition of fuzzy topological space by replacing the first condition by the following requirement:

FT1′. All constant fuzzy sets belong to δ .

Notation :

(i). (X,δ) denotes a set X together with Chang fuzzy topology δ.

(ii). (X,δ) [Lowen] denotes a set X together with Lowen fuzzy topology δ.

Definition : 1.1.9

Let (X, δ) be a fuzzy topological space. The **interior** and **closure** of a fuzzy set f in (X, δ) are defined respectively as follows :

interior of $f = \text{Int} f = \sup \{ g \mid g \le f,\ g \in \delta \}$.

closure of $f = \text{Cl} f = \inf \{ g \mid g \ge f,\ 1 - g \in \delta \}$.

Definition 1.1.10

Let (X , δ) be a fuzzy topological space. A subset $\mathscr{B} \subset \delta$ is a **basis** for δ if, given $f \in \delta$, there exists $\{ f_\lambda \mid \lambda \in \Lambda \} \subset \mathscr{B}$ such that $f = \sup \{ f_\lambda \mid \lambda \in \Lambda \}$.

A subset $\mathscr{B}' \subset \delta$ is a subbasis for δ if the family of finite infima of members of $\mathscr{B}'$ is a basis for δ.

Definition : 1.1.11

Let (X, δ) be a fuzzy topological space. Then **i(δ)** is the topology generated by $\{ f^{-1} (\varepsilon,1] \mid \varepsilon > 0,\ f \in \delta \}$.

Definition : 1.1.12

Let τ be a topology on X . Then $\omega(\tau) = \{ f \mid f^{-1}(\varepsilon,1] \in \tau$ for $\varepsilon > 0\}$ is a fuzzy topology on X.

Definition : 1.1.13

If $\delta = \omega(\tau)$ for some topology τ, then δ is said to be **topologically generated**.

Result : 1.1.14

The operators 'i' and 'ω' are related as follows :

(i). $i(\omega(\tau)) = \tau$ for every topology τ .

(ii). $\omega(i(\delta))$ is the smallest topologically generated fuzzy topology which contains δ.

Definition : 1.1.15.

Let (X,δ) and (Y, σ) be two fuzzy topological spaces. A mapping $\theta : X \rightarrow Y$ is called **fuzzy continuous** if for all $f \in \sigma$, $\theta^{-1}(f) \in \delta$.

Definition : 1.1.16

Let f and g be fuzzy sets on X and Y respectively. The **cartesian product** $f \times g$ of f and g is a fuzzy set on X x Y defined by $(f \times g)(x,y) = \min(f(x), g(y))$, for each $(x, y) \in X \times Y$.

Definition 1.1.17

Let $\{(X_\lambda , \delta_\lambda) \mid \lambda \in \Lambda\}$ be a family of fuzzy topological spaces and let $X = \Pi_\lambda X_\lambda$ be the cartesian product of X_λ's and let P_λ be the projection of the product X into the λ^{th} coordinate set X_λ. Let $\mathcal{P}(\Lambda)$ deonote the family of all finite subsets of Λ and $\mathcal{B} = \{ \bigcap_{\lambda\in\Lambda_0} P_\lambda^{-1}(f_\lambda) \mid f_\lambda \in \delta_\lambda , \Lambda_0 \in \mathcal{P}(\Lambda)\}$. Then the fuzzy topology δ which takes $\mathcal{B}$ as a basis is called the **product fuzzy topology** for X and $\mathcal{B}$ is called the defining basis for the product fuzzy topology. The pair (X, δ) is called the **fuzzy product space** of the fuzzy topological spaces $(X_\lambda, \delta_\lambda)$, $\lambda \in \Lambda$.

Section : 1. 2

Upper and Lower Finite Approximations of Fuzzy Sets

Definition : 1.2.1

A fuzzy set which assumes only a finite set of values is called a finite fuzzy set.

Definition : 1.2.2

With every fuzzy set f defined on a set X and with every positive integer n, we associate a finite fuzzy set ${}^nf$ with values in $I_n = \left\{0, \frac{1}{n}, \frac{2}{n}, ...1\right\}$ as follows :

For $x \in X$,

(i). If $f(x) = 0$, define ${}^nf(x) = 0$.

(ii). If $\frac{i}{n} < f(x) \leq \frac{i+1}{n}$, define ${}^{n}f(x) = \frac{i+1}{n}$ for $i = 0,1,2, \ldots, n-1$.

${}^{n}f$ is called the n^{th} upper approximation of f.

Remark : 1.2.3

(i). $f(x) = \frac{i}{n} \Rightarrow {}^{n}f(x) = \frac{i}{n}$ for $i = 0,1,2. \ldots n$.

(ii). $f(x) \leq {}^{n}f(x)$ for all n.

Proposition : 1.2.4

1. $f \leq g \Rightarrow {}^{n}f \leq {}^{n}g$.
2. $f \leq {}^{n}g \Rightarrow {}^{n}f \leq {}^{n}g$.
3. ${}^{n}({}^{n}f) = {}^{n}f$.
4. ${}^{m}f \leq {}^{n}f$ if $n \mid m$.
5. $f(x) = \inf \{ {}^{n}f(x) \mid n = 1,2,\ldots.. \}$.
6. ${}^{n}f = {}^{n}g$ for all $n \Rightarrow f = g$.
7. ${}^{m}({}^{n}f) = {}^{n}({}^{m}f)$ if $n \mid m$ or $m \mid n$.

Proof :

(1), (2), (3), (4) follow immediately from the definition of ${}^{n}f$.

Proof of (5) :

Since $f(x) \leq {}^{n}f(x)$ for all n, $f(x) \leq \inf {}^{n}f(x)$.

Given $\varepsilon > 0$, choose N such that $\varepsilon > \frac{1}{N}$.

$\therefore \quad | f(x) - {}^{N}f(x) | < \frac{1}{N} < \varepsilon$

$\Rightarrow \quad f(x) > {}^{N}f(x) - \varepsilon$

$\Rightarrow \quad {}^{N}f(x) < f(x) + \varepsilon$

$\Rightarrow \quad \inf\ {}^{n}f(x) < f(x) + \varepsilon$

$\therefore \quad \inf\ {}^{n}f(x) = f(x)$.

(6) follows immediately from **(5)** .

Proof of (7) :

$n \mid m \Rightarrow m = kn$.

$\therefore\ {}^{kn}({}^{n}f) = {}^{n}({}^{kn}f) = {}^{n}f$.

Remark : 1.2.5

If $(n,m) = 1$, then ${}^{n}f$ and ${}^{m}f$ are not comparable in general.

For Example, consider $n = 3$, $m = 4$.

If $0 < f(x) < \frac{1}{4}$, then ${}^{4}f(x) = \frac{1}{4} < {}^{3}f(x) = \frac{1}{3}$.

If $\frac{1}{4} < f(x) < \frac{1}{3}$, then ${}^{4}f(x) = \frac{1}{2} > {}^{3}f(x) = \frac{1}{3}$.

Proposition : 1.2.6

(i). ${}^{n}(\sup\{ f_\lambda \mid \lambda \in \Lambda \}) = \sup\{ {}^{n}(f_\lambda) \mid \lambda \in \Lambda \}$.

(ii). ${}^{n}(\inf\{ f_j \mid j = 1,2,\ldots,k \}) = \inf\{ {}^{n}(f_j) \mid j = 1,2,\ldots,k \}$.

Proof :

(i). Since $^{n}(\sup\{f_\lambda) \mid \lambda \in \Lambda\}) \geq {}^{n}(f_\lambda)$ for each λ,

$\sup\{{}^{n}(f_\lambda) \mid \lambda \in \Lambda\} \leq {}^{n}(\sup\{f_\lambda \mid \lambda \in \Lambda\})$.

Let $^{n}(\sup\{f_\lambda \mid \lambda \in \Lambda\})(x) = \frac{i+1}{n}$.

Then, $\frac{i}{n} < (\sup\{f_\lambda \mid \lambda \in \Lambda\})(x) \leq \frac{i+1}{n}$

$\Rightarrow$ there exists an f_λ such that

$\frac{i}{n} < f_\lambda(x) \leq (\sup\{f_\lambda \mid \lambda \in \Lambda\})(x) \leq \frac{i+1}{n}$

$\Rightarrow {}^{n}(f_\lambda)(x) = \frac{i+1}{n}$

$\Rightarrow \sup\{{}^{n}(f_\lambda) \mid \lambda \in \Lambda\}(x) = \frac{i+1}{n}$

$\Rightarrow \sup\{{}^{n}(f_\lambda) \mid \lambda \in \Lambda\} = {}^{n}(\sup\{f_\lambda \mid \lambda \in \Lambda\})$.

(ii). $^{n}(\inf\{f_j \mid j = 1,2,\ldots,k\}) \leq {}^{n}(f_j)$ for $j = 1,2,\ldots,k$

$\Rightarrow {}^{n}(\inf\{f_j \mid j = 1,2,\ldots,k\}) \leq \inf\{{}^{n}(f_j) \mid j = 1,2,\ldots,k\}$.

Let $\inf\{{}^{n}(f_j)(x) \mid j = 1,2,\ldots,k\} = \frac{i+1}{n}$.

Then $^{n}(f_j)(x) \geq \frac{i+1}{n}$ for all j

$\Rightarrow {}^{n}(f_j)(x) > \frac{i}{n}$ for all j

$\Rightarrow f_j(x) > \frac{i}{n}$ for all j

$\Rightarrow \inf \{ f_j(x) \mid j = 1,2,\ldots,k \} > \frac{i}{n}$

$\Rightarrow {}^{n}(\inf \{ f_j(x) \mid j = 1,2,\ldots,k \}) \geq \frac{i+1}{n}$

$\Rightarrow {}^{n}(\inf \{ f_j(x) \mid j = 1,2,\ldots,k \}) \geq \inf \{ {}^{n}(f_j)(x) \mid j = 1,2,\ldots,k \}$

$\therefore {}^{n}(\inf \{ f_j \mid j = 1,2,\ldots,k \}) = \inf \{ {}^{n}(f_j) \mid j = 1,2,\ldots,k \}$.

Remark : 1.2.7

Example for ${}^{n}(\inf \{ f_m \mid m = 1,2,\ldots \}) \neq \inf \{ {}^{n}(f_m) \mid m = 1,2,\ldots \}$.

Suppose, $f_m(x) = \frac{i}{n} + \frac{1}{m+n}$ where $m = 1,2,\ldots$.

Then, ${}^{n}(f_m)(x) = \frac{i+1}{n}$

$\Rightarrow \inf \{ {}^{n}(f_m)(x) \mid m = 1,2,\ldots \} = \frac{i+1}{n}$.

But, $\inf \{ f_m(x) \mid m = 1,2,\ldots \} = \frac{i}{n}$

$\Rightarrow {}^{n}(\inf \{ f_m(x) \mid m = 1,2,\ldots \}) = \frac{i}{n}$.

Hence, ${}^{n}(\inf \{ f_m \mid m = 1,2,\ldots \}) \neq \inf \{ {}^{n}(f_m) \mid m = 1,2,\ldots \}$.

Proposition : 1.2.8

If $\frac{i}{n} < \varepsilon < \frac{i+1}{n}$, then

(i). $f^{-1}(\varepsilon,1] \subset ({}^{n}f)^{-1}\left[\frac{i+1}{n},1\right]$.

(ii). $({}^{n}f)^{-1}(\varepsilon,1] = f^{-1}(\frac{i}{n},1] = ({}^{n}f)^{-1}(\frac{i}{n},1]$.

Proof :

(i). Proof is obvious .

(ii). ${}^{n}f(x) > \varepsilon \Leftrightarrow {}^{n}f(x) \geq \frac{i+1}{n} \Leftrightarrow f(x) > \frac{i}{n} \Leftrightarrow {}^{n}f(x) > \frac{i}{n}$.

Definition : 1.2.9

For each fuzzy set f on a set X, the **nth lower approximation $_{n}f$** is defined as follows :

For $x \in X$,

(i). If $f(x) = 1$, define $_{n}f(x) = 1$.

(ii). If $\frac{i}{n} \leq f(x) < \frac{i+1}{n}$, define $_{n}f(x) = \frac{i}{n}$ for $i = 0,1,2, \ldots, n-1$.

Remark : 1.2.10

(i). If $f(x) = \frac{i}{n}$, then $_{n}f(x) = \frac{i}{n}$ for $i = 0,1,\ldots,n$.

(ii). $_{n}f(x) \leq f(x)$ for all n.

Proposition : 1.2.11

1. $f \le g \Rightarrow {}_nf \le {}_ng$.

2. ${}_nf \le g \Rightarrow {}_nf \le {}_ng$.

3. ${}_n({}_nf) = {}_nf$.

4. ${}_mf(x) \ge {}_nf(x)$ if $n \mid m$.

5. $f(x) = \sup \{ {}_nf(x) \mid n = 1, 2. \ldots \}$.

6. ${}_nf = {}_ng$ for all n $\Rightarrow f = g$.

7. ${}_m({}_nf) = {}_n({}_mf)$ if $n \mid m$ or $m \mid n$.

8. ${}_n(\inf \{ f_\lambda \mid \lambda \in \Lambda \}) = \inf \{ {}_n(f_\lambda) \mid \lambda \in \Lambda \}$.

9. ${}_n(\sup \{ f_j \mid j = 1, 2, \ldots, k \} = \sup \{ {}_n(f_j) \mid j = 1, 2, \ldots, k \}$.

Proposition : 1.2.12

1. ${}_n(1 - f) = 1 - {}^nf$.

2. ${}^nf \le g \Rightarrow {}^nf \le {}_ng$.

3. $f \le {}_ng \Rightarrow {}^nf \le {}_ng$.

4. ${}_n({}^nf) = {}^nf$.

5. ${}^n({}_nf) = {}_nf$.

Proof :

Suppose $\frac{i}{n} < f(x) \leq \frac{i+1}{n}$,

${}^{n}f(x) = \frac{i+1}{n}$ and $1 - \frac{i+1}{n} \leq 1 - f(x) < 1 - \frac{i}{n}$

$\therefore \quad {}_{n}(1-f)(x) = 1 - \frac{i+1}{n}$ and $1 - {}^{n}f(x) = 1 - \frac{i+1}{n}$

Hence ${}_{n}(1-f) = 1 - {}^{n}f$.

Section : 1.3

n^{th} Degree Approximation of a Fuzzy Topology - Fundamental Properties

Definition :1.3.1

A fuzzy topology in which all open fuzzy sets are finite valued is called a **finite fuzzy topology**.

Definition :1.3.2

A finite fuzzy topology δ is called a **strong finite fuzzy topology** if there exists a finite set $\mathbf{J(\delta)}$ contained in I such that every member of δ assumes values from $\mathbf{J(\delta)}$.

Remark :1.3.3

Any topology can be considered as a strong finite fuzzy topology. If τ is a topology on X, $\delta = \{\chi_A \mid A \in \tau\}$ is a strong finite fuzzy topology.

Definition :1.3.4

We shall associate with every fuzzy topology δ , **a strong finite fuzzy topology** ${}^{n}\delta$ as follows :

Given $\delta = \{ f_\lambda \mid \lambda \in \Lambda \}$, ${}^{n}\delta = \{ {}^{n}(f_\lambda) \mid \lambda \in \Lambda \}$.

Remark :1.3.5

Using the fundamental properties of the correspondence $f \to {}^{n}f$, it can be easily verified that ${}^{n}\delta$ is a strong fuzzy topology where **J(${}^{n}\delta$)** $= \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots\ldots\ldots \frac{n-1}{n}, 1\right\}$. We shall refer to ${}^{n}\delta$ as the **n^th^ degree approximation to δ .**

Remark :1.3.6

Closed sets of ${}^{n}\delta$ are of the form ${}_{n}(1-f)$ where $f \in \delta$.

Remark :1.3.7

In general, ${}^{n}\delta$ need not be contained in δ.

Consider the set I with the fuzzy topology $\delta = \{ 0, 1, \imath \}$ where $\imath$ is the identity function. Then, ${}^{n}\delta = \{ 0, 1, {}^{n}\imath \}$. Clearly, ${}^{n}\delta$ is not contained in δ.

In the following example ${}^{n}\delta \subseteq \delta$ for each n.

Define $\delta = \{ f \mid \text{supp}(1 - f) \text{ is finite} \}$.

Then ${}^{n}\delta = \{ {}^{n}f \mid \text{supp}(1 - f) \text{ is finite} \}$.

Here, ${}^{n}\delta \subseteq \delta$, since $\text{supp}(1 - {}^{n}f) \subseteq \text{supp}(1 - f)$.

Proposition :1.3.8

If $\mathscr{B}$ is a basis for δ, then ${}^{n}\mathscr{B}$ is a basis for ${}^{n}\delta$.

Proof : Obvious .

Properties of ${}^{n}\delta$

Proposition : 1.3.9

Given two fuzzy topologies δ and σ on a set X, we have the following implications :

(i). $\delta \subseteq \sigma \Rightarrow {}^{n}\delta \subseteq {}^{n}\sigma$.

(ii). ${}^{n}(\delta \vee \sigma) = {}^{n}\delta \vee {}^{n}\sigma$.

(iii). ${}^{n}(\delta \wedge \sigma) \subseteq {}^{n}\delta \wedge {}^{n}\sigma$.

(iv). $\delta \subseteq {}^{n}\delta \Rightarrow \delta = {}^{n}\delta$.

Proof :

(i). Proof is obvious .

(ii). By proposition 1.3.8, the set

$\{ {}^{n}(f \wedge g) \mid f \in \delta , g \in \sigma \}$ forms a basis for ${}^{n}(\delta \vee \sigma)$.

But, ${}^{n}(f \wedge g) = {}^{n}f \wedge {}^{n}g \in {}^{n}\delta \vee {}^{n}\sigma$.

Therefore, ${}^{n}(\delta \vee \sigma) \subseteq {}^{n}\delta \vee {}^{n}\sigma$.

Also, ${}^{n}\delta \vee {}^{n}\sigma \subseteq {}^{n}(\delta \vee \sigma)$.

Hence, ${}^{n}(\delta \vee \sigma) = {}^{n}\delta \vee {}^{n}\sigma$.

(iii). and (iv). are clear .

Proposition : 1.3.10

Given two fuzzy topological spaces (X , δ) and (Y , σ) , let $\delta \times \sigma$ be the product fuzzy topology on X x Y. Then $^n(\delta \times \sigma) = {}^n\delta \times {}^n\sigma$.

Proof :

To prove $^n(\delta \times \sigma) = {}^n\delta \times {}^n\sigma$, it is enough to prove that

$^n(f \times g) = {}^nf \times {}^ng$ where $f \in \delta$ and $g \in \sigma$.

$(f \times g)(x, y) = \min\{f(x), g(y)\}$.

Suppose $f(x) \leq g(y)$, $\min\{f(x), g(y)\} = f(x)$.

$\min\{{}^nf(x), {}^ng(y)\} = {}^nf(x)$.

$\therefore\ {}^n(f \times g)(x, y) = ({}^nf \times {}^ng)(x, y)$.

The above result holds for infinite products also.

Proposition : 1.3.11

$i({}^n\delta) \subseteq i({}^{kn}\delta)$ where $k = 1, 2, \ldots .$

Proof :

Let $\frac{i}{n} \leq \varepsilon < \frac{i+1}{n}$

Since $({}^nf)^{-1}(\varepsilon, 1] = f^{-1}\left(\frac{i}{n}, 1\right]$

$$= f^{-1}\left(\frac{ki}{kn}, 1\right] = ({}^{kn}f)^{-1}\left(\frac{ki}{kn}, 1\right],$$

$$i({}^{n}\delta) \subseteq i({}^{kn}\delta) \quad \text{for } k = 1, 2, \ldots .$$

Remark : 1.3.12

$i({}^{2n}\delta)$ need not be contained in $i({}^{n}\delta)$.

Let $X = I$, $\delta = \{0, 1, i\}$.

Then, ${}^{4}i^{-1}\left(\frac{1}{4}, 1\right] = \left(\frac{1}{4}, 1\right] \in i({}^{4}\delta)$.

But, $\left(\frac{1}{4}, 1\right] \notin i({}^{2}\delta)$.

$\therefore \quad i({}^{4}\delta) \not\subseteq i({}^{2}\delta)$.

Theorem : 1.3.13

Let (X, δ) be a fuzzy topological space and let ${}^{n}\delta$ be the n^{th} degree approximation of δ. The topologies $i(\delta)$ and $i({}^{n}\delta)$ are related as follows:

(i). $i({}^{n}\delta) \subseteq i(\delta)$ for all n, and

(ii). $\sup\{ i({}^{n}\delta) \mid n = 1, 2, 3, \ldots \} = i(\delta)$.

Proof :

(i). Proof is clear from proposition 1.2.8

(ii). To prove $\sup\{ i({}^{n}\delta) \mid n = 1,2,3, \ldots \} = i(\delta)$, we have only to show that " Every $f^{-1}(\varepsilon, 1]$ can be expressed as the union of members of $i({}^{n}\delta)$ contained in $f^{-1}(\varepsilon, 1]$ ".

It is enough to prove that, for given $x \in f^{-1}(\varepsilon, 1]$, we can find a 'N' such that $x \in ({}^{N}f)^{-1}\left(\frac{i+2}{2N}, 1\right] \subseteq f^{-1}(\varepsilon, 1]$.

Let $x \in f^{-1}(\varepsilon, 1]$. Then $f(x) > \varepsilon$.

Choose N such that $f(x) - \varepsilon > \frac{1}{N}$.

For this N, suppose ε lies between $\frac{i}{2N}$ and $\frac{i+1}{2N}$, we have

$$f(x) > \varepsilon + \frac{1}{N}$$

$$> \frac{i}{2N} + \frac{1}{N} = \frac{i+2}{2N},$$

$$\Rightarrow {}^{N}f(x) \geq {}^{2N}f(x) > \frac{i+2}{2N}$$

$$\Rightarrow x \in ({}^{N}f)^{-1}\left(\frac{i+2}{2N}, 1\right]$$

Hence, for given $x \in f^{-1}(\varepsilon, 1]$, we can find a 'N' such that

$x \in ({}^{N}f)^{-1}\left(\frac{i+2}{2N}, 1\right]$.

Claim :

$$({}^{N}f)^{-1}\left(\frac{i+2}{2N}, 1\right] \subseteq f^{-1}(\varepsilon, 1].$$

Let $y \in ({}^{N}f)^{-1}\left(\frac{i+2}{2N}, 1\right]$.

Then, ${}^{N}f(y) > \frac{i+2}{2N}$

$\Rightarrow \quad f(y) > \varepsilon$

$\Rightarrow \quad y \in f^{-1}(\varepsilon, 1]$

Hence, $({}^{N}f)^{-1}\left(\frac{i+2}{2N}, 1\right] \subseteq f^{-1}(\varepsilon, 1]$.

Definition : 1.3.14

Let (X, τ) be a topological space. $\omega^{n}(\tau)$ is defined as follows :

$\omega^{n}(\tau) = \{ f \in I_{n}^{X} \mid f^{-1}\left(\frac{i}{n}, 1\right] \in \tau$ for $i = 1, 2, \ldots \}$,

where $I_{n}^{X} = \{ f : X \to \left\{0, \frac{1}{n}, \ldots\ldots 1\right\}\}$.

Proposition : 1.3.15

$\omega^{n}(\tau)$ is a fuzzy topology.

Proof : Obvious.

Remark : 1.3.16

${}^{n}(\omega(\tau)) \subseteq \omega^{n}(\tau)$.

Proof :

$f \in \omega(\tau)$

$$\Rightarrow ({}^{n}f)^{-1}\left(\frac{i}{n},1\right] = f^{-1}\left(\frac{i}{n},1\right] \in \tau$$

$$\Rightarrow {}^{n}f \in \omega^{n}(\tau).$$

Proposition : 1.3.17

Let (X, τ) be a topological space. If $\delta = \omega(\tau)$, and if $\tau_n = i({}^{n}\delta)$, then ${}^{n}\delta = {}^{n}(\omega(\tau_n))$.

Proof :

$$\tau_n = i({}^{n}\delta) \subseteq i(\delta) = \tau.$$

$$\omega(\tau_n) = \omega(i({}^{n}\delta)) \supseteq {}^{n}\delta.$$

$$\therefore {}^{n}\delta \subseteq {}^{n}(\omega(\tau_n)) \subseteq {}^{n}(\omega(\tau)) = {}^{n}\delta.$$

Theorem : 1.3.18

Let (X, δ) be a fuzzy topological space . Let $f \in I^{X}$. Then ,

(i). $\mathrm{Cl}\ f$ in ${}^{n}\delta \geq {}_{n}(\mathrm{Cl}\ f \text{ in } \delta)$

(ii). $\mathrm{Cl}\ f$ in $\delta \geq {}_{n}(\mathrm{Cl}\ f \text{ in } \delta) \geq \mathrm{Cl}\ {}_{n}f$ in ${}^{n}\delta$

(iii). $\mathrm{Cl}\ {}^{n}f$ in $\delta \geq \mathrm{Cl}\ {}^{n}f$ in ${}^{n}\delta = \mathrm{Cl}\ f$ in ${}^{n}\delta$.

Proof :

(i). $\mathrm{Cl} f$ in ${}^{n}\delta = \inf\{1 - {}^{n}g \mid f \leq 1 - {}^{n}g,\ g \in \delta\}$

$$
\begin{aligned}
{}_{n}(\mathrm{Cl}\, f \text{ in } \delta) &= {}_{n}(\inf\{1-g \mid f \le 1-g,\ g \in \delta\} \\
&= \inf\{{}_{n}(1-g) \mid f \le 1-g,\ g \in \delta\} \\
&= \inf\{1-{}^{n}g \mid f \le 1-g,\ g \in \delta\}
\end{aligned}
$$

Since $1-{}^{n}g \le 1-g$,

$\mathrm{Cl}\, f \text{ in } {}^{n}\delta \ \ge\ {}_{n}(\mathrm{Cl}\, f \text{ in } \delta)$.

(ii). It is enough to prove that

$${}_{n}(\mathrm{Cl}\, f \text{ in } \delta) \ \ge\ \mathrm{Cl}\ {}_{n}f \text{ in } {}^{n}\delta$$

$$
\begin{aligned}
\mathrm{Cl}\ {}_{n}f \text{ in } {}^{n}\delta &= \inf\{1-{}^{n}g \mid {}_{n}f \le 1-{}^{n}g,\ g \in \delta\} \\
&= \inf\{{}_{n}(1-g) \mid {}_{n}f \le {}_{n}(1-g),\ g \in \delta\}
\end{aligned}
$$

$f \le 1-g \Rightarrow {}_{n}f \le {}_{n}(1-g)$.

$\therefore\ {}_{n}(\mathrm{Cl}\, f \text{ in } \delta) \ \ge\ \mathrm{Cl}\ {}_{n}f \text{ in } {}^{n}\delta$.

(iii). $\mathrm{Cl}\ {}^{n}f \text{ in } \delta = \inf\{1-g \mid {}^{n}f \le 1-g,\ g \in \delta\}$.

$\mathrm{Cl}\ {}^{n}f \text{ in } {}^{n}\delta = \inf\{1-{}^{n}g \mid {}^{n}f \le 1-{}^{n}g,\ g \in \delta\}$.

$\mathrm{Cl}\ f \text{ in } {}^{n}\delta = \inf\{1-{}^{n}g \mid f \le 1-{}^{n}g,\ g \in \delta\}$.

Since ${}^{n}f \le 1-{}^{n}g \Leftrightarrow f \le 1-{}^{n}g$,

$\mathrm{Cl}\ {}^{n}f \text{ in } {}^{n}\delta = \mathrm{Cl}\ f \text{ in } {}^{n}\delta$. ....(1)

Since ${}^{n}f \le 1-{}^{n}g \Leftrightarrow {}^{n}f \le 1-g$ and $1-g \ge 1-{}^{n}g$,

$(\mathrm{Cl}\ {}^{n}f \text{ in } \delta) \ge \mathrm{Cl}\ {}^{n}f \text{ in } {}^{n}\delta$. ....(2)

By (1) and (2), Proof is clear.

Theorem : 1.3.19

${}^{n}f$ is closed (open) in $\delta \Rightarrow {}^{n}f$ is closed (open) in ${}^{n}\delta$.

Proof :

${}^{n}f$ is closed in $\delta \Rightarrow {}_{n}({}^{n}f)$ is closed in ${}^{n}\delta$

$\Rightarrow {}^{n}f$ closed in ${}^{n}\delta$.

By definition of ${}^{n}\delta$, ${}^{n}f \in \delta \Rightarrow {}^{n}f \in {}^{n}\delta$.

Theorem : 1.3.20

Cl χ_A in ${}^{n}\delta = \chi_B$ for all n $\Rightarrow$ Cl χ_A in $\delta = \chi_B$.

Proof :

Cl χ_A in ${}^{n}\delta = \chi_B$ for all n

$\Rightarrow \inf \{ 1 - {}^{n}g \mid \chi_A \leq 1 - {}^{n}g, g \in \delta \} = \chi_B$

Cl χ_A in $\delta = \inf \{ 1 - g \mid \chi_A \leq 1 - g, g \in \delta \}$

We know that

$\chi_A \leq 1 - {}^{n}g \Leftrightarrow \chi_A \leq 1 - g$ and Cl χ_A in $\delta \geq$ Cl χ_A in ${}^{n}\delta = \chi_B$

Claim :

$\mathrm{Cl}\ \chi_A$ in $\delta = \chi_B$

$x \in B \Rightarrow \chi_B(x) = 1$, $(\mathrm{Cl}\ \chi_A \text{ in } \delta)(x) \geq \chi_B(x) = 1$.

$x \notin B \Rightarrow \chi_B(x) = 0$

$\Rightarrow \inf\{(1 - {}^{n}g)(x) \mid \chi_A \leq 1 - {}^{n}g,\ g \in \delta \text{ for every } n\} = 0$.

$\Rightarrow$ for all n, there exists a $g_n \in \delta$ depending on 'n' such that

$(1 - {}^{n}(g_n))(x) < \frac{1}{n}$

$\Rightarrow {}^{n}(g_n)(x) > 1 - \frac{1}{n}$

$\Rightarrow g_n(x) > 1 - \frac{1}{n}$

$\Rightarrow 1 - g_n(x) < \frac{1}{n}$

$\Rightarrow \inf\{(1 - g_n)(x) \mid n = 1, 2, \ldots\} = 0$

$\Rightarrow (\mathrm{Cl}\ \chi_A \text{ in } \delta)(x) = 0$.

Remark : 1.3.21

χ_A is closed in ${}^{n}\delta$ for every n $\Rightarrow$ χ_A is closed in δ

Remark :1.3.22

$\mathrm{Cl}\ {}^{n}f$ in ${}^{n}\delta$ need not be equal to ${}^{n}f$.

Theorem : 1.3.23

Suppose δ is a fuzzy topology such that $\mathrm{Cl}\ f$ in δ = $\mathrm{Cl}\ {}^{n}f$ in δ, then $\delta = {}^{n}\delta$ and conversely.

Proof :

If $\delta = {}^{n}\delta$, then $\mathrm{Cl}\ f$ in δ = $\mathrm{Cl}\ f$ in ${}^{n}\delta$

$$= \mathrm{Cl}\ {}^{n}f \text{ in } {}^{n}\delta = \mathrm{Cl}\ {}^{n}f \text{ in } \delta .$$

Conversely, assume that $\mathrm{Cl}\ f$ in δ = $\mathrm{Cl}\ {}^{n}f$ in δ.

$\mathrm{Cl}\ f \text{ in } \delta = \mathrm{Cl}\ {}^{n}f \text{ in } \delta = \inf\{1 - g \mid {}^{n}f \leq 1 - g,\ g \in \delta\}$

$\mathrm{Cl}\ f \text{ in } {}^{n}\delta = \mathrm{Cl}\ {}^{n}f \text{ in } {}^{n}\delta = \inf\{1 - {}^{n}g \mid {}^{n}f \leq 1 - {}^{n}g,\ g \in \delta\}$

${}^{n}f \leq 1 - g \Leftrightarrow {}^{n}f \leq 1 - {}^{n}g$

$f \leq \mathrm{Cl}\ f \text{ in } {}^{n}\delta \leq \mathrm{Cl}\ f \text{ in } \delta$

f closed in $\delta \Rightarrow f$ closed in ${}^{n}\delta$

$\Rightarrow \delta \subseteq {}^{n}\delta$.

$\therefore \delta = {}^{n}\delta$.

Definition : 1.3.24

A fuzzy set f is **dense** in (X, δ) if $\mathrm{Cl}\ f$ in $\delta = 1$.

Theorem : 1.3.25

A fuzzy set f is dense in $\delta \Rightarrow f$ is dense in ${}^{n}\delta$.

Proof :

$$\text{Cl } f \text{ in } {}^{n}\delta = \inf\{1 - {}^{n}g \mid f \le 1 - {}^{n}g,\ g \in \delta\}.$$

$\text{Cl } f \text{ in } \delta = \inf\{1 - g \mid f \le 1 - g,\ g \in \delta\} = 1$

$\Rightarrow 1 - g = 1$ if $f \le 1 - g,\ g \in \delta$.

$\therefore f \le 1 - g,\ g \in \delta$

$\Rightarrow g = 0 \Rightarrow {}^{n}g = 0 \Rightarrow 1 - {}^{n}g = 1$

$\therefore \text{Cl } f \text{ in } {}^{n}\delta = 1$.

Hence, f is dense in $\delta \Rightarrow f$ is dense in ${}^{n}\delta$.

Section : 1.4

Fuzzy Continuous Mappings and Finite Approximations

Definition : 1.4.1

Let δ be a fuzzy topology defined on a set X. Then $\delta^{\Diamond}$ is defined as follows :

$$\delta^{\Diamond} = \{f \in I^{X} \mid {}^{n}f \in {}^{n}\delta \text{ for every n}\}.$$

Proposition : 1.4.2

$\delta^{\Diamond}$ is a fuzzy topology on X.

Proof :

(i). Obviously 0, 1 belong to $\delta^{\lozenge}$.

(ii). The collection $\{ f_\lambda \mid \lambda \in \Lambda \} \in \delta^{\lozenge}$

$\Rightarrow$ ${}^{n}(f_\lambda) \in {}^{n}\delta$ for every λ and for every n

$\Rightarrow$ $\sup \{ {}^{n}(f_\lambda) \mid \lambda \in \Lambda \} \in {}^{n}\delta$ for every n

$\Rightarrow$ ${}^{n}(\sup \{ f_\lambda \mid \lambda \in \Lambda \} \in {}^{n}\delta$ for every n

$\Rightarrow$ $\sup \{ f_\lambda \mid \lambda \in \Lambda \} \in \delta^{\lozenge}$.

(iii). The finite collection $\{ f_j \mid j = 1,2,\ldots\ldots,k \} \in \delta^{\lozenge}$

$\Rightarrow$ $\{ {}^{n}(f_j) \mid j = 1,2,\ldots\ldots,k \} \in {}^{n}\delta$ for every n

$\Rightarrow$ $\inf \{ {}^{n}(f_j) \mid j = 1,2,\ldots\ldots,k \} \in {}^{n}\delta$ for every n

$\Rightarrow$ ${}^{n}(\inf \{ f_j \mid j = 1,2,\ldots\ldots,k \}) \in {}^{n}\delta$ for every n

$\Rightarrow$ $\inf \{ f_j \mid j = 1,2,\ldots\ldots,k \}) \in \delta^{\lozenge}$.

Proposition : 1.4.3

$\delta^{\lozenge}$ is the largest fuzzy topology containing δ such that ${}^{n}\delta = {}^{n}(\delta^{\lozenge})$ for every n.

Proof :

By definition of $\delta^{\lozenge}$, it is clear that $\delta \subseteq \delta^{\lozenge}$ and ${}^{n}\delta \subseteq {}^{n}(\delta^{\lozenge})$

Claim 1:

${}^{n}(\delta^{\lozenge}) \subseteq {}^{n}\delta$.

$f \in {}^{n}(\delta^{\lozenge}) \Rightarrow f = {}^{n}g$ where $g \in \delta^{\lozenge}$

$\Rightarrow f = {}^{n}g$ where ${}^{n}g \in {}^{n}\delta$ for every n .

$\Rightarrow f \in {}^{n}\delta$

Hence, $^n(\delta^\Diamond) \subseteq {}^n\delta$.

Suppose σ is any other fuzzy topology such that $^n\sigma = {}^n\delta$ for every n .

Claim 2:

$\sigma \subset \delta^\Diamond$.

Let $f \in \sigma$ Then, $^nf \in {}^n\sigma$ for every n.

This implies $^nf \in {}^n\delta$ for every n .

$\therefore \; f \in \delta^\Diamond$.

Hence, $\delta^\Diamond$ is the largest fuzzy topology containing δ such that $^n\delta = {}^n(\delta^\Diamond)$.

Example : 1.4.4

Let $X = I$ and $\delta = \{ f \in I^X \mid \text{supp}(1 - f) \text{ is finite} \}$

Then , $\delta \neq \delta^\Diamond$.

Let $\{ y_1, y_2, \ldots\ldots\ldots \} \subseteq I$ be an infinite set .

Define,

$$f(y_n) = \frac{n+1}{n+2}, \text{ for } n = 1, 2, 3, \ldots\ldots$$

and $f(x) = 1$, for $x \neq y_1, y_2, \ldots\ldots\ldots\ldots, y_{n-1}, \ldots\ldots\ldots\ldots$

Now, $f(x) \neq 1$, for $x = y_1, y_2, \ldots\ldots\ldots\ldots, y_{n-1}, \ldots\ldots\ldots\ldots$

and $\text{supp}(1 - f) = \{ y_1, y_2, \ldots\ldots\ldots\ldots \}$ which is an infinite set.

$\therefore \; f \notin \delta$.

Claim :

$f \in \delta^\Diamond$

$\text{supp}(1 - {}^nf) = \{ x \mid (1 - {}^nf)(x) \neq 0 \}$

$$= \{ x \mid {}^{n}f(x) \neq 1 \}$$

$$= \{ y_1, y_2, \ldots\ldots, y_{n-3}, y_{n-2} \} .$$

Hence, supp $(1 - {}^{n}f)$ is a finite set .

Therefore , ${}^{n}f \in \delta$ for every n .

$\Rightarrow {}^{n}f \in {}^{n}\delta$ for every n.

Hence, $f \in \delta^{\Diamond}$.

Example : 1.4.5

General method of constructing fuzzy topologies δ such that $\delta \neq \delta^{\Diamond}$. Take any non empty set X. Let h be a non constant function defined on X with values in I such that $h(x)$ is irrational for each x.

Define $f_n : X \rightarrow I$ as follows :

If $\frac{i}{n} < h(x) < \frac{i+1}{n}$,

then $f_n(x)$ is a rational number such that

$$\frac{i}{n} < h(x) < f_n(x) < \frac{i+1}{n} .$$

Then ${}^{n}(f_n)(x) = ({}^{n}h)(x)$.

Let δ be the fuzzy topology having $\{ f_n \mid n = 1, 2, \ldots \}$ as a subbasis. By definition of $f_n(x)$, $h(x) < f_n(x)$.

Let $f \in \delta$. Then f can be expressed as arbitrary union of finite intersection of fuzzy sets of the form $f_n(x)$.

$\therefore f \in \delta \Rightarrow f(x) > h(x)$.

Hence, $h \notin \delta$.

By definition of f_n ,

${}^{n}\hbar = {}^{n}(f_n) \in {}^{n}\delta$ for every n .

$\therefore\ \hbar \in \delta^{\Diamond}$.

That is, $\delta \neq \delta^{\Diamond}$.

Properties of $\delta^{\Diamond}$.

Theorem : 1.4.6

(i). $\delta \subseteq \sigma \Rightarrow \delta^{\Diamond} \subseteq \sigma^{\Diamond}$.

(ii). $(\delta^{\Diamond})^{\Diamond} = \delta^{\Diamond}$.

(iii). $\delta \subseteq \sigma^{\Diamond} \Rightarrow \delta^{\Diamond} \subseteq \sigma^{\Diamond}$.

(iv). $(\delta \vee \sigma)^{\Diamond} = \delta^{\Diamond} \vee \sigma^{\Diamond}$.

(v). $(\delta \times \sigma)^{\Diamond} = \delta^{\Diamond} \times \sigma^{\Diamond}$.

(vi). $i(\delta) = i(\delta^{\Diamond})$.

(vii). If δ is topologically generated fuzzy topology, then $\delta = \delta^{\Diamond}$.

Proof :

(i). $\hbar \in \delta^{\Diamond} \Rightarrow {}^{n}\hbar \in {}^{n}\delta$ for every n

$\Rightarrow {}^{n}\hbar \in {}^{n}\sigma$ for every n

$\Rightarrow \hbar \in \sigma^{\Diamond}$.

Hence, $\delta^{\Diamond} \subseteq \sigma^{\Diamond}$ if $\delta \subseteq \sigma$.

(ii). Obvious, since ${}^{n}(\delta^{\Diamond}) = {}^{n}\delta$

(iii). Proof is clear by (ii) .

(iv). $(\delta \vee \sigma)^{\Diamond} = \{f \mid {}^{n}f \in {}^{n}(\delta \vee \sigma) \text{ for all n}\}$

$= \{f \mid {}^{n}f \in {}^{n}\delta \vee {}^{n}\sigma \text{ for all n}\}$

$= \{f \mid {}^{n}f \in {}^{n}(\delta^{\Diamond}) \vee {}^{n}(\sigma^{\Diamond}) \text{ for all n}\}$

$= \{ f \mid {}^{n}f \in {}^{n}(\delta^{\Diamond} \vee \sigma^{\Diamond}) \text{ for all } n\}$

$= \delta^{\Diamond} \vee \sigma^{\Diamond}$.

(v). $f \in (\delta \times \sigma)^{\Diamond} \Rightarrow {}^{n}f \in {}^{n}(\delta \times \sigma)$ for every n

$\Rightarrow {}^{n}f \in {}^{n}\delta \times {}^{n}\sigma$ for every n

$\Rightarrow {}^{n}f \in {}^{n}(\delta^{\Diamond}) \times {}^{n}(\sigma^{\Diamond})$ for every n

$\Rightarrow {}^{n}f \in {}^{n}(\delta^{\Diamond} \times \sigma^{\Diamond})$ for every n

$\Rightarrow f \in \delta^{\Diamond} \times \sigma^{\Diamond}$

$\therefore (\delta \times \sigma)^{\Diamond} \subseteq \delta^{\Diamond} \times \sigma^{\Diamond}$ .... (1)

Conversely, let $f \times g$ be a basic element of $\delta^{\Diamond} \times \sigma^{\Diamond}$, where $f \in \delta^{\Diamond}$, $g \in \sigma^{\Diamond}$.

Then ${}^{n}f \in {}^{n}\delta$, ${}^{n}g \in {}^{n}\sigma$ for all n

$\Rightarrow {}^{n}f \times {}^{n}g \in {}^{n}\delta \times {}^{n}\sigma$ for all n

$\Rightarrow {}^{n}(f \times g) \in {}^{n}(\delta \times \sigma)$ for all n

$\Rightarrow f \times g \in (\delta \times \sigma)^{\Diamond}$

$\therefore \delta^{\Diamond} \times \sigma^{\Diamond} \subseteq (\delta \times \sigma)^{\Diamond}$ . . . . (2)

By (1) and (2), $(\delta \times \sigma)^{\Diamond} = \delta^{\Diamond} \times \sigma^{\Diamond}$.

(vi). $i (\delta) = \sup \{ i({}^{n}\delta) \}$

$= \sup \{ i ({}^{n}(\delta^{\Diamond})) \}$

$= i (\delta^{\Diamond})$.

(vii). δ is topologically generated

$\Rightarrow \delta = \omega (i (\delta)) = \omega (i (\delta^{\Diamond})) \supseteq \delta^{\Diamond} \supseteq \delta$

Hence, $\delta = \delta^{\Diamond}$.

Theorem : 1.4.7

Let $\theta : X \to Y$ be a function. Then,

(i). For all $f \in I^X$, ${}^n(\theta(f)) = \theta({}^n f)$.

(ii). For given $f \in I^Y$, ${}^n(\theta^{-1}(f)) = \theta^{-1}({}^n f)$.

Proof :

Let $\theta : X \to Y$ be a function

(i). For all $f \in I^X$,

$$\begin{aligned} \theta({}^n f)(y) &= \sup\{{}^n f(x) \mid \theta(x) = y\} \\ &= {}^n(\sup\{f(x) \mid \theta(x) = y\}) \\ &= {}^n(\theta(f)(y)) \\ &= {}^n(\theta(f))(y) \end{aligned}$$

$\Rightarrow \theta({}^n f) = {}^n(\theta(f))$.

(ii). For given $f \in I^Y$,

$(\theta^{-1}(f))(x) = f(\theta(x))$ for all $x \in X$

$$\begin{aligned} \Rightarrow {}^n(\theta^{-1}(f))(x) &= {}^n(f(\theta(x)) \\ &= {}^n f(\theta(x)) \\ &= (\theta^{-1}({}^n f))(x) \end{aligned}$$

Hence, ${}^n(\theta^{-1}(f)) = \theta({}^n f)$.

Theorem : 1.4.8

If $\theta : (X, \delta) \to (Y, \sigma)$ is fuzzy continuous (open), then $\theta : (X, {}^n\delta) \to (Y, {}^n\sigma)$ is fuzzy continuous (open) for all n.

Proof :

Let $\theta : (X, \delta) \to (Y, \sigma)$ be a fuzzy continuous map.

Let ${}^{n}f \in {}^{n}\sigma$.

Then there exists a $g \in \sigma$ such that ${}^{n}f = {}^{n}g$.

$$g \in \sigma \Rightarrow \theta^{-1}(g) \in \delta$$

$$\Rightarrow {}^{n}(\theta^{-1}(g)) \in {}^{n}\delta$$

$$\Rightarrow \theta^{-1}({}^{n}g) \in {}^{n}\delta$$

$$\Rightarrow \theta^{-1}({}^{n}f) \in {}^{n}\delta$$

Hence, $\theta : (X, {}^{n}\delta) \to (Y, {}^{n}\sigma)$ is fuzzy continuous .

Let $\theta : (X, \delta) \to (Y, \sigma)$ be an open map.

Given ${}^{n}f \in {}^{n}\delta$, assume $f \in \delta$.

Then $\theta(f) \in \sigma$.

$\therefore\ \theta({}^{n}f) = {}^{n}(\theta(f)) \in {}^{n}\sigma$.

Definition : 1.4.9

Let $\theta : X \to Y$ be a map. Let σ be a fuzzy topology on Y. The smallest fuzzy topology δ on X which makes $\theta : (X, \delta) \to (Y, \sigma)$ fuzzy continuous is given by $\delta = \{\theta^{-1}(f) \mid f \in \sigma\}$.

Theorem : 1.4.10

Let (Y, σ) be a fuzzy topological space and $\theta : X \to Y$ be a map. If δ is the smallest fuzzy topology on X which makes $\theta : (X, \delta) \to (Y, \sigma)$

fuzzy continuous, then ${}^{n}\delta$ is the smallest fuzzy topology on X which makes $\theta : (X, {}^{n}\delta) \to (Y, {}^{n}\sigma)$ fuzzy contin$\overset{u}{\wedge}$ous .

Proof :

To prove the theorem , it is enough to show that

${}^{n}f \in {}^{n}\delta \Leftrightarrow {}^{n}f = \theta^{-1}({}^{n}h)$ for ${}^{n}h \in {}^{n}\sigma$.

Assume ${}^{n}f \in {}^{n}\delta$. Then ${}^{n}f = {}^{n}g$ for some $g \in \delta$.

But, by hypothesis, $g \in \delta \Rightarrow g = \theta^{-1}(h)$ for some $h \in \sigma$.

$\therefore\ {}^{n}f = {}^{n}g = {}^{n}(\theta^{-1}(h)) = \theta^{-1}({}^{n}h)$ for ${}^{n}h \in {}^{n}\sigma$.

Conversely, let ${}^{n}f = \theta^{-1}({}^{n}h)$ for ${}^{n}h \in {}^{n}\sigma$.

Without loss of generality we can assume that $h \in \sigma$.

Therefore, $\theta^{-1}(h) \in \delta$.

Hence, ${}^{n}(\theta^{-1}(h)) = \theta^{-1}({}^{n}h) \in {}^{n}\delta$.

Theorem : 1.4.11

If $\theta : (X, {}^{n}\delta) \to (Y, {}^{n}\sigma)$ is fuzzy continuous for every n, then $\theta : (X, \delta^{\Diamond}) \to (Y, \sigma^{\Diamond})$ is fuzzy continuous

Proof :

Let $\theta : (X, {}^{n}\delta) \to (Y, {}^{n}\sigma)$ be fuzzy continuous for all n .

$f \in \sigma^{\Diamond} \Rightarrow {}^{n}f \in {}^{n}\sigma$ for all n

$\Rightarrow \theta^{-1}({}^{n}f) \in {}^{n}\delta$ for all n

$\Rightarrow {}^{n}(\theta^{-1}(f)) \in {}^{n}\delta$ for all n

$\Rightarrow (\theta^{-1}(f)) \in \delta^{\Diamond}$

Hence, $\theta : (X, \delta^{\Diamond}) \to (Y, \sigma^{\Diamond})$ is fuzzy continuous.

Corollary : 1.4.12

If $\theta : (X,\delta) \to (Y,\sigma)$ is fuzzy continuous , then $\theta : (X,\delta^{\Diamond}) \to (Y,\sigma^{\Diamond})$ is fuzzy continuous .

Proof :

Proof follows by Theorem 1.4.8 and Theorem 1.4.11.

Section : 1.5

Two Interesting Fuzzy Topologies Using Finite Approximations

Definition : 1.5.1

Let (X,δ) be a fuzzy topological space. Then $\boxed{\delta(n)}$ is defined as follows:

$\boxed{\delta(n)} = \{ f \mid {}^{n}f \in \delta \}$.

Theorem : 1.5.2

Let (X,δ) be a fuzzy topological spaces. Then $\boxed{\delta(n)}$ is a fuzzy topology on X.

Proof :

FT1. Obviously, 0 and 1 belong to $\boxed{\delta(n)}$.

FT2. Let $f_j \in \boxed{\delta(n)}$ for $j = 1,2$.

Then ${}^{n}(f_j) \in \delta$ for $j = 1,2$.

$\therefore\ {}^{n}(f_1 \wedge f_2) = {}^{n}(f_1) \wedge {}^{n}(f_2) \in \delta$.

Hence $f_1 \wedge f_2 \in \boxed{\delta(n)}$

FT3. Let $f_\lambda \in \boxed{\delta(n)}$ for each $\lambda \in \Lambda$.

Then ${}^{n}(f_\lambda) \in \delta$ for each $\lambda \in \Lambda$.

$\therefore {}^{n}(\sup\{ f_\lambda \mid \lambda \in \Lambda \}) = \sup\{{}^{n}(f_\lambda) \mid \lambda \in \Lambda\} \in \delta$.

Hence, $\sup\{ f_\lambda \mid \lambda \in \Lambda \} \in \boxed{\delta(n)}$.

Remark : 1.5.3

(i). ${}^{n}(\boxed{\delta(n)}) = \delta \wedge {}^{n}\delta$.

(ii). ${}^{n}\delta \subseteq \delta \Rightarrow \delta \subseteq \boxed{\delta(n)}$.

(iii). ${}^{n}\delta \subseteq \delta \Rightarrow {}^{n}\delta = {}^{n}(\boxed{\delta(n)})$.

Theorem : 1.5.4

If $\theta : (X,\delta) \to (Y,\sigma)$ is fuzzy continuous, then $\theta : (X, \boxed{\delta(n)}) \to (Y, \boxed{\sigma(n)})$ is fuzzy continuous.

Proof :

Let $\theta : (X,\delta) \to (Y,\sigma)$ be a fuzzy continuous map.

$$f \in \boxed{\sigma(n)} \Rightarrow {}^{n}f \in \sigma \Rightarrow \theta^{-1}({}^{n}f) \in \delta$$

$$\Rightarrow {}^{n}(\theta^{-1}(f)) \in \delta$$

$$\Rightarrow \theta^{-1}(f) \in \boxed{\delta(n)}$$

Definition : 1.5.5

Let (X,δ) be a fuzzy topological space. Then $\textcircled{\delta(n)}$ is defined as follows :

$\textcircled{\delta(n)} = \{ f \mid {}^{n}f \in {}^{n}\delta \}$.

Theorem : 1.5.6

Let δ be a fuzzy topology defined on X. Then $\textcircled{\delta(n)}$ is the largest fuzzy topology on X containing δ such that ${}^{n}\delta = {}^{n}(\textcircled{\delta(n)})$.

Proof :

FT1. Obviously, 0 and 1 belong to $\textcircled{\delta(n)}$.

FT2. $f_i \in \textcircled{\delta(n)}$ for $i = 1,2$

$\Rightarrow {}^{n}(f_i) \in {}^{n}\delta$ for $i = 1,2$.

$\therefore {}^{n}(f_1 \wedge f_2) = {}^{n}(f_1) \wedge {}^{n}(f_2) \in {}^{n}\delta$

$\Rightarrow f_1 \wedge f_2 \in \textcircled{\delta(n)}$

FT3. $f_\lambda \in \textcircled{\delta(n)}$ for each $\lambda \in \Lambda$.

$\Rightarrow {}^{n}(f_\lambda) \in {}^{n}\delta$ for each $\lambda \in \Lambda$.

$\therefore {}^{n}(\sup \{ f_\lambda \mid \lambda \in \Lambda \}) = \sup \{ {}^{n}(f_\lambda) \mid \lambda \in \Lambda \} \in {}^{n}\delta$

$\Rightarrow \sup \{ f_\lambda \mid \lambda \in \Lambda \} \in \textcircled{\delta(n)}$

Hence, $\textcircled{\delta(n)}$ is a fuzzy topology on X.

Obviously, $\delta \subseteq \textcircled{\delta(n)}$.

Suppose $\delta \subseteq \sigma$ and ${}^{n}\delta = {}^{n}\sigma$.

Then, $\sigma \subseteq \textcircled{\sigma(n)} = \textcircled{\delta(n)}$.

$\therefore$ $\textcircled{\delta(n)}$ is the largest fuzzy topology containing δ such that ${}^{n}\delta = {}^{n}(\textcircled{\delta(n)})$.

Theorem : 1.5.7

If $\theta : (X,\delta) \to (Y,\sigma)$ is fuzzy continuous, then $\theta : (X,\textcircled{\delta(n)}) \to (Y,\textcircled{\sigma(n)})$ is fuzzy continuous.

Proof :

Let $\theta : (X, \delta) \to (Y, \sigma)$ be a fuzzy continuous map. Then, by Theorem 1.4.8 , $\theta : (X, {}^{n}\delta) \to (Y, {}^{n}\sigma)$ is fuzzy continuous .

$$
\begin{aligned}
f \in \textcircled{\delta(n)} &\Rightarrow {}^{n}f \in {}^{n}\sigma \\
&\Rightarrow \theta^{-1}({}^{n}f) \in {}^{n}\delta \\
&\Rightarrow {}^{n}(\theta^{-1}(f)) \in {}^{n}\delta \\
&\Rightarrow \theta^{-1}(f) \in \textcircled{\delta(n)}
\end{aligned}
$$

Hence, $\theta : (X, \textcircled{\delta(n)}) \to (Y, \textcircled{\delta(n)})$ is fuzzy continuous.

CHAPTER II

FUZZY TOPOLOGICAL CONCEPTS AND FINITE APPROXIMATIONS

CHAPTER II

FUZZY TOPOLOGICAL CONCEPTS AND FINITE APPROXIMATIONS

"A fuzzy topological space (X,δ) is Lowen fuzzy compact iff $(X, \delta^{\Diamond})$ is Lowen fuzzy compact". This is an important result on compactness obtained in the first section of this chapter.

In section 2, different types of fuzzy separation axioms are studied. The results obtained are of the following types :

(i). (X,δ) has the property **P** $\Rightarrow$ $(X,{}^{n}\delta)$ has the property **P**.
(ii). $(X,{}^{n}\delta)$ has the property **P** for every n $\Rightarrow$ (X,δ) has the property **P**.
(iii). (X,δ) has the property **P** $\Rightarrow$ $(X,\delta^{\Diamond})$ has the property **P**,

where **P** stands for suitable separation axioms.

In section 3, with every fuzzy convering uniformity Γ, a fuzzy covering uniformity ${}^{n}\Gamma$ is constructed such that ${}^{n}\Gamma$ has a basis consisting of finite valued fuzzy coverings. Further ${}^{n}\Gamma$ and Γ induce the same fuzzy topology on the underlying set.

Section : 2.1

Fuzzy Compactness and Finite Approximations

Definition [49] : 2.1.1

Let (X, δ) be a fuzzy topological space. (X, δ) is **Chang fuzzy compact** if for all family $\eta \subseteq \delta$ such that $\sup\{ f \mid f \in \eta \} = 1$, there exists a finite sub family $\eta_0 \subseteq \eta$ such that $\sup \{ f \mid f \in \eta_0 \} = 1$.

Definition [47] : 2.1.2

Let (X, δ) be a fuzzy topological space. A **fuzzy set** f is **Lowen fuzzy compact** if given a family $\eta \subseteq \delta$ such that $\sup \{ g \mid g \in \eta \} \geq f$ and given $\varepsilon > 0$, there exists a finite sub family $\eta_0 \subseteq \eta$ such that $\sup \{ g \mid g \in \eta_0 \} \geq f - \varepsilon$.

Definition : 2.1.3

A fuzzy topological space **(X, δ) is Lowen fuzzy compact** if each constant fuzzy set in (X, δ) is Lowen fuzzy compact.

Definition [49] : 2.1.4

A fuzzy topological space (X,δ) is called **ultra fuzzy compact** if (X, i(δ)) is compact.

Theorem : 2.1.5

Let (X, δ) be a fuzzy topological space. If ${}^{n}\delta$ is Lowen fuzzy compact for every n, then δ is Lowen fuzzy compact.

Proof :

Let (X, δ) be a fuzzy topological space and η be a subfamily of δ. Assume that , $\sup \{ f \mid f \in \eta \subseteq \delta \} \geq \alpha$, where α is a constant.

To prove :

Given $\varepsilon > 0$, there exists a finite sub family η_0 of η such that

$\sup \{ f \mid f \in \eta_0 \subseteq \eta \} \geq \alpha - \varepsilon$.

Let $\varepsilon' = \frac{2}{3} \varepsilon$. Now, choose N such that ${}^{N}f(x) \leq f(x) + \frac{1}{2} \varepsilon'$.

By hypothesis, ${}^{N}\delta$ is Lowen fuzzy compact. Therefore, there exists a finite subfamily $\eta_0 \subseteq \eta$ such that $\sup \{ {}^{N}f \mid f \in \eta_0 \} \geq \alpha - \varepsilon'$.

$$\begin{aligned} \text{Hence, } \sup \{ f \mid f \in \eta_0 \} &\geq \sup \{ {}^{N}f \mid f \in \eta_0 \} - \frac{1}{2} \varepsilon'. \\ &\geq \alpha - \varepsilon' - \frac{1}{2}\varepsilon'. \\ &= \alpha - \varepsilon. \end{aligned}$$

Theorem : 2.1.6

A fuzzy topological space (X, δ) is Lowen fuzzy compact iff (X , $\delta^{\Diamond}$) is Lowen fuzzy compact.

Proof :

Since $\delta \subseteq \delta^{\Diamond}$,

(X, $\delta^{\Diamond}$) is Lowen fuzzy compact $\Rightarrow$ (X , δ) is Lowen fuzzy compact.

Conversely, assume that δ is Lowen fuzzy compact .

To prove :

$\delta^{\Diamond}$ is Lowen fuzzy compact .

Consider a family $\{ f_\lambda \mid \lambda \in \Lambda \} \subseteq \delta^{\Diamond}$ such that $\sup\{ f_\lambda \mid \lambda \in \Lambda \} \geq \alpha$ ......(1)

Given $\varepsilon > 0$, choose N such that ${}^{N}g \leq g + \frac{1}{3}\varepsilon$.

Given $f_\lambda \in \delta^{\Diamond}$, choose $g_\lambda \in \delta$ such that ${}^{N}(f_\lambda) = {}^{N}(g_\lambda)$.

(1) implies that $\sup \{ {}^{N}(f_\lambda) \mid \lambda \in \Lambda \} \geq \alpha$.

$\therefore \sup \{ {}^{N}(g_\lambda) \mid \lambda \in \Lambda \} \geq \alpha$.

$\sup \{ g_\lambda \mid \lambda \in \Lambda \} \geq \sup \{ {}^{N}(g_\lambda) \mid \lambda \in \Lambda \} - \frac{1}{3}\varepsilon \geq \alpha - \frac{1}{3}\varepsilon$.

Since δ is Lowen fuzzy compact, there exists a finite $\Lambda_0 \subseteq \Lambda$ such that

$\sup \{ g_\lambda \mid \lambda \in \Lambda_0 \} \geq \alpha - \frac{2}{3}\varepsilon$.

$$\begin{aligned}
\sup \{ f_\lambda \mid \lambda \in \Lambda_0 \} &\geq \sup \{ {}^{N}(f_\lambda) \mid \lambda \in \Lambda_0 \} - \frac{1}{3}\varepsilon \\
&= \sup \{ {}^{N}(g_\lambda) \mid \lambda \in \Lambda_0 \} - \frac{1}{3}\varepsilon \\
&\geq \sup \{ g_\lambda \mid \lambda \in \Lambda_0 \} - \frac{1}{3}\varepsilon \\
&\geq \alpha - \frac{2}{3}\varepsilon - \frac{1}{3}\varepsilon \geq \alpha - \varepsilon .
\end{aligned}$$

Theorem : 2.1.7

A fuzzy topological space (X, δ) is ultra fuzzy compact

$\Rightarrow (X, {}^{n}\delta)$ is ultra fuzzy compact.

Proof :

A fuzzy topological space (X, δ) is ultra fuzzy compact

$\Rightarrow (X, i(\delta))$ is compact

$\Rightarrow$ (X, i (${}^{n}\delta$)) is compact

$\Rightarrow$ (X, ${}^{n}\delta$) is ultra fuzzy compact.

Section : 2.2

Fuzzy Separation Axioms and Finite Approximations

Definition [24] : 2.2.1

A fuzzy topological space (X, δ) is said to be **fuzzy W-Hausdorff** (fuzzy W-H) if $\forall$ x, y $\in$ X, x $\neq$ y, there exist f, g $\in \delta$ such that $f(x) = 1$, $g(y) = 1$ and $f \wedge g = 0$.

Theorem : 2.2.2

Let (X, δ) be a fuzzy topological space. Then,

(i). (X, δ) fuzzy W-H $\Rightarrow$ (X, ${}^{n}\delta$) fuzzy W-H for every n.

(ii). (X, ${}^{kn}\delta$) fuzzy W-H $\Rightarrow$ (X, ${}^{n}\delta$) fuzzy W-H.

(iii). (X, δ) fuzzy W-H $\Rightarrow$ (X, $\delta^{\lozenge}$) fuzzy W-H.

Proof :

To prove (i) it is enough to observe that $f \wedge g = 0 \Leftrightarrow {}^{n}f \wedge {}^{n}g = 0$.

To prove (ii) it is enough to note that ${}^{kn}f \leq {}^{n}f$ and ${}^{kn}f \wedge {}^{kn}g = 0$

$\Leftrightarrow f \wedge g = 0 \Leftrightarrow {}^{n}f \wedge {}^{n}g = 0$.

Since $\delta \subseteq \delta^{\lozenge}$, (iii) is obvious.

Corollary : 2.2.3

A topological space (X,τ) is Hausdorff $\Rightarrow$ (X, ${}^{n}(w(\tau))$) is fuzzy W-H.

Proof :

This follows readily from the following result of Srivastava, A.K. and Ali, D.M. [83] :

(X, τ) Hausdorff $\Rightarrow$ $(X, w(\tau))$ fuzzy W-H.

Example: 2.2.4

In this example (X, δ) is not fuzzy W-H, but $(X, {}^{n}\delta)$ is fuzzy W-H. for $n = 1,2,3$.

Let $X = \{ x, y\}$ be a two element set. Define the fuzzy sets f and g on X by $f(x) = \frac{3}{4}$, $f(y) = 0$, $g(y) = \frac{3}{4}$, $g(x) = 0$.

Let $\delta = \{ 0, 1, f, g, f \vee g\}$.

Obviously, δ is not fuzzy W-H but ${}^{n}\delta$ is fuzzy W-H for $n = 1, 2, 3$.

Definition [86] : 2.2.5

A fuzzy topological space (X, δ) is said to be **fuzzy S-Hausdorff** (fuzzy S-H) if for any pair of distinct fuzzy points x_t, y_s in X, there exist $f, g \in \delta$ such that $x_t \in f$, $y_s \in g$ and $f \wedge g = 0$.

Theorem : 2.2.6

Let (X, δ) be a fuzzy topological space . Then

(i). (X, δ) fuzzy S-H $\Rightarrow$ $(X, {}^{n}\delta)$ fuzzy S-H for every n.

(ii). $(X, {}^{n}\delta)$ fuzzy S-H for every n $\Rightarrow$ (X, δ) fuzzy S-H .

(iii). (X, δ) fuzzy S-H $\Leftrightarrow$ (X, δ^{0}) fuzzy S-H.

Proof :

(i). Proof is obvious from the definition of ${}^{n}\delta$.

(ii). Let the fuzzy topological space (X ,$^n\delta$) be fuzzy S-H for every n .

Let x_t and y_s be two fuzzy points in X such that $x_t \neq y_s$.

Choose 'n' such that $^nt \neq 1$, $^ns \neq 1$. Now $^n(x_t) \neq {}^n(y_s)$.

Since (X, $^n\delta$) is fuzzy S-H, there exist ^nf and ^ng in $^n\delta$ with $^nf \wedge {}^ng = 0$ such that $^n(x_t) \in {}^nf$ and $^n(y_s) \in {}^ng$.

$\therefore {}^nf(x) > {}^nt$ and $^ng(y) > {}^ns$.

$\Rightarrow f(x) > t$, $g(y) > s$.

Without loss of generality , we can assume that $f, g \in \delta$.

$\therefore$ (X, δ) is fuzzy S-H.

(iii). (X, $\delta^\Diamond$) fuzzy S-H

$\Rightarrow$ (X, $^n(\delta^\Diamond)$) fuzzy S-H for every n

$\Rightarrow$ (X, $^n\delta$) fuzzy S-H for every n

$\Rightarrow$ (X, δ) fuzzy S-H.

Corollary : 2.2.7

A topological space (X,τ) is Hausdorff $\Leftrightarrow$ (X, $^n(w(\tau))$) is fuzzy S-H.

Proof :

(X,τ) is Hausdorff $\Leftrightarrow$ (X, w(τ)) is S-H by a result of Srivastava, R., Lal, S.N. and Srivastava, A.K.[86].

$\therefore$ (X,τ) is Hausdorff $\Leftrightarrow$ (X, $^n(w(\tau))$) is S-H.

Corollary : 2.2.8

Let (X, δ) be fuzzy S-H. Then (X, i($^n\delta$)) is Hausdorff.

Proof :

(X, δ) fuzzy S - H $\Rightarrow (X, i(\delta))$ Hausdorff [86].

$\therefore$ (X, δ) fuzzy S - H $\Rightarrow (X, {}^n\delta)$ fuzzy S-H

$\Rightarrow (X, i({}^n\delta))$ Hausdorff.

Definition [35] : 2.2.9

A fuzzy topological space (X, δ) is said to be **fuzzy K-Hausdorff** (fuzzy K-H) if for all $x, y \in X$, $x \neq y$, there exist $f, g \in \delta$ such that $f(x) > 0$, $g(y) > 0$ and $f \wedge g = 0$.

Theorem : 2.2.10

Let (X, δ) be a fuzzy topological space. Then

(i). (X, δ) fuzzy K-H $\Rightarrow (X, {}^n\delta)$ fuzzy K-H for every n.

(ii). $(X, {}^n\delta)$ fuzzy K-H for some n $\Rightarrow (X, \delta)$ fuzzy K-H.

(iii). For given m,n positive integers, $(X, {}^m\delta)$ fuzzy K-H $\Rightarrow (X, {}^n\delta)$ fuzzy K-H.

(iv). (X, δ) fuzzy K-H $\Leftrightarrow (X, \delta^0)$ fuzzy K-H.

Proof :

Similar to the proof of Theorem 2.2.2

Definition[35] : 2.2.11

A fuzzy topological space (X, δ) is called a **fuzzy K-T_1 space** if for any two distinct points $x, y \in X$, there exist $f, g \in \delta$ such that $f(x) > 0$, $g(y) > 0$, $f(y) = g(x) = 0$.

Theorem : 2.2.12

Let (X, δ) be a fuzzy topological space. Then

(i). (X, δ) fuzzy K-$T_1 \Rightarrow (X, {}^n\delta)$ fuzzy K-T_1 for every n.

(ii). $(X, {}^n\delta)$ fuzzy K-T_1 for some n $\Rightarrow (X, \delta)$ fuzzy K-T_1

(iii). (X, δ) fuzzy K-$T_1 \Leftrightarrow (X, \delta^\Diamond)$ fuzzy K-T_1

(iv). For any two positive integers m and n , $(X, {}^m\delta)$ fuzzy K-T_1

$\Leftrightarrow (X, {}^n\delta)$ fuzzy K-T_1.

Proof :

Obvious.

Definition [88] : 2.2.13

A fuzzy topological space (X, δ) is called a **fuzzy SS-T_1 space** if $\forall$ x, y $\in$ X, x $\neq$ y, there exist f, $g \in \delta$ with $f(x) = 1$, $f(y) = 0$ and $g(y) = 1$, $g(x) = 0$.

Theorem : 2.2.14

Let (X, δ) be a fuzzy topological space. Then

(i). (X, δ) fuzzy SS-T_1, $\Rightarrow (X, {}^n\delta)$ fuzzy SS-T_1 for every n .

(ii). $(X, {}^{kn}\delta)$ fuzzy SS-$T_1 \Rightarrow (X, {}^n\delta)$ fuzzy SS-T_1 .

(iii). (X, δ) fuzzy SS-$T_1 \Rightarrow (X, \delta^\Diamond)$ fuzzy SS-T_1 .

(iv). $(X, {}^n\delta)$ fuzzy SS-T_1 for some n $\Rightarrow (X, i(\delta))$ is T_1 .

Proof :

Proof of (i), (ii) , (iii) follow immediately by the definition of ${}^n\delta$.

(iv). $(X, {}^n\delta)$ fuzzy SS-$T_1 \Rightarrow (X, i({}^n\delta))$ is T_1 by a result of Srivastava, R., Lal, S.N and Srivastava, A.K.[89].

$\therefore (X, {}^n\delta)$ fuzzy SS-$T_1 \Rightarrow (X, i(\delta))$ is T_1 , since $i({}^n\delta) \subseteq i(\delta)$.

Definition [19]: 2.2.15

A fuzzy topological space (X,δ) is said to be a **fuzzy F-T_1 space** if for any two distinct fuzzy points x_t, y_s in X, there exist $f, g \in \delta$ such that $x_t \in f$, $f(y) = 0$, $y_s \in g$, $g(x) = 0$.

Theorem : 2.2.16

Let (X, δ) be a fuzzy topological space. Then,

(i). (X, δ) is fuzzy F-$T_1 \Leftrightarrow (X, {}^n\delta)$ is fuzzy F-T_1 for every n .

(ii). (X, δ) is fuzzy F-$T_1 \Leftrightarrow (X, \delta^\Diamond)$ is fuzzy F-T_1

Proof :

Similar to the proof of Theorem 2.2.6

Definition [88] : 2.2.17

A fuzzy topological space (X, δ) is said to be a **fuzzy S-T_0 space** if $\forall$ $x, y \in X$, $x \neq y$, there exists $f \in \delta$ such that either $f(x) = 1$ and $f(y) = 0$ or $f(y) = 1$ and $f(x) = 0$.

Theorem:2.2.18

Let (X, δ) be a fuzzy topological space . Then,

(i). (X, δ) fuzzy S-T_0 space $\Rightarrow (X, {}^n\delta)$ fuzzy S-T_0 space for every n.

(ii). $(X, {}^{kn}\delta)$ fuzzy S-T_0 space $\Rightarrow$ $(X, {}^{n}\delta)$ fuzzy S-T_0 space .

(iii). (X, δ) fuzzy S-T_0 space $\Rightarrow$ $(X, \delta^{\Diamond})$ fuzzy S-T_0 space .

Proof :

By the definition of ${}^{n}\delta$, the proof is obvious .

Definition[19] : 2.2.19

A fuzzy topological space (X, δ) is said to be a **fuzzy F- T_0 space** if for any two distinct fuzzy points x_t, y_s in X, there exists an open fuzzy set f in δ such that $x_t \in f$ and $f(y) = 0$ or $y_s \in f$ and $f(x) = 0$.

Theorem : 2.2.20

Let (X, δ) be a fuzzy topological space. Then

(i). (X, δ) fuzzy F-T_0 space $\Leftrightarrow$ $(X, {}^{n}\delta)$ fuzzy F- T_0 space for every n .

(ii). (X, δ) fuzzy F-T_0 space $\Leftrightarrow$ $(X, \delta^{\Diamond})$ fuzzy F-T_0 space

Proof :

The proof is similar to that of Theorem 2.2.16.

Section : 2.3

Fuzzy Covering Uniformity and Finite Approximations

In this section we shall study the covering uniformity defined by Chandrika and Meenakshi [9] with reference to finite approximations.

Definition : 2.3.1

A collection $\mathcal{U} = \{ f_\lambda \mid \lambda \in \Lambda \}$ of fuzzy sets in a set X is called a **covering** if $\sup \{ f_\lambda \mid \lambda \in \Lambda \} = 1$

Note : 2.3.2

If $\mathcal{U} = \{ f_\lambda \mid \lambda \in \Lambda \}$ and $\mathcal{V} = \{ g_\alpha \mid \alpha \in \Lambda' \}$ are two coverings of a set X, then, $\mathcal{U} \wedge \mathcal{V} = \{ f_\lambda \wedge g_\alpha \mid \lambda \in \Lambda, \alpha \in \Lambda' \}$ is a covering of X.

Definition : 2.3.3

If $\mathcal{U}$ and $\mathcal{V}$ are coverings of a set X, then $\mathcal{U}$ is a **refinement** of $\mathcal{V}$ written as $\mathcal{U} < \mathcal{V}$, if for every $f \in \mathcal{U}$ there is a $g \in \mathcal{V}$ such that $f(x) \leq g(x)$ for every $x \in X$.

Definition : 2.3.4

A covering $\mathcal{U}$ of a set X, **separates** two points x and y in X if $f(x) \wedge f(y) = 0$ for every $f \in \mathcal{U}$. .

Notation :

$< \mathcal{U} > (x) = \{y \mid x \text{ and } y \text{ are not separated by } \mathcal{U}\}$.

Definition : 2.3.5

If $\mathcal{U}$ is a covering of X and f any fuzzy set in X, then **star of f with respect to $\mathcal{U}$**, denoted by $\mathrm{St}(f, \mathcal{U})$ is the fuzzy set in X, defined by

$\mathrm{St}(f, \mathcal{U})(x) = \sup \{ f(y) \mid y \in < \mathcal{U} > (x) \}$.

Definition : 2.3.6

A covering $\mathcal{U}$ is a **star refinement** of a covering $\mathcal{V}$ of X if the following conditions are satisfied :

ST1. For every $f \in \mathcal{U}$, there is a $g \in \mathcal{V}$, such that $\mathrm{St}(f, \mathcal{U}) \leq g$.

ST2. If x and y are not separated by $\mathcal{U}$ and y and z are not separated by $\mathcal{U}$ then x and z are not separated by $\mathcal{V}$.

Definition : 2.3.7

Let X be a non empty set. A family Γ of coverings of X is called a **fuzzy uniformity** on X if the follwing conditions are satisfied :

U1. For $\mathcal{U}, \mathcal{V}$ in Γ, $\mathcal{U} \wedge \mathcal{V}$ is in Γ.

U2. If $\mathcal{U} < \mathcal{V}$ and if $\mathcal{U} \in \Gamma$, then $\mathcal{V} \in \Gamma$.

U3. Every member of Γ has a star refinement in Γ.

A fuzzy uniformity Γ is called a **Hausdorff fuzzy uniformity** if the following condition is satisfied :

U4. For every pair $x, y \in X$ with $x \neq y$, there is a covering in Γ which separates x and y.

Definition : 2.3.8

Let X be a non empty set. A collection $\mathcal{B}$ of coverings is called a **basis for a fuzzy uniformity** if the following conditions are satisfied :

UB1. For $\mathcal{U}, \mathcal{V}$ in $\mathcal{B}$, $\mathcal{U} \wedge \mathcal{V}$ is refined by a member of $\mathcal{B}$.

UB2. Every $\mathcal{U}$ in $\mathcal{B}$ has a star refinement $\mathcal{V}$ in $\mathcal{B}$.

$\mathscr{B}$ is said to form a **basis for a Hausdorff fuzzy uniformity**, if in addition it satisfies the following condition :

UB3. Given x, y in X with $x \neq y$, there is a covering $\mathcal{U}$ in $\mathscr{B}$ which separates x and y.

Definition : 2.3.9

Let (X, Γ) be a fuzzy uniform space. The **fuzzy topology induced by** Γ is defined as follows :

For any f in I^X, define Cl f in $\Gamma = \inf \{ \mathrm{St}(f, \mathcal{U}) \mid \mathcal{U} \in \Gamma \}$.

Theorem 2.3.10 follows immediately from the definitions:

Theorem : 2.3.10

(i). If $\mathcal{U} = \{ f_\lambda \in \Lambda \}$ is a covering of X ,
then ${}^n\mathcal{U} = \{ {}^n(f_\lambda) \mid \lambda \in \Lambda \}$ is a covering of X.

(ii). Let $\mathcal{U}$ and $\mathcal{V}$ be coverings of a set X
If $\mathcal{U}$ is a refinement of $\mathcal{V}$, then ${}^n\mathcal{U}$ is a refinement of ${}^n\mathcal{V}$.

(iii). A covering $\mathcal{U}$ of X separates x and y in X iff the covering ${}^n\mathcal{U}$ of X separates x and y in X .

Theorem : 2.3.11.

Let $\mathcal{U}$ be a convering of X. Then $\mathrm{St}(f, \mathcal{U}) = \mathrm{St}(f, {}^n\mathcal{U})$.

Proof :

$$\begin{aligned} \mathrm{St}(f, \mathcal{U})(x) &= \sup \{ f(y) \mid y \in \langle \mathcal{U} \rangle (x) \} \\ &= \sup \{ f(y) \mid y \in \langle {}^n\mathcal{U} \rangle (x) \} \\ &= \mathrm{St}(f, {}^n\mathcal{U}). \end{aligned}$$

Theorem : 2.3.12.

If a covering $\mathcal{U}$ of X is a star refinement of a covering $\mathcal{V}$ of X, then ${}^n\mathcal{U}$ is a star refinement of ${}^n\mathcal{V}$.

Proof :

$\mathcal{U}$ is a star refinement of $\mathcal{V}$ of X

$\Rightarrow$ for every $f \in \mathcal{U}$, there is a $g \in \mathcal{V}$ such that $\mathrm{St}(f, \mathcal{U}) \le g$.

$$\begin{aligned}
\mathrm{St}({}^nf, {}^n\mathcal{U})(x) &= \mathrm{St}({}^nf, \mathcal{U})(x) \\
&= \sup\{{}^nf(y) \mid y \in \langle \mathcal{U} \rangle (x)\} \\
&= {}^n(\sup\{f(y) \mid y \in \langle \mathcal{U} \rangle (x)\}) \\
&= {}^n(\mathrm{St}(f, \mathcal{U})(x)) \\
&\le {}^n(g(x)) \\
&= ({}^ng)(x).
\end{aligned}$$

Second condition of the definition of star refinement follows by observing that for every $x \in X$, $\langle \mathcal{U} \rangle (x) = \langle {}^n\mathcal{U} \rangle (x)$.

Theorem : 2.3.13

Let Γ be a fuzzy uniformity. Then $\{{}^n\mathcal{U} \mid \mathcal{U} \in \Gamma\}$ is a basis for a fuzzy uniformity ${}^n\Gamma$ such that ${}^n\Gamma \subseteq \Gamma$. If Γ is a Hausdorff fuzzy uniformity, then ${}^n\Gamma$ is also a Hausdorff fuzzy uniformity.

Proof :

${}^n\Gamma \subseteq \Gamma$, since $\mathcal{U} \le {}^n\mathcal{U}$.

UB1. Let $\mathcal{U}$, $\mathcal{V} \in \Gamma$. Then $\mathcal{U} \wedge \mathcal{V} \in \Gamma$.

${}^n\mathcal{U} \wedge {}^n\mathcal{V} = {}^n(\mathcal{U} \wedge \mathcal{V}) \in {}^n\Gamma$.

UB2. If $\mathcal{V} \in \Gamma$ is a star refinement of $\mathcal{U}$ then ${}^{n}\mathcal{V}$ is a star refinement of ${}^{n}\mathcal{U}$.

The collection $\{ {}^{n}\mathcal{U} \mid \mathcal{U} \in \Gamma \}$ is a basis for a fuzzy uniformity ${}^{n}\Gamma$.

Given $x \neq y$, there exists $\mathcal{U} \in \Gamma$ such that $\mathcal{U}$ separates x and y

$\Rightarrow {}^{n}\mathcal{U} \in {}^{n}\Gamma$ separates x and y.

Theorem : 2.3.14

Let (X, Γ) be a fuzzy uniform space.

Then Γ and ${}^{n}\Gamma$ define the same fuzzy topology on X.

Proof :

It is enough to prove that ,

Cl f with respect to Γ = Cl f with respect to ${}^{n}\Gamma$.

For any fuzzy set $f \in I^{X}$,

$$\begin{aligned} \mathrm{Cl}\, f \text{ in } \Gamma &= \inf \{ \mathrm{St}(f, \mathcal{U}) \mid \mathcal{U} \in \Gamma \} \\ &= \inf \{ \mathrm{St}(f, {}^{n}\mathcal{U}) \mid {}^{n}\mathcal{U} \in {}^{n}\Gamma \} \\ &= \mathrm{Cl}\, f \text{ in } {}^{n}\Gamma. \end{aligned}$$

CHAPTER III

SOME INTERESTING SUBCATEGORIES OF THE CATEGORY OF FUZZY TOPOLOGICAL SPACES

CHAPTER III

SOME INTERESTING SUB CATEGORIES OF THE CATEGORY OF FUZZY TOPOLOGICAL SPACES

In this chapter, some interesting subcategories of the category of fuzzy topological spaces are studied. This includes the subcategory of fuzzy topological spaces of the form $(X,{}^n\delta)$, (X, δ^0) and (X,δ) where δ is topologically generated. Few interesting functorial properties are established. Generalizing a result of Srivastava[81], Sierpinski spaces for two subcategories are constructed .

Section : 3.1

Definitions and Notations

Definition : 3.1.1

A category $\mathscr{C}$ consists of two classes, a class $\mathbf{o}\mathscr{C}$ called the objects of $\mathscr{C}$ and a class $\mathbf{m}\mathscr{C}$ called the morphisms of $\mathscr{C}$, together with the following axioms which link the two classes :

(i). The composition $\theta \circ \psi \circ \varphi$ of three morphisms is defined whenever the compositions $\theta \circ \psi$ and $\psi \circ \varphi$ are defined.

(ii). Composition of morphisms is associative. That is, $(\theta \circ \psi) \circ \varphi = \theta \circ (\psi \circ \varphi)$ and both compositions are defined if either is defined.

(iii). There is a bijection which assigns to each object X an identity morphism $(\mathbf{id})_X : X \to X$ such that for each morphism $\theta : X \to Y$, $\theta = \theta \circ (\mathbf{id})_X = (\mathbf{id})_Y \circ \theta$.

Definition : 3.1.2

A category $\mathscr{R}$ is a subcategory of a category $\mathscr{C}$ if every object (morphism) of $\mathscr{R}$ is also an object (morphism) of $\mathscr{C}$. The subcategory $\mathscr{R}$ is said to be a full sub category if every morphism in $\mathscr{C}$ between two objects of $\mathscr{R}$ is also a morphism in $\mathscr{R}$.

Definition : 3.1.3

A functor from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a rule ζ which assigns to each object X and morphism θ of $\mathscr{C}$ an object $\zeta(X)$ and morphism $\zeta(\theta)$ of $\mathscr{D}$ such that

(i). ζ preserves identities, That is, $\zeta(\mathbf{id}_X) = \mathbf{id}_{\zeta(X)}$.

(ii). ζ preserves composition. That is, if $\theta \circ \psi$ is defined in $\mathscr{C}$, then $\zeta(\theta) \circ \zeta(\psi)$ is defined in $\mathscr{D}$ and is equal to $\zeta(\theta \circ \psi)$.

Definition : 3.1.4.

A functor ζ from a category $\mathscr{C}$ to a subcategory $\mathscr{R}$ of $\mathscr{C}$ is a reflective functor if there is a morphism $\theta : X \to \zeta(X)$ and every morphism ψ from X to an object Y of $\mathscr{R}$ factors uniquely through $\zeta(X)$ via θ so that the following diagram commutes. That is, given a morphism $\psi : X \to Y$, there exists a unique morphism $\varphi : \zeta(X) \to Y$ such that $\psi = \varphi \circ \theta$.

$$\begin{tikzcd}
 & \zeta(X) \arrow[dr, "\varphi"] & \\
X \arrow[ur, "\theta"] \arrow[rr, "\psi"'] & & Y
\end{tikzcd}$$

If $\zeta : \mathscr{C} \to \mathscr{R}$ is a reflective functor, the sub category $\mathscr{R}$ is called a reflective subcategory. The object $\zeta(X)$ is called the reflection of X in $\mathscr{R}$.

Definition : 3.1.5

A functor $\zeta : \mathscr{C} \to \mathscr{R}$ is called a coreflective functor if $\mathscr{R}$ is a subcategory of $\mathscr{C}$ and there is a morphism $\theta : \zeta(X) \to X$ such that any morphism $\psi : Y \to X$ with Y an object of $\mathscr{R}$ factors uniquely through θ such that the diagram commutes. That is, given a morphism $\psi : Y \to X$ there exists a unique map $\varphi : Y \to \zeta(X)$ such that $\psi = \theta \circ \varphi$.

$$\begin{tikzcd}
\zeta(X) \arrow[rr, "\theta"] & & X \\
& Y \arrow[ul, "\phi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

$\mathscr{R}$ is called a coreflective subcategory of $\mathscr{C}$ and $\zeta(X)$ is called the coreflection of X in $\mathscr{R}$.

Definition [47] : 3.1.6

Let (X,δ) and (Y,σ) be two fuzzy topological spaces. A function $\theta : (X,\delta) \to (Y,\sigma)$ is said to be **Lowen continuous** if $\theta : (X,i(\delta)) \to (Y,i(\sigma))$ is continuous.

Remark : 3.1.7

(i). Fuzzy continuity $\Rightarrow$ Lowen continuity.

(ii). **id** : $(X,\delta) \to (X, \delta^0)$ is Lowen continuous, since $i(\delta) = i(\delta^0)$.

(iii). If $\theta : (X,\delta) \rightarrow (Y,\sigma)$ is Lowen continuous then $\theta : (X, \delta^{\Diamond}) \rightarrow (Y,\sigma^{\Diamond})$ is Lowen continuous.

Definition: 3.1.8

Let (X,δ) and (Y,σ) be two fuzzy topological spaces. The function $\theta : (X,\delta) \rightarrow (Y,\sigma)$ is said to be **weak -n- continuous** if $\theta : (X,{}^{n}\delta) \rightarrow (Y,{}^{n}\sigma)$ is fuzzy continuous.

Definition : 3.1.9

A fuzzy topology δ on a set X is said to be **saturated** if $\delta = \delta^{\Diamond}$.

Definition : 3.1.10.

(i). **$\mathscr{C}$FTOP** : Category of fuzzy topological spaces with fuzzy continuous functions as morphisms.

(ii). **$\mathscr{C}_n$ FTOP** : Subcategory of $\mathscr{C}$FTOP of fuzzy topological spaces of the form (X,δ) where δ is a strong finite fuzzy topology with **$J(\delta)$** $= \left\{0, \frac{1}{n}, \ldots. 1\right\}$.

(iii). **$\mathscr{C}$ FFTOP** : Category of finite fuzzy topological spaces with fuzzy continuous functions as morphisms.

(iv). **$\mathscr{C}$SFTOP** : Category of strong finite fuzzy topological spaces with fuzzy continuous functions as morphisms.

(v). **$\mathscr{C}_{w_n}$FTOP** : Category of fuzzy topological spaces with weak -n-continuous functions as morphisms.

(vi). **$\mathscr{C}^{\Diamond}$FTOP** : Sub category of $\mathscr{C}$FTOP with objects (X,δ) where δ is a saturated fuzzy topology on X.

(vii). **$\mathscr{C}_c$ FTOP** : Category of fuzzy topological spaces with continuous functions as morphisms.

(viii). **$\mathscr{C}^{\Diamond}_c$ FTOP** : Category of saturated fuzzy topologtical spaces with continuous functions as morphisms.

(ix). **$\mathscr{C}_L$FTOP** : Sub category of $\mathscr{C}$ FTOP with topologically generated fuzzy topological spaces as objects.

Remark : 3.1.11.

(i). $(X,\delta) \in \mathscr{C}_n$FTOP $\Rightarrow \delta = {}^n\delta$.

(ii). $(X,\delta) \in \mathscr{C}^{\Diamond}$FTOP $\Rightarrow \delta = \delta^{\Diamond}$.

(iii). In $\mathscr{C}_L$FTOP, continuity and fuzzy continuity are equivalent. Therefore, $\mathscr{C}_L$FTOP is a subcategory of $\mathscr{C}_c$FTOP.

(iv). Since every topologically generated fuzzy topology is saturated $\mathscr{C}_L$FTOP is also contained in $\mathscr{C}^{\Diamond}$FTOP.

Section : 3.2

Functorial Properties

Proposition : 3.2.1

The corespondence $(X,\delta) \rightarrow (X, {}^n\delta)$ defines a functor ζ_n from $\mathscr{C}$FTOP to $\mathscr{C}_n$FTOP.

Proof :

Proof follows from the following result :

"If a map $\theta : (X,\delta) \rightarrow (Y,\sigma)$ is fuzzy continuous then $\theta : (X, {}^{n}\delta) \rightarrow (Y, {}^{n}\sigma)$ is fuzzy continuous".

Proposition : 3.2.2

The sub category $\mathscr{C}_n$FTOP is closed with respect to product. The functor ζ_n preserves product.

Proof :

Proof follows from Theorem 1.3.10

Proposition : 3.2.3

(i). $\mathscr{C}_n$FTOP $\subseteq$ $\mathscr{C}$ SFTOP $\subseteq$ $\mathscr{C}$ FFTOP $\subseteq$ $\mathscr{C}$ FTOP.

(ii). $\mathscr{C}$SFTOP is closed with respect to finite product.

(iii). Given (X,δ), (Y,σ) in $\mathscr{C}$FFTOP, the fuzzy topology $\delta \times \sigma$ on $X \times Y$ has a basis consisting of finite fuzzy sets.

Proof :

(ii). Let δ be a strong finite fuzzy topology on X such that every member of δ assumes values from $\mathbf{J}(\delta)$. Let σ be a strong finite fuzzy topology on Y such that every member of σ assumes values from $\mathbf{J}(\sigma)$. It is easy to prove that $\mathbf{J}(\delta \times \sigma) \subseteq \mathbf{J}(\delta) \cup \mathbf{J}(\sigma)$ and hence $\delta \times \sigma$ is a strong finite fuzzy topology on $X \times Y$.

(iii). To prove (iii) it is enough to note that if f and g are finite valued fuzzy sets on X and Y, $f \times g$ is finite valued on $X \times Y$.

Proposition : 3.2.4

$\mathscr{C}_n$FTOP is a full subcategory of $\mathscr{C}_{w_n}$ FTOP.

Proof :

It is enough to note that fuzzy continuity in $\mathscr{C}_n$FTOP is the same as fuzzy weak -n-continuity.

Proposition : 3.2.5

$\mathscr{C}_n$FTOP is a reflective subcategory of $\mathscr{C}_{w_n}$FTOP.

Proof :

Obviously, the map $(X,\delta) \rightarrow (X,{}^n\delta)$ defines a functor from $\mathscr{C}_{w_n}$FTOP to $\mathscr{C}_n$FTOP.

The identity function **id** : $(X,\delta) \rightarrow (X, {}^n\delta)$ is a weak -n- continuous map. Let $\theta : (X,\delta) \rightarrow (Y,{}^n\sigma)$ be a weak -n- continuous map. Then $\theta : (X , {}^n\delta) \rightarrow (Y,{}^n\sigma)$ is fuzzy continuous.

$$\begin{tikzcd}
(X, \delta) \arrow[rr, "\mathbf{id}"] \arrow[dr, "\theta"'] & & (X,{}^n\delta) \arrow[dl, "\theta"] \\
& (Y,{}^n\sigma) &
\end{tikzcd}$$

Hence , the map : $(X,\delta) \rightarrow (X, {}^n\delta)$ defines a reflective functor from $\mathscr{C}_{w_n}$FTOP to $\mathscr{C}_n$FTOP.

Proposition : 3.2.6

The correspondence $(X, \delta) \rightarrow (X, \delta^{\Diamond})$ defines a coreflective functor $\zeta^{\Diamond} : \mathscr{C}\,\mathrm{FTOP} \rightarrow \mathscr{C}^{\Diamond}\,\mathrm{FTOP}$ and $\zeta^{\Diamond}$ preserves product.

Proof :

By corollary 1.4.12, the map $(X,\delta) \rightarrow (X,\delta^{\Diamond})$ defines a functor $\zeta^{\Diamond}$ from $\mathscr{C}\mathrm{FTOP}$ to $\mathscr{C}^{\Diamond}\mathrm{FTOP}$. So it is enough to prove that the functor is coreflective. The identity function **id :** $(X,\delta^{\Diamond}) \rightarrow (X,\delta)$ is fuzzy continuous.

Let $\theta : (Y, \sigma^{\Diamond}) \rightarrow (X,\delta)$ be fuzzy continuous. Consider $\theta : (Y,\sigma^{\Diamond}) \rightarrow (X,\delta^{\Diamond})$.

Let $f \in \delta^{\Diamond}$. Then ${}^{n}f \in {}^{n}\delta$ for all n.

$${}^{n}(\theta^{-1}(f)) = \theta^{-1}({}^{n}f) = \theta^{-1}({}^{n}g) \text{ for some } g \in \delta$$

$$= {}^{n}(\theta^{-1}(g)) \text{ for } \theta^{-1}(g) \in \sigma^{\Diamond}$$

$$\in {}^{n}(\sigma^{\Diamond}) = {}^{n}\sigma.$$

$\therefore\ \theta^{-1}(f) \in \sigma^{\Diamond}$

$$\begin{tikzcd}
(X,\delta) & & (X,\delta^{\Diamond}) \arrow[ll, "\mathbf{id}"'] \\
& (Y,\sigma^{\Diamond}) \arrow[ul, "\theta"] \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Hence $\theta : (Y,\sigma^{\Diamond}) \rightarrow (X,\delta^{\Diamond})$ is fuzzy continuous.

Proposition : 3.2.7

The map $(X,\delta) \rightarrow (X,\delta^{\Diamond})$ defines a reflective functor $\zeta^{\Diamond}{}_{c} : \mathscr{C}_{c}\mathrm{FTOP} \rightarrow \mathscr{C}^{\Diamond}{}_{c}\,\mathrm{FTOP}$.

Proof :

The map $(X,\delta) \to (X,\delta^{\lozenge})$ defines a functor $\zeta^{\lozenge}_{c}$. The identity map $(X,\delta) \to (X,\delta^{\lozenge})$ is continuous.

Let $\theta : (X,\delta) \to (Y,\sigma^{\lozenge})$ be continuous. Then $\theta : (X, \delta^{\lozenge}) \to (Y,\sigma^{\lozenge})$ is continuous.

$\therefore$ $\zeta^{\lozenge}_{c} : \mathscr{C}_{c}\mathrm{FTOP} \to \mathscr{C}^{\lozenge}_{c}\mathrm{FTOP}$ is a reflective functor.

Section : 3.3

Fuzzy Sierpinski Space

Definition : 3.3.1

A category $\mathscr{C}$ of sets with structure is defined by the following two data and two axioms :

(i). $\mathscr{C}$ assigns to each set X a class $\mathscr{C}(X)$ of $\mathscr{C}$-structures on X. A $\mathscr{C}$-structured set is a pair (X,δ) with $\delta \in \mathscr{C}(X)$.

(ii). $\mathscr{C}$ assigns to each pair (X,δ), (Y,σ) of $\mathscr{C}$-structured sets a subset $\mathscr{C}(\delta,\sigma)$ of the set of all functions from X to Y; We write, $\theta : (X,\delta) \to (Y,\sigma)$ in case $\theta \in \mathscr{C}(\delta,\sigma)$ and say that ‘θ is admissible in $\mathscr{C}$’.

Axiom : 1

If $\theta : (X,\delta) \to (Y,\sigma)$ and $\psi : (Y, \sigma) \to (Z, \rho)$ then $\psi \circ \theta : (X,\delta) \to (Z, \rho)$.

Axiom : 2

Given a bijection $\theta : X \to Y$ and $\delta \in \mathscr{C}(X)$, there exists a unique $\sigma \in \mathscr{C}(Y)$ such that $\theta : (X,\delta) \to (Y,\sigma)$ and $\theta^{-1} : (Y,\sigma) \to (X,\delta)$.

Definition : 3.3.2

Let $\mathscr{C}$ be a category of sets with structure. A family $\theta_j : (X, \delta) \rightarrow (X_j, \delta_j)$ of $\mathscr{C}$-admissible maps is optimal if for every $\mathscr{C}$-structured set (Y,σ) and function $\theta : Y \rightarrow X$, $\theta \in \mathscr{C}(\sigma, \delta)$ provided that $\theta_j \circ \theta \in \mathscr{C}(\sigma, \delta)$ for each j. An optimal lift of $\{ \theta_j : X \rightarrow (X_j, \delta_j) \}$ is a structure $\delta \in \mathscr{C}(X)$ such that $\theta_j : (X,\delta) \rightarrow (X_j, \delta_j)$ is an optimal family.

An object (S, ρ) in $\mathscr{C}$ is a Sierpinski object if for every object (X,δ) in $\mathscr{C}$. the family of all $\mathscr{C}$-admissible maps $(X,\delta) \rightarrow (S, \rho)$ is optimal.

Theorem : 3.3.3

(i). $\mathscr{C}$ FTOP, (ii). $\mathscr{C}$ $_n$FTOP, (iii). $\mathscr{C}$ $^{\Diamond}$FTOP , (iv). $\mathscr{C}$ FFTOP and (v). $\mathscr{C}$SFTOP are $\mathscr{C}$-structured categories.

Proof :

To prove this theorem, it is enough to verify the Axiom 2 of definition 3.3.1.

(i).Let X and Y be two sets and let $\theta : X \rightarrow Y$ be a bijective map.

Let δ be a fuzzy topolgy on X. Define σ on Y as follows :

$\sigma = \{ f \in I^Y \mid \theta^{-1}(f) \in \delta) \}$ . . . . . . (**I**)

Then σ is the unique fuzzy topology on Y such that $\theta : (X,\delta) \rightarrow (Y,\sigma)$ and $\theta^{-1} : (Y,\sigma) \rightarrow (X, \delta)$ are fuzzy continuous .

$\therefore$ $\mathscr{C}$ FTOP is a $\mathscr{C}$-structured category.

(ii). Let $(X,\delta) \in \mathscr{C}_n$FTOP. Then $\delta = {}^n\delta$.

Let $\sigma = \{ f \in I^Y \mid \theta^{-1}(f) \in \delta = {}^n\delta \}$.

Then it is obvious that $\sigma = {}^{n}\sigma$.

$\therefore$ $\mathscr{C}_n$FTOP is a $\mathscr{C}$-structured space.

(iii). Let $(X,\delta^{\Diamond}) \in \mathscr{C}^{\Diamond}$FTOP

Define, $\sigma = \{ f \in I^{Y} \mid \theta^{-1}(f) \in \delta^{\Diamond} \}$.

To show that $\sigma = \sigma^{\Diamond}$.

Let $f \in \sigma^{\Diamond}$. Then ${}^{n}f \in {}^{n}\sigma$ for all n .

$\therefore \theta^{-1}({}^{n}f) = {}^{n}(\theta^{-1}(f) \in {}^{n}(\delta^{\Diamond}) = {}^{n}\delta$.

$\therefore \theta^{-1}(f) \in \delta^{\Diamond} \Rightarrow f \in \sigma$.

Hence $\mathscr{C}^{\Diamond}$FTOP is a $\mathscr{C}$-structured space.

In the case of $\mathscr{C}$FFTOP and $\mathscr{C}$SFTOP, it is enough to note that when $(X,\delta) \in \mathscr{C}$FFTOP ($\mathscr{C}$SFTOP), the topology σ defined by (**I**) is such that $(Y,\sigma) \in \mathscr{C}$FFTOP ($\mathscr{C}$SFTOP).

Remark : 3.3.4.

$\mathscr{C}_{w_n}$FTOP does not satisfy Axiom 2 of definition 3.3.1.

Consider the identity map $(X,\delta) \rightarrow (X, \delta^{\Diamond})$ where $\delta \neq \delta^{\Diamond}$.

Definition : 3.3.5

Let S = I be the unit interval. Then the space S together with the fuzzy topology $\rho = (0, 1, \imath)$, where $\imath \in I^{I}$ is the identity function, is called the fuzzy Sierpinski space .

Srivastava, A.K.[81] has obtained the following Theorem :

Theorem : 3.3.6

The fuzzy Sierpinski space (S, ρ) is a Sierpinski object in $\mathscr{C}$FTOP.

Generalising the above result we get,

Theorem : 3.3.7

$(S, {}^n\rho)$ is a Sierpinski object in $\mathscr{C}_n$FTOP

Proof :

Let $(X, {}^n\delta)$ be a fuzzy topological space.

Claim 1:

${}^n\theta \in {}^n\delta \Leftrightarrow \theta : (X, {}^n\delta) \rightarrow (X, {}^n\rho)$ is fuzzy continuous.

Let ${}^n\theta \in {}^n\delta$.

$\theta^{-1}({}^n\boldsymbol{i})(x) = ({}^n\boldsymbol{i})(\theta(x))$

$= {}^n(\boldsymbol{i}(\theta(x)))$

$= {}^n(\theta(x))$

$\therefore \theta^{-1}({}^n\boldsymbol{i}) = {}^n\theta \in {}^n\delta$

Hence $\theta : (X, {}^n\delta) \rightarrow (X, {}^n\rho)$ is fuzzy continuous .

Now, Assume $\theta : (X, {}^n\delta) \rightarrow (X, {}^n\rho)$ is fuzzy continuous.

Then $\theta^{-1}({}^n\boldsymbol{i}) \in {}^n\delta$ for ${}^n\boldsymbol{i} \in {}^n\rho$.

But $\theta^{-1}({}^n\boldsymbol{i}) = {}^n\theta$.

$\therefore {}^n\theta \in {}^n\delta$.

Hence ${}^n\theta \in {}^n\delta \Leftrightarrow \theta : (X, {}^n\delta) \rightarrow (S, {}^n\rho)$ is fuzzy continuous.

Let $\{ \theta_j : (X, {}^n\delta) \rightarrow (X, {}^n\rho) \}$ be the family of all fuzzy continuous functions. Then ${}^n(\theta_j) \in {}^n\delta$ for all j .

Claim : 2 :

The above collection is optimal.

Let $(Y, {}^n\sigma) \in \mathscr{C}_n$FTOP and $\theta : X \rightarrow Y$ be a map.

Assume $\theta_j \circ \theta : (Y, {}^n\sigma) \rightarrow (S, {}^n\rho)$ is fuzzy continuous for all j .

To prove :

$\theta : (Y, {}^n \sigma) \rightarrow (X , {}^n\delta)$ is fuzzy continuous.

Any element of ${}^n\delta$ must be equal to ${}^n(\theta_j)$ for some θ_j.

$$
\begin{aligned}
(\theta^{-1}({}^n(\theta_j))) (y) &= ({}^n (\theta_j)) (\theta(y)) \\
&= {}^n(\theta_j \circ \theta(y)) \\
&= ((\theta_j \circ \theta)^{-1} ({}^n\boldsymbol{i})) (y) \\
\therefore \quad \theta^{-1}({}^n(\theta_j)) &= (\theta_j \circ \theta)^{-1}\, {}^n\boldsymbol{i} \\
&\in {}^n\sigma \text{ , since } \theta_j \circ \theta \text{ is fuzzy continuous .}
\end{aligned}
$$

Hence, $\theta : (X, {}^n\delta) \rightarrow (Y, {}^n\sigma)$ is fuzzy continuous.

Theorem : 3.3.8

$(S, \rho^{\Diamond})$ is a Sierpinski object in $\mathscr{C}^{\Diamond}$FTOP.

Proof :

To prove this theorem, it is enough to note that $\theta : (X,\delta^{\Diamond}) \rightarrow (S, \rho^{\Diamond})$ is fuzzy continuous iff $\theta \in \delta^{\Diamond}$.

$\theta : (X, \delta^{\Diamond}) \to (S, \rho^{\Diamond})$ is fuzzy continuous

$\Rightarrow \theta : (X, \delta^{\Diamond}) \to (X, \rho)$ is fuzzy continuous

$\Rightarrow \theta \in \delta^{\Diamond}$, since $\theta = \theta^{-1}(i)$.

Conversely, assume $\theta \in \delta^{\Diamond}$.

Then $\theta : (X, \delta^{\Diamond}) \to (S, \rho)$ is fuzzy continuous.

Let $g \in \rho^{\Diamond}$. Then ${}^{n}g \in {}^{n}\rho$.

$\Rightarrow {}^{n}g = {}^{n}h$ for some $h \in \rho$

$h \in \rho \Rightarrow \theta^{-1}(h) \in \delta^{\Diamond}$.

$\Rightarrow {}^{n}(\theta^{-1}(h)) \in {}^{n}(\delta^{\Diamond})$

$\Rightarrow \theta^{-1}({}^{n}h) \in {}^{n}(\delta^{\Diamond})$

$\Rightarrow \theta^{-1}({}^{n}g) \in {}^{n}\delta$

$\Rightarrow {}^{n}(\theta^{-1}(g)) \in {}^{n}\delta$

This is true for all n. Therefore, $\theta^{-1}(g) \in \delta^{\Diamond}$.

Hence, $\theta : (X, \delta^{\Diamond}) \to (S, \rho^{\Diamond})$ is fuzzy continuous.

CHAPTER IV

$\mathscr{F}$- STRUCTURED SPACES

CHAPTER IV

$\mathscr{F}$- STRUCTURED SPACES

In this chapter, the problem of extending the concept of fuzzy topology by considering intuitionistic fuzzy sets on a set X is discussed. A new structure called $\mathscr{F}$-Structure is introduced. The concepts of continuity, separation axioms, compactness etc. are introduced and studied.

Section : 4 .1

$\mathscr{F}$- Structure - Continuity- Products

Definition : 4.1.1

An **intuitionistic fuzzy set** on a set X is a pair (f, g) of fuzzy sets defined on X such that $0 \leq f(x) + g(x) \leq 1$ for all $x \in X$.

Definition : 4.1.2

Given a collection $\{ (f_\lambda, g_\lambda) \mid \lambda \in \Lambda \}$ of intuitionistic fuzzy sets, their **$\mathscr{F}$-union** and **$\mathscr{F}$-intersection** are defined as follows :

$$\vee \{ (f_\lambda, g_\lambda) \mid \lambda \in \Lambda \} = (\vee f_\lambda, \wedge g_\lambda).$$

$$\wedge \{ (f_\lambda, g_\lambda) \mid \lambda \in \Lambda \} = (\wedge f_\lambda, \vee g_\lambda),$$

where $\vee f_\lambda$ stands for $\vee\{ f_\lambda \mid \lambda \in \Lambda \}$ and

$\wedge g_\lambda$ stands for $\wedge\{ g_\lambda \mid \lambda \in \Lambda \}$.

We introduce a new structure called $\mathscr{F}$-structure defined as follows :

Definition : 4.1.3

An $\mathscr{F}$ - **structure $\mathfrak{F}$ on a set X** is a set $\mathfrak{F}$ of intuitionistic fuzzy sets satisfying the following conditions :

$\mathscr{F}_1$. $(1, 0) \in \mathfrak{F}$, $(0, 1) \in \mathfrak{F}$.

$\mathscr{F}_2$. $(f_\lambda, g_\lambda) \in \mathfrak{F}$, $\lambda \in \Lambda \Rightarrow \vee\{(f_\lambda, g_\lambda) \mid \lambda \in \Lambda\} \in \mathfrak{F}$.

$\mathscr{F}_3$. $(f_j, g_j) \in \mathfrak{F}$, for $j = 1, 2, \ldots, k \Rightarrow \wedge\{(f_j, g_j) \mid j = 1, 2, \ldots, n\} \in \mathfrak{F}$.

Remark : 4.1.4

A set X together with an $\mathscr{F}$-structure $\mathfrak{F}$ is denoted by $(X, \mathfrak{F})$ and it is called an **$\mathscr{F}$ - space** or **$\mathscr{F}$-structured space**.

Example : 4.1.5

Let $\delta = \{ f_\lambda \mid \lambda \in \Lambda \}$ be a fuzzy topology on X. Then the following sets of intuitionistic fuzzy sets form $\mathscr{F}$-structures :

(i). $\{ (f_\lambda, 1 - f_\lambda) \mid \lambda \in \Lambda \}$

(ii). $\{ ({}^{n}(f_\lambda), {}_{n}(1 - f_\lambda)) \mid \lambda \in \Lambda \}$

(iii). $\{ (f_\lambda, {}_{n}(1 - f_\lambda)) \mid \lambda \in \Lambda \}$

(iv). $\{ (0, 1), (1, 0), \{ (\alpha, (1 - f_\lambda) \wedge (1 - \alpha)) \mid \lambda \in \Lambda \}\}$

where α is a fixed constant in $(0, 1)$.

Definition : 4.1.6

Given a fuzzy topology $\delta = \{ f_\lambda \mid \lambda \in \Lambda \}$ on X, the $\mathscr{F}$-structure, $\mathfrak{F}_\delta = \{ (f_\lambda, 1 - f_\lambda) \mid \lambda \in \Lambda \}$ is called the **$\mathscr{F}$-structure induced by** δ.

Definition : 4.1.7

With every $\mathscr{F}$-structure $\mathscr{T} = \{ (f_\lambda, g_\lambda) \mid \lambda \in \Lambda \}$, two fuzzy topologies $(\mathscr{T}(\delta))^1$, $(\mathscr{T}(\delta))^2$ are associated as follows :

$(\mathscr{T}(\delta))^1$ = Distinct elements of the collection $\{ f_\lambda \mid \lambda \in \Lambda \}$.

$(\mathscr{T}(\delta))^2$ = Distinct elements of the collection $\{ 1 - g_\lambda \mid \lambda \in \Lambda \}$.

Remark : 4.1.8

$((\mathscr{T}_\delta)(\delta))^1 = \delta = ((\mathscr{T}_\delta)(\delta))^2$

Remark : 4.1.9

If $\mathscr{T} = \{ (0,1), (1,0), \{ (\alpha, (1 - f_\lambda) \wedge (1 - \alpha)) \mid \lambda \in \Lambda \}\}$

where $\{ f_\lambda \mid \lambda \in \Lambda \}$ is a fuzzy topology δ, then

$(\mathscr{T}(\delta))^1 = (1, 0, \alpha)$.

$(\mathscr{T}(\delta))^2 = \{ f_\lambda \vee \alpha \mid \lambda \in \Lambda \} \cup \{ 0 \}$.

Definition : 4.1.10

Basis for an $\mathscr{F}$-structure $\mathscr{T}$ is a subcollection $\mathscr{B}_{\mathscr{T}}$ of $\mathscr{T}$ such that every member (f, g) of $\mathscr{T}$ can be expressed as $(f, g) = \vee\{(f_\lambda, g_\lambda) \mid \lambda \in \Lambda_0\}$ where $(f_\lambda, g_\lambda) \in \mathscr{B}_{\mathscr{T}}$.

Definition : 4.1.11

A collection $\mathcal{S}$ of intuitionistic fuzzy sets forms a **subbasis for $\mathscr{T}$** if the collection of finite $\mathscr{T}$-intersections $\wedge \{(f_j, g_j) \mid j = 1, 2, \ldots, k\}$ where $(f_j, g_j) \in \mathcal{S}$ forms a basis for $\mathscr{T}$.

Remark : 4.1.12

Let $\delta = \{ f_\lambda \mid \lambda \in \Lambda\}$ be a fuzzy topology. Let $\mathscr{B}$ be a basis for δ. Consider the $\mathscr{F}$-structure $\mathfrak{F}_\delta = \{ (f_\lambda, 1- f_\lambda) \mid \lambda \in \Lambda \}$, the collection $\{ (g_\mu, 1- g_\mu) \mid g_\mu \in \mathscr{B} \}$ forms a basis $\mathscr{B}_{\mathfrak{F}}$ for $\mathfrak{F}_\delta$.

Definition : 4.1.13

Given an $\mathscr{F}$- structure $\mathfrak{F} = \{ (f_\lambda, g_\lambda) \mid \lambda \in \Lambda\}$ and a positive integer n, an $\mathscr{F}$- structure ${}^n\mathfrak{F}$ is defined as follows :

$${}^n\mathfrak{F} = \{ ({}^n(f_\lambda), {}_n(g_\lambda)) \mid \lambda \in \Lambda \} .$$

Remark : 4.1.14

$${}^n(\mathfrak{F}_\delta) = \mathfrak{F}_{{}^n\delta}$$

Proof :

Let $\delta = \{f_\lambda \mid \lambda \in \Lambda\}$. Then ${}^n\delta = \{ {}^n(f_\lambda) \mid \lambda \in \Lambda\}$.

$$\mathfrak{F}_\delta = \{ (f_\lambda , 1-f_\lambda) \mid f_\lambda \in \delta\}$$

$$\mathfrak{F}_{{}^n\delta} = \{ ({}^n(f_\lambda) , 1 - {}^n(f_\lambda)) \mid f_\lambda \in \delta \}$$

$$= \{ ({}^n(f_\lambda) , {}_n(1-f_\lambda)) \mid f_\lambda \in \delta \}$$

$${}^n(\mathfrak{F}_\delta) = \{ ({}^n(f_\lambda) , {}_n(1-f_\lambda)) \mid f_\lambda \in \delta \}$$

Hence the proof.

Definition : 4.1.15

Let $(X, \mathfrak{F}_1)$ and $(Y, \mathfrak{F}_2)$ be two $\mathscr{F}$- structures. Then $\theta : (X, \mathfrak{F}_1) \to (Y, \mathfrak{F}_2)$ is said to be **$\mathscr{F}$- continuous** if $\theta^{-1}(f , g) \in \mathfrak{F}_1$ for every $(f , g) \in \mathfrak{F}_2$. Here $\theta^{-1} (f , g) = (\theta^{-1}(f) , \theta^{-1}(g))$.

Theorem : 4.1.16

Let (X, δ) and (Y, σ) be two fuzzy topological spaces. Let $(X, \mathcal{F}_\delta)$, $(Y, \mathcal{F}_\sigma)$ be the associated $\mathcal{F}$- structured spaces. Then a map $\theta : (X, \mathcal{F}_\delta) \to (Y, \mathcal{F}_\sigma)$ is $\mathcal{F}$- continuous iff $\theta : (X, \delta) \to (Y, \sigma)$ is fuzzy continuous.

Proof :

Assume that $\theta : (X, \delta) \to (Y, \sigma)$ is fuzzy continuous.

To prove :

$\theta : (X, \mathcal{F}_\delta) \to (Y, \mathcal{F}_\sigma)$ is $\mathcal{F}$- continuous.

Let $(f_\lambda, 1 - f_\lambda) \in \mathcal{F}_\delta$ where $f_\lambda \in \sigma$.

Then $\theta^{-1}(f_\lambda) \in \delta$

$\Rightarrow (\theta^{-1}(f_\lambda), 1 - \theta^{-1}(f_\lambda)) \in \mathcal{F}_\delta$

$\Rightarrow (\theta^{-1}(f_\lambda), \theta^{-1}(1 - f_\lambda)) \in \mathcal{F}_\delta$.

Conversely, assume that $\theta : (X, \mathcal{F}_\delta) \to (Y, \mathcal{F}_\sigma)$ is $\mathcal{F}$-continuous.

To prove :

$\theta : (X.\delta) \to (Y, \sigma)$ is fuzzy continuous.

$f_\lambda \in \sigma \Rightarrow (f_\lambda, 1 - f_\lambda) \in \mathcal{F}_\sigma$

$\Rightarrow (\theta^{-1}(f_\lambda), \theta^{-1}(1 - f_\lambda)) \in \mathcal{F}_\delta$

$\Rightarrow \theta^{-1}(f_\lambda) \in \delta$.

Theorem : 4.1.17

Let $(X, \mathfrak{F}_1)$ and $(Y, \mathfrak{F}_2)$ be two $\mathscr{F}$-structured spaces. Let $(\mathfrak{F}_1(\delta))^1$, $(\mathfrak{F}_1(\delta))^2$, $(\mathfrak{F}_2(\delta))^1$, $(\mathfrak{F}_2(\delta))^2$ be the fuzzy topologies induced by $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Then $\theta : (X , \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ is $\mathscr{F}$-continuous implies that $\theta : (X, (\mathfrak{F}_1(\delta))^1, (\mathfrak{F}_1(\delta))^2) \rightarrow (Y, (\mathfrak{F}_2(\delta))^1, (\mathfrak{F}_2(\delta))^2)$ is pairwise continuous.

Proof :

Assume that $\theta : (X , \mathfrak{F}_1) \rightarrow (Y , \mathfrak{F}_2)$ is $\mathscr{F}$-continuous .

Then $\theta^{-1}(f_\lambda , g_\lambda) \in \mathfrak{F}_1 , \forall (f_\lambda, g_\lambda) \in \mathfrak{F}_2$.

Claim : 1

$\theta : (X , (\mathfrak{F}_1(\delta))^1) \rightarrow (Y, (\mathfrak{F}_2(\delta))^1)$ is fuzzy continuous.

Let $f \in (\mathfrak{F}_2(\delta))^1$.

Then there exists $(f_\lambda, g_\lambda) \in \mathfrak{F}_2$ such that $f = f_\lambda$.

This implies $\theta^{-1}(f_\lambda , g_\lambda) \in \mathfrak{F}_1$

$\Rightarrow (\theta^{-1}(f_\lambda), \theta^{-1}(g_\lambda)) \in \mathfrak{F}_1$

$\Rightarrow \theta^{-1}(f_\lambda) \in (\mathfrak{F}_1 (\delta))^1$

That is, $\theta^{-1}(f) \in (\mathfrak{F}_1(\delta))^1$

Claim : 2

$\theta : (X , (\mathfrak{F}_1(\delta))^2) \rightarrow (Y,(\mathfrak{F}_2(\delta))^2)$ is fuzzy continuous.

Let $h \in (\mathfrak{F}_2(\delta))^2$.

Then there exists $(f_\lambda , g_\lambda) \in \mathfrak{F}_2$ and $h = 1 - g_\lambda$.

This implies $\theta^{-1}(f_\lambda, g_\lambda) \in \mathcal{F}_1$

$\Rightarrow (\theta^{-1}(f_\lambda), \theta^{-1}(g_\lambda)) \in \mathcal{F}_1$

$\Rightarrow 1\text{-}\theta^{-1}(g_\lambda) \in (\mathcal{F}_1(\delta))^2$

$\Rightarrow \theta^{-1}(1\text{-}g_\lambda) \in (\mathcal{F}_1(\delta))^2$

$\Rightarrow \theta^{-1}(h) \in (\mathcal{F}_1(\delta))^2$.

Theorem : 4.1.18

Let $(X, \mathcal{F}_1)$ and $(Y, \mathcal{F}_2)$ be two $\mathscr{F}$-structured spaces.

If $\theta : (X, \mathcal{F}_1) \rightarrow (Y, \mathcal{F}_2)$ is $\mathscr{F}$- continuous ,

then $\theta : (X, {}^{n}(\mathcal{F}_1)) \rightarrow (Y, {}^{n}(\mathcal{F}_2))$ is $\mathscr{F}$- continuous.

Proof :

Assume that $\theta : (X, \mathcal{F}_1) \rightarrow (Y, \mathcal{F}_2)$ is $\mathscr{F}$- continuous.

Then $\theta^{-1}(f_\lambda, g_\lambda) \in \mathcal{F}_1$ for $(f_\lambda, g_\lambda) \in \mathcal{F}_2$ such that $({}^{n}(f_\lambda), {}_{n}(g_\lambda)) \in {}^{n}(\mathcal{F}_2)$.

To Prove :

$\theta^{-1}({}^{n}(f_\lambda), {}_{n}(g_\lambda)) \in {}^{n}(\mathcal{F}_1)$

$\theta^{-1}(f_\lambda, g_\lambda) \in \mathcal{F}_1 \Rightarrow (\theta^{-1}(f_\lambda), \theta^{-1}(g_\lambda)) \in \mathcal{F}_1$

$\Rightarrow ({}^{n}(\theta^{-1}(f_\lambda)), {}_{n}(\theta^{-1}(g_\lambda))) \in {}^{n}(\mathcal{F}_1)$

$\Rightarrow (\theta^{-1}({}^{n}(f_\lambda)), \theta^{-1}({}_{n}(g_\lambda))) \in {}^{n}(\mathcal{F}_1)$

$\Rightarrow \theta^{-1}({}^{n}(f_\lambda), {}_{n}(g_\lambda)) \in {}^{n}(\mathcal{F}_1)$.

Hence , $\theta : (X, {}^{n}(\mathcal{F}_1)) \rightarrow (Y, {}^{n}(\mathcal{F}_2))$ is $\mathscr{F}$- continuous.

Definition : 4.1.19

Let $(X,\mathcal{F}_1)$ and $(Y,\mathcal{F}_2)$ be two $\mathcal{F}$-spaces where $\mathcal{F}_1 = \{(f_\lambda, g_\lambda) \mid \lambda \in \Lambda\}$ and $\mathcal{F}_2 = \{(h_\mu, \ell_\mu) \mid \mu \in \Lambda\}$. **The product $\mathcal{F}$-structure** $\mathcal{F}_1 \times \mathcal{F}_2$ on $X \times Y$ is defined as the $\mathcal{F}$- structure having $(f_\lambda * h_\mu , g_\lambda \oplus \ell_\mu)$ as basis where

$(f_\lambda * h_\mu)(x, y) = \min(f_\lambda(x), h_\mu(y))$ and

$(g_\lambda \oplus \ell_\mu)(x, y) = \sup(g_\lambda(x), \ell_\mu(y))$.

Theorem : 4.1.20

Let (X,δ) and (Y,σ) be two fuzzy topological spaces. Let $(X,\mathcal{F}_\delta)$ and $(Y,\mathcal{F}_\sigma)$ be the induced $\mathcal{F}$-spaces. Then the two $\mathcal{F}$- structures $\mathcal{F}_\delta \times \mathcal{F}_\sigma$ and $\mathcal{F}_{\delta \times \sigma}$ on $X \times Y$ coincide. That is , $\mathcal{F}_\delta \times \mathcal{F}_\sigma = \mathcal{F}_{\delta \times \sigma}$.

Proof :

$\mathcal{F}_\delta = \{ (f, 1-f) \mid f \in \delta \}$.

$\mathcal{F}_\sigma = \{ (g, 1-g) \mid g \in \sigma \}$.

$\mathcal{F}_\delta \times \mathcal{F}_\sigma$ has basis $(f * g, (1-f) \oplus (1-g))$.

$\mathcal{F}_{\delta \times \sigma}$ has basis $\{(f * g, 1-(f*g)) \mid f \in \delta, g \in \sigma\}$

$= \{ f*g, (1-f) \oplus (1-g) \mid f \in \delta, g \in \sigma \}$.

Hence , $\mathcal{F}_\delta \times \mathcal{F}_\sigma = \mathcal{F}_{\delta \times \sigma}$.

Theorem : 4.1.21

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two $\mathcal{F}$-structures. The projection maps

$P_1 : (X \times Y, \mathcal{F}_1 \times \mathcal{F}_2) \rightarrow (X, \mathcal{F}_1)$ and

$P_2 : (X \times Y, \mathcal{F}_1 \times \mathcal{F}_2) \rightarrow (Y, \mathcal{F}_2)$ are $\mathcal{F}$- continuous.

Proof :

Let $(f_\lambda, g_\lambda) \in \mathcal{F}_1$.

$$
\begin{aligned}
P_1^{-1}(f_\lambda, g_\lambda)(x, y) &= (f_\lambda, g_\lambda)(P_1(x, y)) \\
&= (f_\lambda, g_\lambda)(x) \\
&= (f_\lambda(x), g_\lambda(x)) \\
&= (f_\lambda * 1, g_\lambda \oplus 0)(x, y).
\end{aligned}
$$

$\therefore P_1^{-1}(f_\lambda, g_\lambda) \in \mathcal{F}_1 \times \mathcal{F}_2$.

Let $(h_\mu, l_\mu) \in \mathcal{F}_2$.

$$
\begin{aligned}
P_2^{-1}(h_\mu, l_\mu)(x, y) &= (h_\mu, l_\mu)(P_2(x, y)) \\
&= (h_\mu, l_\mu)(y) \\
&= (h_\mu(y), l_\mu(y)) \\
&= (1 * h_\mu, 0 \oplus l_\mu)(x, y).
\end{aligned}
$$

$\therefore \quad P_2^{-1}(h_\mu, l_\mu) \in \mathcal{F}_1 \times \mathcal{F}_2$.

Hence , P_1 and P_2 are $\mathscr{F}$- continuous.

Section : 4. 2

Separation Axioms and Compactness in $\mathscr{F}$-Structured Spaces

Definition : 4.2.1

An $\mathscr{F}$- structure $\mathcal{F}$ on a set X is said to be $\mathscr{F}$-Hausdorff$_1$ ($\mathscr{F}$- H_1) if $\forall$ x , y $\in$ X, x $\neq$ y, there exists $(f, g) \in \mathcal{F}$ such that $f(x) = 1$, $g(y) = 0$ and $f \wedge g = 0$.

Theorem : 4.2.2

Let $(X, \mathfrak{F})$ be an $\mathscr{F}$-space. Then $(X, \mathfrak{F})$ is $\mathscr{F}$-$H_1 \Rightarrow (X, {}^n\mathfrak{F})$ is $\mathscr{F}$-H_1.

Proof :

Let $(X, \mathfrak{F})$ be an $\mathscr{F}$-space and $(X, \mathfrak{F})$ is $\mathscr{F}$-H_1.

Then $\forall\, x, y \in X,\ x \neq y$, there exists $(f, g) \in \mathfrak{F}$ such that $f(x) = 1$, $g(y) = 0$ and $f \wedge g = 0$.

$\Rightarrow \forall\, x, y \in X,\ x \neq y$, there exists $({}^nf, {}_ng) \in {}^n\mathfrak{F}$ such that ${}^nf(x) = 1$, ${}_ng(y) = 0$ and ${}^nf \wedge {}_ng = 0$

$\Rightarrow (X, {}^n\mathfrak{F})$ is $\mathscr{F}$-H_1.

Definition : 4.2.3

An $\mathscr{F}$- structure $\mathfrak{F}$ on a set X is said to be $\mathscr{F}$-Hausdorff$_2$ ($\mathscr{F}$-H_2) if $\forall\, x, y \in X,\ x \neq y$, there exists $(f, g) \in \mathfrak{F}$ such that $f(x) > 0$, $g(y) \neq 1$ and $f \wedge g = 0$.

Theorem : 4.2.4

Let $(X, \mathfrak{F})$ be an $\mathscr{F}$-structured space.

$(X, \mathfrak{F})$ is $\mathscr{F}$-$H_2 \Leftrightarrow (X, {}^n\mathfrak{F})$ is $\mathscr{F}$-H_2.

Proof :

An $\mathscr{F}$-structured space $(X, \mathfrak{F})$ is $\mathscr{F}$-H_2.

$\Leftrightarrow \forall\, x, y \in X,\ x \neq y$, there exists $(f, g) \in \mathfrak{F}$ such that $f(x) > 0$, $g(y) \neq 1$ such that $f \wedge g = 0$.

$\Leftrightarrow \forall\, x, y \in X, x \neq y$, there exists $(f, g) \in \mathfrak{F}$ such that ${}^nf(x) > 0$, ${}_ng(y) \neq 1$ such that ${}^nf \wedge {}_ng = 0 \Leftrightarrow (X, {}^n\mathfrak{F})$ is $\mathscr{F}$-H_2.

Definition : 4.2.5

An $\mathscr{F}$-structure on a set X is said to be $\mathscr{F}$- Hausdorff$_3$ ($\mathscr{F}$-H_3) if $\forall$ x , y $\in$ X, x $\neq$ y, there exist $(f_\lambda, g_\lambda), (f_\mu, g_\mu) \in \mathcal{F}$ such that $f_\lambda(x) > 0$, $f_\mu(y) > 0$ and $(1 - g_\lambda) \wedge (1 - g_\mu) = 0$.

Theorem : 4.2.6

Let δ be a fuzzy topology on a set X and $\mathcal{F}$ be an $\mathscr{F}$-structure on X. Then ,

(i). (X, δ) is fuzzy K-Hausdorff $\Rightarrow (X, \mathcal{F}_\delta)$ is $\mathscr{F}$-H_3 .

(ii). $(X, \mathcal{F})$ is $\mathscr{F}$-$H_3 \Rightarrow (X, (\mathcal{F}(\delta))^1), (X, (\mathcal{F}(\delta))^2)$ are fuzzy K-Hausdorff.

(iii). $(X, \mathcal{F})$ is $\mathscr{F}$-$H_3 \Rightarrow (X, {}^n\mathcal{F})$ is $\mathscr{F}$-H_3.

Proof :

(i). Assume (X, δ) is fuzzy K-Hausdorff.

Given x , y $\in$ X , x $\neq$ y, there exist f_λ, f_μ such that $f_\lambda(x) > 0$, $f_\mu(y) > 0$ and $f_\lambda \wedge f_\mu = 0$.

Then the two elements $(f_\lambda, 1 - f_\lambda), (f_\mu, 1 - f_\mu)$ in $\mathcal{F}_\delta$ are such that $f_\lambda(x) > 0$, $f_\mu(y) > 0$ and $(1 - (1 - f_\lambda) \wedge (1 - (1 - f_\mu)) = f_\lambda \wedge f_\mu = 0$.

Hence , $(X, \mathcal{F}_\delta)$ is $\mathscr{F}$-H_3 .

(ii). Assume $(X, \mathcal{F})$ is $\mathscr{F}$-H_3

Given x , y $\in$ X , x $\neq$ y, there exist $(f_\lambda, g_\lambda), (f_\mu, g_\mu) \in \mathcal{F}$ such that $f_\lambda(x) > 0$, $f_\mu(y) > 0$ and $(1 - g_\lambda) \wedge (1 - g_\mu) = 0$.

$\therefore\ 1 - g_\lambda(x) \geq f_\lambda(x) > 0,\ 1 - g_\mu(y) \geq f_\mu(y) > 0$ and

$f_\lambda \wedge f_\mu \leq (1-g_\lambda) \wedge (1-g_\mu) = 0$

Hence, $(X, (\mathcal{F}(\delta)^1)$ and $(X, (\mathcal{F}(\delta))^2)$ are fuzzy K-Hausdorff.

(iii). Assume $(X, \mathcal{F})$ is $\mathscr{F}$-H_3 .

Given $x, y \in X, x \neq y$, there exist $(f_\lambda, g_\lambda), (f_\mu, g_\mu) \in \mathcal{F}$

such that $f_\lambda(x) > 0,\ f_\mu(y) > 0$ and $(1-g_\lambda) \wedge (1-g_\mu) = 0$.

Therefore , ${}^{n}(f_\lambda)(x) > 0, {}^{n}(f_\mu)(y) > 0$ and

$(1-{}_{n}(g_\lambda)) \wedge (1-{}_{n}(g_\mu)) = {}^{n}(1-g_\lambda) \wedge {}^{n}(1-g_\mu) = 0$.

Hence, $(X, {}^{n}\mathcal{F})$ is $\mathscr{F}$-H_3.

Definition : 4.2.7

Let $\mathcal{F} = \{ (f_\lambda, g_\lambda) \mid \lambda \in \Lambda \}$ be an $\mathscr{F}$-structure on X. Then $\mathcal{F}$ is Chang $\mathscr{F}$-compact if the following condition is satisfied :

Given $(f_\lambda, g_\lambda) \in \mathcal{F}$ such that $\vee \{ f_\lambda \mid \lambda \in \Lambda \} = 1$ there exists a finite subset Λ_0 of Λ such that $\vee \{(1-g_\lambda) \mid \lambda \in \Lambda_0 \subseteq \Lambda \} = 1$.

Example : 4.2.8

Let (X,δ) be a Chang compact fuzzy topological space where $\delta = \{ f_\lambda \mid \lambda \in \Lambda_0 \}$. Then, $\mathcal{F} = \{ (f_\lambda, {}_{n}(1-f_\lambda) \mid \lambda \in \Lambda \}$ is Chang $\mathscr{F}$- compact.

Proof :

Assume that, $\vee \{ f_\lambda \mid \lambda \in \Lambda \} = 1$. Then by hypothesis, there exists a finite $\Lambda_0 \subseteq \Lambda$ such that $\vee \{ f_\lambda \mid \lambda \in \Lambda_0 \subseteq \Lambda \} = 1$.

$\therefore \vee \{ {}^{n}(f_\lambda) \mid \lambda \in \Lambda_0 \subseteq \Lambda \} = 1.$

That is, $\vee \{ 1 -_{n} (1-f_\lambda) \mid \lambda \in \Lambda_0 \subseteq \Lambda \} = 1$

Hence $\mathfrak{F}$ is Chang $\mathscr{F}$- compact.

Theorem : 4.2.9

Let (X , δ) be a fuzzy topological space where $\delta = \{ f_\lambda \mid \lambda \in \Lambda \}$.

Then (X,δ) is Chang fuzzy compact $\Leftrightarrow$ $(X, \mathfrak{F}_\delta)$ is Chang $\mathscr{F}$-compact .

Proof :

Obvious.

Theorem : 4.2.10

Let $(X , \mathfrak{F})$ be an $\mathscr{F}$-structred space where $\mathfrak{F} = \{ (f_\lambda, g_\lambda) \mid \lambda \in \Lambda \}$.

(i).$(X, (\mathfrak{F}(\delta))^1)$ is Chang fuzzy compact $\Rightarrow$ $(X, \mathfrak{F})$ is Chang $\mathscr{F}$-compact.

(ii).$(X, (\mathfrak{F}(\delta))^2)$ is Chang fuzzy compact $\Rightarrow$ $(X ,\mathfrak{F})$is Chang $\mathscr{F}$-compact.

Proof :

(i). Obvious.

(ii). Assume that $\vee \{ f_\lambda \mid \lambda \in \Lambda \} = 1$.

Then, $\vee \{ 1- g_\lambda \mid \lambda \in \Lambda \} = 1$.

Then by hypothesis, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that

$\vee \{ 1- g_\lambda \mid \lambda \in \Lambda_0 \subseteq \Lambda \} = 1$.

Hence, $\mathfrak{F}$ is Chang $\mathscr{F}$-compact .

Note : 4.2.11

$(\mathcal{F}(\delta))^j$ Chang fuzzy compact need not imply $(\mathcal{F}(\delta))^k$ Chang fuzzy compact for $j \neq k$.

Definition : 4.2.12

Let $\mathcal{F} = \{ (f_\lambda, g_\lambda) \mid \lambda \in \Lambda \}$ be an $\mathscr{F}$-structure on X. A fuzzy set f is Lowen $\mathscr{F}$-compact if given a family $\eta \subseteq \Lambda$ such that $\vee \{ f_\lambda \mid \lambda \in \eta \} \geq f$ and given $\varepsilon > 0$, there exists a finite subfamily η_0 of η such that

$\vee \{1 - g_\lambda \mid \lambda \in \eta_0 \} \geq f - \varepsilon$.

Definition : 4.2.13

An $\mathscr{F}$-structured space $(X, \mathcal{F})$ is Lowen $\mathscr{F}$-compact if every constant fuzzy set is Lowen $\mathscr{F}$-compact.

Theorem : 4.2.14

Let $\mathcal{F} = \{(f_\lambda, g_\lambda) \mid \lambda \in \Lambda \}$ be an $\mathscr{F}$-structure on X. Then,

(i). (X,δ) Lowen fuzzy compact $\Leftrightarrow$ $(X, \mathcal{F}_\delta)$ is Lowen $\mathscr{F}$--compact.

(ii). $(X, (\mathcal{F}(\delta))^1)$ Lowen fuzzy compact $\Rightarrow$ $(X, \mathcal{F})$ is Lowen $\mathscr{F}$-compact.

(iii). $(X, (\mathcal{F}(\delta))^2)$ Lowen fuzzy compact $\Rightarrow$ $(X, \mathcal{F})$ is Lowen $\mathscr{F}$-compact.

(iv). $(X , {}^n\mathcal{F})$ is Lowen $\mathscr{F}$-compact for every n $\Rightarrow$ $(X , \mathcal{F})$ is Lowen $\mathscr{F}$-compact.

Proof :

Proofs of (i),(ii) and (iii) follow immediately from the definition. Proof of (iv) is similar to that of Theorem 2.1.5.

Section : 4.3

Functorial Properties

Definition : 4.3.1

(i). **$\mathcal{C}$ $\mathcal{F}$STR** : Category of $\mathcal{F}$-structured spaces with $\mathcal{F}$-continuous functions as morphisms.

(ii). **$\mathcal{C}$ T$\mathcal{F}$STR** : Subcategory of $\mathcal{F}$-structured spaces of the form $(X,\mathcal{T}_\delta)$.

(iii). **$\mathcal{C}_n\mathcal{F}$STR** : Subcategory of $\mathcal{F}$-structured spaces of the form $(X,{}^{n}\mathcal{T})$.

Theorem : 4.3.2

The correspondence $(X,\delta) \to (X, \mathcal{T}_\delta)$ defines an embedding ζ_e of $\mathcal{C}$FTOP into a full subcategory of $\mathcal{C}\mathcal{F}$STR.

Proof :

Proof follows by Theorem 4.1.16

Theorem : 4.3.3

The correspondence $(X, \mathcal{T}) \to (X, {}^{n}\mathcal{T})$ defines a functor $\zeta^{n}_{\mathcal{T}}$: $\mathcal{C}\mathcal{F}\text{STR} \to \mathcal{C}_n\,\mathcal{F}\text{STR}$.

Proof :

Proof follows by Theorem 4.1.18.

Theorem : 4.3.4.

The subcategory $\mathcal{C}$ T$\mathcal{F}$STR is closed with respect to products.

Proof :

Since , $\mathcal{T}_\delta \times \mathcal{T}_\sigma = \mathcal{T}_{\delta \times \sigma}$, the proof is clear.

CHAPTER V

STUDY OF GRADATION OF OPENNESS

CHAPTER V

STUDY OF GRADATION OF OPENNESS

In 1992, Hazra, Samanta and Chattopadhyay [28] introduced the concept of gradation of openness and gave a new definition of fuzzy topology.

In this chapter, two interesting methods of associating finite valued gradations with a given gradation are introduced. The third method associates with each gradation, a gradation which depends only on finite valued functions.

In the fifth section of this chapter, we have given a new method of associating a gradation $({}^{n}\mathcal{G})^{*}$ with a gradation $\mathcal{G}$ such that if $\mathcal{G}$ induces the fuzzy topology δ then $({}^{n}\mathcal{G})^{*}$ will induce the fuzzy topology ${}^{n}\delta$.

Section : 5.1

Preliminary Definitions

Definition [28] : 5.1.1

Let X be a non empty set and $\mathcal{G} : I^{X} \rightarrow I$ be a mapping satisfying the following properties :

$\mathcal{G}$01. $\mathcal{G}(0) = \mathcal{G}(1) = 1$ where 0 and 1 denote the constant maps with values 0 and 1.

$\mathcal{G}$02. $\mathcal{G}(f_i) > 0$, for $i = 1, 2$, implies $\mathcal{G}(f_1 \wedge f_2) > 0$.

$\mathcal{G}$03. $\mathcal{G}(f_\lambda) > 0$, for $\lambda \in \Lambda$, implies $\mathcal{G}(\sup \{f_\lambda \mid \lambda \in \Lambda\}) > 0$.

Then $\mathcal{G}$ is called a **gradation of openness** on X.

Definition : 5.1.2

A set X together with a gradation of openness $\mathcal{G}$ is called a **gradation space** and it is denoted by (X, $\mathcal{G}$).

Definition : 5.1.3

Let (X, $\mathcal{G}$) be a gradation space. Then the **fuzzy topology induced by (X, $\mathcal{G}$)** is given by $\delta(\mathcal{G}) = \{ f \in I^X \mid \mathcal{G}(f) > 0 \}$.

Definition : 5.1.4

Let (X , $\mathcal{G}$) and (Y, $\mathcal{G}'$) be two gradation spaces. Then a map $\theta : X \to Y$ is called

(i). a **gradation preserving (gp-) map**, if $\mathcal{G}'(f) \leq \mathcal{G}(\theta^{-1}(f))$, for each $f \in I^Y$.

(ii). a **strongly gradation preserving (sgp-) map**, if $\mathcal{G}'(f) = \mathcal{G}(\theta^{-1}(f))$, for each $f \in I^Y$.

(iii). a **weakly gradation preserving (wgp-) map**, if $\mathcal{G}'(f) > 0 \Rightarrow \mathcal{G}(\theta^{-1}(f)) > 0$, for each $f \in I^Y$.

Definition : 5.1.5

Let $\mathcal{G}$ and $\mathcal{G}'$ be two gradations of openness on X. Then $\mathcal{G} \geq \mathcal{G}'$ if $\mathcal{G}(f) \geq \mathcal{G}'(f)$ for all $f \in I^X$.

Definition : 5.1.6

Let X be a set. A mapping $\mathcal{G} : I^X \to I$ satisfying

$\mathcal{G}$C1. $\mathcal{G}(0) = \mathcal{G}(1) = 1$,

$\mathcal{G}$C2. If $\mathcal{G}(f_i) > 0$, for i =1,2, then $\mathcal{G}(f_1 \vee f_2) > 0$,

$\mathcal{G}$C3. If $\mathcal{G}(f_\lambda) > 0$, for $\lambda \in \Lambda$, then $\mathcal{G}(\inf \{f_\lambda \mid \lambda \in \Lambda\}) > 0$

is called a **gradation of closedness** on X.

Definition : 5.1.7

Let (X, $\mathcal{G}$) be a gradation space and $Y \subset X$. Then the mapping $\mathcal{G}_Y : I^Y \to I$ defined by

$\mathcal{G}_Y(f) = \sup \{ \mathcal{G}(h) \mid h \in I^X , h_{|Y} = f\}$ is a gradation of openness on Y.

Chattopadhyay, Hazra and Samanta [11] modified the definition of gradation and introduced the following new definition of gradation :

Definition [11] : 5.1.8

Let X be a non-empty set. A map $\mathcal{G} : I^X \to I$ is called a **M-gradation of openness** if it satisfies the following properties:

M$\mathcal{G}$01. $\mathcal{G}(0) = \mathcal{G}(1) = 1$,

M$\mathcal{G}$02. $\mathcal{G}(f_i) \geq r$, for i = 1,2, implies $\mathcal{G}(f_1 \wedge f_2) \geq r$,

M$\mathcal{G}$03. $\mathcal{G}(f_\lambda) \geq r$, $\lambda \in \Lambda$, implies $\mathcal{G}(\sup \{f_\lambda \mid \lambda \in \Lambda\}) \geq r$,

where $0 < r \leq 1$.

Definition : 5.1.10

A set X together with a M-gradation of openness $\mathcal{G}$ is called a **M-gradation space**.

The definition 5.1.2 to definition 5.1.8 are extended to M-gradation spaces also.

Section : 5.2

First nth Order Approximations

Definition : 5.2.1

Given a gradation of openness $\mathcal{G}$, the nth degree approximation **$^n\mathcal{G}$ of $\mathcal{G}$ is defined by $({}^n\mathcal{G})(f) = {}^n(\mathcal{G}(f))$**.

Theorem : 5.2.2

Let $\mathcal{G}$ be a gradation of openness defined on X. Then $^n\mathcal{G}$ is a gradation of openness on X.

Proof:

Let (X, $\mathcal{G}$) be a gradation space.

$\mathcal{G}$ 01. $({}^n\mathcal{G})(0) = {}^n(\mathcal{G}(0)) = {}^n1 = 1.$

$({}^n\mathcal{G})(1) = {}^n(\mathcal{G}(1)) = {}^n1 = 1.$

$\mathcal{G}$ 02. Let $({}^n\mathcal{G})(f_i) > 0$ for i =1, 2.

Then $^n(\mathcal{G}(f_i)) > 0$ for i =1, 2.

This implies that $\mathcal{G}(f_i) > 0$ for i = 1,2

$\Rightarrow \mathcal{G}(f_1 \wedge f_2) > 0$

$\Rightarrow\ {}^{n}(\mathcal{G}(f_1 \wedge f_2)) > 0$

$\Rightarrow\ ({}^{n}\mathcal{G})(f_1 \wedge f_2) > 0$.

$\mathcal{G}$**03.** Let $({}^{n}\mathcal{G})(f_\lambda) > 0$ for $\lambda \in \Lambda$.

Then ${}^{n}(\mathcal{G}(f_\lambda)) > 0$ for $\lambda \in \Lambda$.

This implies that $\mathcal{G}(f_\lambda) > 0$ for $\lambda \in \Lambda$

$\Rightarrow\ \mathcal{G}(\sup\{f_\lambda \mid \lambda \in \Lambda\}) > 0$

$\Rightarrow\ {}^{n}(\mathcal{G}(\sup\{f_\lambda \mid f_\lambda \in \Lambda\})) > 0$

$\Rightarrow\ ({}^{n}\mathcal{G})(\sup\{f_\lambda \mid \lambda \in \Lambda\}) > 0$.

Hence, $(X, {}^{n}\mathcal{G})$ is a gradation space.

Remark : 5.2.3

$\mathcal{G}$ and ${}^{n}\mathcal{G}$ induce the same fuzzy topology on X.

To prove this, we have only to note that $\mathcal{G}(f) > 0 \Leftrightarrow ({}^{n}\mathcal{G})(f) > 0$.

Theorem : 5.2.4

Let $(X, \mathcal{G})$ be a gradation space. Then **id** : $(X, {}^{n}\mathcal{G}) \rightarrow (X, \mathcal{G})$ is a gradation preserving map.

Proof :

Since $\mathcal{G}(f) \leq ({}^{n}\mathcal{G})(f)$, the proof is clear.

Theorem : 5.2.5

Let $(X,\mathcal{G})$ and $(Y,\mathcal{G}')$ be two gradation spaces. Then a map $\theta : (X,\mathcal{G}) \rightarrow (Y,\mathcal{G}')$ is a gradation preserving (sgp-) map iff $\theta : (X,{}^{n}\mathcal{G}) \rightarrow (Y,{}^{n}(\mathcal{G}'))$ is a gradation preserving (sgp-) map for all n.

Proof :

Assume that $\theta : (X, \mathcal{G}) \to (Y, \mathcal{G}')$ is a gradation preserving map. Then, $\mathcal{G}'(f) \leq \mathcal{G}(\theta^{-1}(f))$ for given $f \in I^Y$.

Given $f \in I^Y$ and for all n,

$$\begin{aligned}({}^{n}(\mathcal{G}'))(f) &= {}^{n}(\mathcal{G}'(f)) \\ &\leq {}^{n}(\mathcal{G}(\theta^{-1}(f)) \\ &= ({}^{n}\mathcal{G})(\theta^{-1}(f)).\end{aligned}$$

Hence $\theta : (X, {}^{n}\mathcal{G}) \to (Y, {}^{n}(\mathcal{G}'))$ is a gradation preserving map for all n.

Conversely , assume that $\theta : (X, {}^{n}\mathcal{G}) \to (Y, {}^{n}(\mathcal{G}'))$ is a gradation preserving map for all n. Then, for each $f \in I^Y$ and for all n, $({}^{n}(\mathcal{G}'))(f) \leq ({}^{n}\mathcal{G})(\theta^{-1}(f))$

$\therefore \inf\{({}^{n}(\mathcal{G}'))(f) \mid f \in I^Y\} \leq \inf\{({}^{n}\mathcal{G})(\theta^{-1}(f)) \mid f \in I^Y\}$

$\Rightarrow \mathcal{G}'(f) \leq \mathcal{G}(\theta^{-1}(f))$.

Hence, $\theta : (X, \mathcal{G}) \to (Y, \mathcal{G}')$ is a gradation preserving map.

Similary, the result can be proved for strongly gradation preserving map.

Theorem : 5.2.6

If $\theta : (X, \mathcal{G}) \to (y, \mathcal{G}')$ is a weakly gradation preserving map, then $\theta : (X, {}^{n}\mathcal{G}) \to (Y, {}^{n}(\mathcal{G}'))$ is a weakly gradation preserving map for all n. Conversely, if $\theta : (X, {}^{n}\mathcal{G}) \to (Y, {}^{n}(\mathcal{G}'))$ is weakly gradation preserving map for some n, then $\theta : (X, \mathcal{G}) \to (Y, \mathcal{G}')$ is a weakly gradation preserving map.

Proof :

Let $\theta : (X, \mathcal{G}) \to (Y, \mathcal{G}')$ be a wgp-map. Then,

$\mathcal{G}'(f) > 0 \Rightarrow \mathcal{G}(\theta^{-1}(f)) > 0$ for each $f \in I^Y$.

$({}^{n}(\mathcal{G}'))(f) > 0 \Rightarrow \mathcal{G}'(f) > 0$

$\Rightarrow \mathcal{G}(\theta^{-1}(f)) > 0$

$\Rightarrow ({}^{n}\mathcal{G})(\theta^{-1}(f)) > 0$

Hence, $\theta : (X, {}^{n}\mathcal{G}) \rightarrow (Y, {}^{n}(\mathcal{G}'))$ is a wgp - map.

Conversely, assume that $\theta : (X, {}^{n}\mathcal{G}) \rightarrow (Y, {}^{n}(\mathcal{G}'))$ is a wgp - map for some n.

Then $({}^{n}(\mathcal{G}'))(f) > 0 \Rightarrow ({}^{n}\mathcal{G})(\theta^{-1}(f)) > 0$.

$\mathcal{G}'(f) > 0 \Rightarrow ({}^{n}(\mathcal{G}'))(f) > 0$

$\Rightarrow ({}^{n}\mathcal{G})(\theta^{-1}(f)) > 0$

$\Rightarrow \mathcal{G}(\theta^{-1}(f)) > 0$

Hence, $\theta : (X, \mathcal{G}) \rightarrow (Y, \mathcal{G}')$ is a wgp-map.

Theorem : 5.2.7

Let $\theta : X \rightarrow Y$ be a map. Let $\mathcal{G}$ be a gradation of openness on X. Then the largest gradation of openness $\mathcal{G}'$ on Y which makes $\theta : (X,\mathcal{G}) \rightarrow (Y,\mathcal{G}')$ a gradation preserving map is given by $\mathcal{G}'(f) = \mathcal{G}(\theta^{-1}(f))$ for each $f \in I^{Y}$.

Proof :

For $f \in I^{Y}$, define $\mathcal{G}'(f) = \mathcal{G}(\theta^{-1}(f))$.

Claim : 1

$\mathcal{G}'$ is a gradation of openness on Y.

$\mathcal{G}$01. $\mathcal{G}'(0) = \mathcal{G}(\theta^{-1}(0)) = \mathcal{G}(0) = 1$.

$\mathcal{G}(1) = \mathcal{G}(\theta^{-1}(1)) = \mathcal{G}(1) = 1$.

$\mathcal{G}$02. Let $\mathcal{G}'(f_i) > 0$ for i = 1, 2.

Then $\mathcal{G}(\theta^{-1}(f_i)) > 0$ for i = 1, 2 .

$$\mathcal{G}'(f_1 \wedge f_2) = \mathcal{G}(\theta^{-1}(f_1 \wedge f_2))$$
$$= \mathcal{G}(\theta^{-1}(f_1) \wedge \theta^{-1}(f_2)) > 0 .$$

$\mathcal{G}$03. Let $\mathcal{G}'(f_\lambda) > 0$ for $\lambda \in \Lambda$.

Then $\mathcal{G}(\theta^{-1}(f_\lambda)) > 0$ for each $\lambda \in \Lambda$.

$\Rightarrow \mathcal{G}(\sup\{\theta^{-1}(f_\lambda) \mid \lambda \in \Lambda\}) > 0$

$$\mathcal{G}'(\sup\{f_\lambda \mid \lambda \in \Lambda\}) = \mathcal{G}(\theta^{-1}(\sup\{f_\lambda \mid \lambda \in \Lambda\}))$$
$$= \mathcal{G}(\sup\{\theta^{-1}(f_\lambda) \mid \lambda \in \Lambda\}) > 0 .$$

Claim : 2

$\mathcal{G}'$ is the largest gradation of openness on Y such that $\theta : (X,\mathcal{G}) \rightarrow (Y,\mathcal{G}')$ is a gp-map.

If $\mathcal{G}''$ is a gradation of openness on Y such that $\theta : (X,\mathcal{G}) \rightarrow (Y, \mathcal{G}'')$ is a gp- map , then $\mathcal{G}''(f) \leq \mathcal{G}(\theta^{-1}(f))$ for each $f \in I^Y$.

Therefore, $\mathcal{G}''(f) \leq \mathcal{G}'(f)$ by definition of $\mathcal{G}'$.

Hence, $\mathcal{G}'$ is the largest gradation of openness on Y such that θ is a gp-map.

Note : 5.2.8

We call the gradation of openness $\mathcal{G}'$ defined in the above theorem as the quotient gradation with respect to $\mathcal{G}$ denote this by $\mathbf{Q}_{\mathcal{G}}$.

Theorem : 5.2.9

Let $\theta : X \rightarrow Y$ be a map. Let $\mathcal{G}$ be a gradation of openness on X and $\mathcal{G}'$ be the largest gradation of openness on Y such that $\theta : (X,\mathcal{G}) \rightarrow (Y,\mathcal{G}')$ is a gp-map. Then ${}^{n}(\mathcal{G}')$ is the largest gradation of openness on Y such that $\theta : (X,{}^{n}\mathcal{G}) \rightarrow (Y, {}^{n}(\mathcal{G}'))$ is a gp-map.

Proof :

Since $\mathcal{G}'$ is the largest gradation on Y such that θ is gradation preserving, $\mathcal{G}'(f) = \mathcal{G}(\theta^{-1}(f))$.

$$\begin{aligned} ({}^{n}(\mathcal{G}'))(f) &= {}^{n}(\mathcal{G}'(f)) \\ &= {}^{n}(\mathcal{G}(\theta^{-1}(f))). \end{aligned}$$

Hence, $({}^{n}\mathcal{G}')$ is the largest gradation on Y such that

$\theta : (X, {}^{n}\mathcal{G}) \to (Y, {}^{n}(\mathcal{G}'))$ is a gp-map.

Theorem : 5.2.10

Let $\theta : X \to Y$ be a one-one onto map. Let $\mathcal{G}'$ be a gradation of openness on Y. Then the smallest gradation of openness $\mathcal{G}$ on X which makes $\theta : (X, \mathcal{G}) \to (Y, \mathcal{G}')$ a gradation preserving map is given by $\mathcal{G}(f) = \mathcal{G}'(\theta(f))$.

Proof :

Obvious.

Theorem : 5.2.11

Let $\theta : X \to Y$ be a 1-1 onto map and $\mathcal{G}'$ be a gradation of openness on Y. Let $\mathcal{G}$ be the smallest gradation of openness on X such that $\theta : (X, \mathcal{G}) \to (Y, \mathcal{G}')$ is a gp-map. Then ${}^{n}\mathcal{G}$ is the smallest gradation of openness on X such that $\theta : (X, {}^{n}\mathcal{G}) \to (Y, {}^{n}(\mathcal{G}'))$ is a gp-map.

Proof :

$$({}^n\mathcal{G})(f) = {}^n(\mathcal{G}(f))$$

$$= {}^n(\mathcal{G}'(\theta(f)))$$

$$= ({}^n\mathcal{G}')(\theta(f)).$$

Hence the theorem.

Theorem : 5.2.12

Let (X, $\mathcal{G}$) be a gradation space . Let $Y \subseteq X$. Then $({}^n\mathcal{G})_Y = {}^n(\mathcal{G}_Y)$

Proof :

$$\mathcal{G}_Y(f) = \sup \{ \mathcal{G}(h) \mid h \in I^X, h|_Y = f \}.$$

$$({}^n\mathcal{G})_Y(f) = \sup \{ ({}^n\mathcal{G})(h) \mid h|_Y = f \}$$

$$= \sup \{ {}^n(\mathcal{G}(h)) \mid h|_Y = f \}$$

$$= {}^n \{ \sup \{ \mathcal{G}(h) \mid h|_Y = f \}$$

$$= {}^n(\mathcal{G}_Y)(f).$$

Theorem : 5.2.13

If $\mathcal{G}$ is a gradation of closedness, then ${}^n\mathcal{G}$ is a gradation of closedness.

Proof :

$\mathcal{G}$C1. $({}^n\mathcal{G})(0) = {}^n(\mathcal{G}(0)) = 1$ and $({}^n\mathcal{G})(1) = {}^n(\mathcal{G}(0)) = 1$

$\mathcal{G}$C2. $({}^n\mathcal{G})(f_i) > 0$ for $i = 1, 2$

$\Rightarrow {}^n(\mathcal{G}(f_i)) > 0$ for $i = 1, 2$

$\Rightarrow \mathcal{G}(f_i) > 0$ for $i = 1, 2$

$\Rightarrow \mathcal{G}(f_1 \vee f_2) > 0$

$\Rightarrow {}^{n}(\mathcal{G}(f_1 \vee f_2) > 0$

$\Rightarrow ({}^{n}\mathcal{G})(f_1 \vee f_2) > 0$.

$\mathcal{G}$**C3.** $({}^{n}\mathcal{G})(f_\lambda) > 0$ for $\lambda \in \Lambda$

$\Rightarrow \mathcal{G}(f_\lambda) > 0$ for $\lambda \in \Lambda$

$\Rightarrow \mathcal{G}(\inf\{f_\lambda \mid \lambda \in \Lambda\}) > 0$

$\Rightarrow {}^{n}(\mathcal{G}(\inf\{f_\lambda \mid \lambda \in \Lambda\})) > 0$

$\Rightarrow ({}^{n}\mathcal{G})(\inf\{f_\lambda \mid \lambda \in \Lambda\})) > 0$.

Hence , ${}^{n}\mathcal{G}$ is a gradation of closedness.

Note : 5.2.14

If $\mathcal{G}$ is a M-gradation of openness (closedness) on X, ${}^{n}\mathcal{G}$ need not satisfy the conditions of M-gradation of openness (closedness).

Section : 5.3

Second n^{th} Order Approximations

Definition : 5.3.1

Given a gradation of openness $\mathcal{G}$, **$({}^{n}\mathcal{G})^*$ is defined as $({}^{n}\mathcal{G})^*(f) = \mathcal{G}({}^{n}f)$.**

Theorem : 5.3.2

Let (X,$\mathcal{G}$) be a gradation space. Then (X,$({}^{n}\mathcal{G})^*$) is a gradation space.

Proof :

Let $\mathcal{G}$ be a gradation of openness on X.

$\mathcal{G}$01. $({}^{n}\mathcal{G})^{*}(0) = \mathcal{G}({}^{n}0) = \mathcal{G}(0) = 1.$

$({}^{n}\mathcal{G})^{*}(1) = \mathcal{G}({}^{n}1) = \mathcal{G}(1) = 1.$

$\mathcal{G}$02. Let $({}^{n}\mathcal{G})^{*}(f_i) > 0, \quad i = 1, 2.$

Then $\mathcal{G}({}^{n}(f_i)) > 0, \ i = 1, 2.$

This implies that $\mathcal{G}({}^{n}(f_1) \wedge {}^{n}(f_2)) > 0$

$\Rightarrow \mathcal{G}({}^{n}(f_1 \wedge f_2)) > 0$

$\Rightarrow ({}^{n}\mathcal{G})^{*}(f_1 \wedge f_2) > 0 .$

$\mathcal{G}$03. Let $({}^{n}\mathcal{G})^{*}(f_\lambda) > 0$ for $\lambda \in \Lambda$.

Then $\mathcal{G}({}^{n}(f_\lambda)) > 0$ for $\lambda \in \Lambda$.

This implies that $\mathcal{G}(\sup \{{}^{n}(f_\lambda) \mid \lambda \in \Lambda\} > 0$

$\Rightarrow \mathcal{G}({}^{n}(\sup \{f_\lambda \mid \lambda \in \Lambda\})) > 0$

$\Rightarrow ({}^{n}\mathcal{G})^{*}(\sup \{f_\lambda \mid \lambda \in \Lambda\}) > 0.$

Hence , $({}^{n}\mathcal{G})^{*}$ is a gradation of openness on X.

Note : 5.3.3

$$\begin{aligned} f \in \delta(({}^{n}\mathcal{G})^{*}) &\Leftrightarrow ({}^{n}\mathcal{G})^{*}(f) > 0 \\ &\Leftrightarrow \mathcal{G}({}^{n}f) > 0 \\ &\Leftrightarrow {}^{n}f \in \delta(\mathcal{G}) \end{aligned}$$

Hence, $f \in \delta(({}^{n}\mathcal{G})^{*}) \Leftrightarrow {}^{n}f \in \delta(\mathcal{G})$.

$$\therefore \boxed{\delta(\mathcal{G})(n)} = \delta(({}^{n}\mathcal{G})^{*})$$

Note : 5.3.4

$$({}^{n}\mathcal{G})^{*}(f) = ({}^{n}\mathcal{G})^{*}({}^{n}f)$$

$$\therefore {}^{n}(\delta({}^{n}\mathcal{G})^{*}) \subseteq \delta(({}^{n}\mathcal{G})^{*}).$$

Theorem : 5.3.5

$${}^{n}(\delta(({}^{n}\mathcal{G})^{*})) = \delta(\mathcal{G}) \wedge {}^{n}(\delta(\mathcal{G})).$$

Proof :

$$\delta(\mathcal{G}) = \{ f \mid \mathcal{G}(f) > 0 \}.$$

$\delta(({}^{n}\mathcal{G})^{*}) = \{ f \mid ({}^{n}\mathcal{G})^{*}(f) > 0 \}.$

Let ${}^{n}f \in {}^{n}(\delta(({}^{n}\mathcal{G})^{*}))$.

Then $({}^{n}\mathcal{G})^{*}(f) > 0$

$\Rightarrow \mathcal{G}({}^{n}f) > 0$

$\Rightarrow {}^{n}f \in \delta(\mathcal{G})$.

Also, ${}^{n}f \in {}^{n}(\delta(\mathcal{G}))$.

$\therefore {}^{n}(\delta(({}^{n}\mathcal{G})^{*})) \subseteq \delta(\mathcal{G}) \wedge {}^{n}(\delta(\mathcal{G}))$.

Let ${}^{n}f \in \delta(\mathcal{G}) \wedge {}^{n}(\delta(\mathcal{G}))$.

Then ${}^{n}f \in \delta(\mathcal{G})$ and ${}^{n}f \in {}^{n}(\delta(\mathcal{G}))$.

$\Rightarrow \mathcal{G}({}^{n}f) > 0$

$\Rightarrow ({}^{n}\mathcal{G})^{*}(f) > 0$

$\Rightarrow {}^{n}f \in {}^{n}(\delta(({}^{n}\mathcal{G})^{*}))$.

$\therefore \delta(\mathcal{G}) \wedge {}^{n}(\delta(\mathcal{G})) \subseteq {}^{n}(\delta(({}^{n}\mathcal{G})^{*}))$.

Hence the proof.

Theorem : 5.3.6

Let $\theta : (X, \mathcal{G}) \to (Y, \mathcal{G}')$ be a gp (sgp) map.

Then $\theta : (X, (\mathcal{G})^*) \to (Y, ({}^n(\mathcal{G}'))^*)$ is a gp (sgp-) map.

Proof :

Let $\theta : (X, \mathcal{G}) \to (Y, \mathcal{G}')$ be a gp-map.

Then $\mathcal{G}'(f) \le \mathcal{G}(\theta^{-1}(f)) \quad \forall f \in I^Y$.

$$\begin{aligned}({}^n(\mathcal{G}')^*(f) &= \mathcal{G}'({}^nf) \\ &\le \mathcal{G}(\theta^{-1}({}^nf)) \\ &= \mathcal{G}({}^n(\theta^{-1}(f))) \\ &= ({}^n\mathcal{G})^*(\theta^{-1}(f)).\end{aligned}$$

Hence, $\theta : (X, ({}^n\mathcal{G})^*) \to (Y, ({}^n(\mathcal{G}'))^*)$ is a gp - map.

The proofs of the following three Theorems are similar to the proofs of the corresponding Theorems in the previous section.

Theorem : 5.3.7

Let $(X, \mathcal{G}) \to (Y, \mathcal{G}')$ be a wgp - map.

Then $\theta : (X, ({}^n\mathcal{G})^*) \to (Y, ({}^n(\mathcal{G}'))^*)$ is a wgp-map.

Theorem : 5.3.8

Let $\theta : X \to Y$ be a map. Let $\mathcal{G}$ be a gradation of openness on X, and $\mathcal{G}'$ be the largest gradation of openness on Y such that $\theta : (X, \mathcal{G}) \to (Y, \mathcal{G}')$ is a gp-map. Then $({}^n(\mathcal{G}'))^*$ is the largest gradation of openness on Y such that $\theta : (X, ({}^n\mathcal{G})^*) \to (Y, ({}^n(\mathcal{G}'))^*)$ is a gp-map.

Theorem : 5.3.9

Suppose $\theta : X \to Y$ is a 1- 1 onto map. Let $\mathcal{G}'$ be the gradation of openness on Y, and $\mathcal{G}$ be the smallest gradation of openness on X such that $\theta : (X, \mathcal{G}) \to (Y, \mathcal{G}')$ is a gp-map. Then $({}^{n}\mathcal{G})^*$ is the smallest gradation on X such that $\theta : (X, ({}^{n}\mathcal{G})^*) \to (Y, ({}^{n}(\mathcal{G}'))^*)$ is a gp-map.

Theorem : 5.3.10

Let $\mathcal{G}$ be M-gradation of openness on X. Then $({}^{n}\mathcal{G})^*$ defined by $({}^{n}\mathcal{G})^*(f) = \mathcal{G}({}^{n}f)$ for each $f \in I^X$ is a M-gradation of opennesson X.

Proof :

For each $f \in I^X$, $({}^{n}\mathcal{G})^*$ is defined by $({}^{n}\mathcal{G})^*(f) = \mathcal{G}({}^{n}f)$

M$\mathcal{G}$01. $({}^{n}\mathcal{G})^*(0) = ({}^{n}\mathcal{G})^*(1) = 1$.

M$\mathcal{G}$02. Let $({}^{n}\mathcal{G})^*(f_i) \geq r$ for $i = 1, 2$.

Then $\mathcal{G}({}^{n}(f_i)) \geq r$ for $i = 1, 2$.

This implies that $\mathcal{G}({}^{n}(f_1) \wedge {}^{n}(f_2)) \geq r$

$\Rightarrow \mathcal{G}({}^{n}(f_1 \wedge f_2)) \geq r$

$\Rightarrow ({}^{n}\mathcal{G})^*(f_1 \wedge f_2) \geq r$.

M$\mathcal{G}$03. Let $({}^{n}\mathcal{G})^*(f_\lambda) \geq r$ for $\lambda \in \Lambda$.

Then $\mathcal{G}({}^{n}(f_\lambda)) \geq r$ for $\lambda \in \Lambda$.

This implies that $\mathcal{G}(\sup\{{}^{n}(f_\lambda) \mid \lambda \in \Lambda\}) \geq r$

$\Rightarrow \mathcal{G}({}^{n}(\sup\{f_\lambda \mid \lambda \in \Lambda\})) \geq r$

$\Rightarrow ({}^{n}\mathcal{G})^*(\sup\{f_\lambda \mid \lambda \in \Lambda\}) \geq r$.

Hence , $({}^{n}\mathcal{G})^*$ is a M-gradation of openness on X.

Note : 5.3.11

If $\mathcal{G}$ is a M-gradation of openness on X, results parallel to Theorem 5.3.6 to Theorem 5.3.9 follow immediately.

Section : 5.4

Third nth Order Approximations

Definition : 5.4.1

Let $\mathcal{G}$ be a M-gradation of openness on X. Then $_n\mathcal{G}$ **is defined by** $(_n\mathcal{G})(f) = {}_n(\mathcal{G}(f))$ **for each** $f \in I^X$.

Theorem : 5.4.2

If (X, $\mathcal{G}$) be a M-gradation space, then (X, $_n\mathcal{G}$) is a M-gradation space.

Proof :

Let (X,$\mathcal{G}$) be a M- gradation of openness on X.

M$\mathcal{G}$01. $(_n\mathcal{G})(0) = (_n\mathcal{G})(1) = 1$.

M$\mathcal{G}$02. Let $(_n\mathcal{G})(f_i) \geq r$ for i = 1,2.

Suppose $\frac{j}{n} < r \leq \frac{j+1}{n}$,

$(_n\mathcal{G})(f_i) \geq \frac{j+1}{n}$, i =1,2

$\Rightarrow \mathcal{G}(f_i) \geq \frac{j+1}{n}$, i = 1,2

$\Rightarrow \mathcal{G}(f_1 \wedge f_2) \geq \frac{j+1}{n}$

$$\Rightarrow ({}_n\mathcal{G})(f_1 \wedge f_2) \geq \frac{j+1}{n}$$

$$\Rightarrow ({}_n\mathcal{G})(f_1 \wedge f_2) \geq r.$$

If $\frac{j}{n} = r$, then $({}_n\mathcal{G})(f_i) \geq r = \frac{j}{n}$, for $i = 1,2$.

$$\Rightarrow \mathcal{G}(f_i) \geq \frac{j}{n}, \text{ for } i = 1,2.$$

$$\Rightarrow \mathcal{G}(f_1 \wedge f_2) \geq \frac{j}{n}$$

$$\Rightarrow {}_n(\mathcal{G}(f_1 \wedge f_2)) \geq \frac{j}{n}$$

$$\Rightarrow ({}_n\mathcal{G})(f_1 \wedge f_2) \geq r.$$

M$\mathcal{G}$03. Let $({}_n\mathcal{G})(f_\lambda) \geq r$ for $\lambda \in \Lambda$.

Suppose $\frac{j}{n} < r \leq \frac{j+1}{n}$,

$$({}_n\mathcal{G})(f_\lambda)) \geq \frac{j+1}{n} \text{ for } \lambda \in \Lambda$$

$$\Rightarrow {}_n(\mathcal{G}(f_\lambda)) \geq \frac{j+1}{n} \text{ for } \lambda \in \Lambda$$

$$\Rightarrow \mathcal{G}(f_\lambda) \geq \frac{j+1}{n} \text{ for } \lambda \in \Lambda$$

$$\Rightarrow \mathcal{G}(\sup\{f_\lambda \mid \lambda \in \Lambda\}) \geq \frac{j+1}{n}$$

$$\Rightarrow {}_n(\mathcal{G}(\sup\{f_\lambda \mid \lambda \in \Lambda\})) \geq \frac{j+1}{n}$$

$$\Rightarrow ({}_n\mathcal{G})(\sup\{f_\lambda \mid \lambda \in \Lambda\}) \geq \frac{j+1}{n}$$

$\Rightarrow ({}_n\mathcal{G})(\sup\{f_\lambda \mid \lambda \in \Lambda\}) \geq r.$

If $r = \frac{j}{n}$, then $({}_n\mathcal{G})(f_\lambda) \geq \frac{j}{n}$ for $\lambda \in \Lambda$.

$\Rightarrow \mathcal{G}(f_\lambda) \geq \frac{j}{n}$ for $\lambda \in \Lambda$

$\Rightarrow \mathcal{G}(\sup\{f_\lambda \mid \lambda \in \Lambda\}) \geq \frac{j}{n}$

$\Rightarrow {}_n(\mathcal{G}(\sup\{f_\lambda \mid \lambda \in \Lambda\})) \geq \frac{j}{n}$

$\Rightarrow ({}_n\mathcal{G})(\sup\{f_\lambda \mid \lambda \in \Lambda\}) \geq r.$

Hence, ${}_n\mathcal{G}$ is a M-gradation of openness on X.

Remark : 5.4.3

Results parallel to Theorem 5.2.4 to Theorem 5.2.11 of section 5.2 are true for the gradation introduced in this section immediately. An interesting fact about this new definition is stated in the next Theorem.

Theorem : 5.4.4

If $\mathcal{G}$ is a M-gradation of closedness. Then ${}_n\mathcal{G}$ is a M-gradation of closedness.

Note : 5.4.5

If $\mathcal{G}$ is a gradation of openness (closedness), then ${}_n\mathcal{G}$ need not be a gradation of openness (closedness).

Theorem : 5.4.6

Let $\mathcal{G}$ be a gradation (M-gradation) of closedness. Then $({}_n\mathcal{G})^*$ defined by $({}_n\mathcal{G})^*(f) = \mathcal{G}({}_n f)$ is a gradation (M-gradation) of closedness.

Proof :

$\mathcal{G}$C1. $({}_n\mathcal{G})^*(0) = 1$ and $({}_n\mathcal{G})^*(1) = 1$.

$\mathcal{G}$C2. $({}_n\mathcal{G})^*(f_i) > 0$ for $i = 1,2$

$\Rightarrow \mathcal{G}({}_n(f_i)) > 0$ for $i = 1, 2$

$\Rightarrow \mathcal{G}({}_n(f_1) \vee {}_n(f_2)) > 0$

$\Rightarrow \mathcal{G}({}_n(f_1 \vee f_2)) > 0$

$\Rightarrow ({}_n\mathcal{G})^*(f_1 \vee f_2)) > 0.$

$\mathcal{G}$C3. $({}_n\mathcal{G})^*(f_\lambda) > 0$ for $\lambda \in \Lambda$

$\Rightarrow \mathcal{G}({}_n(f_\lambda)) > 0$ for $\lambda \in \Lambda$

$\Rightarrow \mathcal{G}(\inf\{{}_n(f_\lambda) \mid \lambda \in \Lambda\}) > 0$

$\Rightarrow \mathcal{G}({}_n(\inf\{f_\lambda \mid \lambda \in \Lambda\})) > 0$

$\Rightarrow ({}_n\mathcal{G})^*(\inf\{f_\lambda \mid \lambda \in \Lambda\}) > 0.$

Hence, $({}_n\mathcal{G})^*$ is a gradation of closedness.

Note : 5.4.7

Since $\sup\{{}_n(f_\lambda) \mid \lambda \in \Lambda\} \neq {}_n(\sup\{f_\lambda \mid \lambda \in \Lambda\})$, $({}_n\mathcal{G})^*$ need not be a gradation (M-gradation) of openness , where $\mathcal{G}$ is a gradaion (M-gradation) of openness.

Section : 5.5

Fourth n^{th} Order Approximations

Definition : 5.5.1

Given a gradation of openness $\mathcal{G}$ on X, $({}^{n}\mathcal{G})^{*}$ **is defined as** follows :

$$({}^{n}\mathcal{G})^{*}(f) = 0 \quad \text{if } f \neq {}^{n}f.$$

$$({}^{n}\mathcal{G})^{*}({}^{n}f) = \sup\{\, \mathcal{G}(g) \mid {}^{n}g = {}^{n}f \,\}.$$

Theorem : 5.5.2

Let $\mathcal{G}$ be a gradation of openness defined on X. Then $({}^{n}\mathcal{G})^{*}$ is a gradation of openness on X. If $\mathcal{G}$ induces the fuzzy topology δ, then $({}^{n}\mathcal{G})^{*}$ induces the fuzzy topology ${}^{n}\delta$.

Proof :

$\mathcal{G}$01. $({}^{n}\mathcal{G})^{*}(0) = \sup\{\mathcal{G}(g) \mid {}^{n}g = 0\} = 1$.

$({}^{n}\mathcal{G})^{*}(1) = \sup\{\mathcal{G}(g) \mid {}^{n}g = 1\} = 1$.

$\mathcal{G}$02. Let $({}^{n}g)^{*}(f_i) > 0$ for i = 1, 2.

Then $f_i = {}^{n}(f_i)$ for i=1,2 and there exist g_i such that ${}^{n}(g_i) = {}^{n}(f_i)$, $\mathcal{G}(g_i) > 0$ for i = 1,2.

$f_1 \wedge f_2 = {}^{n}(f_1) \wedge {}^{n}(f_2) = {}^{n}(f_1 \wedge f_2)$, and ${}^{n}(g_1 \wedge g_2) = {}^{n}(f_1 \wedge f_2)$ such that $\mathcal{G}(g_1 \wedge g_2) > 0$.

$\therefore\ ({}^{n}\mathcal{G})^{*}(f_1 \wedge f_2) > 0$.

$\mathcal{G}$03. Let $({}^{n}\mathcal{G})^{*}(f_\lambda) > 0$ for each $\lambda \in \Lambda$.

Then $f_\lambda = {}^{n}(f_\lambda)$ and there exist g_λ such that ${}^{n}(g_\lambda) = {}^{n}(f_\lambda)$, $\mathcal{G}(g_\lambda) > 0$ for each $\lambda \in \Lambda$.

$(\sup \{ f_\lambda \mid \lambda \in \Lambda\}) = {}^n(\sup\{ f_\lambda \mid \lambda \in \Lambda\}) = {}^n(\sup\{ g_\lambda \mid \lambda \in \Lambda\})$

such that $\mathcal{G} (\sup \{ g_\lambda \mid \lambda \in \Lambda \}) > 0$.

$\therefore ({}^n\mathcal{G})^{\clubsuit} (\sup \{ f_\lambda \mid \lambda \in \Lambda\}) > 0$.

Hence , $({}^n\mathcal{G})^{\clubsuit}$ is a gradation of openness on X.

$\mathcal{G}$ induces δ implies that $\delta = \{ f \mid \mathcal{G}(f) > 0 \}$.

$\delta (({}^n\mathcal{G})^{\clubsuit}) = \{ {}^nf \mid ({}^n\mathcal{G})^{\clubsuit} ({}^nf) > 0 \}$

$= \{ {}^nf \mid \exists g , \mathcal{G}(g) > 0, {}^nf = {}^ng \}$.

${}^nf \in \delta (({}^n\mathcal{G})^{\clubsuit}) \Rightarrow \exists g$ such that ${}^nf = {}^ng$, $g \in \delta$

$\Rightarrow {}^nf = {}^ng \in {}^n\delta$.

${}^nf \in {}^n\delta \Rightarrow \exists g \in \delta$ such that ${}^nf = {}^ng$

$\Rightarrow \exists g , \mathcal{G}(g) > 0, {}^nf = {}^ng$

$\Rightarrow {}^nf \in \delta (({}^n\mathcal{G})^{\clubsuit})$.

Hence, $({}^n\mathcal{G})^{\clubsuit}$ induces ${}^n\delta$.

Theorem : 5.5.3

Let $\mathcal{G}$ be a M-gradation of openness defined on X. Then $({}^n\mathcal{G})^{\clubsuit}$ is a M-gradation of openness on X.

Proof :

M$\mathcal{G}$01. Obvious .

M$\mathcal{G}$02. Let $({}^n\mathcal{G})^{\clubsuit} (f_i) \geq r$ for $i = 1,2$ and $0 < r \leq 1$.

Then $f_i = {}^n(f_i)$ for i =1, 2. Given $\varepsilon > 0$, there exist g_i such that

${}^n(g_i) = {}^n(f_i)$ and $\mathcal{G}(g_i) \geq r - \varepsilon$ for i =1,2.

$f_1 \wedge f_2 = {}^{n}(f_1 \wedge f_2)$ and ${}^{n}(g_1 \wedge g_2) = {}^{n}(f_1 \wedge f_2)$

such that $\mathcal{G}(g_1 \wedge g_2) \geq r - \varepsilon$.

$\therefore ({}^{n}\mathcal{G})^{*}(f_1 \wedge f_2) \geq r$.

M$\mathcal{G}$03. Let $({}^{n}\mathcal{G})^{*}(f_\lambda) \geq r$ for each $\lambda \in \Lambda$ and $0 \leq r \leq 1$.

Then $f_\lambda = {}^{n}(f_\lambda)$ for each $\lambda \in \Lambda$.

Given $0 < \varepsilon < r$, for each λ, there exist g_λ such that ${}^{n}(g_\lambda) = {}^{n}(f_\lambda)$

and $\mathcal{G}(g_\lambda) \geq r - \varepsilon$.

$\sup\{f_\lambda \mid \lambda \in \Lambda\} = {}^{n}(\sup\{f_\lambda \mid \lambda \in \Lambda\})$ and

${}^{n}(\sup\{f_\lambda \mid \lambda \in \Lambda\}) = {}^{n}(\sup\{g_\lambda \mid \lambda \in \Lambda\})$

such that $\mathcal{G}(\sup\{g_\lambda \mid \lambda \in \Lambda\}) \geq r - \varepsilon$.

$\therefore ({}^{n}\mathcal{G})^{*}(\sup\{f_\lambda \mid \lambda \in \Lambda\}) \geq r$.

Section : 5.6

Functorial Properties

Definition : 5.6.1

(i). $\mathscr{C}\mathcal{G}$: Category with objects gradation spaces and morphisms gradation preserving maps.

(ii). $\mathscr{C}_n\mathcal{G}$: Subcategory of gradation spaces of the form $(X, \mathcal{G})$ where $\mathcal{G}(f) \in I_n$.

Theorem : 5.6.2

(i). The correspondence $(X,\mathcal{G}) \rightarrow (X,{}^{n}\mathcal{G})$ defines a coreflective functor $\zeta^{n}_{\mathcal{G}}$ from $\mathcal{C}\mathcal{G}$ to $\mathcal{C}_{n}\mathcal{G}$.

(ii). The correspondence $(X,\mathcal{G}) \rightarrow (X,{}_{n}\mathcal{G})$ defines a reflective functor from $\mathcal{C}\mathcal{G}$ to $\mathcal{C}_{n}\mathcal{G}$.

Proof :

(i). **id** : $(X, {}^{n}\mathcal{G}) \rightarrow (X,\mathcal{G})$ is a morphism in $\mathcal{C}\mathcal{G}$. Given a morphism $\theta : (Y, {}^{n}(\mathcal{G}')) \rightarrow (X,\mathcal{G})$, it is easy to note that $\theta : (Y, {}^{n}(\mathcal{G}')) \rightarrow (X, {}^{n}\mathcal{G})$ is also a morphism.

$$\begin{tikzcd}
(X, \mathcal{G}) & & (X, {}^{n}\mathcal{G}) \arrow[ll, "\mathbf{id}"'] \\
& (Y, {}^{n}(\mathcal{G}')) \arrow[ul, "\theta"] \arrow[ur, "\theta"'] &
\end{tikzcd}$$

(ii). **id** : $(X,\mathcal{G}) \rightarrow (X , {}_{n}\mathcal{G})$ is a morphism in $\mathcal{C}\mathcal{G}$. Given a morphism $\theta : (X, \mathcal{G}) \rightarrow (Y, {}_{n}(\mathcal{G}'))$, it is easy to note that $\theta : (X, {}_{n}\mathcal{G}) \rightarrow (Y, {}_{n}(\mathcal{G}'))$ is a morphism.

$$\begin{tikzcd}
(X, \mathcal{G}) \arrow[rr, "\mathbf{id}"] \arrow[dr, "\theta"'] & & (X, {}_{n}\mathcal{G}) \arrow[dl, "\theta"] \\
& (Y, {}_{n}(\mathcal{G}')) &
\end{tikzcd}$$

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NOTATIONS

NOTATIONS

				PAGE NO.
1.	I	-	closed unit interval [0,1]	22
2.	I_n	-	$\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots\ldots 1\right\}$	6
3.	$X, Y, Z, \ldots$	-	non empty sets	23
4.	$f, g, h, \ldots$	-	fuzzy sets	23
5.	i	-	identity function	35
6.	I^X	-	all functions $f: X \to I$	46
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The above list of notations and terminology is not applicable to the review of literature.

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