Chapter VI

Soft Connected Spaces

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SOFT CONNECTED SPACES

Definition : 6.1

Let (X, τ, E) be a soft topological space over X. A **soft separation** of $\tilde{X}$ is a pair (F, E), (G, E) of non-null soft open sets over X such that

$$\tilde{X} = (F, E) \cup (G, E) \text{ and } (F, E) \cap (G, E) = \Phi_E$$

A soft topological space (X, τ, E) is said to be **soft connected** if there does not exists a soft separation of $\tilde{X}$.

Theorem : 6.2

Let (F, E) be a soft set in $SS(X)_E$. Then the following hold

(i) $(F, E) \cup (F, E)^c = \tilde{X}$;

(ii) $(F, E) \cap (F, E)^c = \Phi_E$;

(iii) $(F, E) \cap \tilde{X} = (F, E)$.

Proof

We prove (ii), Let $(F, E) \cap (F, E)^c = (H, E)$. Then

$H(e) = F(e) \cap F^c(e) = F(e) \cap (X - F(e)) = \Phi$

Therefore, $(H, E) = \Phi_E$

Using theorem 6.2, we prove the following.

Theorem : 6.3

A soft topological space (X, τ, E) is soft connected if and only if the only soft sets in $SS(X)_E$ that are both soft open and soft closed over X are Φ_E and $\tilde{X}$.

Proof

Let (X, τ, E) be soft connected. Suppose to the contrary that (F, E) is both soft open and soft closed in X different from Φ_E and $\tilde{X}$. Clearly, $(F, E)^c$ is a soft open set in X different from Φ_E and $\tilde{X}$. By theorem 6.2, (F, E), $(F, E)^c$ is a soft separation of $\tilde{X}$. This is a contradiction.

Thus the only soft closed and open sets in X are Φ_E and $\tilde{X}$. Conversely, let (F, E), (G, E) be a soft separation of $\tilde{X}$. Let $(F, E) = \tilde{X}$. Then Theorem 6.2 implies that $(G, E) = \Phi_E$. This is a contradiction.

Hence $(F, E) \neq \tilde{X}$. Since $F(e) \cap G(e) = \Phi$ and $F(e) \cup G(e) = X$, for each $e \in E$, then we have $G^c(e) = X - G(e) = F(e)$. Therefore $(F, E) = (G, E)^c$. This shows that (F, E) is both soft open and soft closed in X different from Φ_E and $\tilde{X}$. This is a contradiction. Therefore, (X, τ, E) is soft connected.

Theorem : 6.4

Let $SS(U)_A$ and $SS(V)_B$ be families of soft sets. For a function $f_{pu} : SS(U)_A \to SS(V)_B$, the following hold

i. $f_{pu}^{-1}((F, B) \cup (G, B)) = f_{pu}^{-1}(F, B) \cup f_{pu}^{-1}(G, B)$;

ii. $f_{pu}^{-1}(\tilde{V}) = \tilde{U}$;

iii. $f_{pu}((F, A) \cap (G, A)) \tilde{\subseteq} f_{pu}(F, A) \cap f_{pu}(G, A)$;

iv. $f_{pu}^{-1}((F, B) \cap (G, B)) = f_{pu}^{-1}(F, B) \cap f_{pu}^{-1}(G, B)$;

v. $f_{pu}^{-1}(\Phi_B) = \Phi_A$

Proof

(i) Let (F, B) $\cup$ (G, B) = (H, B). Then $f_{pu}^{-1}(H, B) = (f_{pu}^{-1}(H), A)$, where $f_{pu}^{-1}(H)(x) = u^{-1}(H(p(x)))$, for each $x \in A$. On the other hand, let $f_{pu}^{-1}(F, B) \cup f_{pu}^{-1}(G, B) = (O, A)$, where

$O(x) = f_{pu}^{-1}(F)(x) \cup f_{pu}^{-1}(G)(x) = u^{-1}(F(p(x)) \cup G(p(x))) = u^{-1}(H(p(x)))$, for each $x \in A$. Therefore $f_{pu}^{-1}(H, B) = (O, A)$.

(ii) $f_{pu}^{-1}(\tilde{V}) = f_{pu}^{-1}(V, B) = (f_{pu}^{-1}(V), A)$, where $f_{pu}^{-1}(V)\,(x) = u^{-1}(V(p(x))) = u^{-1}(V)$ $= U = U(x)$

(iii) Let (F, A) $\cap$ (G, A) = (H, A). Then $f_{pu}(H, A) = (f_{pu}(H), B)$, where

$$f_{pu}(H)(y) \;=\; \begin{cases} \bigcup\limits_{x \in p^{-1}(y) \cap A} u(H(x)) & p^{-1}(y) \cap A \neq \Phi \\ \Phi & p^{-1}(y) \cap A = \Phi \end{cases}$$

For each $y \in B$. On the other hand, let $f_{pu}(F, A) \cap f_{pu}(G, A) = (O, B)$, where $O(y) = f_{pu}(F)(y) \cap f_{pu}(G)(y)$. for each $y \in B$. We have

$$O(y) \;=\; \begin{cases} (\bigcup\limits_{x \in p^{-1}(y) \cap A} u(F(x))) \cap (\bigcup\limits_{x \in p^{-1}(y) \cap A} u(G(x))) & p^{-1}(y) \cap A \neq \Phi \\ \Phi & p^{-1}(y) \cap A = \Phi \end{cases}$$

For each $y \in B$. Since $H(x) = F(x) \cap G(x)$, for each $x \in A$, then it is easy to see that $f_{pu}(H)(y) \subseteq O(y)$ for each $y \in B$. This implies that $f_{pu}(H, A) \subseteq (O, B)$

(iv) Let (F, B) $\cap$ (G, B) = (H, B). Then $f_{pu}^{-1}(H, B) = (f_{pu}^{-1}(H), A)$, where $f_{pu}^{-1}(H)(x) = u^{-1}(H(p(x)))$, for each $x \in A$.

On the other hand, let $f_{pu}^{-1}(F, B) \cap f_{pu}^{-1}(G, B) = (O, A)$, where

$O(x) = f_{pu}^{-1}(F)(x) \cap f_{pu}^{-1}(G)(x) = u^{-1}(F(p(x))) \cap u^{-1}(G(p(x)))$

$= u^{-1}(H(p(x)))$, for each $x \in A$. Therefore, $f_{pu}^{-1}(H, B) = (O, A)$.

Note : 6.5

Let (U, τ, A) and (V, τ', B) be soft topological spaces. Let $f_{pu} : SS(U)_A \rightarrow SS(V)_B$ be a function. Then f_{pu} is said to be soft pu-continuous if for each $(F, B) \in \tau'$ we have $f_{pu}^{-1}(F, B) \in \tau$.

Theorem : 6.6

Let (F, E), (G, E) and (H, E) be soft sets in $SS(X)_E$. Then,

(i) $(F, E) \cap ((G, E) \cup (H, E)) = ((F, E) \cap (G, E)) \cup ((F, E) \cap (H, E))$;

(ii) $(F, E) \tilde{\subseteq} (G, E)$ iff $(F, E) \cap (G, E) = (F, E)$;

(iii) $(F, E) \tilde{\subseteq} (G, E)$ iff $(F, E) \cup (G, E) = (G, E)$.

Proof

(i) Let $(G, E) \cup (H, E) = (A, E)$ and $(F, E) \cap (A, E) = (B, E)$. Then $B(e) = F(e) \cap A(e) = F(e) \cap (G(e) \cup H(e)) = (F(e) \cup G(e)) \cup (F(e) \cap H(e))$, for each $e \in E$.

On the other hand, if $(F, E) \cap (G, E) = (C, E)$, $(F, E) \cap (H, E) = (D, E)$ and $(C, E) \cup (D, E) = (I, E)$, then $I(e) = C(e) \cup D(e) = (F(e) \cap G(e)) \cup (F(e) \cap H(e))$ for each $e \in E$. Therefore, $(B, E) = (I, E)$.

The proof of (ii) and (iii) are similar.

Theorem : 6.7

Let f_{pu} be a soft pu-continuous function carrying the soft connected space (U, τ, A) onto the soft space (V, τ', B). Then (V, τ', B) is soft connected.

Proof

Suppose to the contrary there exists a soft separation (F, B), (G, B) of $\tilde{V}$. Then Theorem 6.4, implies that

$\tilde{U} = f_{pu}^{-1}((F, B) \cup (G, B)) = f_{pu}^{-1}(F, B) \cup f_{pu}^{-1}(G, B)$,

$f_{pu}^{-1}(F, B) \cap f_{pu}^{-1}(G, B) = f_{pu}^{-1}(\Phi_B) = \Phi_A$.

Let $f_{pu}^{-1}(F, B) = \Phi_A$. Since f_{pu} is surjective, then by Theorem 6.4, we have $(F, B) = \Phi_B$. This is contradiction.

Therefore, $f_{pu}^{-1}(F, B)$ and by similar reason $f_{pu}^{-1}(G, B)$ are differet from Φ_A. Now, Theorem 6.6 shows that $f_{pu}^{-1}(F, B)$, $f_{pu}^{-1}(G, B)$ is soft separation of $\tilde{U}$. This is a contradiction and this completes the proof.

Definition : 6.8

Let (F, E) be a soft set over X and Y be a non-empty subset of X. Then the sub soft set of (F, E) over Y denoted by $({}^{Y}F, E)$ is defined as follows :

$${}^{Y}F(e) = Y \cap F(e), \text{ for each } e \in E.$$

In otherword, $({}^{Y}F, E) = \tilde{Y} \cap (F, E)$.

Now suppose that (X, τ, E) be a soft topological space over X and Y be a non empty subset of X. Then,

$\tau_Y = \{({}^{Y}F, E) / (F, E) \in \tau\}$, is said to be the **soft relative topology** on Y and (Y, τ_Y, E) is called a **soft subspace** of (X, τ, E).

Theorem : 6.9

If the soft sets (F, E) and (G, E) form a soft separation of $\tilde{X}$, and (Y, τ_Y, E) is a soft connected subspace of (X, τ, E) then $\tilde{Y}$ lies entirely within either (F, E) or (G, E).

Proof

Since $\tilde{Y} \tilde{\subseteq} (F, E) \cup (G, E)$, then by Theorem 6.6, we have $\tilde{Y} = (\tilde{Y} \cap (F, E)) \cup (\tilde{Y} \cap (G, E))$.

This means that $\tilde{Y} \cap (F, E)$ and $\tilde{Y} \cap (G, E)$ are soft open sets over Y. Suppose to the contrary $\tilde{Y}$ does not lie entirely within either (F, E) or (G, E). By the hypothesis and Theorem 6.6. $\tilde{Y} \cap (F, E)$ and $\tilde{Y} \cap (G, E)$ are different from $\tilde{Y}$ and Φ_E. But $Y(e) \cap F(e) \cap G(e) = \Phi$, for each $e \in E$. Therefore, $(\tilde{Y} \cap (F, E)) \cap (\tilde{Y} \cap (G, E)) = \Phi_E$. Since $(\tilde{Y} \cap (F, E))$ and $(\tilde{Y} \cap (G, E))$ are soft open sets over $\tilde{Y}$, then we have a soft separation of $\tilde{Y}$. This is a contradiction.

This completes the proof.

Theorem : 6.10

Let (F, E), (G, E) and (H, E) be soft sets in $SS(X)_E$. Then the following hold.

i. $(F, E) \cap ((G, E) \cap (H, E)) = ((F, E) \cap (G, E)) \cap (H, E)$;

ii. $(F, E) \cup ((G, E) \cup (H, E)) = ((F, E) \cup (G, E)) \cup (H, E)$.

Theorem : 6.11

Let $\{(F_\alpha, E)\}_{\alpha \in J}$ be a family of soft sets in $SS(X)_E$. Then the following hold.

i. $(F, E) \cap (\bigcup_{\alpha \in J} (F_\alpha, E)) = \bigcup_{\alpha \in J} ((F, E) \cap (F_\alpha, E))$;

ii If $(F, E) = (G, E) \cup (H, E)$ then $(G, E), (H, E) \tilde{\subseteq} (F, E)$.

Lemma : 6.12

Let (Y, τ', E) and (Z, τ'', E) be soft subspaces of (X, τ, E) and (Y, E) $\tilde{\subseteq}$ (Z, E). Then (Y, τ, E) is a soft subspaces of (Z, τ'', E).

Proof

By Theorem 6.6, we have $\tilde{Y} = \tilde{Y} \cap \tilde{Z}$. Moreover each soft open set of (Y, τ', E) is of the form $\tilde{Y} \cap$ (F, E), where (F, E) is a soft open set of (X, τ, E). Therefore, by Theorem 6.9 we have

$$\tilde{Y} \cap (F, E) = (\tilde{Y} \cap \tilde{Z}) \cap (F, E) = \tilde{Y} \cap [\tilde{Z} \cap (F, E)]$$

Conversely, it is clear that each soft open set in Y as a soft subspace of (Z, τ'', E) is of the form $\tilde{Y} \cap (\tilde{Z} \cap (F, E)) = \tilde{Y} \cap (F, E)$.

This completes the proof.

Theorem : 6.13

The union of a collection of soft connected subpsace of (X, τ, E) that have non-null intersection is soft connected.

Proof

Let $\{(Y_\alpha, \tau_{Y_\alpha}, E)\}_{\alpha \in J}$ be an arbitrary collection of soft connected soft subspace of (X, τ, E), such that $(\cap_{\alpha \in J} Y_\alpha, E) \neq \Phi_E$. Suppose to the contrary that there exists a soft separation $\tilde{Y} \cap (F, E)$, $\tilde{Y} \cap (G, E)$ of $\tilde{Y} = \cup_{\alpha \in J} \tilde{Y}_\alpha$. By Theorem 6.10, we have $\tilde{Y} = (\cup_{\alpha \in J} (F_\alpha, E)) \cup (\cup_{\alpha \in J}(G_\alpha, E))$ where $F_\alpha(e) = F(e) \cap Y_\alpha$ and $G_\alpha(e) = G(e) \cap Y_\alpha$, for each $\alpha \in J$ and $e \in E$.

Since $\cap_{\alpha \in J} \tilde{Y}_\alpha \neq \Phi_E$, it is easy to see that $\cap_{\alpha \in J} Y_\alpha \neq \Phi$,

and $x \in \cap_{\alpha \in J} Y_\alpha$. On the other hand, Lemma 6.11 implies that $(Y_\alpha, \tau_\alpha, E)$ is a soft subspace of (Y, τ_Y, E), for each $\alpha \in J$. By Theorem 6.9, we can assume that $\tilde{Y}_\alpha$ lies entirely within $\tilde{Y} \cap (F, E)$.