Classes of Fuzzy Ideals in Z-Algebras

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In 1994, Jun and Meng [32] introduced the notion of fuzzy p-ideals and in 1999, Khalid and Ahmad [40] introduced the concept of fuzzy H-ideals in BCI-algebras and studied their properties. In 1997, Meng et al. [49] and Mostafa [50] fuzzified the concept of implicative ideals in BCK-algebras, independently.

In this Chapter, we introduce the notions of **Fuzzy H-Ideals, Fuzzy p-Ideals and Fuzzy Implicative Ideals in Z-Algebras**. This chapter is divided into three sections. In the first section, we obtained some interesting results in Fuzzy H-ideals of Z-algebras while the second section deals with the study of Fuzzy p-ideals in Z-algebras. In the third section, we discuss the notion of Fuzzy implicative ideals in Z-algebras. Further, the relationship between fuzzy Z-ideal, fuzzy implicative ideal and fuzzy sub-implicative ideal of a Z-algebra are also obtained.

3.1 Fuzzy H-Ideals in Z-algebras

In this section, we introduce the notion of Fuzzy H-ideals in Z-algebras and prove some simple but elegant results.

Definition 3.1.1: Let $(X,*,0)$ be a Z-algebra and I be a subset of X. Then, I is called an **H-ideal** of X, if it satisfies the following conditions: For all x, y, z in X,

(i) $0 \in I$

(ii) $x*(y*z) \in I$ and $y \in I \Rightarrow x*z \in I$

Example 3.1.2: Consider a Z-algebra X= {0, 1, 2, 3} be a set with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	0	1	1	3
2	0	1	2	2
3	0	3	2	3

Then, $I = \{0,1,2\} \subset X$ is an H-Ideal of X.

Definition 3.1.3: Let $(X,*,0)$ be a Z-algebra. A fuzzy set A in X with membership function μ_A is said to be a **fuzzy H-ideal** of a Z-algebra X if it satisfies the following conditions: For all x, y , z in X,

(i) $\mu_A(0) \geq \mu_A(x)$

(ii) $\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$

Example 3.1.4: Let X= {0, 1, 2, 3} be a set with the following Cayley table:

*	**0**	**1**	**2**	**3**
0	0	1	2	3
1	0	1	3	3
2	0	3	2	2
3	0	3	2	3

Then $(X,*,0)$ is a Z-algebra.

Define a fuzzy set A with membership function μ_A as $\mu_A(x) = \begin{cases} 0.6 & \text{if } x = 0 \\ 0.2 & \text{if } x = 1,2,3 \end{cases}$.

Then A is a fuzzy H-ideal of X .

Definition 3.1.5: Let $(X,*,0)$ be a Z-algebra. Then X is called an **associative Z-algebra** if

$x * (y * z) = (x * y) * z, \ \forall\ x, y, z \in X.$

Remark 3.1.6: In an associative Z-algebra X, $(x * y) * z = (x * z) * y, \ \forall\ x, y, z \in X.$

Since $(x * y) * z = x * (y * z) = x * (z * y)$ by (Z4)

$$= (x * z) * y$$

Theorem 3.1.7: In an associative Z-algebra X, every fuzzy H-ideal of X is a fuzzy Z-ideal of X. Converse is also true.

Proof: Let A be a fuzzy H-ideal of an associative Z-algebra X.

Then, $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$ (1)

For all $x, y, z \in X$,

$$\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$$
$$= \min\{\mu_A((x * y) * z), \mu_A(y)\}$$

$= \min\{\mu_A((x*z)*y), \mu_A(y)\}$ by Remark 3.1.6

Put z = x,

$\mu_A(x*x) \geq \min\{\mu_A((x*x)*y), \mu_A(y)\}$

$\mu_A(x) \geq \min\{\mu_A(x*y), \mu_A(y)\}$ (2)

From (1) and (2), A is a fuzzy Z-ideal of X.

Conversely, assume that A is a fuzzy Z-ideal of an associative Z-algebra X.

Then, $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$ (3)

For all $x, y, z \in X$,

$\min\{\mu_A(x*(y*z)), \mu_A(y)\} = \min\{\mu_A((x*y)*z), \mu_A(y)\}$

$= \min\{\mu_A((x*z)*y), \mu_A(y)\}$ by Remark 3.1.6

$\leq \mu_A(x*z)$ (4)

From (3) and (4), A is a fuzzy H-ideal of X.

Remark 3.1.8: If a Z-algebra $(X,*,0)$ is not an associative then every fuzzy H-ideal of X is not a fuzzy Z-ideal of X. This is justified by the following example.

Let X= {0, 1, 2, 3} be a set with the following Cayley table:

*	**0**	**1**	**2**	**3**
0	0	1	2	3
1	0	1	3	3
2	0	3	2	1
3	0	3	1	3

Then $(X,*,0)$ is a Z-algebra with $x*(y*z) \neq (x*y)*z$.

Define a fuzzy set A with membership function μ_A in X as $\mu_A(x) = \begin{cases} 0.7 & \text{if} \quad x = 0 \\ 0.1 & \text{if} \quad x = 1,2,3 \end{cases}$.

Then A is a fuzzy H-ideal of a Z-algebra X .

But , it is not a fuzzy Z-ideal of X. Since $\mu_A(1) = 0.1 \ngeq 0.7 = \min\{\mu_A(1*0), \mu_A(0)\}$.

Theorem 3.1.9: Arbitrary intersection of fuzzy H-ideals of a Z-algebra X is also a fuzzy H-ideal.

Proof: Let $\{A_i | i \in \Omega\}$ be a family of fuzzy H-ideals of a Z-algebra X.

For any x, y $\in$ X,

$\mu_{\underset{i\in\Omega}{\cap} A_i}(0) = \inf_{i\in\Omega}(\mu_{A_i}(0)) \geq \inf_{i\in\Omega}(\mu_{A_i}(x)) = \mu_{\underset{i\in\Omega}{\cap} A_i}(x)$

and $\mu_{\underset{i\in\Omega}{\cap} A_i}(x * z) = \inf_{i\in\Omega}(\mu_{A_i}(x * z)) \geq \inf_{i\in\Omega}(\min\{\mu_{A_i}(x * (y * z)), \mu_{A_i}(y)\})$

$= \min\{\inf_{i\in\Omega}(\mu_{A_i}(x*(y*z))), \inf_{i\in\Omega}(\mu_{A_i}(y))\}$

$= \min\{\mu_{\underset{i\in\Omega}{\cap} A_i}(x*(y*z)), \mu_{\underset{i\in\Omega}{\cap} A_i}(y)\}$

Hence $\underset{i\in\Omega}{\cap} A_i$ is a fuzzy H-ideal of a Z-algebra X.

Hence the proof.

Theorem 3.1.10: A fuzzy set A of a Z-algebra $(X,*,0)$ is a fuzzy H-ideal of X if and only if for any $t \in [0,1]$, $U(\mu_A;t) = \{x \in X \mid \mu_A(x) \geq t\}$ is an H-ideal of X where $U(\mu_A;t) \neq \phi$.

Proof: Suppose A is a fuzzy H-ideal of a Z-algebra X and $U(\mu_A;t) \neq \phi$ for any $t \in [0,1]$.

Let $x \in U(\mu_A;t)$, then $\mu_A(x) \geq t$ and $\mu_A(0) \geq \mu_A(x) \geq t$. Thus $0 \in U(\mu_A;t)$.

If $x*(y*z) \in U(\mu_A;t)$ and $y \in U(\mu_A;t)$, then $\mu_A(x*(y*z)) \geq t$ and $\mu_A(y) \geq t$.

$\mu_A(x*z) \geq \min\{\mu_A(x*(y*z)), \mu_A(y)\} \geq \min\{t,t\} = t$.

Therefore $x*z \in U(\mu_A;t)$. Hence $U(\mu_A;t)$ is an H-ideal of a Z-algebra X.

Conversely, suppose that for each $t \in [0,1]$, $U(\mu_A;t)$ is either empty or an H-ideal of a Z-algebra X.

For any $x \in X$, let $\mu_A(x) = t$. Then $x \in U(\mu_A;t)$.

Since $U(\mu_A;t) \neq \phi$ is an H-ideal of X, we have $0 \in U(\mu_A;t)$.

Thus $\mu_A(0) \geq t = \mu_A(x)$ for all $x \in X$.

Assume $\mu_A(x*z) \geq \min\{\mu_A(x*(y*z)), \mu_A(y)\}$ for all $x, y, z \in X$ is not true.

Then there exists $x_0, y_0, z_0 \in X$ such that

$\mu_A(x_0 * z_0) < \min\{\mu_A(x_0*(y_0*z_0)), \mu_A(y_0)\}$

Let $t_0 = \frac{1}{2}[\mu_A(x_0*z_0) + \min\{\mu_A(x_0*(y_0*z_0)), \mu_A(y_0)\}]$

Then $\mu_A(x_0 * z_0) < t_0 < \min\{\mu_A(x_0 * (y_0 * z_0)), \mu_A(y_0)\}$

This implies $x_0 * (y_0 * z_0)$ and $y_0 \in U(\mu_A; t_0)$ but $x_0 * z_0 \notin U(\mu_A; t_0)$

This is a contradiction.

Therefore $\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$, for all $x, y, z \in X$.

Hence A is a fuzzy H-ideal of a Z-algebra X.

Theorem 3.1.11: A fuzzy set A of a Z-algebra $(X,*,0)$ is a fuzzy H-ideal if and only if every nonempty upper q-level subset $U(\mu_A; q)$ for $q \in \mathrm{Im}(A)$ is an H-ideal.

Proof : Let A be a fuzzy H-ideal of a Z-algebra X.

Since $U(\mu_A; q) \neq \phi$ there exists $x \in U(\mu_A; q)$ such that $\mu_A(x) \geq q$.

For this $x \in U(\mu_A; q)$, $\mu_A(0) \geq \mu_A(x) \geq q$, which shows that $0 \in U(\mu_A; q)$.

Now, for any $x, y \in X$, assume that $x * (y * z) \in U(\mu_A; q)$ and $y \in U(\mu_A; q)$

Then $\mu_A(x * (y * z)) \geq q$ and $\mu_A(y) \geq q$ and $\min\{\mu_A(x * (y * z)), \mu_A(y)\} \geq q$.

Hence $\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\} \geq q$,

Thus $x * z \in U(\mu_A; q)$, this proves that $U(\mu_A; q)$ is an H-ideal of a Z-algebra X.

Conversely, let $U(\mu_A; q)$ for $q \in \mathrm{Im}(A)$ be an H-ideal of a Z-algebra X.

Let $x, y, z \in X$. For any $q \in \mathrm{Im}(A)$, let $q = \min\{\mu_A(x * (y * z)), \mu_A(y)\}$.

Therefore, $\mu_A(x * (y * z)) \geq q$ and $\mu_A(y) \geq q$.

This shows that $x * (y * z), y \in U(\mu_A; q)$.

Since $U(\mu_A; q)$ is an H-ideal , we have $x * z \in U(\mu_A; q)$.

This implies that $\mu_A(x * z) \geq q = \min\{\mu_A(x * (y * z)), \mu_A(y)\}$

Hence A is a fuzzy H-ideal of a Z-algebra X.

Theorem 3.1.12: Let $h : (X,*,0) \to (Y,*',0')$ be a Z-homomorphism of Z-algebras. If B is a fuzzy H-ideal of Y, then $h^{-1}(B)$ is a fuzzy H-ideal of X.

Proof: For any x $\in$ X, we have

(i) $\mu_{h^{-1}(B)}(x) = \mu_B(h(x)) \leq \mu_B(0') = \mu_B(h(0)) = \mu_{h^{-1}(B)}(0)$

(ii) Let x, y ,z$\in$ X. Then

$$\min\{\mu_{h^{-1}(B)}(x*(y*z)),\mu_{h^{-1}(B)}(y)\} = \min\{\mu_B(h(x*(y*z))),\mu_B(h(y))\}$$

$$= \min\{\mu_B(h(x)*'(h(y)*'h(z))),\mu_B(h(y))\}$$

$$\leq \mu_B(h(x*z))$$

$$= \mu_{h^{-1}(B)}(x*z)$$

$$\Rightarrow \quad \mu_{h^{-1}(B)}(x*z) \geq \min\{\mu_{h^{-1}(B)}(x*(y*z)),\mu_{h^{-1}(B)}(y)\}$$

From (i) and (ii) we get, $h^{-1}(B)$ is a fuzzy H-ideal of a Z-algebra X.

Theorem 3.1.13: Let $h:(X,*,0)\to(Y,*',0')$ be an Z-epimorphism of Z-algebras. Let B be a fuzzy set of Y. If $h^{-1}(B)$ is a fuzzy H-ideal of X then B is a fuzzy H-ideal of Y.

Proof: Let y $\in$ Y, there exists x $\in$ X such that h (x) = y. Then

$$\mu_B(y) = \mu_B(h(x)) = \mu_{h^{-1}(B)}(x) \leq \mu_{h^{-1}(B)}(0) = \mu_B(h(0)) = \mu_B(0')$$

This implies, $\mu_B(0') \geq \mu_B(y)$

Let x, y, z $\in$ Y. Then there exists a, b, c $\in$ X such that h(a) = x , h(b)=y and h (c) = z. It follows that

$$\mu_B(x*'z) = \mu_B(h(a)*'h(c)) = \mu_B(h(a*c)) = \mu_{h^{-1}(B)}(a*c)$$

$$\geq \min\{\mu_{h^{-1}(B)}(a*(b*c)),\mu_{h^{-1}(B)}(b)\}$$

$$= \min\{\mu_B(h(a*(b*c))),\mu_B(h(b))\}$$

$$= \min\{\mu_B(h(a)*'(h(b)*'h(c))),\mu_B(h(b))\}$$

$$= \min\{\mu_B(x*'(y*'z)),\mu_B(y)\}$$

Hence B is a fuzzy H-ideal of a Z-algebra Y.

Analogously, we can prove the following result.

Theorem 3.1.14: Let h be an Z-endomorphism of Z-algebra X and A be a fuzzy set in X. Then we define a **new fuzzy set** A^h in X as $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$ is a fuzzy H-ideal of X if A is a fuzzy H-ideal .

Theorem 3.1.15: If A and B be fuzzy H-ideals in a Z-algebra X then $A \times B$ is a fuzzy H-ideal in $X \times X$.

Proof: Let $(x_1, x_2) \in X\times X$,

$$\mu_{A\times B}(0,0) = \min\{\mu_A(0), \mu_B(0)\} \geq \min\{\mu_A(x_1), \mu_B(x_2)\} = \mu_{A\times B}(x_1, x_2) \quad (1)$$

Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X\times X$. Then,

$$\begin{aligned}
\mu_{A\times B}(x * z) &= \mu_{A\times B}((x_1, x_2) * (z_1, z_2)) \\
&= \mu_{A\times B}(x_1 * z_1, x_2 * z_2) \\
&= \min\{\mu_A(x_1 * z_1), \mu_B(x_2 * z_2)\} \\
&\geq \min\{\min\{\mu_A(x_1 * (y_1 * z_1)), \mu_A(y_1)\}, \min\{\mu_B(x_2 * (y_2 * z_2)), \mu_B(y_2)\}\} \\
&= \min\{\min\{\mu_A(x_1 * (y_1 * z_1)), \mu_B(x_2 * (y_2 * z_2))\}, \min\{\mu_A(y_1), \mu_B(y_2)\}\} \\
&= \min\{\mu_{A\times B}(x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)), \mu_{A\times B}(y_1, y_2)\} \\
&= \min\{\mu_{A\times B}((x_1, x_2) * ((y_1 * z_1), (y_2 * z_2))), \mu_{A\times B}(y_1, y_2)\} \\
&= \min\{\mu_{A\times B}((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))), \mu_{A\times B}(y_1, y_2)\} \quad (2)
\end{aligned}$$

By (1) and (2) we get, $A\times B$ is a fuzzy H-ideal in $X\times X$.

Theorem 3.1.16: Let A and B be fuzzy sets in a Z-algebra X such that $A\times B$ is a fuzzy H-ideal of $X\times X$. Then,

(i) Either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$.

(ii) If $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$, then either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$

(iii) If $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$ then either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$

Proof: (i) If $\mu_A(0) < \mu_A(x_1)$ and $\mu_B(0) < \mu_B(x_2)$ for some $x_1, x_2 \in X$.

Then, $\mu_{A\times B}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\} > \min\{\mu_A(0), \mu_B(0)\}$

$= \mu_{A\times B}(0,0)$, which is a contradiction.

Hence, either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$, for all $x \in X$.

(ii) Let $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$.

Assume that there exists $x_1, x_2 \in X$ such that $\mu_B(0) < \mu_A(x_1)$ and $\mu_B(0) < \mu_B(x_2)$.

Then, $\mu_{A\times B}(0,0) = \min\{\mu_A(0), \mu_B(0)\} = \mu_B(0)$

$\mu_{A\times B}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\} > \mu_B(0) = \mu_{A\times B}(0,0)$

$\Rightarrow \mu_{A\times B}(x_1, x_2) > \mu_{A\times B}(0,0)$, which is a contradiction.

Hence either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$, for all $x \in X$.

(iii) will obtain by interchanging the roles of A and B in part (ii).

Theorem 3.1.17: Let A and B be fuzzy sets in a Z-algebra X and $A \times B$ is a fuzzy H-ideal of $X \times X$ then either A or B is a fuzzy H-ideal of X.

Proof : By Theorem 3.1.16(i), we can assume that $\mu_B(0) \geq \mu_B(x)$, for all $x \in X$. Then, by Theorem 3.1.16(iii), either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$.

Let $\mu_A(0) \geq \mu_B(x)$ for any $x \in X$, then

$\mu_B(x) = \min\{\mu_A(0), \mu_B(x)\} = \mu_{A\times B}(0, x)$

$$
\begin{aligned}
\mu_B(x * z) &= \min\{\mu_A(0), \mu_B(x * z)\} \\
&= \mu_{A\times B}(0, x * z) \\
&= \mu_{A\times B}(0 * 0, x * z) \\
&= \mu_{A\times B}((0, x) * (0, z)) \\
&\geq \min\{\mu_{A\times B}((0, x) * ((0, y) * (0, z))), \mu_{A\times B}(0, y)\} \\
&= \min\{\mu_{A\times B}((0, x) * (0 * 0, y * z)), \mu_{A\times B}(0, y)\} \\
&= \min\{\mu_{A\times B}(0 * (0 * 0), x * (y * z)), \mu_{A\times B}(0, y)\} \\
&= \min\{\mu_{A\times B}(0, x * (y * z)), \mu_{A\times B}(0, y)\} \\
&= \min\{\min\{\mu_A(0), \mu_B(x * (y * z))\}, \min\{\mu_A(0), \mu_B(y)\}\} \\
&= \min\{\mu_B(x * (y * z)), \mu_B(y)\}
\end{aligned}
$$

Therefore, $\mu_B(x * z) \geq \min\{\mu_B(x * (y * z)), \mu_B(y)\}$, for all $x, y, z \in X$.

Hence B is a fuzzy H-ideal of a Z-algebra X.

By Theorem 3.1.16 (i) and (ii), assume that $\mu_A(0) \geq \mu_A(x)$, for all $x \in X$ and $\mu_B(0) \geq \mu_A(x)$, for any $x \in X$.

Then A is a fuzzy H-ideal of a Z-algebra X.

This completes the proof.

3.2 Fuzzy p-Ideals in Z-algebras

In this section, we introduce the notion of Fuzzy p-ideals in Z-algebras and prove some interesting results.

Definition 3.2.1: A Z-algebra $(X,*,0)$ is called **medial** if $x*(x*y)=y$, for all $x,y \in X$.

Example 3.2.2: Consider a Z-algebra X= {0, 1, 2, 3} be a set with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

Then $(X,*,0)$ is a medial Z-algebra. .

Definition 3.2.3: Let $(X,*,0)$ be a Z-algebra and I be a subset of X. Then, I is called an **p-ideal** of X, if it satisfies the following conditions: For all x, y, z in X,

(i) $0 \in I$

(ii) $(x*z)*(y*z) \in I$ and $y \in I \Rightarrow x \in I$

Example 3.2.4: Consider a Z-algebra X= {0, 1, 2, 3} be a set with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	0	1	1	3
2	0	1	2	1
3	0	3	1	3

Then, $I=\{0,1,2\} \subset X$ is an p-ideal of X.

Definition 3.2.5: Let $(X,*,0)$ be a Z-algebra. A fuzzy set A in X with membership function μ_A is said to be a **fuzzy p-ideal** of a Z-algebra X if it satisfies the following conditions: For all x, y,z in X,

(i) $\mu_A(0) \geq \mu_A(x)$

(ii) $\mu_A(x) \geq \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$

Example 3.2.6: Let X= {0, 1, 2, 3} be a set with the following Cayley table:

*	**0**	**1**	**2**	**3**
0	0	1	2	3
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

Then $(X,*,0)$ is a Z-algebra.

Define a fuzzy set A in X with membership function μ_A is given by $\mu_A(x)$=0.6 for all x = 0,1,2,3. Then A is a fuzzy p-ideal of a Z-algebra X.

Theorem 3.2.7: Let X be a medial Z-algebra then every fuzzy p-ideal of X is a fuzzy Z- ideal of X.

Proof: Assume that A is a fuzzy p-ideal of a medial Z-algebra X. Then,

$\mu_A(0) \geq \mu_A(x)$ for all $x \in X$.

Let $x, y \in X$. Then,

$\mu_A(x) \geq \min\{\mu_A((x*(x*y))*(y*(x*y))), \mu_A(y)\}$

$= \min\{\mu_A(y*(y*(x*y))), \mu_A(y)\}$

$= \min\{\mu_A(y*(y*(y*x))), \mu_A(y)\}$ (Z4)

$= \min\{\mu_A(y*x), \mu_A(y)\}$

$= \min\{\mu_A(x*y), \mu_A(y)\}$ (Z4)

$\Rightarrow \mu_A(x) \geq \min\{\mu_A(x*y), \mu_A(y)\}$

Hence A is a fuzzy Z-ideal of X.

Theorem 3.2.8: Let X be a Z-algebra satisfying the condition: $x * y = (x * z) * (y * z)$ for all $x, y, z \in X$ then every fuzzy Z-ideal of X is a fuzzy p-ideal of X.

Proof: Assume that A is a fuzzy Z-ideal of a Z-algebra X. Then, for all $x, y, z \in X$

$\mu_A(0) \geq \mu_A(x)$ (1)

$\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$

$= \min\{\mu_A((x * z) * (y * z)), \mu_A(y)\}$

$\Rightarrow \mu_A(x) \geq \min\{\mu_A((x * z) * (y * z)), \mu_A(y)\}$ (2)

From (1) and (2) we get, A is a fuzzy p-ideal of a Z-algebra X.

Theorem 3.2.9: Let X be a medial Z-algebra then a fuzzy set A of X is a fuzzy p-ideal of X if and only if A is a fuzzy H-ideal of X.

Proof: Let A be a fuzzy p-ideal of a medial Z-algebra X. Then,

$\mu_A(0) \geq \mu_A(x)$ for all $x \in X$ (1)

Let $x, y, z \in X$. Then,

$\mu_A(x) \geq \min\{\mu_A((x * z) * (y * z)), \mu_A(y)\}$

Put $x = x * z$,

$\mu_A(x * z) \geq \min\{\mu_A(((x * z) * z) * (y * z)), \mu_A(y)\}$

$= \min\{\mu_A((z * (z * x)) * (y * z)), \mu_A(y)\}$ by (Z4)

$= \min\{\mu_A(x * (y * z)), \mu_A(y)\}$

$\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$ (2)

From (1) and (2), we get A is a fuzzy H-ideal of X.

Conversely, assume that A is a fuzzy H-ideal of a medial Z-algebra X. Then,

$\mu_A(0) \geq \mu_A(x)$ for all $x \in X$ (3)

Let $x, y, z \in X$. Then,

$\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$

Put $x = x * z$,

$\mu_A((x * z) * z) \geq \min\{\mu_A((x * z) * (y * z)), \mu_A(y)\}$

$\Rightarrow \quad \mu_A(z*(z*x)) \geq \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$ by (Z4)

$\Rightarrow \quad \mu_A(x) \geq \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$ (4)

From (3) and (4) we get, A is a fuzzy p-ideal of X.

Theorem 3.2.10: A fuzzy set A of a Z-algebra $(X,*,0)$ is a fuzzy p-ideal if and only if for any $t \in [0,1]$, $U(\mu_A;t) = \{x \in X \mid \mu_A(x) \geq t\}$ is an p-ideal of X where $U(\mu_A;t) \neq \phi$.

Proof: Suppose A is a fuzzy p-ideal of a Z-algebra X and $U(\mu_A;t) \neq \phi$ for any $t \in [0,1]$.

Let $x \in U(\mu_A;t)$, then $\mu_A(x) \geq t$.

By definition of fuzzy p-ideal, we have $\mu_A(0) \geq \mu_A(x) \geq t$. Thus $0 \in U(\mu_A;t)$.

If $(x*z)*(y*z) \in U(\mu_A;t)$ and $y \in U(\mu_A;t)$, then $\mu_A((x*z)*(y*z)) \geq t$ and $\mu_A(y) \geq t$.

By definition, we have $\mu_A(x) \geq \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\} \geq \min\{t,t\} = t$

Therefore $x \in U(\mu_A;t)$. Hence $U(\mu_A;t)$ is an p-ideal of a Z-algebra X.

Conversely, suppose that for each $t \in [0,1]$, $U(\mu_A;t)$ is either empty or an p-ideal of a Z-algebra X.

For any $x \in X$, let $\mu_A(x) = t$. Then $x \in U(\mu_A;t)$.

Since $U(\mu_A;t) \neq \phi$ is an p-ideal of X, we have $0 \in U(\mu_A;t)$ and

hence $\mu_A(0) \geq t = \mu_A(x)$ for all $x \in X$.

Assume $\mu_A(x) \geq \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$ for all $x, y, z \in X$ is not true.

Then there exists $x_0, y_0, z_0 \in X$ such that

$\mu_A(x_0) < \min\{\mu_A((x_0*z_0)*(y_0*z_0)), \mu_A(y_0)\}$

Let $t_0 = \frac{1}{2}[\mu_A(x_0) + \min\{\mu_A((x_0*z_0)*(y_0*z_0)), \mu_A(y_0)\}]$

Then $\mu_A(x_0) < t_0 < \min\{\mu_A((x_0*z_0)*(y_0*z_0)), \mu_A(y_0)\}$

This implies $(x_0*z_0)*(y_0*z_0), y_0 \in U(\mu_A;t_0)$ and $x_0 \notin U(\mu_A;t_0)$

But $U(\mu_A;t_0)$ is an p-ideal of X. So $x_0 \in U(\mu_A;t_0)$ by the definition of p-ideal.

This implies $\mu_A(x_0) \geq t_0$.

This is a contradiction.

Therefore $\mu_A(x) \geq \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$, for all $x, y, z \in X$.

Hence A is a fuzzy p-ideal of Z-algebra X.

Theorem 3.2.11: A fuzzy set A of a Z-algebra $(X,*,0)$ is a fuzzy p-ideal if and only if every nonempty upper level subset $U(\mu_A;q)$ of A, $q \in Im(A)$ is an p-ideal.

Proof : Let A be a fuzzy p-ideal of a Z-algebra X.

Claim: $U(\mu_A;q)$, $q \in Im(A)$ is an p-ideal.

Since $U(\mu_A;q) \neq \phi$ there exists $x \in U(\mu_A;q)$ such that $\mu_A(x) \geq q$.

Since A is a fuzzy p-ideal of X, $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$.

Hence for this $x \in U(\mu_A;q)$, $\mu_A(0) \geq q$, which shows that $0 \in U(\mu_A;q)$.

Now, for any $x, y, z \in X$, assume that $(x*z)*(y*z) \in U(\mu_A;q)$ and $y \in U(\mu_A;q)$

Then $\mu_A((x*z)*(y*z)) \geq q$ and $\mu_A(y) \geq q$.

This shows that, $\min\{\mu_A((x*z)*(y*z)), \mu_A(y)\} \geq q$.

Since A is a fuzzy p-ideal of a Z-algebra X, $\mu_A(x) \geq \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\} \geq q$,

Thus $x \in U(\mu_A;q)$, this proves that $U(\mu_A;q)$ is an p-ideal of a Z-algebra X.

Conversely, let $U(\mu_A;q)$, $q \in Im(A)$ be an p-ideal of a Z-algebra X.

Claim: A is a fuzzy p-ideal of a Z-algebra X.

Let $x, y, z \in X$. For any $q \in Im(A)$, let $q = \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$.

Therefore, $\mu_A((x*z)*(y*z)) \geq q$ and $\mu_A(y) \geq q$.

This shows that $(x*z)*(y*z), y \in U(\mu_A;q)$.

Since $U(\mu_A;q)$ is an p-ideal, we have $x \in U(\mu_A;q)$.

This proves that $\mu_A(x) \geq q = \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$

This shows that A is a fuzzy p-ideal of a Z-algebra X.

Theorem 3.2.12: Let A be a fuzzy p-ideal of a Z-algebra X and let $x \in X$. Then $\mu_A(x) = t$ if and only if $x \in U(\mu_A;t)$ but $x \notin U(\mu_A;q)$ for all $q > t$.

Proof: Assume $\mu_A(x) = t$, so that $x \in U(\mu_A;t)$.

If possible, let $x \in U(\mu_A;q)$ for $q > t$.

Then $\mu_A(x) \geq q > t$.

This contradicts the fact that $\mu_A(x) = t$. Hence $x \notin U(\mu_A; q)$ for all q > t.

Conversely, let $x \in U(\mu_A; t)$, but $x \notin U(\mu_A; q)$ for all q > t.

$x \in U(\mu_A; t) \Rightarrow \mu_A(x) \geq t$.

Since $x \notin U(\mu_A; q)$ for all q > t, $\mu_A(x) = t$.

Theorem 3.2.13: Let $h : (X,*,0) \rightarrow (Y,*',0')$ be a Z-homomorphism of Z-algebras. If B is a fuzzy p-ideal of Y, then $h^{-1}(B)$ is a fuzzy p-ideal of X.

Proof: Since B is a fuzzy p-ideal of a Z-algebra Y. For any x $\in$ X, we have

(i) $\mu_{h^{-1}(B)}(x) = \mu_B(h(x)) \leq \mu_B(0') = \mu_B(h(0)) = \mu_{h^{-1}(B)}(0)$

(ii) Let x, y, z $\in$ X. Then

$$\min\{\mu_{h^{-1}(B)}((x*z)*(y*z)), \mu_{h^{-1}(B)}(y)\} = \min\{\mu_B(h(x*z)*(y*z)), \mu_B(h(y))\}$$

$$= \min\{\mu_B(((h(x)*'h(z))*'(h(y)*'h(z))), \mu_B(h(y))\}$$

$$\leq \mu_B(h(x))$$

$$= \mu_{h^{-1}(B)}(x)$$

From (i) and (ii) we get, $h^{-1}(B)$ is a fuzzy p-ideal of a Z-algebra X.

Theorem 3.2.14: Let $h : (X,*,0) \rightarrow (Y,*',0')$ be an Z-epimorphism of Z-algebras. Let B be a fuzzy set of Y. If $h^{-1}(B)$ is a fuzzy p-ideal of X then B is a fuzzy p-ideal of Y.

Proof: Let y $\in$ Y, there exists x $\in$ X such that h (x) = y. Then

$$\mu_B(y) = \mu_B(h(x)) = \mu_{h^{-1}(B)}(x) \leq \mu_{h^{-1}(B)}(0) = \mu_B(h(0)) = \mu_B(0')$$

This implies, $\mu_B(0') \geq \mu_B(y)$

Let x, y, z $\in$ Y. Then there exists a, b,c $\in$ X such that h(a) = x , h (b) = y and h(c) = z. It follows that

$$\mu_B(x) = \mu_B(h(a)) = \mu_{h^{-1}(B)}(a)$$

$$\geq \min\{\mu_{h^{-1}(B)}((a*c)*(b*c)), \mu_{h^{-1}(B)}(b)\}$$

$$= \min\{\mu_B(h((a*c)*(b*c))), \mu_B(h(b))\}$$

$$= \min\{\mu_B((h(a) *' h(c)) *' (h(b) *' h(c))), \mu_B(h(b))\}$$

$$= \min\{\mu_B((x *' z) *' (y *' z)), \mu_B(y)\}$$

This implies, $\mu_B(x) \geq \min\{\mu_B((x *' z) *' (y *' z)), \mu_B(y)\}$

Hence B is a fuzzy p-ideal of a Z-algebra Y.

We can easily prove the following result.

Theorem 3.2.15: Let h be an Z-endomorphism of Z-algebra X and A be a fuzzy set in X. Then $A^h : X \to [0,1]$ defined by $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$ is a fuzzy p-ideal of X if A is a fuzzy p-ideal.

Theorem 3.2.16: If A and B be fuzzy p-ideals in a Z-algebra X then $A \times B$ is a fuzzy p-ideal in $X \times X$.

Proof: Let $(x_1, x_2) \in X \times X$,

$\mu_{A\times B}(0,0) = \min\{\mu_A(0), \mu_B(0)\} \geq \min\{\mu_A(x_1), \mu_B(x_2)\} = \mu_{A\times B}(x_1, x_2)$

Hence $\mu_{A\times B}(0,0) \geq \mu_{A\times B}(x_1, x_2)$ (1)

Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$. Then

$$\mu_{A\times B}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\}$$

$$\geq \min\{\min\{\mu_A((x_1 * z_1) * (y_1 * z_1)), \mu_A(y_1)\}, \min\{\mu_B((x_2 * z_2) * (y_2 * z_2)), \mu_B(y_2)\}\}$$

$$= \min\{\min\{\mu_A((x_1 * z_1) * (y_1 * z_1)), \mu_B((x_2 * z_2) * (y_2 * z_2))\}, \min\{\mu_A(y_1), \mu_B(y_2)\}\}$$

$$= \min\{\mu_{A\times B}(((x_1 * z_1) * (y_1 * z_1)), ((x_2 * z_2) * (y_2 * z_2))), \mu_{A\times B}(y_1, y_2)\}$$

$$= \min\{\mu_{A\times B}(((x_1 * z_1), (x_2 * z_2)) * ((y_1 * z_1), (y_2 * z_2))), \mu_{A\times B}(y_1, y_2)\}$$

$$= \min\{\mu_{A\times B}(((x_1, x_2) * (z_1, z_2)) * ((y_1, y_2) * (z_1, z_2))), \mu_{A\times B}(y_1, y_2)\} \quad (2)$$

By (1) and (2), $A \times B$ is a fuzzy p-ideal in $X \times X$.

Theorem 3.2.17: Let A and B be fuzzy sets in a Z-algebra X such that $A \times B$ is a fuzzy p-ideal of $X \times X$. Then,

(i) Either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$.

(ii) If $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$, then either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$

(iii) If $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$ then either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$

Proof: (i) If $\mu_A(0) < \mu_A(x_1)$ and $\mu_B(0) < \mu_B(x_2)$ for some $x_1, x_2 \in X$.

Then, $\mu_{A\times B}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\} > \min\{\mu_A(0), \mu_B(0)\}$

$= \mu_{A\times B}(0,0)$, which is a contradiction.

Hence, either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$.

(ii) Let $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$.

Assume that there exists $x_1, x_2 \in X$ such that $\mu_B(0) < \mu_A(x_1)$ and $\mu_B(0) < \mu_B(x_2)$.

Then, $\mu_{A\times B}(0,0) = \min\{\mu_A(0), \mu_B(0)\} = \mu_B(0)$

$\mu_{A\times B}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\} > \mu_B(0) = \mu_{A\times B}(0,0)$

$\Rightarrow \mu_{A\times B}(x_1, x_2) > \mu_{A\times B}(0,0)$, which is a contradiction.

Hence either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$.

(iii) will obtain by interchanging the roles of A and B in part (ii).

Theorem 3.2.18: Let A and B be fuzzy sets in a Z-algebra X and $A \times B$ is fuzzy p-ideal of $X \times X$ then either A or B is a fuzzy p-ideal of X.

Proof : By Theorem 3.2.17 (i), we can assume that $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$. Then, by Theorem 3.2.17 (iii), either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$.

Let $\mu_A(0) \geq \mu_B(x)$ for any $x \in X$, then

$$
\begin{aligned}
\mu_B(x) &= \min\{\mu_A(0), \mu_B(x)\} \\
&= \mu_{A\times B}(0, x) \\
&\geq \min\{\mu_{A\times B}(((0,x) * (0,z)) * ((0,y) * (0,z))), \mu_{A\times B}(0,y)\} \\
&= \min\{\mu_{A\times B}(((0*0),(x*z)) * ((0*0),(y*z))), \mu_{A\times B}(0,y)\} \\
&= \min\{\mu_{A\times B}((0*0),((x*z)*(y*z))), \mu_{A\times B}(0,y)\} \\
&= \min\{\mu_{A\times B}(0,(x*z)*(y*z)), \mu_{A\times B}(0,y)\} \\
&= \min\{\min\{\mu_A(0), \mu_B((x*z)*(y*z))\}, \min\{\mu_A(0), \mu_B(y)\}\} \\
&= \min\{\mu_B((x*z)*(y*z)), \mu_B(y)\}
\end{aligned}
$$

Therefore, $\mu_B(x) \geq \min\{\mu_B((x*z)*(y*z)), \mu_B(y)\}$ for all $x, y, z \in X$.

Hence B is a fuzzy p-ideal of a Z-algebra X.

By Theorem 3.2.17 (i) and (ii), assume that $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$ and $\mu_B(0) \geq \mu_A(x)$ for any $x \in X$.

Then A is a fuzzy p-ideal of a Z-algebra X.

This completes the proof.

Theorem 3.2.19: Let A be a fuzzy relation on a Z-algebra X and A_B be the strongest fuzzy relation on X, where B is a fuzzy set of X. If B is a fuzzy p-ideal of a Z-algebra X, then A_B is a fuzzy p-ideal of X x X.

Proof: Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$. Then

$$\begin{aligned}\mu_{A_B}(0,0) &= \min\{\mu_B(0), \mu_B(0)\} \\ &\geq \min\{\mu_B(x_1), \mu_B(x_2)\} \\ &= \mu_{A_B}(x_1, x_2)\end{aligned}$$

and

$$\begin{aligned}\mu_{A_B}(x_1, x_2) &= \min\{\mu_B(x_1), \mu_B(x_2)\} \\ &\geq \min\{\min\{\mu_B((x_1 * z_1) * (y_1 * z_1)), \mu_B(y_1)\}, \min\{\mu_B((x_2 * z_2) * (y_2 * z_2)), \mu_B(y_2)\}\} \\ &= \min\{\min\{\mu_B((x_1 * z_1) * (y_1 * z_1)), \mu_B((x_2 * z_2) * (y_2 * z_2))\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\} \\ &= \min\{\mu_{A_B}(((x_1 * z_1) * (y_1 * z_1)), ((x_2 * z_2) * (y_2 * z_2))), \mu_{A_B}(y_1, y_2)\} \\ &= \min\{\mu_{A_B}(((x_1 * z_1), (x_2 * z_2)), ((y_1 * z_1), (y_2 * z_2))), \mu_{A_B}(y_1, y_2)\} \\ &= \min\{\mu_{A_B}(((x_1, x_2) * (z_1, z_2)) * ((y_1, y_2) * (z_1, z_2))), \mu_{A_B}(y_1, y_2)\}\end{aligned}$$

Therefore A_B is a fuzzy p-ideal of X x X.

Theorem 3.2.20: Let A be a fuzzy relation on a Z-algebra X and B be a fuzzy set of X. If the strongest fuzzy relation A_B is a fuzzy p-ideal of X x X, then B is a fuzzy p-ideal of a Z-algebra X.

Proof : Let $x \in X$. Then,

$$\min\{\mu_B(0), \mu_B(0)\} = \mu_{A_B}(0,0) = \mu_{A_B}(x, x) = \min\{\mu_B(x), \mu_B(x)\}$$

$\Rightarrow \mu_B(0) \geq \mu_B(x)$.

For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$,

Also,

$$\begin{aligned}\min\{\mu_B(x_1), \mu_B(x_2)\} &= \mu_{A_B}(x_1, x_2) \\ &\geq \min\{\mu_{A_B}(((x_1, x_2) * (z_1, z_2)) * ((y_1, y_2) * (z_1, z_2))), \mu_{A_B}(y_1, y_2)\} \\ &= \min\{\mu_{A_B}(((x_1 * z_1), (x_2 * z_2)) * ((y_1 * z_1), (y_2 * z_2))), \mu_{A_B}(y_1, y_2)\}\end{aligned}$$

$$= \min\{\mu_{A_B}(((x_1 * z_1) * (y_1 * z_1)), ((x_2 * z_2) * (y_2 * z_2))), \mu_{A_B}(y_1, y_2)\}$$

$$= \min\{\min\{\mu_B((x_1 * z_1) * (y_1 * z_1)), \mu_B((x_2 * z_2) * (y_2 * z_2))\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\}$$

$$\geq \min\{\min\{\mu_B((x_1 * z_1) * (y_1 * z_1)), \mu_B(y_1)\}, \min\{\mu_B((x_2 * z_2) * (y_2 * z_2)), \mu_B(y_2)\}\}$$

Put $x_2 = y_2 = z_2 = 0$, we get $\mu_B(x_1) \geq \min\{\mu_B((x_1 * z_1) * (y_1 * z_1)), \mu_B(y_1)\}$

Hence B is a fuzzy p-ideal of a Z-algebra X.

Remark 3.2.21: If $\{I_n\}$ be a family of ascending sequence of p-ideals of a Z-algebra X, then $\bigcup I_n$ is also an p-ideal of X.

Theorem 3.2.22: If every fuzzy p-ideal A of a Z-algebra X has only finite values, then every descending chain of p-ideals of X terminates after a finite stage.

Proof: Let A be any fuzzy p-ideal of a Z-algebra X which has finite number of values. Suppose there exists a strictly descending chain $X = I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$ of p-ideals of X which does not terminate after a finite stage. Now define a fuzzy set A in X by

$$\mu_A(x) = \begin{cases} \dfrac{n}{n+1} \text{ if } x \in I_n - I_{n+1} \\ 1 \text{ if } x \in \cap I_n \end{cases}$$

where $n \in \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the set of natural numbers.

Every p-ideals I_n contains 0 implies $0 \in \bigcap I_n \Rightarrow \mu_A(0) \geq \mu_A(x)$ for all $x \in X$.

Let $x, y, z \in X$.

Case(1) If $(x * z) * (y * z) \in I_t - I_{t+1}$ and $y \in I_k - I_{k+1}$ for $t, k \in \mathbb{N} \cup \{0\}$.

Without loss of generality, we can assume that $t \leq k$.

Then $(x * z) * (y * z) \in I_t$ and $y \in I_t$ implies $x \in I_t$.

Hence $\mu_A(x) \geq \dfrac{t}{t+1} = \min\{\mu_A((x * z) * (y * z)), \mu_A(y)\}$

Case (2) If $(x * z) * (y * z) \in \bigcap I_n$ and $y \in \bigcap I_n$ then $x \in \bigcap I_n$

Hence $\mu_A(x) = 1 = \min\{\mu_A((x * z) * (y * z)), \mu_A(y)\}$.

Case(3) If $(x * z) * (y * z) \notin \bigcap I_n$ and $y \in \bigcap I_n$, then $(x * z) * (y * z) \in I_k - I_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$.

Then $x \in I_k$. Hence $\mu_A(x) \geq \frac{k}{k+1} = \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$

Case(4) If $(x*z)*(y*z) \in \bigcap I_n$ and $y \notin \bigcap I_n$ then $y \in I_m - I_{m+1}$ for some $m \in \mathbb{N} \cup \{0\}$.

Hence $x \in I_m$. Therefore $\mu_A(x) \geq \frac{m}{m+1} = \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$.

Therefore A is a fuzzy p-ideal of X and A has an infinite number of different values.

This contradicts our assumption.

Therefore every descending chain of p-ideals in X terminates after a finite stage.

Lemma 3.2.23: Let $I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots$ be a strictly ascending sequence of p-ideals in a Z-algebra X and let (t_n) be a strictly decreasing sequence in (0,1). Let A be a fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N} \\ t_n & \text{if } x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N} \end{cases}$$ where $I_0 = \phi$. Then A is a fuzzy p-ideal of X.

Proof: Let $I = \bigcup_{n \in \mathbb{N}} I_n$. By remark 3.2.21, I is an p-ideal of X. Obviously, $\mu_A(0) = t_1 \geq \mu_A(x)$ for all $x \in X$.

Let $x, y, z \in X$.

Case:i If $(x*z)*(y*z), y \in I_n - I_{n-1}$ for some $n \in \mathbb{N}$ then $(x*z)*(y*z), y \in I_n$.

Since I_n is an p-ideal of X, $x \in I_n$.

That is, $x \in I_n - I_{n-1}$ or $x \in I_{n-1}$.

$\therefore \mu_A(x) \geq t_n = \mu_A((x*z)*(y*z)) = \mu_A(y) = \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$.

Case:ii For $n > m$, $t_n < t_m$. Also $I_m \subset I_n$.

If $(x*z)*(y*z) \in I_n - I_{n-1}$ and $y \in I_m - I_{m-1}$ then $\mu_A((x*z)*(y*z)) = t_n$ and $\mu_A(y) = t_m$.

$\min\{\mu_A((x*z)*(y*z)), \mu_A(y)\} = \min\{t_n, t_m\} = t_n$.

Also $(x*z)*(y*z) \in I_n$ and $y \in I_m \subset I_n$ implies $x \in I_n$, since I_n is an p-ideal of X.

$\Rightarrow \mu_A(x) \geq t_n = \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$.

Consequently, A is a fuzzy p-ideal of a Z-algebra X.

Definition 3.2.24: A Z-algebra $(X,*,0)$ is called **Noetherian** if for every ascending sequence $I_1 \subseteq I_2 \subseteq \ldots$ of p-ideals of X there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geq k$, where $\mathbb{N}$ be the set of natural numbers.

Definition 3.2.25: A Z-algebra $(X,*,0)$ is called **Artinian** if for every descending sequence $I_1 \supseteq I_2 \supseteq \ldots$ of p-ideals of X there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geq k$.

Theorem 3.2.26: Let X be a Z-algebra. The following statements are equivalent:

(a) X is Noetherian

(b) for each fuzzy p-ideal A of X, $Im(A) = \{\mu_A(x) : x \in X\}$ is a well-ordered subset of [0,1].

Proof: (a) $\Rightarrow$ (b)

Assume that X is Noetherian and A is a fuzzy p-ideal of X such that Im(A) is not a well-ordered subset of [0,1].

Then there exists a strictly decreasing sequence $(\mu_A(x_n))$ such that $\mu_A(x_n) = t_n$ where $x_n \in X$.

Then by Theorem 3.2.10, $U_n = U(\mu_A; t_n) = \{x \in X : \mu_A(x) \geq t_n\}$ is an p-ideal of X for every $n \in \mathbb{N}$. It follows that, $U_1 \subset U_2 \subset \ldots$ is a strictly ascending sequence of p-ideals of X.

This is a contradiction with the assumption that X is Noetherian.

Therefore, Im(A) is a well-ordered set for each fuzzy p-ideal A of X.

(b) $\Rightarrow$ (a)

Assume that (b) is true. Suppose that the Z-algebra X is not Noetherian. Then there exists a strictly ascending sequence $I_1 \subset I_2 \subset \ldots I_n \subset \ldots$ of p-ideals of X.

Let A be a fuzzy set in X such that

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N} \\ \frac{1}{n} & \text{if } x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N} \end{cases}$$

where $I_0 = \phi$. By Lemma 3.2.23, A is a fuzzy p-ideal of a Z-algebra X.

But Im(A) is not a well-ordered set, which is impossible.

Therefore, X is Noetherian.

Corollary 3.2.27: Let X be a Z-algebra. If, for every fuzzy p-ideal A of X, Im(A) is a finite set, then X is Noetherian.

Theorem 3.2.28: Let X be a Z-algebra and let $T = \{t_1, t_2, ...,\} \cup \{0\}$ where (t_n) is strictly descending sequence in (0,1). Then the following conditions are equivalent:

(a) X is Noetherian

(b) for each fuzzy p-ideal A of X, if $Im(A) \subseteq T$, then there exists k ∈ ℕ such that $Im(A) \subseteq \{t_1, t_2, ..., t_k\} \cup \{0\}$.

Proof: (a) ⇒(b)

Assume that X is Noetherian. Let A be a fuzzy p-ideal of X such that $Im(A) \subseteq T$. From Theorem 3.2.26, we know that Im(A) is a well-ordered subset of [0,1].

Then, since $1 > t_1 > t_2 > ... > t_n > ... > 0$ and $Im(A) \subseteq \{t_1, t_2, ...\} \cup \{0\}$, there exists k ∈ ℕ such that $Im(A) \subseteq \{t_1, t_2, ..., t_k\} \cup \{0\}$.

(b) ⇒(a)

Assume that (b) is true. Suppose that X is not Noetherian. Then there exists a strictly ascending sequence $I_1 \subset I_2 \subset \ldots$ of an p-ideals of a Z-algebra X.

Define a fuzzy set A in X by

$$\mu_A(x) = \begin{cases} 0 \text{ if } x \notin I_n \text{ for each } n \in \mathbb{N} \\ t_n \text{ if } x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N} \end{cases}$$

where $I_0 = \phi$. By Lemma 3.2.23, A is a fuzzy p-ideal of a Z-algebra X. This is a contradiction with our assumption. Thus X is Noetherian.

Theorem 3.2.29: Let X be a Z-algebra and let $T = \{t_1, t_2, ...\} \cup \{0,1\}$, where (t_n) is a strictly increasing sequence in (0,1). Then the following are equivalent:

(a) X is Artinian

(b) for each fuzzy p-ideal A of X, if $Im(A) \subseteq T$, then there exists k ∈ ℕ such that $Im(A) \subseteq \{t_1, t_2, ..., t_k\} \cup \{0,1\}$.

Proof: (a) ⇒(b)

Suppose that $t_{i_1} < t_{i_2} < ... < t_{i_m} < ...$ is strictly increasing sequence of elements of Im(A). Let $U_m = U(\mu_A; t_{i_m})$ for each m ∈ ℕ. Then $U_1 \supset U_2 \supset ... \supset U_m ...$ is a strictly descending sequence of p-ideals of X. This contradicts the assumption that X is Artinian.

(b) $\Rightarrow$(a)

Assume that (b) is true. Suppose that X is not Artinian. Then there exists a strictly descending sequence $I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$ of p-ideals of X.

Define a fuzzy set A in X by

$$\mu_A(x) = \begin{cases} 0 \text{ if } x \notin I_1 \\ t_n \text{ if } x \in I_n - I_{n+1} \text{ for } n = 1,2,\ldots \\ 1 \text{ if } x \in \cap\{I_n : n \in \mathbb{N}\} \end{cases}$$

Obviously, $\mu_A(0) = 1 \geq \mu_A(x)$ for all $x \in X$. Let $x, y, z \in X$. We have three cases:

Case 1: $x \notin I_1$

Then $(x * z) * (y * z) \notin I_1$ or $y \notin I_1$.

Therefore, $\min\{\mu_A((x * z) * (y * z)), \mu_A(y)\} = 0 = \mu_A(x)$.

Case 2: $x \in I_n - I_{n+1}$ for some $n \in \mathbb{N}$

Then $(x * z) * (y * z) \notin I_{n+1}$ or $y \notin I_{n+1}$.

Hence $\mu_A((x * z) * (y * z)) \leq t_n$ or $\mu_A(y) \leq t_n$

Therefore, $\min\{\mu_A((x * z) * (y * z)), \mu_A(y)\} \leq \{t_n, t_n\} = t_n = \mu_A(x)$.

Case 3: $x \in \cap\{I_n : n \in \mathbb{N}\}$

It is obvious.

Thus A is a fuzzy p-ideal of a Z-algebra X.

This contradicts our assumption. Thus X is Artinian.

Corollary 3.2.30: Let X be a Z-algebra. If, for every fuzzy p-ideal A of X, Im(A) is finite set, then X is Artinian.

3.3 Fuzzy Implicative Ideals in Z-Algebras

In this section, we introduce the notions of fuzzy implicative ideals and fuzzy sub-implicative ideals in Z-algebras. Also, the relationship between fuzzy implicative ideal, fuzzy sub-implicative ideal and fuzzy Z-ideal of a Z-algebra are discussed.

Definition 3.3.1: A Z-algebra $(X,*,0)$ is said to be an **implicative** if it satisfies the condition $(x*(x*y))*(y*x) = y*(y*x)$, for all $x, y \in X$.

Example 3.3.2: Consider the Z-algebra X={0,1,2,3} with the binary operation $*$ defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	0	1	2	2
2	0	2	2	1
3	0	2	1	3

Then, $(X,*,0)$ is an implicative Z-algebra.

Proposition 3.3.3: Every medial Z-algebra is an implicative Z-algebra.

Proof: Let X be a medial Z-algebra. Then, $((x*(x*y))*(y*x)) = y*(y*x)$, for all $x, y \in X$. Thus, X is an implicative Z-algebra.

Definition 3.3.4: A nonempty subset I of a Z-algebra $(X,*,0)$ is called an **implicative ideal** of X if it satisfies the following conditions:

(i) $0 \in I$

(ii) $(x*(y*x))*z \in I$ and $z \in I$ imply $x \in I$, for all $x, y, z \in X$.

Example 3.3.5: Consider the Z-algebra X={0,1,2,3} with the binary operation $*$ defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	0	1	1	2
2	0	1	2	3
3	0	2	3	3

Then $I = \{0,1,2\} \subset X$ is an implicative ideal of a Z-algebra X.

Proposition 3.3.6: Every implicative ideal of a Z-algebra X is an Z-ideal of X.

Proof: Let I be an implicative ideal of a Z-algebra X. Then,

$(x*(y*x))*z \in I$ and $z \in I$ imply $x \in I$ for all $x, y, z \in X$.

Now, $z*(x*(y*x)) \in I$ and $z \in I$ imply $x \in I$ for all $x, y, z \in X$. by (Z4)

Put z = y and y = x,

$y*(x*(x*x)) \in I$ and $y \in I$ imply $x \in I$

$y*x \in I$ and $y \in I$ imply $x \in I$

$\Rightarrow$ $x*y \in I$ and $y \in I$ imply $x \in I$ by (Z4)

Therefore, I is an Z-ideal of a Z-algebra X.

Definition 3.3.7: A fuzzy set A of a Z-algebra $(X,*,0)$ with membership function μ_A is called a **fuzzy implicative ideal** of X if it satisfies the following conditions:

(i) $\mu_A(0) \geq \mu_A(x)$

(ii) $\mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$, for all $x, y, z \in X$.

Example 3.3.8: Consider a Z-algebra X as in example 3.3.5. Define a fuzzy set A of X with membership function μ_A is given by $\mu_A(x) = 0.5$ for all $x \in X$. Then A is a fuzzy implicative ideal of X.

Proposition 3.3.9: In a Z-algebra X, every fuzzy implicative ideal is a fuzzy Z-ideal.

Proof: Let A be a fuzzy implicative ideal of a Z-algebra X. Then,

$\mu_A(0) \geq \mu_A(x)$ (1)

$\mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$ for all $x, y, z \in X$. (2)

Put y=x,

$\mu_A(x) \geq \min\{\mu_A((x*(x*x))*z), \mu_A(z)\}$

$= \min\{\mu_A(x*z), \mu_A(z)\}$ by (Z3)

Hence, A is a fuzzy Z-ideal of a Z-algebra X.

Theorem 3.3.10: A be a fuzzy Z-ideal of a Z-algebra X. Then A is a fuzzy implicative ideal of X if and only if A satisfies the following inequality: for all $x, y \in X$, $\mu_A(x) \geq \mu_A(x*(y*x))$.

Proof: Let A be a fuzzy implicative ideal of a Z-algebra X. Let $x, y, z \in X$. Then,

$\mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$

Put $z = x*(y*x)$,

$\mu_A(x) \geq \min\{\mu_A((x*(y*x))*(x*(y*x))), \mu_A((x*(y*x))\}$

$\Rightarrow \mu_A(x) \geq \min\{\mu_A(x*(y*x)), \mu_A(x*(y*x))\}$ by (Z3)

$\Rightarrow \mu_A(x) \geq \mu_A(x*(y*x))$

Conversely, $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$, since A is a fuzzy Z-ideal of a Z-algebra X.

and $\mu_A(x) \geq \mu_A(x*(y*x))$

$\geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$

Hence, A is a fuzzy implicative ideal of a Z-algebra X.

Proposition 3.3.11: Let X be a Z-algebra and let A be a fuzzy set of X. Then A is a fuzzy implicative ideal of X if and only if $\mu_{A^{\#}}(x) = \mu_A(x) + 1 - \mu_A(0)$ is a fuzzy implicative ideal of X.

Proof: Let A be a fuzzy implicative ideal of a Z-algebra X. Then,

$\mu_{A^{\#}}(0) = \mu_A(0) + 1 - \mu_A(0)$

$\Rightarrow \quad \mu_{A^{\#}}(0) = 1$

$\Rightarrow \quad \mu_{A^{\#}}(0) \geq \mu_{A^{\#}}(x)$ for all $x \in X$. (1)

Let $x, y, z \in X$. Then,

$\mu_{A^{\#}}(x) = \mu_A(x) + 1 - \mu_A(0) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\} + 1 - \mu_A(0)$

$= \min\{\mu_A((x*(y*x))*z) + 1 - \mu_A(0), \mu_A(z) + 1 - \mu_A(0)\}$

$= \min\{\mu_{A^{\#}}((x*(y*x))*z), \mu_{A^{\#}}(z)\}$ (2)

Hence by (1) and (2), $A^{\#}$ is a fuzzy implicative ideal of a Z-algebra X.

Conversely, let $A^{\#}$ be a fuzzy implicative ideal of a Z-algebra X.

For $x \in X$. Then, we have $\mu_A(0) = \mu_{A^{\#}}(0) - 1 + \mu_A(0) \geq \mu_{A^{\#}}(x) - 1 + \mu_A(0) = \mu_A(x)$ (3)

For all $x, y, z \in X$,

$\mu_A(x) = \mu_{A^{\#}}(x) - 1 + \mu_A(0)$

$\geq \min\{\mu_{A^{\#}}((x*(y*x))*z), \mu_{A^{\#}}(z)\} - 1 + \mu_A(0)$

$\geq \min\{\mu_{A^{\#}}((x*(y*x))*z)-1+\mu_A(0), \mu_{A^{\#}}(z)-1+\mu_A(0)\}$

$= \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$ (4)

Hence by (3) and (4), A is a fuzzy implicative ideal of a Z-algebra X.

Proposition 3.3.12: Let A be a fuzzy Z-ideal of a Z-algebra X and $w \in X$. If A satisfies the condition: for all $x, y \in X$, $\mu_A(x) \geq \mu_A(x*(y*x))$ then $X_{A^w} = \{x \in X \mid \mu_A(w) \leq \mu_A(x)\}$ is an implicative ideal of X.

Proof: For all $x \in X$, we have $\mu_A(0) \geq \mu_A(x)$.

Since $w \in X$, we have $\mu_A(0) \geq \mu_A(w)$.

Then $0 \in X_{A^w}$. (1)

Let $x, y, z \in X$ such that $(x*(y*x))*z \in X_{A^w}$ and $z \in X_{A^w}$.

$\Rightarrow \mu_A(w) \leq \mu_A((x*(y*x))*z)$ and $\mu_A(w) \leq \mu_A(z)$

$\Rightarrow \mu_A(w) \leq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\} \leq \mu_A(x*(y*x))$

But $\mu_A(x*(y*x)) \leq \mu_A(x)$

$\Rightarrow \mu_A(w) \leq \mu_A(x)$

$\Rightarrow x \in X_{A^w}$ (2)

From (1) and (2) we get, X_{A^w} is an implicative ideal of a Z-algebra X.

Proposition 3.3.13: Let X be a Z-algebra and $w \in X$. If A is a fuzzy implicative ideal of X, then X_{A^w} is an implicative ideal of X.

Proof: Let A be a fuzzy implicative ideal of a Z-algebra X. Then, $\mu_A(0) \geq \mu_A(x)$, for all $x \in X$.

Since $w \in X$, we have $\mu_A(0) \geq \mu_A(w)$.

Then $0 \in X_{A^w}$. (1)

Let $x, y, z \in X$ such that $(x*(y*x))*z \in X_{A^w}$ and $z \in X_{A^w}$.

$\Rightarrow \mu_A(w) \leq \mu_A((x*(y*x))*z)$ and $\mu_A(w) \leq \mu_A(z)$

$\Rightarrow \mu_A(w) \leq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$

But $\min\{\mu_A((x*(y*x))*z), \mu_A(z)\} \leq \mu_A(x)$

$\Rightarrow \mu_A(w) \leq \mu_A(x)$.

Then $x \in X_{A^w}$ (2)

Thus, from (1) and (2) we get, X_{A^w} is an implicative ideal of a Z-algebra X.

Proposition 3.3.14: Let $\{A_i \mid i \in \Omega\}$ be a family of fuzzy implicative ideals of a Z-algebra X. Then $\bigcap_{i\in\Omega} A_i$ is a fuzzy implicative ideal of X.

Proof: Let $x \in X$. Then $\mu_{\underset{i\in\Omega}{\cap} A_i}(0) = \inf_{i\in\Omega}(\mu_{A_i}(0)) \geq \inf_{i\in\Omega}(\mu_{A_i}(x)) = \mu_{\underset{i\in\Omega}{\cap} A_i}(x)$

Let $x, y, z \in X$. Then, we have

$$\mu_{\underset{i\in\Omega}{\cap} A_i}(x) = \inf_{i\in\Omega}(\mu_{A_i}(x)) \geq \inf_{i\in\Omega}\{\min\{\mu_{A_i}((x*(y*x))*z), \mu_{A_i}(z)\}\}$$

$$= \min\{\inf_{i\in\Omega} \mu_{A_i}((x*(y*x))*z), \inf_{i\in\Omega} \mu_{A_i}(z)\}$$

$$= \min\{\mu_{\underset{i\in\Omega}{\cap} A_i}((x*(y*x))*z), \mu_{\underset{i\in\Omega}{\cap} A_i}(z)\}$$

Hence $\bigcap_{i\in\Omega} A_i$ is a fuzzy implicative ideal of a Z-algebra X.

Theorem 3.3.15: Let $\{A_i \mid i \in \Omega\}$ be a family of fuzzy Z-ideals of a Z-algebra X satisfies the condition $\mu_A(x) \geq \mu_A(x*(y*x))$ for all $x, y \in X$. Then $\bigcap_{i\in\Omega} A_i$ is a fuzzy implicative ideal of X.

Proof: It follows directly from Theorem 3.3.13 and Theorem 3.3.10.

Proposition 3.3.16: Let $\{A_i \mid i \in \Omega\}$ be a chain of fuzzy implicative ideals of a Z-algebra X. Then $\bigcup_{i\in\Omega} A_i$ is a fuzzy implicative ideal of X.

Proof: Let $x \in X$. Then $\mu_{\underset{i\in\Omega}{\cup} A_i}(0) = \sup_{i\in\Omega}(\mu_{A_i}(0)) \geq \sup_{i\in\Omega}(\mu_{A_i}(x)) = \mu_{\underset{i\in\Omega}{\cup} A_i}(x)$

Let $x, y, z \in X$. Then, we have

$$\mu_{\underset{i\in\Omega}{\cup} A_i}(x) = \sup_{i\in\Omega}(\mu_{A_i}(x)) \geq \sup_{i\in\Omega}\{\min\{\mu_{A_i}((x*(y*x))*z), \mu_{A_i}(z)\}\}$$

$$= \min\{\sup_{i\in\Omega} \mu_{A_i}((x*(y*x))*z), \sup_{i\in\Omega} \mu_{A_i}(z)\}$$

$$= \min\{\mu_{\underset{i\in\Omega}{\cup} A_i}((x*(y*x))*z), \mu_{\underset{i\in\Omega}{\cup} A_i}(z)\}$$

Hence $\bigcup_{i\in\Omega} A_i$ is a fuzzy implicative ideal of a Z-algebra X.

Theorem 3.3.17: Let $\{A_i \mid i \in \Omega\}$ be a chain of fuzzy Z-ideals of a Z-algebra X satisfies the condition $\mu_A(x) \geq \mu_A(x*(y*x))$ for all $x, y \in X$. Then $\bigcup_{i\in\Omega} A_i$ is a fuzzy implicative ideal of X.

Proof: It follows directly from Theorem 2.2.4 and Theorem 3.3.16.

Theorem 3.3.18: A fuzzy set A of a Z-algebra $(X,*,0)$ is a fuzzy implicative ideal if and only if for any $t \in [0,1]$, $U(\mu_A;t) = \{x \in X \mid \mu_A(x) \geq t\}$ is an implicative ideal of X where $U(\mu_A;t) \neq \phi$.

Proof: Suppose A is a fuzzy implicative ideal of a Z-algebra X and $U(\mu_A;t) \neq \phi$ for any $t \in [0,1]$. Then $\mu_A(0) \geq \mu_A(x) \;\; \forall\, x \in X$.

Let $x \in U(\mu_A;t)$, then $\mu_A(x) \geq t$. Hence $\mu_A(0) \geq \mu_A(x) \geq t$. Thus $0 \in U(\mu_A;t)$.

Let $x, y, z \in X$ such that $(x*(y*x))*z \in U(\mu_A;t)$ and $z \in U(\mu_A;t)$.

$\Rightarrow \mu_A((x*(y*x))*z) \geq t$ and $\mu_A(z) \geq t$

$\Rightarrow \min\{\mu_A((x*(y*x))*z), \mu_A(z)\} \geq \min\{t,t\} = t.$

But, $\mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$

$\Rightarrow \quad \mu_A(x) \geq t \quad \Rightarrow x \in U(\mu_A;t)$. (2)

From (1) and (2) we get, $U(\mu_A;t)$ is an implicative ideal of a Z-algebra X.

Conversely, suppose that $U(\mu_A;t)$ is either empty or an implicative ideal of a Z-algebra X , for each $t \in [0,1]$.

For any $x \in X$, let $\mu_A(x) = t$. Then $x \in U(\mu_A;t)$.

Since $U(\mu_A;t) \neq \phi$ is an implicative ideal of X, we have $0 \in U(\mu_A;t)$ and hence $\mu_A(0) \geq t = \mu_A(x)$.

Thus $\mu_A(0) \geq \mu_A(x)$, for all $x \in X$.

Let $x, y, z \in X$ such that $\min\{\mu_A((x*(y*x))*z), \mu_A(z)\} = t$.

$\Rightarrow \mu_A((x*(y*x))*z) \geq t$ and $\mu_A(z) \geq t$

$\Rightarrow (x*(y*x))*z \in U(\mu_A;t)$ and $z \in U(\mu_A;t)$

$\Rightarrow x \in U(\mu_A;t)$

$\Rightarrow \quad \mu_A(x) \geq t$

$\Rightarrow \mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$

Therefore, A is a fuzzy implicative Ideal of X.

Note: Hereafter, $U(\mu_A;t)$ is called level implicative ideal of a Z-algebra X.

Theorem 3.3.19: Let A be a fuzzy implicative ideal of a Z-algebra X then two level implicative ideals $U(\mu_A;t_1)$ and $U(\mu_A;t_2)$ (with $t_1 < t_2$) of A are equal if and only if there is no $x \in X$ such that $t_1 \le \mu_A(x) < t_2$.

Proof : Let A be a fuzzy implicative ideal of a Z-algebra X.

Assume that $U(\mu_A;t_1) = U(\mu_A;t_2)$ for some $t_1 < t_2$ and there exists $x \in X$ such that $t_1 \le \mu_A(x) < t_2$.

Then $U(\mu_A;t_2)$ is a proper subset of $U(\mu_A;t_1)$ which is a contradiction.

Hence there is no $x \in X$ such that $t_1 \le \mu_A(x) < t_2$.

Conversely, Suppose that there is no $x \in X$ such that $t_1 \le \mu_A(x) < t_2$. Since $t_1 < t_2$, we get

$U(\mu_A;t_2) \subseteq U(\mu_A;t_1)$ (1)

If $x \in U(\mu_A;t_1)$ then $\mu_A(x) \ge t_1$ and so $\mu_A(x) > t_2$, because $\mu_A(x)$ does not lie between t and s.

Hence $x \in U(\mu_A;t_2)$.

Hence $U(\mu_A;t_1) \subseteq U(\mu_A;t_2)$ (2)

From (1) and (2) we get $U(\mu_A;t_1) = U(\mu_A;t_2)$

Remark 3.3.20: As a consequence of **Theorem 3.3.19**, the level implicative ideals of a fuzzy implicative ideal A of a finite Z-algebra X form a chain

$U(\mu_A;t_0) \subset U(\mu_A;t_1) \subset \cdots \subset U(\mu_A;t_r) = X$, where $t_0 > t_1 > t_2 > ... > t_r$.

Theorem 3.3.21: Let X be a finite Z-algebra and A be a fuzzy implicative ideal of X. If $Im(A) = \{t_1, \cdots, t_n\}$, then the family of implicative ideals $U(\mu_A;t_i), i = 1,2,...,n$, constitutes all the level implicative ideals of A.

Proof: Let $t \in [0,1]$ and $t \notin Im(A)$. Suppose $t_1 < t_2 < \cdots < t_n$ without loss of generality.

If $t \le t_1$, then $U(\mu_A;t_1) \subseteq U(\mu_A;t)$. Since $U(\mu_A;t_1) = X$, $U(\mu_A;t) = X$ and $U(\mu_A;t_1) = U(\mu_A;t)$.

If $t > t_n$, then $U(\mu_A;t) = \phi$ obviously.

If $t_i < t < t_{i+1}$ $(1 \le i \le n-1)$, then there is no $x \in X$ such that $t \le \mu_A(x) < t_{i+1}$. It follows from Theorem 3.3.19 that $U(\mu_A;t) = U(\mu_A;t_{i+1})$. This shows that for any $t \in [0,1]$, the level implicative ideal $U(\mu_A;t)$ is in $\{U(\mu_A;t_i) \mid i = 1,2,\cdots,n\}$.

Lemma 3.3.22: Let X be a Z-algebra and A be a fuzzy implicative ideal of X. If Im(A) is finite, say $\{t_1,t_2,...,t_n\}$ then for any t_i, $t_j \in \mathrm{Im}(A)$, $U(\mu_A;t_i) = U(\mu_A;t_j)$ implies $t_i=t_j$.

Proof: Assume that $t_i \ne t_j$ and $t_i < t_j$.

Then there is $x \in X$ such that $\mu_A(x) = t_i < t_j$, and so $x \in U(\mu_A;t_i)$ and $x \notin U(\mu_A;t_j)$

Thus $U(\mu_A;t_j) \ne U(\mu_A;t_i)$ a contradiction.

Then, $U(\mu_A;t_i) = U(\mu_A;t_j)$. Therefore $t_i = t_j$.

Theorem 3.3.23: Let A and B be two fuzzy implicative ideals of a Z-algebra X with identical family of level implicative ideals. If $\mathrm{Im}(A) = \{t_1, t_2,..., t_m\}$ and $\mathrm{Im}(B) = \{q_1, q_2,..., q_n\}$, where $t_1 > t_2 > ... > t_m$ and $q_1 > q_2 > ... > q_n$, then

(i) $m = n$

(ii) $U(\mu_A;t_i) = U(\mu_B;q_i)$ $i = 1,\dots,m$.

(iii) If $x \in X$ such that $\mu_A(x) = t_i$ then $\mu_B(x) = q_i$, $i = 1,\dots,m$.

Proof: To prove (i): Using Theorem 3.3.21, we know that the only level implicative ideals of A and B are $U(\mu_A;t_i)$ and $U(\mu_B;q_i)$ respectively. Since A and B have the identical family of level implicative ideals, it follows that $m = n$. Thus (i) holds.

To prove (ii): Using Theorem 3.3.21 again, we get that

$\{U(\mu_A;t_1), U(\mu_A;t_2),\dots, U(\mu_A;t_m)\} = \{U(\mu_B;q_1), U(\mu_B;q_2),\dots, U(\mu_B;q_n)\}$, and by lemma we have $U(\mu_A;t_1) \subset U(\mu_A;t_2) \subset \dots \subset U(\mu_A;t_m) = X$

and $U(\mu_B;q_1) \subset U(\mu_B;q_2) \subset \dots \subset U(\mu_B;q_n) = X$.

Hence $U(\mu_A;t_i) = U(\mu_B;q_i)$, $i = 1,\dots,m$ and (ii) holds.

To prove (iii): Let $x \in X$ be such that $\mu_A(x) = t_i$ and let $\mu_B(x) = q_j$. Noticing that $x \in U(\mu_B;q_i)$, that is, $\mu_B(x) \ge q_i$, we obtain $q_j \ge q_i$. Thus $U(\mu_B;q_j) \subseteq U(\mu_B;q_i)$. Since

$x \in U(\mu_B;q_j)$ and $U(\mu_B;q_j) = U(\mu_A;t_j)$, therefore $x \in U(\mu_A;t_j)$ and so $t_i = \mu_A(x) \geq t_j$. It follows that $U(\mu_A;t_i) \subseteq U(\mu_A;t_j)$. By (ii), $U(\mu_B;q_i) = U(\mu_A;t_i) \subseteq U(\mu_A;t_j) = U(\mu_B;q_j)$. Consequently $U(\mu_B;q_i) = U(\mu_B;q_j)$, and by lemma 3.3.22 we conclude that $q_i = q_j$. Thus $\mu_B(x) = q_i$.

Hence the proof.

Corollary 3.3.24: Let A and B be two fuzzy implicative ideals of a Z-algebra X with identical family of level implicative ideals. Then Im(A)=Im(B) if and only if A = B.

Proof : Let Im(A) = Im(B) = $\{q_1, q_2, ..., q_m\}$ where $q_1 > q_2 > ... > q_m$.

By Theorem 3.3.23, for any $x \in X$ there exists q_i such that $\mu_A(x) = q_i = \mu_B(x)$.

Thus $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

This implies A=B.

Theorem 3.3.25: Let A be a fuzzy set in a Z-algebra X with $Im(A) = \{t_0, t_1, ..., t_k\}$ where $t_0 > t_1 > t_2 > ... > t_k$. If there exists a chain of implicative ideals of X: $I_0 \subset I_1 \subset ... \subset I_k = X$ such that $\mu_A(I_n^*) = t_n$ where $I_n^* = I_n - I_{n-1}$, $I_{-1} = \phi$, $n = 0,1,...,k$, then A is a fuzzy implicative ideal of X.

Proof: Since $0 \in I_0$, we have $\mu_A(0) = t_0 \geq \mu_A(x)$ for all $x \in X$. For all $x, y, z \in X$,

(i) If $(x*(y*x))*z \in I_n^*$ and $z \in I_n^*$ then $x \in I_n^*$ because I_n is an implicative ideal of a Z-algebra X. Thus $\mu_A(x) \geq t_n = \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$.

(ii) If $(x*(y*x))*z \notin I_n^*$ and $z \notin I_n^*$ then the following four cases arise:

Case i : $(x*(y*x))*z \in X - I_n$ and $z \in X - I_n$

Case ii : $(x*(y*x))*z \in I_{n-1}$ and $z \in I_{n-1}$

Case iii : $(x*(y*x))*z \in X - I_n$ and $z \in I_{n-1}$

Case iv : $(x*(y*x))*z \in I_{n-1}$ and $z \in X - I_n$

But, in either case, we know that $\mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$.

(iii) If $(x*(y*x))*z \in I_n^*$ and $z \notin I_n^*$, then either $z \in I_{n-1}$ or $z \in X - I_n$. It follows that either $x \in I_n$ or $x \in X - I_n$. Thus, $\mu_A(x) \geq t_n = \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$.

(iv) If $(x*(y*x))*z \notin I_n^*$ and $z \in I_n^*$, then clearly $\mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$.

Consequently A is a fuzzy implicative ideal of X.

Proposition 3.3.26: Let h be a Z-homomorphism from a Z –algebra $(X,*,0)$ onto a Z-algebra $(Y,*',0')$ and A be a fuzzy implicative ideal of X with the supremum property. Then the image of A denoted by h(A) is a fuzzy implicative ideal of Y.

Proof: Let $a, b, c \in Y$ with $x_0 \in h^{-1}(a)$, $y_0 \in h^{-1}(b)$ and $z_0 \in h^{-1}(c)$ such that

$\mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t)$; $\mu_A(y_0) = \sup_{t \in h^{-1}(b)} \mu_A(t)$ and $\mu_A(z_0) = \sup_{t \in h^{-1}(c)} \mu_A(t)$.

$$\mu_{h(A)}(0') = \sup_{t \in h^{-1}(0')} \mu_A(t) \geq \mu_A(0) \geq \mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t) = \mu_{h(A)}(a)$$

$$\begin{aligned}
\min\{\mu_{h(A)}((a*'(b*'a))*'c), \mu_{h(A)}(c)\} &= \min\left\{\sup_{t \in h^{-1}((a*'(b*'a))*'c)} \mu_A(t), \sup_{t \in h^{-1}(c)} \mu_A(t)\right\} \\
&= \min\{\mu_A((x_0*(y_0*x_0))*z_0), \mu_A(z_0)\} \\
&\leq \mu_A(x_0) \\
&= \sup_{t \in h^{-1}(a)} \mu_A(t) \\
&= \mu_{h(A)}(a)
\end{aligned}$$

Hence h(A) is a fuzzy implicative ideal of a Z-algebra Y.

Proposition 3.3.27: Let $h:(X,*,0) \to (Y,*',0')$ be a Z-homomorphism of Z-algebras. If B is a fuzzy implicative ideal of Y, then $h^{-1}(B)$ is a fuzzy implicative ideal of X.

Proof: For any x $\in$ X, we have

$$\mu_{h^{-1}(B)}(x) = \mu_B(h(x)) \leq \mu_B(0') = \mu_B(h(0)) = \mu_{h^{-1}(B)}(0)$$

Let x, y, z $\in$ X. Then

$$\begin{aligned}
\min\{\mu_{h^{-1}(B)}((x*(y*x))*z), \mu_{h^{-1}(B)}(z)\} &= \min\{\mu_B(h((x*(y*x))*z)), \mu_B(h(z))\} \\
&= \min\{\mu_B((h(x)*'(h(y)*'h(x)))*'h(z)), \mu_B(h(z))\} \\
&\leq \mu_B(h(x))
\end{aligned}$$

$= \mu_{h^{-1}(B)}(x)$

$\Rightarrow \quad \mu_{h^{-1}(B)}(x) \geq \min\{\mu_{h^{-1}(B)}((x*(y*x))*z), \mu_{h^{-1}(B)}(z)\}$

Hence $h^{-1}(B)$ is a fuzzy implicative ideal of a Z-algebra X.

Corollary 3.3.28: Let X be a Z-algebra . Then A is a fuzzy implicative ideal of X if and only if the set X_A is an implicative ideal of X, where $X_A = \{x \in X \mid \mu_A(x) = \mu_A(0)\}$.

Proof: Let A be a fuzzy implicative ideal of a Z-algebra X.

If x=0, then $\mu_A(x) = \mu_A(0)$. Then $0 \in X_A$. (1)

Let $x, y, z \in X$ such that $(x*(y*x))*z \in X_A$ and $z \in X_A$.

$\Rightarrow \mu_A((x*(y*x))*z) = \mu_A(0)$ and $\mu_A(z) = \mu_A(0)$.

$\mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\} = \min\{\mu_A(0), \mu_A(0)\} = \mu_A(0)$.

$\Rightarrow \mu_A(x) \geq \mu_A(0)$ (2)

Since A is a fuzzy implicative ideal of a Z-algebra X, $\mu_A(0) \geq \mu_A(x)$. (3)

By (2) and (3), $\mu_A(x) = \mu_A(0)\ \forall\, x \in X$.

$\Rightarrow x \in X_A$.

Hence, X_A is an implicative ideal of a Z-algebra X.

Conversely, let X_A be an implicative Ideal of a Z-algebra X.

Since $X_A = U(\mu_A; t)$ where $t = \mu_A(0)$.

Therefore, A is a fuzzy implicative ideal of a Z-algebra X.

Proposition 3.3.29: Let X be a Z-algebra and A be a fuzzy set of X defined by

$\mu_A(x) = \begin{cases} t_1 \text{ if } x \in X_A \\ t_2 \text{ otherwise} \end{cases}$, where $t_1, t_2 \in [0,1]$ such that $t_1 > t_2$. Then A is a fuzzy implicative ideal of X if and only if X_A is an implicative ideal of X.

Proof: Let A be a fuzzy implicative ideal of a Z-algebra X.

Since $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$, $\mu_A(0) = t_1 \Rightarrow 0 \in X_A$.

Let $x, y, z \in X_A$ such that $(x*(y*x))*z \in X_A$ and $z \in X_A$

$\Rightarrow \mu_A((x*(y*x))*z) = \mu_A(0) = t_1$ and $\mu_A(z) = \mu_A(0) = t_1$

$\Rightarrow \mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\} = \min\{t_1, t_1\} = t_1$

$\Rightarrow \mu_A(x) = t_1$

$\Rightarrow x \in X_A$

Then X_A is an implicative ideal of a Z-algebra X.

Conversely, let X_A be an implicative ideal of a Z-algebra X.

Since $0 \in X_A$ then $\mu_A(0) = t_1 \geq \mu_A(x)$ for all $x \in X$. (1)

Let $x, y, z \in X$,

Case 1: Suppose $(x*(y*x))*z \in X_A$ and $z \in X_A$ then $x \in X_A$.

Since X_A is an implicative ideal of X,

$\Rightarrow \mu_A((x*(y*x))*z) = t_1$, $\mu_A(z) = t_1$ and $\mu_A(x) = t_1$

$\Rightarrow \min\{\mu_A((x*(y*x))*z), \mu_A(z)\} = \min\{t_1, t_1\} = t_1$

Hence $\mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$

Case 2: Suppose $(x*(y*x))*z \in X_A$ and $z \notin X_A$.

Then $\mu_A((x*(y*x))*z) = t_1$ and $\mu_A(z) = t_2$

$\Rightarrow \min\{\mu_A((x*(y*x))*z), \mu_A(z)\} = \min\{t_1, t_2\} = t_2$

$\Rightarrow \mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$

Case 3: Suppose $(x*(y*x))*z \notin X_A$ and $z \in X_A$.

Then $\mu_A((x*(y*x))*z) = t_2$ and $\mu_A(z) = t_1$

$\Rightarrow \min\{\mu_A((x*(y*x))*z), \mu_A(z)\} = \min\{t_2, t_1\} = t_2$

$\Rightarrow \mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$

Case 4: Suppose $(x*(y*x))*z \notin X_A$ and $z \notin X_A$.

Then $\mu_A((x*(y*x))*z) = t_2$ and $\mu_A(z) = t_2$

$\Rightarrow \min\{\mu_A((x*(y*x))*z), \mu_A(z)\} = \min\{t_2, t_2\} = t_2$

$\Rightarrow \mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$

Hence for all $x, y, z \in X$, $\mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$ (2)

Therefore, from (1) and (2), A is a fuzzy implicative ideal of a Z-algebra X.

Theorem 3.3.30: If A is a fuzzy implicative ideal of a Z-algebra X and h is a mapping from X into itself, define $A^h : X \to [0,1]$ by $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$. Then A^h is a fuzzy implicative ideal of X.

Proof: For any $x, y, z \in X$, we have

$\mu_{A^h}(x) = \mu_A(h(x)) \leq \mu_A(0) = \mu_A(h(0)) = \mu_{A^h}(0)$ and

$\min\{\mu_{A^h}((x*(y*x))*z), \mu_{A^h}(z)\} = \min\{\mu_A((h(x)*(h(y)*h(x)))*h(z)), \mu_A(h(z))\} \leq \mu_A(h(x)) = \mu_{A^h}(x)$

Hence A^h is a fuzzy implicative ideal of X.

Definition 3.3.31: An implicative ideal I of a Z-algebra $(X,*,0)$ is said to be **characteristic** if $h(I) = I$ for all $h \in Aut(X)$.

Definition 3.3.32: A fuzzy implicative ideal A of a Z-algebra X is said to be **fuzzy characteristic** if $A^h = A$ for all $h \in Aut(X)$ where $A^h : X \to [0,1]$ is defined by $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$.

Theorem 3.3.33 : If A is a fuzzy characteristic implicative ideal of a Z-algebra X, then each upper level implicative ideal of A is a characteristic implicative ideal of X.

Proof: Assume that A is a fuzzy characteristic implicative ideal of X.

Let $t \in Im(A)$, $h \in Aut(X)$ and $x \in U(\mu_A ; t)$. Then $\mu_A(h(x)) = \mu_{A^h}(x) \geq t$, and so $h(x) \in U(\mu_A ; t)$. This implies $h(U(\mu_A ; t)) \subseteq U(\mu_A ; t)$.

To prove the reverse inclusion, let $x \in U(\mu_A ; t)$ and let $y \in X$ be such that $h(y) = x$. Then $\mu_A(y) = \mu_{A^h}(y) = \mu_A(h(y)) = \mu_A(x) \geq t$, hence $y \in U(\mu_A ; t)$, which implies that $x = h(y) \in h(U(\mu_A ; t))$. Thus $U(\mu_A ; t) \subseteq h(U(\mu_A ; t))$.

Consequently $U(\mu_A ; t)$, $t \in Im(A)$ is a characteristic implicative ideal of X, completing the proof.

Lemma 3.3.34: Let A is a fuzzy implicative ideal of a Z-algebra X and let $x \in X$. Then $\mu_A(x) = t$ if and only if $x \in U(\mu_A ; t)$ and $x \notin U(\mu_A ; s)$ for all $s > t$.

Proof: Since $U(\mu_A ; s) \subseteq U(\mu_A ; t)$ for all $s > t$, result follows.

Now we provides the converse of Theorem 3.3.33.

Theorem 3.3.35: If A is a fuzzy implicative ideal of a Z-algebra X in which every upper level implicative ideal is characteristic, then A is a fuzzy characteristic implicative ideal of X.

Proof: Let $x \in X$, $h \in Aut(X)$ and $\mu_A(x) = t$. It follows from Lemma 3.3.34 that $x \in U(\mu_A;t)$ and $x \notin U(\mu_A;s)$ for all $s > t$. Since $h(U(\mu_A;t)) = U(\mu_A;t)$ by hypothesis, therefore $h(x) \in h(U(\mu_A;t)) = U(\mu_A;t)$ and hence $\mu_{A^h}(x) = \mu_A(h(x)) \geq t$. Let $\mu_{A^h}(x) = s$. Then we have $t = s$. For $s > t$, then $h(x) \in U(\mu_A;s) = h(U(\mu_A;t))$. Since h is one-one, it follows that $x \in U(\mu_A;s)$, which is a contradiction. Hence $\mu_{A^h}(x) = t = \mu_A(x)$ or $A^h = A$, showing that A is a fuzzy characteristic implicative ideal of X.

Definition 3.3.36: A nonempty subset I of a Z-algebra $(X,*,0)$ is called a **sub-implicative ideal** of X if it satisfies the following conditions :

(i) $0 \in I$

(ii) $((x*(x*y))*(y*x))*z \in I$ and $z \in I$ imply $y*(y*x) \in I$ for all $x, y, z \in X$.

Example 3.3.37: Consider the Z-algebra X={0,1,2,3} with the binary operation $*$ defined by the following table:

*	0	1	2	3
0	0	1	2	3
1	0	1	1	2
2	0	1	2	2
3	0	2	2	3

Then $I = \{0,1,2\} \subset X$ is a sub-implicative ideal of X.

Proposition 3.3.38: Let X be a medial and an associative Z-algebra.Then, every implicative ideal of X is a sub-implicative ideal of X.

Proof : Let I be a implicative ideal of a medial and an associative Z-algebra.

Then, for all $x, y, z \in X$,

$0 \in I$ (1)

Now, $(x*(y*x))*z \in I$ and $z \in I$ imply $x \in I$

$((y*(y*x))*(y*x))*z \in I$ and $z \in I$ imply $y*(y*x) \in I$

$(((x*(x*y))*(y*x))*(y*x))*z \in I$ and $z \in I$ imply $y*(y*x) \in I$

$(x*(x*y))*((y*x)*(y*x))*z \in I$ and $z \in I$ imply $y*(y*x) \in I$ by assumption

$((x*(x*y))*(y*x))*z \in I$ and $z \in I$ imply $y*(y*x) \in I$ by (Z4) (2)

Hence by (1) and (2) , I is a sub-implicative ideal of X.

Definition 3.3.39: A fuzzy set A of a Z-algebra $(X,*,0)$ with membership function μ_A is said to be a **fuzzy sub-implicative ideal** of X if it satisfies the following conditions :

(i) $\mu_A(0) \geq \mu_A(x)$

(ii) $\mu_A(y*(y*x)) \geq \min\{\mu_A(((x*(x*y))*(y*x))*z), \mu_A(z)\}$, for all $x, y, z \in X$.

Example 3.3.40: Consider a Z-algebra X as in example 3.3.37. Define a fuzzy set A of X with membership function μ_A is given by $\mu_A(x) = 0.6$ for all $x \in X$. Then A is a fuzzy sub-implicative ideal of X.

Proposition 3.3.41: Let X be a Z-algebra. Then every fuzzy sub-implicative ideal of X is a fuzzy Z-ideal of X.

Proof: Let $x, y, z \in X$, Then

$\mu_A(0) \geq \mu_A(x)$ (1)

$\mu_A(x) = \mu_A(x*x)$ By (Z3)

$= \mu_A(x*(x*x))$

$\geq \min\{\mu_A(((x*(x*x))*(x*x))*z), \mu_A(z)\}$

$= \min\{\mu_A(((x*x)*x)*z), \mu_A(z)\}$

$= \min\{\mu_A((x*x)*z), \mu_A(z)\}$

$= \min\{\mu_A(x*z), \mu_A(z)\}$ (2)

Therefore, by (1) and (2) A is a fuzzy Z-ideal of a Z-algebra X.

Theorem 3.3.42: Let X be an implicative Z-algebra. Then every fuzzy Z-ideal of X is a fuzzy sub-implicative ideal of X.

Proof: Let A be a fuzzy Z-ideal of X. Let $x, y, z \in X$,

Then, $\mu_A(0) \geq \mu_A(x)$ (1)

and $\mu_A(y*(y*x)) \geq \min\{\mu_A((y*(y*x))*z), \mu_A(z)\}$

Since X is an implicative, $\mu_A(y*(y*x)) \geq \min\{\mu_A((x*(x*y)*(y*x))*z), \mu_A(z)\}$ (2)

From (1) and (2) , A is a fuzzy sub-implicative ideal of X.

Corollary 3.3.43: In a medial Z-algebra X, every fuzzy Z-ideal of X is a fuzzy sub-implicative ideal of X.

Proof: Let A be a fuzzy Z-ideal of a medial Z-algebra X. Let $x, y, z \in X$,

Then, $\mu_A(0) \geq \mu_A(x)$

and $\mu_A(y*(y*x)) = \mu_A(x) \geq \min\{\mu_A(x*z), \mu_A(z)\}$

$= \min\{\mu_A((y*(y*x))*z), \mu_A(z)\}$

$= \min\{\mu_A(((x*(x*y))*(y*x))*z), \mu_A(z)\}$

Hence A is a fuzzy sub- implicative ideal of X.

Theorem 3.3.44: If X is a Z-algebra satisfies the condition: for all $x, y, z \in X$; $\mu_A(y*z) \geq \mu_A((x*(x*y))*z)$ then every fuzzy Z-ideal of X is a fuzzy sub-implicative ideal of X.

Proof: Let X be a Z-algebra satisfies the condition:

$\mu_A(y*z) \geq \mu_A((x*(x*y))*z)$ for all $x, y, z \in X$. (1)

Let A be a fuzzy Z-ideal of X. For $x, y, z \in X$,

$\mu_A((x*(x*y)*(y*x)) \geq \min\{\mu_A((x*(x*y)*(y*x))*z), \mu_A(z)\}$

Put $z = y*x$ in (1),

$\mu_A(y*(y*x)) \geq \mu_A((x*(x*y))*(y*x))$

$\Rightarrow \mu_A(y*(y*x)) \geq \min\{\mu_A((x*(x*y)*(y*x))*z), \mu_A(z)\}$

Therefore, A is a fuzzy sub-implicative ideal of X.

Theorem 3.3.45: Let X be a medial Z-algebra satisfies the condition: for all $x, y \in X$,

$\mu_A((x*(x*y))*(y*x)) \geq \mu_A(x*(y*x))$

Then every fuzzy sub-implicative ideal of X is a fuzzy implicative ideal of X.

Proof: Let A be a fuzzy sub-implicative ideal of X . Then A is a fuzzy Z-ideal of X.

$\mu_A(0) \geq \mu_A(x)$, for all $x \in X$. (1)

For $x, y, z \in X$,

$$\mu_A(x) = \mu_A(y*(y*x)) \geq \mu_A((x*(x*y))*(y*x)) \geq \mu_A(x*(y*x))$$
$$\geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$$

Put $z = x*(y*x)$,

$$\mu_A(x) \geq \min\{\mu_A((x*(y*x))*(x*(y*x))), \mu_A(x*(y*x))\}$$
$$\geq \min\{\mu_A(x*(y*x)), \mu_A(x*(y*x))\} \quad \text{by (Z3)}$$
$$= \mu_A(x*(y*x))$$

$\Rightarrow \mu_A(x) \geq \mu_A(x*(y*x))$

Therefore, by Theorem 3.3.10 A is a fuzzy implicative ideal of X.

Theorem 3.3.46: Let X be an implicative Z-algebra. Then every fuzzy implicative ideal of X is a fuzzy sub-implicative ideal of X.

Proof: A is a fuzzy implicative ideal of X implies A is a fuzzy Z-ideal of X. Then,

$\mu_A(0) \geq \mu_A(x)$ for all $x \in X$. (1)

For $x, y, z \in X$, $\mu_A((x*(x*y)*(y*x)) \geq \min\{\mu_A((x*(x*y)*(y*x))*z), \mu_A(z)\}$

Since X is implicative, $\mu_A((x*(x*y)*(y*x)) = \mu_A(y*(y*x))$.

This implies, $\mu_A(y*(y*x)) \geq \min\{\mu_A((x*(x*y)*(y*x))*z), \mu_A(z)\}$ (2)

From (1) and (2) , A is a fuzzy sub-implicative ideal of X.

The following diagram gives the relations between fuzzy Z-ideal, fuzzy implicative ideal and fuzzy sub-implicative ideal of a Z-algebra