

## *CHAPTER - II*

## CHAPTER II

### FUZZY IRRESOLUTE FUNCTIONS

#### Definition: 2.1

Let  $\lambda$  be fuzzy set in a fts  $(X, \tau)$ . Then  $\lambda$  is called

- i. a **fuzzy semi-open (Fs-open)**, for short) set of  $X$  iff there is a  $\beta \in \tau$  such that  $\beta \leq \lambda \leq \text{cl } \beta$ ,
- ii. a **fuzzy semi-closed (Fs-closed)**, for short) set of  $X$  iff there is a closed set  $\beta$  in  $(X, \tau)$  such that  $\text{int } \beta \leq \lambda \leq \beta$ ,
- iii. a **fuzzy regular open** set of  $X$  iff  $\text{Int cl } \lambda = \lambda$ , and
- iv. a **fuzzy regular closed** set of  $X$  iff  $\text{cl Int } \lambda = \lambda$ .

The fuzzy **semi-closure** of a fuzzy set  $\beta$  in a fts  $(X, \tau)$ , denoted by **scl**  $\beta$ , is the intersection of all fuzzy semi-closed sets containing  $\beta$ . A fuzzy set  $\lambda$  is a fuzzy semi-closed iff  $\lambda = \text{scl } \lambda$  holds in  $X$ .

#### Notation: 2.2

The collection of all **fuzzy semi-open** sets in  $X$  is denoted by **FSO(X)**.

#### Result: 2.3

For a fuzzy set  $\lambda$  in a fts  $(X, \tau)$  the following are equivalent:

- a.  $\lambda$  is a fuzzy semi-closed set,

- b.  $\lambda'$  is a fuzzy semi-open set,
- c.  $\text{Int cl } \lambda \leq \lambda$ , and
- d.  $\text{cl Int } \lambda' \geq \lambda'$ .

**Definition: 2.4**

A fuzzy set  $\lambda$  in  $(X, \tau)$  is said to be a **fuzzy semi-neighbourhood (f.semi-nbd**, in short) of a fuzzy point  $\beta$  iff there is a fuzzy semi-open set  $\alpha$  in  $X$  such that  $\beta \leq \alpha \leq \lambda$ .

**Definition: 2.5**

Let  $f: (X, \tau) \rightarrow (Y, \tau_1)$  be a mapping from a fts  $(X, \tau)$  to another fts  $(Y, \tau_1)$ . Then

$f$  is called :

- a. a **fuzzy continuous mapping (f-continuous**, for short) if  $f^{-1}(\lambda) \in \tau$  for each  $\lambda \in \tau_1$ ,
- b. a **fuzzy semi-continuous mapping (f-s-continuous**, for short) if  $f^{-1}(\lambda)$  is a fuzzy semi-open (semi-closed) set of  $X$ , for each fuzzy open (closed) set  $\lambda$  in  $Y$ .

**Theorem: 2.6**

Let  $f: (X, \tau) \rightarrow (Y, \tau_1)$  be a mapping from a fts  $(X, \tau)$  to another fts  $(Y, \tau_1)$ . Then the following are equivalent:

- i.  $f$  is a fuzzy semi-continuous
- ii.  $\text{Int}(\text{cl}(f^{-1}(\lambda))) \leq f^{-1}(\text{cl}(\lambda))$  for each fuzzy set  $\lambda$  in  $Y$ .

iii.  $f(\text{Int}(\text{cl}(\mu))) \leq \text{cl}(f(\mu))$  for each fuzzy set  $\mu$  in  $X$ .

**Proof :**

**(i)  $\Rightarrow$  (ii):**

Suppose  $f$  is fuzzy semi-continuous and  $\lambda$  be any fuzzy set in  $Y$ . Then

$f^{-1}(\text{cl}(\lambda))$  is a fuzzy semi-closed set in  $X$ .

Since  $\text{cl}(\lambda) \geq \lambda$  and  $\text{Int} \text{cl } \nu \leq \nu$  for any fuzzy semi-closed  $\nu$  in  $X$ , we have

$$f^{-1}(\text{cl}(\lambda)) \geq \text{int}(\text{cl}(f^{-1}(\text{cl}(\lambda)))) \geq \text{int}(\text{cl}(f^{-1}(\lambda))).$$

**(ii)  $\Rightarrow$  (iii):**

Suppose (ii) holds and let  $\mu$  be an arbitrary fuzzy set in  $X$ . Let us put  $\lambda = f(\mu)$ ,

then  $\mu < f^{-1}(\lambda)$ .

Therefore, by hypothesis we have

$$\text{Int}(\text{cl}(\mu)) \leq \text{int}(\text{cl}(f^{-1}(\lambda))) \leq f^{-1}(\text{cl}(\lambda)).$$

So,  $f(\text{int}(\text{cl}(\mu))) \leq \text{cl}(\lambda) = \text{cl}(f(\mu))$ ,

**(iii)  $\Rightarrow$  (i):**

Suppose (iii) holds and let  $\nu$  be an arbitrary fuzzy closed set in  $Y$ . Let us put

$\mu = f^{-1}(\nu)$ , then  $f(\mu) \leq \nu$ . Therefore by hypothesis we have

$$f(\text{int}(\text{cl}(\mu))) \leq \text{cl}(f(\mu)) \leq \text{cl}(v) = v$$

since we have always  $f^{-1}(f(\beta)) \geq \beta$  for any fuzzy set  $\beta$  in  $X$ . We obtain

$$\text{Int}(\text{cl}(f^{-1}(v))) \leq f^{-1}(v).$$

Therefore by Result 2.3,  $f^{-1}(v)$  is fuzzy semi-closed and hence  $f$  is fuzzy

semi-continuous.

**Definition: 2.7**

A function  $f: X \rightarrow Y$  from a fts  $(X, \tau)$  to a fts  $(Y, \tau_1)$  is said to be **fuzzy irresolute** if  $f^{-1}(\alpha)$  is fuzzy semi-open in  $X$  for each fuzzy semi-open set  $\alpha$  in  $Y$ .

**Note: 2.8**

It is evident that every fuzzy irresolute function is fuzzy semi-continuous.

However, the converse may be false as shown by the following example.

**Example: 2.9**

Let  $u_1, u_2, u_3$  be fuzzy sets of  $I$  defined as

$$u_1(x) = \begin{cases} 0 & , 0 \leq x \leq 1/2 \\ 2x - 1 & , 1/2 \leq x \leq 1 \end{cases}$$

$$u_2(x) = \begin{cases} 1 & , 0 \leq x \leq 1/4 \\ -4x + 2 & , 1/4 \leq x \leq 1/2 \\ 0 & , 1/2 \leq x \leq 1 \end{cases}$$

$$u_3(x) = \begin{cases} 0 & , 0 \leq x \leq 1/4 \\ \frac{1}{3}(4x - 1) & , 1/4 \leq x \leq 1 \end{cases}$$

Clearly  $T = \{0, u_1, u_2, u_1 \cup u_2, 1\}$  is a fuzzy topology on  $I$ . Let  $f : I \rightarrow I$  be defined by

$f(x) = 1/2 x$  for each  $x \in I$ . Simple computations give  $f^{-1}(0) = 0, f^{-1}(1) = 1,$

$$f^{-1}(u_1) = 0, f^{-1}(u_2) = u_1' = f^{-1}(u_1 \cup u_2) .$$

$$\text{Again, } \text{cl int } u_1' = \text{cl } u_2 = 1 - \text{int } (1 - u_2) = 1 - u_1 = u_1'.$$

Hence,  $u_1'$  is fuzzy semi-open and consequently  $f$  is fuzzy semi-continuous.

$$\text{Again } \text{cl } u_1 = u_2', \text{cl } u_2 = u_1', \text{cl } (u_1 \cup u_2) = 1,$$

$$\text{Int } u_1' = u_2, \text{Int } u_2' = u_1, \text{ and } \text{int } (u_1 \cup u_2)' = 0.$$

Since  $u_1 \leq u_3 \leq \text{cl } u_1, u_3$  is a fuzzy semi-open set which is not a fuzzy open set.

Let  $\beta(x) = f^{-1}(u_3)(x) = u_3(1/2 x), x \in I$ .

$$= \begin{cases} 0 & , 0 \leq x \leq 1/2 \\ \frac{1}{3}(2x - 1) & , 1/2 \leq x \leq 1 \end{cases}$$

Then  $\text{cl int } \beta = 0$ . Hence  $\beta \not\subseteq \text{cl Int } \beta$

Therefore  $f^{-1}(u_3)$  is not semi-open in  $I$  and thus  $f$  is not fuzzy irresolute.

**Theorem: 2.10**

Let  $(X, \tau)$  and  $(Y, \tau_1)$  be two fts and  $f: X \rightarrow Y$ . Then the following are equivalent:

- a)  $f$  is fuzzy irresolute.
- b) for each fuzzy point  $p$  in  $X$  and each fuzzy semi-open set  $\alpha$  in  $Y$  such that  $f(p) \leq \alpha$ , there exists a fuzzy semi-open set  $\beta$  in  $X$  such that  $p \leq \beta$  and  $f(\beta) \leq \alpha$ .
- c) The inverse of each fuzzy semi-closed set in  $Y$  is fuzzy semi-closed in  $X$ .
- d) For each fuzzy point  $p$  of  $X$  the inverse of each fuzzy semi-nbd  $\beta$  of  $f(p)$  in  $Y$  is a fuzzy semi-nbd of  $p$  in  $X$ .
- e) For each fuzzy point  $p$  of  $X$  and each fuzzy semi-nbd  $\beta$  of  $f(p)$ , there exists a fuzzy semi-nbd  $\gamma$  of  $p$  in  $X$  such that  $f(\gamma) \leq \beta$ .
- f) For each fuzzy set  $\lambda$  of  $X$ ,  $f[\text{scl}(\lambda)] \leq \text{scl}(f(\lambda))$ .
- g) For each fuzzy set  $\beta$  of  $Y$ ,  $\text{scl}[f^{-1}(\beta)] \leq f^{-1}(\text{scl}(\beta))$ .

**Proof:**

**(b) $\Rightarrow$ (a) :**

Let  $\alpha$  be a fuzzy semi- open set in  $Y$  and  $p$  be a fuzzy point in  $f^{-1}(\alpha)$ .

Then  $p \leq f^{-1}(\alpha)$ .

That is,  $f(p) \leq \alpha$  .

Thus there exists a fuzzy semi- open set  $\beta$  in  $X$  such that  $p \leq \beta$  and  $f(\beta) \leq \alpha$  .

Then  $\beta \leq f^{-1}(\alpha)$  .

Thus  $p \leq \beta \leq \text{cl int } \beta \leq \text{cl int } f^{-1}(\alpha)$ .

Since  $p$  is arbitrary and  $f^{-1}(\alpha)$  is the union of a fuzzy points in  $f^{-1}(\alpha)$ ,

$f^{-1}(\alpha) \leq \text{cl int } f^{-1}(\alpha)$  , which implies that

$f^{-1}(\alpha)$  is fuzzy semi-open.

Thus  $f$  is fuzzy irresolute.

**(a) $\Leftrightarrow$ (c) :**

Let (a) hold and  $\beta$  be a fuzzy semi-closed set in  $Y$ . Then  $1 - \beta$  is fuzzy

semi-open in  $Y$  and hence  $f^{-1}(1 - \beta)$  is fuzzy semi- open in  $X$ , that is,

$1 - f^{-1}(\beta)$  is fuzzy semi- open in  $X$ .

Thus  $f^{-1}(\beta)$  is a fuzzy semi- closed set in  $X$ .

The proof of the converse is easy and thus omitted.

**(a) $\Rightarrow$ (d) :**

Let  $p$  be a fuzzy point in  $X$  and  $\beta$  be a fuzzy semi- nbd of  $f(p)$  in  $Y$ . Then there is a fuzzy semi-open set  $\alpha$  such that  $f(p) \leq \alpha \leq \beta$ .

Now  $f^{-1}(\alpha)$  is fuzzy semi- open in  $X$  and  $p$  is a fuzzy point in  $X$ .

Then  $p \leq f^{-1}(\alpha) \leq f^{-1}(\beta)$

which implies that  $f^{-1}(\beta)$  is a fuzzy semi-nbd of  $p$  in  $X$ .

**(d) $\Rightarrow$ (e) :**

Let  $p$  be a fuzzy point in  $X$  and  $\beta$  be a fuzzy semi- nbd of  $f(p)$  in  $Y$ .

By (d),  $f^{-1}(\beta)$  is a fuzzy semi-nbd of  $p$  in  $X$ .

Let  $f^{-1}(\beta) = \gamma$ . Then  $f(f^{-1}(\beta)) = f(\gamma)$ .

But  $f(f^{-1}(\beta)) \leq \beta$ .

Hence  $f(\gamma) \leq \beta$ .

**(e) $\Rightarrow$ (b) :**

Let  $p$  be a fuzzy point in  $X$  and let  $\alpha$  be a fuzzy semi-open set in  $Y$  such that  $f(p) \leq \alpha$ .

Then by (e) there exists a fuzzy semi-nbd  $\gamma$  of  $p$  in  $X$  such that  $f(\gamma) \leq \alpha$ .

Then there exists a fuzzy semi – open set  $\beta$  such that  $p \leq \beta \leq \gamma$ .

That is  $f(p) \leq f(\beta) \leq f(\gamma) \leq \alpha$ . Hence  $f(\beta) \leq \alpha$ .

**(c)  $\Rightarrow$  (f) :**

Let (c) hold and  $\lambda$  be a fuzzy set in  $X$ . Now  $\text{scl } f(\lambda)$  is fuzzy semi- closed in  $Y$ .

Thus  $f^{-1}(\text{scl } f(\lambda))$  is fuzzy semi-closed in  $X$ .

Now,  $f(\lambda) \leq \text{scl } (f(\lambda))$ .

That is  $\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(\text{scl } f(\lambda))$ .

That is  $\text{scl } (\lambda) \leq \text{scl } f^{-1}(\text{scl } f(\lambda)) = f^{-1}(\text{scl } f(\lambda))$ .

That is  $f(\text{scl } (\lambda)) \leq f(f^{-1}(\text{scl } f(\lambda))) \leq \text{scl } (f(\lambda))$ .

**(f)  $\Rightarrow$  (c) :**

Suppose (f) holds and  $\beta$  be a fuzzy semi- closed set in  $Y$ .

Let  $\lambda = f^{-1}(\beta)$ .

By (f),  $f(\text{scl}(\lambda)) \leq \text{scl } (f(\lambda)) = \text{scl}[f(f^{-1}(\beta))] \leq \text{scl}(\beta) = \beta$ .

Thus  $f^{-1}[f(\text{scl } (\lambda))] \leq f^{-1}(\beta) = \lambda$ .

That is,  $\text{scl}(\lambda) \leq f^{-1}[f(\text{scl}(\lambda))] \leq \lambda$ ,

or,  $\text{scl}(\lambda) = \lambda$ .

Hence,  $\lambda = f^{-1}(\beta)$  is fuzzy semi-closed in X.

**(f)  $\Rightarrow$  (g) :**

Let  $\beta$  be a fuzzy set in Y and let  $\lambda = f^{-1}(\beta)$  be a fuzzy set in X.

By (f),  $f[\text{scl}(\lambda)] \leq \text{scl}[f(\lambda)]$

That is,  $\text{scl}(\lambda) \leq f^{-1}[f(\text{scl}(\lambda))] \leq f^{-1}[\text{scl}(f(\lambda))]$ .

or  $\text{scl}[f^{-1}(\beta)] \leq f^{-1}[\text{scl}[f(f^{-1}(\beta))]] \leq f^{-1}(\text{scl}(\beta))$ .

**(g)  $\Rightarrow$  (f) :**

Let  $\lambda$  be a fuzzy set in X and  $\beta = f(\lambda)$  be a fuzzy set in Y.

By (g),  $\text{scl}(f^{-1}(\beta)) \leq f^{-1}(\text{scl}(\beta))$

or  $f(\text{scl}(f^{-1}(\beta))) \leq f[f^{-1}(\text{scl}(\beta))] \leq \text{scl}(\beta)$ .

Then  $f[\text{scl}(f^{-1}(f(\lambda)))] \leq \text{scl}(f(\lambda))$ .

That is,  $f[\text{scl}(\lambda)] \leq f[\text{scl}(f^{-1}(f(\lambda)))] \leq \text{scl} f(\lambda)$ .

That is,  $f[\text{scl} f(\lambda)] \leq \text{scl} f(\lambda)$ .