Chapter VIII

CHAPTER - VIII

α-CUTS OF TRIANGULAR FUZZY NUMBERS AND α-CUTS OF TRIANGULAR FUZZY NUMBER MATRICES

Definition : 8.1 [3]

For all x, $\alpha \in [0, 1]$ the **upper α-cut of x** is denoted as $x^{(\alpha)}$ and is defined as

$$x^{(\alpha)} = \begin{cases} 1, & \text{if } x \geq \alpha \\ 0, & \text{if } x < \alpha \end{cases}$$

and the **lower α-cut of x** is denoted as $x_{(\alpha)}$ and is defined as

$$x_{(\alpha)} = \begin{cases} x, & \text{if } x \geq \alpha \\ 0, & \text{if } x < \alpha \end{cases}$$

Now we define α-cuts of the Triangular Fuzzy Numbers and Triangular Fuzzy Number Matrices.

Definition : 8.2

For $\alpha \in [0, 1]$, the **upper α-cut of the Triangular Fuzzy Number** $\tilde{M} = < m, \rho, \beta >$ is defined as

$$\tilde{M}^{(\alpha)} = < m^{(\alpha)}, \rho^{(\alpha)}, \beta^{(\alpha)} >$$

and the **lower α-cut of** $\tilde{M}$ is defined as

$$\tilde{M}_{(\alpha)} = < m_{(\alpha)}, \rho_{(\alpha)}, \beta_{(\alpha)} >$$

Example : 8.3

Consider the triangular fuzzy number $\tilde{M}$ as follows :

$\tilde{M}$ = < 0.5, 0.1, 0.4 >. By taking $\alpha = 0.5$, we get

$\tilde{M}_{(\alpha)}$ = < 0.5, 0, 0 > and

$\tilde{M}^{(\alpha)}$ = < 1, 0, 0 >

Definition : 8.4

The **upper α-cut of a Triangular Fuzzy Number Matrix** $M = (\tilde{M}_{ij})_{mxn}$ is defined as

$$M^{(\alpha)} = (\tilde{M}_{ij}^{(\alpha)})_{mxn}$$

and the **lower α-cut of M** is defined as

$$M_{(\alpha)} = (\tilde{M}_{ij(\alpha)})_{mxn}$$

Example : 8.5

Consider the triangular fuzzy number matrix M as follows :

$$M = \begin{bmatrix} <0.5, 0.1, 0.4> & <0.1, 0.7, 0.6> \\ <0.3, 0.5, 0.2> & <0.6, 0.3, 0.65> \end{bmatrix}$$

Then by taking $\alpha = 0.5$, we get

$$M_{(\alpha)} = \begin{bmatrix} <0.5, 0, 0> & <0, 0.7, 0.6> \\ <0, 0.5, 0> & <0.6, 0, 0.65> \end{bmatrix} \text{ and}$$

$$M^{(\alpha)} = \begin{bmatrix} <1, 0, 0> & <0, 1, 1> \\ <0, 1, 0> & <1, 0, 1> \end{bmatrix}$$

Theorem : 8.6

For any two triangular fuzzy number matrices M and N,

(i) $(M \ominus N)^{(\alpha)} \geq M^{(\alpha)} \ominus N^{(\alpha)}$

(ii) $(M \vee N)^{(\alpha)} = M^{(\alpha)} \vee N^{(\alpha)}$

(iii) $(M \oplus N)^{(\alpha)} \geq M^{(\alpha)} \oplus N^{(\alpha)}$

Proof

Let $M = (\tilde{M}_{ij})_{mxn}$ where $\tilde{M}_{ij} = < m_{ij}, \rho_{ij}, \beta_{ij} >$ and

$N = (\tilde{N}_{ij})_{mxn}$ where $\tilde{N}_{ij} = < n_{ij}, \gamma_{ij}, \delta_{ij} >$

(i) Let $\tilde{E}_{ij}$ and $\tilde{D}_{ij}$ be the ij^{th} elements of $M^{(\alpha)} \ominus N^{(\alpha)}$ and $(M \ominus N)^{(\alpha)}$.

$\therefore \tilde{E}_{ij} = \tilde{M}_{ij}^{(\alpha)} \ominus \tilde{N}_{ij}^{(\alpha)}$ and $\tilde{D}_{ij} = (\tilde{M}_{ij} \ominus \tilde{N}_{ij})^{(\alpha)}$

Case 1 : $\tilde{M}_{ij} \geq \tilde{N}_{ij} \geq \alpha$

$< m_{ij}, \rho_{ij}, \beta_{ij} > \geq < n_{ij}, \gamma_{ij}, \delta_{ij} > \geq \alpha$

$\therefore m_{ij} \geq n_{ij} \geq \alpha, \rho_{ij} \geq \gamma_{ij} \geq \alpha, \beta_{ij} \geq \delta_{ij} \geq \alpha.$

Therefore, $\tilde{D}_{ij} = (\tilde{M}_{ij} \ominus \tilde{N}_{ij})^{(\alpha)}$

$= < m_{ij}, \rho_{ij}, \beta_{ij} >^{(\alpha)}$

$= < m_{ij}^{(\alpha)}, \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} >$

$= < 1, 1, 1 >$

and $\tilde{E}_{ij} = \tilde{M}_{ij}^{(\alpha)} \ominus \tilde{N}_{ij}^{(\alpha)}$

$= < m_{ij}^{(\alpha)} \ominus n_{ij}^{(\alpha)}, \ \rho_{ij}^{(\alpha)} \ominus \gamma_{ij}^{(\alpha)}, \ \beta_{ij}^{(\alpha)} \ominus \delta_{ij}^{(\alpha)} >$

$= < 1 \ominus 1, 1 \ominus 1, 1 \ominus 1 >$

$= < 0, 0, 0 >$

i.e., $\tilde{D}_{ij} > \tilde{E}_{ij}$

Case 2 : $\tilde{M}_{ij} \geq \alpha \geq \tilde{N}_{ij}$

$\therefore < m_{ij}, \rho_{ij}, \beta_{ij} > \geq \alpha \geq < n_{ij}, \gamma_{ij}, \delta_{ij} >$

$\therefore m_{ij} \geq \alpha \geq n_{ij}, \rho_{ij} \geq \alpha \geq \gamma_{ij}, \beta_{ij} \geq \alpha \geq \delta_{ij}.$

Then

$\tilde{D}_{ij} = (\tilde{M}_{ij} \ominus \tilde{N}_{ij})^{(\alpha)}$

$= < m_{ij}, \rho_{ij}, \beta_{ij} >^{(\alpha)}$

$= < m_{ij}^{(\alpha)}, \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} >$

$= < 1, 1, 1 >$

and $\tilde{E}_{ij} = \tilde{M}_{ij}^{(\alpha)} \ominus \tilde{N}_{ij}^{(\alpha)}$

$= < m_{ij}^{(\alpha)} \ominus n_{ij}^{(\alpha)}, \ \rho_{ij}^{(\alpha)} \ominus \gamma_{ij}^{(\alpha)}, \ \beta_{ij}^{(\alpha)} \ominus \delta_{ij}^{(\alpha)} >$

$= < 1 \ominus 0, 1 \ominus 0, 1 \ominus 0 >$

$= < 1, 1, 1 >$

$\therefore \tilde{D}_{ij} = \tilde{E}_{ij}$

Case 3 : $\alpha \geq \tilde{M}_{ij} \geq \tilde{N}_{ij}$

$\alpha \geq < m_{ij}, \rho_{ij}, \beta_{ij} > \geq < n_{ij}, \gamma_{ij}, \delta_{ij} >$

$\therefore \alpha \geq m_{ij} \geq n_{ij}, \alpha \geq \rho_{ij} \geq \gamma_{ij}, \alpha \geq \beta_{ij} \geq \delta_{ij}$.

Here

$$\tilde{D}_{ij} = (\tilde{M}_{ij} \ominus \tilde{N}_{ij})^{(\alpha)}$$

$$= < m_{ij}^{(\alpha)}, \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} >$$

$$= < 0, 0, 0 >$$

and $\tilde{E}_{ij} = \tilde{M}_{ij}^{(\alpha)} \ominus \tilde{N}_{ij}^{(\alpha)}$

$$= < m_{ij}^{(\alpha)} \ominus n_{ij}^{(\alpha)}, \ \rho_{ij}^{(\alpha)} \ominus \gamma_{ij}^{(\alpha)}, \ \beta_{ij}^{(\alpha)} \ominus \delta_{ij}^{(\alpha)} >$$

$$= < 0 \ominus 0, 0 \ominus 0, 0 \ominus 0 >$$

$$= < 0, 0, 0 >$$

i.e., $\tilde{D}_{ij} = \tilde{E}_{ij}$

$\therefore$ In all the cases, $\tilde{D}_{ij} \geq \tilde{E}_{ij}$

$\therefore (\tilde{M}_{ij} \ominus \tilde{N}_{ij})^{(\alpha)} \geq \tilde{M}_{ij}^{(\alpha)} \ominus \tilde{N}_{ij}^{(\alpha)}$

$\therefore (M \ominus N)^{(\alpha)} \geq M^{(\alpha)} \ominus N^{(\alpha)}$.

(ii) Let $\tilde{C}_{ij}$ and $\tilde{D}_{ij}$ be the ij[th] elements of $M^{(\alpha)} \vee N^{(\alpha)}$ and $(M \vee N)^{(\alpha)}$.

$\therefore \tilde{C}_{ij} = \tilde{M}_{ij}^{(\alpha)} \vee \tilde{N}_{ij}^{(\alpha)}$ and $\tilde{D}_{ij} = (\tilde{M}_{ij} \vee \tilde{N}_{ij})^{(\alpha)}$

Case 1 : $\tilde{M}_{ij} \geq \tilde{N}_{ij} \geq \alpha$

i.e., $< m_{ij}, \rho_{ij}, \beta_{ij} > \geq < n_{ij}, \gamma_{ij}, \delta_{ij} > \geq \alpha$

$\therefore m_{ij} \geq n_{ij} \geq \alpha, \rho_{ij} \geq \gamma_{ij} \geq \alpha$ and $\beta_{ij} \geq \delta_{ij} \geq \alpha$.

$$\therefore \tilde{D}_{ij} = (\tilde{M}_{ij} \vee \tilde{N}_{ij})^{(\alpha)}$$

$$= < m_{ij}, \rho_{ij}, \beta_{ij} >^{(\alpha)}$$

$$= < m_{ij}^{(\alpha)}, \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} >$$

$$= < 1, 1, 1 >$$

and $\tilde{C}_{ij} = \tilde{M}_{ij}^{(\alpha)} \vee \tilde{N}_{ij}^{(\alpha)}$

$= < m_{ij}^{(\alpha)} \vee n_{ij}^{(\alpha)},\ \rho_{ij}^{(\alpha)} \vee \gamma_{ij}^{(\alpha)},\ \beta_{ij}^{(\alpha)} \vee \delta_{ij}^{(\alpha)} >$

$= < 1 \vee 1, 1 \vee 1, 1 \vee 1 >$

$= < 1, 1, 1 >$

i.e., $\tilde{D}_{ij} = \tilde{C}_{ij}$.

Case 2 : $\tilde{\mathbf{M}}_{ij} \geq \alpha \geq \tilde{\mathbf{N}}_{ij}$

$\therefore < m_{ij}, \rho_{ij}, \beta_{ij} > \geq \alpha \geq < n_{ij}, \gamma_{ij}, \delta_{ij} >$

$\therefore m_{ij} \geq \alpha \geq n_{ij}, \rho_{ij} \geq \alpha \geq \gamma_{ij}, \beta_{ij} \geq \alpha \geq \delta_{ij}$.

Then

$\tilde{D}_{ij} = (\tilde{M}_{ij} \vee \tilde{N}_{ij})^{(\alpha)}$

$= < m_{ij}, \rho_{ij}, \beta_{ij} >^{(\alpha)}$

$= < m_{ij}^{(\alpha)}, \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} >$

$= < 1, 1, 1 >$

and $\tilde{C}_{ij} = \tilde{M}_{ij}^{(\alpha)} \vee \tilde{N}_{ij}^{(\alpha)}$

$= < m_{ij}^{(\alpha)} \vee n_{ij}^{(\alpha)},\ \rho_{ij}^{(\alpha)} \vee \gamma_{ij}^{(\alpha)},\ \beta_{ij}^{(\alpha)} \vee \delta_{ij}^{(\alpha)} >$

$= < 1 \vee 0, 1 \vee 0, 1 \vee 0 >$

$= < 1, 1, 1 >$

$\therefore \tilde{D}_{ij} = \tilde{C}_{ij}$

Case 3 : $\alpha \geq \tilde{\mathbf{M}}_{ij} \geq \tilde{\mathbf{N}}_{ij}$

i.e., $\alpha \geq < m_{ij}, \rho_{ij}, \beta_{ij} > \geq < n_{ij}, \gamma_{ij}, \delta_{ij} >$

$\therefore \alpha \geq m_{ij} \geq n_{ij}, \alpha \geq \rho_{ij} \geq \gamma_{ij}, \alpha \geq \beta_{ij} \geq \delta_{ij}$.

Then

$\tilde{D}_{ij} = (\tilde{M}_{ij} \vee \tilde{N}_{ij})^{(\alpha)}$

$= < m_{ij}, \rho_{ij}, \beta_{ij} >^{(\alpha)}$

$= < m_{ij}^{(\alpha)}, \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} >$

$= < 0, 0, 0 >$

and $\tilde{C}_{ij} = (\tilde{M}_{ij}^{(\alpha)} \vee \tilde{N}_{ij}^{(\alpha)})$

$= < m_{ij}^{(\alpha)} \vee n_{ij}^{(\alpha)},\ \rho_{ij}^{(\alpha)} \vee \gamma_{ij}^{(\alpha)},\ \beta_{ij}^{(\alpha)} \vee \delta_{ij}^{(\alpha)} >$

$= < 0 \vee 0, 0 \vee 0, 0 \vee 0 >$

$= < 0, 0, 0 >$

i.e., $\tilde{D}_{ij} = \tilde{C}_{ij}$.

$\therefore$ In all the cases, $\tilde{D}_{ij} = \tilde{C}_{ij}$

$\therefore (\tilde{M}_{ij} \vee \tilde{N}_{ij})^{(\alpha)} = \tilde{M}_{ij}^{(\alpha)} \vee \tilde{N}_{ij}^{(\alpha)}$

$\therefore (M \vee N)^{(\alpha)} = M^{(\alpha)} \vee N^{(\alpha)}$.

(iii) Let $\tilde{P}_{ij}$ and $\tilde{Q}_{ij}$ be the ij^th elements of $M^{(\alpha)} \oplus N^{(\alpha)}$ and $(M \oplus N)^{(\alpha)}$.

$\therefore \tilde{P}_{ij} = \tilde{M}_{ij}^{(\alpha)} \oplus \tilde{N}_{ij}^{(\alpha)}$ and $\tilde{Q}_{ij} = (\tilde{M}_{ij} \oplus \tilde{N}_{ij})^{(\alpha)}$.

Case 1 : $\tilde{\mathbf{M}}_{ij} \geq \tilde{\mathbf{N}}_{ij} \geq \alpha$

$\therefore < m_{ij}, \rho_{ij}, \beta_{ij} > \geq < n_{ij}, \gamma_{ij}, \delta_{ij} > \geq \alpha$

$\therefore m_{ij} \geq n_{ij} \geq \alpha$, $\rho_{ij} \geq \gamma_{ij} \geq \alpha$ and $\beta_{ij} \geq \delta_{ij} \geq \alpha$.

Then $\tilde{Q}_{ij} = (\tilde{M}_{ij} \oplus \tilde{N}_{ij})^{(\alpha)}$

$= < m_{ij} + n_{ij} - m_{ij} . n_{ij},\ \rho_{ij} + \gamma_{ij} - \rho_{ij} . \gamma_{ij},\ \beta_{ij} + \delta_{ij} - \beta_{ij} . \delta_{ij} >^{(\alpha)}$

$= < m_{ij} + n_{ij} (1 - m_{ij}),\ \rho_{ij} + \gamma_{ij} (1 - \rho_{ij}),\ \beta_{ij} + \delta_{ij} (1 - \beta_{ij}) >^{(\alpha)}$

$\geq < m_{ij}, \rho_{ij}, \beta_{ij} >^{(\alpha)}$

$> < m_{ij}^{(\alpha)}, \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} >$

$= < 1, 1, 1 >$

and $\tilde{P}_{ij} = \tilde{M}_{ij}^{(\alpha)} \oplus \tilde{N}_{ij}^{(\alpha)}$

$= < m_{ij}^{(\alpha)} \oplus n_{ij}^{(\alpha)},\ \rho_{ij}^{(\alpha)} \oplus \gamma_{ij}^{(\alpha)},\ \beta_{ij}^{(\alpha)} \oplus \delta_{ij}^{(\alpha)} >$

$= < m_{ij}^{(\alpha)} + n_{ij}^{(\alpha)} - m_{ij}^{(\alpha)} . n_{ij}^{(\alpha)},\ \rho_{ij}^{(\alpha)} + \gamma_{ij}^{(\alpha)} - \rho_{ij}^{(\alpha)} . \gamma_{ij}^{(\alpha)},$

$\beta_{ij}^{(\alpha)} + \delta_{ij}^{(\alpha)} - \beta_{ij}^{(\alpha)} . \delta_{ij}^{(\alpha)} >$

$= \; <1+1-1,\ 1+1-1,\ 1+1-1>$

$= \; <1, 1, 1>$

$\therefore\ \tilde{Q}_{ij} > \tilde{P}_{ij}.$

Case 2 : $\tilde{\mathbf{M}}_{ij} \geq \alpha \geq \tilde{\mathbf{N}}_{ij}$

$\therefore\ <m_{ij}, \rho_{ij}, \beta_{ij}> \ \geq\ \alpha\ \geq\ <n_{ij}, \gamma_{ij}, \delta_{ij}>$

$\therefore\ m_{ij} \geq \alpha \geq n_{ij},\ \rho_{ij} \geq \alpha \geq \gamma_{ij},\ \beta_{ij} \geq \alpha \geq \delta_{ij}.$

Then

$$\begin{aligned}
\tilde{Q}_{ij} &= (\tilde{M}_{ij} \oplus \tilde{N}_{ij})^{(\alpha)} \\
&= <m_{ij} + n_{ij} - m_{ij}\,.\,n_{ij},\ \ \rho_{ij} + \gamma_{ij} - \rho_{ij}\,.\,\gamma_{ij},\ \ \beta_{ij} + \delta_{ij} - \beta_{ij}\,.\,\delta_{ij}>^{(\alpha)} \\
&= <m_{ij} + n_{ij}\,(1 - m_{ij}),\ \ \rho_{ij} + \gamma_{ij}\,(1 - \rho_{ij}),\ \beta_{ij} + \delta_{ij}\,(1 - \beta_{ij})>^{(\alpha)} \\
&\geq <m_{ij}, \rho_{ij}, \beta_{ij}>^{(\alpha)} \\
&> <m_{ij}^{(\alpha)}, \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}> \\
&= <1, 1, 1>
\end{aligned}$$

and $\tilde{P}_{ij} = \tilde{M}_{ij}^{(\alpha)} \oplus \tilde{N}_{ij}^{(\alpha)}$

$$\begin{aligned}
&= <m_{ij}^{(\alpha)} \oplus n_{ij}^{(\alpha)},\ \ \rho_{ij}^{(\alpha)} \oplus \gamma_{ij}^{(\alpha)},\ \ \beta_{ij}^{(\alpha)} \oplus \delta_{ij}^{(\alpha)}> \\
&= <m_{ij}^{(\alpha)} + n_{ij}^{(\alpha)} - m_{ij}^{(\alpha)}\,.\,n_{ij}^{(\alpha)},\ \ \rho_{ij}^{(\alpha)} + \gamma_{ij}^{(\alpha)} - \rho_{ij}^{(\alpha)}\,.\,\gamma_{ij}^{(\alpha)}, \\
&\quad\ \beta_{ij}^{(\alpha)} + \delta_{ij}^{(\alpha)} - \beta_{ij}^{(\alpha)}\,.\,\delta_{ij}^{(\alpha)}> \\
&= <1+0-0,\ 1+0-0,\ 1+0-0> \\
&= <1, 1, 1>
\end{aligned}$$

$\therefore\ \tilde{Q}_{ij} > \tilde{P}_{ij}.$

Case 3 : $\alpha \geq \tilde{\mathbf{M}}_{ij} \geq \tilde{\mathbf{N}}_{ij}$

i.e., $\alpha\ \geq <m_{ij}, \rho_{ij}, \beta_{ij}> \ \geq\ <n_{ij}, \gamma_{ij}, \delta_{ij}>$

$\therefore\ \alpha \geq m_{ij} \geq n_{ij},\ \alpha \geq \rho_{ij} \geq \gamma_{ij},\ \alpha \geq \beta_{ij} \geq \delta_{ij}.$

Then

$$\begin{aligned}
\tilde{Q}_{ij} &= (\tilde{M}_{ij} \oplus \tilde{N}_{ij})^{(\alpha)} \\
&= <m_{ij} + n_{ij} - m_{ij}\,.\,n_{ij},\ \ \rho_{ij} + \gamma_{ij} - \rho_{ij}\,.\,\gamma_{ij},\ \ \beta_{ij} + \delta_{ij} - \beta_{ij}\,.\,\delta_{ij}>^{(\alpha)}
\end{aligned}$$

$= < m_{ij} + n_{ij} (1 - m_{ij}),\ \rho_{ij} + \gamma_{ij} (1 - \rho_{ij}),\ \beta_{ij} + \delta_{ij} (1 - \beta_{ij}) >^{(\alpha)}$

$\geq < m_{ij}, \rho_{ij}, \beta_{ij} >^{(\alpha)}$

$> < m_{ij}^{(\alpha)}, \rho_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)} >$

$= < 0, 0, 0 >$

and $\tilde{P}_{ij} = \tilde{M}_{ij}^{(\alpha)} \oplus \tilde{N}_{ij}^{(\alpha)}$

$= < m_{ij}^{(\alpha)} \oplus n_{ij}^{(\alpha)},\ \rho_{ij}^{(\alpha)} \oplus \gamma_{ij}^{(\alpha)},\ \beta_{ij}^{(\alpha)} \oplus \delta_{ij}^{(\alpha)} >$

$= < m_{ij}^{(\alpha)} + n_{ij}^{(\alpha)} - m_{ij}^{(\alpha)}.\ n_{ij}^{(\alpha)},\ \rho_{ij}^{(\alpha)} + \gamma_{ij}^{(\alpha)} - \rho_{ij}^{(\alpha)}.\gamma_{ij}^{(\alpha)},$

$\beta_{ij}^{(\alpha)} + \delta_{ij}^{(\alpha)} - \beta_{ij}^{(\alpha)}.\delta_{ij}^{(\alpha)} >$

$= < 0 + 0 - 0, 0 + 0 - 0, 0 + 0 - 0 >$

$= < 0, 0, 0 >$

$\therefore\ \tilde{Q}_{ij} = \tilde{P}_{ij}$.

$\therefore$ In all the cases, $\tilde{Q}_{ij} \geq \tilde{P}_{ij}$

$\therefore\ (\tilde{M}_{ij} \oplus \tilde{N}_{ij})^{(\alpha)} \geq \tilde{M}_{ij}^{(\alpha)} \oplus \tilde{N}_{ij}^{(\alpha)}$

Thus, $(M \oplus N)^{(\alpha)} \geq M^{(\alpha)} \oplus N^{(\alpha)}$.

In a similar manner, for the lower cut of triangular fuzzy number matrices, one can prove the following theorem :

Theorem : 8.7

For any two triangular fuzzy number matrices M and N

(i) $(M \vee N)_{(\alpha)} = M_{(\alpha)} \vee N_{(\alpha)}$

(ii) $(M \ominus N)_{(\alpha)} = M_{(\alpha)} \ominus N_{(\alpha)}$

(iii) $(M \oplus N)_{(\alpha)} \geq M_{(\alpha)} \oplus N_{(\alpha)}$