CHAPTER IV

SOFT TERNARY SEMIGROUPS

Definition : 4.1

A **ternary semigroup** is an algebraic structure (S, f) such that S is a non-empty set and $f : S^3 \rightarrow S$ is a ternary operation satisfying the following associative law:

$$f(f(a, b, c), d, e) = f(a, f(b, c, d), e) = f(a, b, f(c, d, e)) \text{ for all } a, b, c, d, e \in S.$$

For simplicity we write $f(a, b, c)$ as abc and consider the ternary operation f as " · ". A ternary semigroup $(S, \cdot)$ is called **commutative** if

$$abc = bac = bca \ (abc = cba) \text{ for all } a, b, c \in S.$$

Definition : 4.2

A non-empty subset T of a ternary semigroup S is said to be a **ternary subsemigroup** of S if

$$TTT = T^3 \subseteq T.$$

Definition : 4.3

By a **left** (**right**, **lateral**) **ideal** of a ternary semigroup S we mean a non-empty subset A of S such that $SSA \subseteq A \ (ASS \subseteq A, SAS \subseteq A)$. By a two sided ideal, we mean a subset of S which is both a left and a right ideal of S.

If a non-empty subset of S is a left, right and a lateral ideal of S, then it is called an **ideal** of S. A left (right, lateral, two sided) ideal I of a ternary semigroup S is idempotent if $I^3 = I$.

Definition : 4.4

A non-empty subset Q of a ternary semigroup S is called a **quasi-ideal** of S if $(QSS) \cap (SQS) \cap (SSQ) \subseteq Q, (QSS) \cap (SSQSS) \cap (SSQ) \subseteq Q$.

Definition : 4.5

Let (F,A), (G,B) and (H,C) be any three soft sets over a ternary semigroup S. Then, the **restricted product** (K,D) is defined as the soft set $(K,D) = (F,A)\ \hat{o}\ (G,B)\ \hat{o}\ (H,C)$, where $D = A \cap B \cap C$ is non-empty and K is a set valued function from D to $P(S)$ defined as $K(d) = F(d)G(d)H(d)$ for all $d \in D$.

Definition : 4.6

A soft set (F,A) over a ternary semigroup S is called a **soft ternary semigroup** over S if,

$$(F,A)\hat{o}(F,A)\hat{o}(F,A) \subseteq (F,A)$$

Theorem : 4.7

A soft set (F,A) over a ternary semigroup S is a soft ternary semigroup over S if and only if for all $a \in A$, $F(a) \neq \emptyset$ is a ternary subsemigroup of S.

Proof :

Suppose that (F,A) is a soft ternary semigroup over S. We prove that $F(a) \neq \emptyset$ is a ternary subsemigroup of S. By definition

$$(F,A)\hat{o}(F,A)\hat{o}(F,A) = (H, A \cap A \cap A) = (H,A)$$

where H is defined by

$$H(a) = F(a)F(a)F(a), \text{ for all } a \in A.$$

As $(F,A)\hat{o}(F,A)\hat{o}(F,A) \subseteq (F,A)$, so $(H,A) \subseteq (F,A)$.

That is, $H(a) \subseteq F(a)$ for all $a \in A$, and so $F(a)F(a)F(a) \subseteq F(a)$. This implies that $F(a)$ is a ternary subsemigroup of S.

Conversely, suppose that $F(a) \neq \emptyset$ is a ternary subsemigroup of S. We show that (F,A) is a soft ternary semigroup over S. By definition

$$(F,A)\hat{o}(F,A)\hat{o}(F,A) = (H, A \cap A \cap A) = (H,A)$$

where $H(a) = F(a)F(a)F(a)$, for all $a \in A$. Since $F(a)$ is a ternary subsemigroup of S, we have $F(a)F(a)F(a) \subseteq F(a)$, that is $H(a) \subseteq F(a)$.

Thus, $(H, A) \subseteq (F, A)$. This implies that

$$(F, A)\hat{o}(F, A)\hat{o}(F, A) \subseteq (F, A)$$

Hence, (F, A) is a soft ternary semigroup over S.

Definition : 4.8

A soft set (F, A) over a ternary semigroup S is called a **soft left (right, lateral) ideal** over S if

$$(S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A)((F, A)\hat{o}(S, E)\hat{o}(S, E) \subseteq (F, A), (S, E)\hat{o}(F, A)\hat{o}(S, E) \subseteq (F, A)),$$

respectively.

Definition : 4.9

A soft set (F, A) over S is called a **soft ideal** over S, if it is a soft left, soft right and a soft lateral ideal over S.

Theorem : 4.10

A soft set (F, A) over S is a **soft left (right, lateral) ideal** over S if and only if for all $a \in A$, $F(a) \neq \emptyset$ is a left (right, lateral) ideal of S.

Proof :

Suppose that (F, A) is a soft left ideal over S. We show that $F(a) \neq \emptyset$ is a left ideal of S.

By definition,

$$(S, E)\hat{o}(S, E)\hat{o}(F, A) = (H, E \cap E \cap A) = (H, A)$$

$$\Rightarrow \quad S(a)S(a)F(a) = H(a), \quad \text{for all } a \in A$$

That is,

$$SSF(a) = H(a).$$

As $(S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A)$

$\Rightarrow$ $(H, A) \subseteq (F, A)$

Thus, $H(a) \subseteq F(a)$ for all $a \in A$.

Therefore, $SSF(a) \subseteq F(a)$.

Hence F(a) is a left ideal of S.

Conversely, assume that $F(a) \neq \emptyset$ is a left ideal of S. We show that (F, A) is a soft left ideal over S. By definition,

$$(S, E)\hat{o}(S, E)\hat{o}(F, A) = (H, E \cap E \cap A) = (H, A)$$

$\Rightarrow$ $S(a)S(a)F(a) = H(a)$, for all $a \in A$

That is,

$$SSF(a) = H(a)$$

But $SSF(a) \subseteq F(a)$

$\Rightarrow$ $H(a) \subseteq F(a)$

$\Rightarrow$ $(H, A) \subseteq (F, A)$

$\Rightarrow (S, E)\ \hat{o}\ (S, E)\hat{o}(F, A) \subseteq (F, A)$. Hence (F, A) is a soft left ideal over S.

Definition : 4.11

Let (F, A), (G, B) and (H, C) be three soft sets over a ternary semigroup $(S, *)$. Then the **ternary operation** $*$ (**extended product**) for soft sets is defined as

$$(K, D) = (F, A) * (G, B) * (H, C)$$

where $D = A \times B \times C$ and $K(a, b, c) = F(a) * G(b) * H(c)$, $a \in A, b \in B, c \in C$, and $A \times B \times C$ is the cartesian product of the sets A, B, C. If there does not arise any ambiguity then we can write $(F, A)(G, B)(H, C)$ instead of $(F, A) * (G, B) * (H, C)$ and $F(a)G(b)H(c)$ for $F(a) * G(b) * H(c)$.

Theorem : 4.12

Let (F, A) and (G, B) be two soft ternary semigroups over S, such that $A \cap B \neq \emptyset$. Then $(F, A) \cap_R (G, B)$ is a soft ternary semigroup over S.

Proof :

By definition,

$$(H, C) = (F, A) \cap_R (G, B),$$

where $C = A \cap B \neq \emptyset$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$. As $F(c)$ and $G(c)$ both are ternary subsemigroups of S, so $H(c)$ is either empty or a ternary subsemigroup of S. Consequently, (H, C) is a soft ternary semigroup over S.

Theorem : 4.13

Let (F, A) and (G, B) be two soft ternary semigroups over S, such that $A \cap B = \emptyset$. Then $(F, A) \cup_E (G, B)$ is a soft ternary semigroup over S.

Proof :

By definition $(H, C) = (F, A) \cup_E (G, B)$, where $C = A \cup B$ and $A \cap B = \emptyset$. Then for all $c \in C$ either $c \in A - B$ or $c \in B - A$. If $c \in A - B$, then $H(c) = F(c)$ and if $c \in B - A$, then $H(c) = G(c)$, in both cases $H(c)$ is a ternary subsemigroup of S. Therefore (H, C) is a soft ternary semigroup over S.

Theorem : 4.14

Let (F, A) and (G, B) be two soft ternary semigroups over S. Then $(F, A) \wedge (G, B)$ is a soft ternary semigroup over S.

Proof :

By definition,

$$(H, C) = (F, A) \wedge (G, B)$$

where $C = A \times B$ and $H(a,b) = F(a) \cap G(b)$ for all $(a,b) \in A \times B$. As $F(a)$ and $G(b)$ are ternary subsemigroup of S, therefore either $F(a) \cap G(b) = \emptyset$ or $F(a) \cap G(b)$ is a ternary subsemigroup of S. Consequently, (H,C) is a soft ternary semigroup over S.

Theorem : 4.15

Let (F,A), (G,B) and (H,C) be any three soft ternary semigroups over a commutative ternary semigroup S. Then $(F,A) * (G,B) * (H,C)$ is a soft ternary semigroup over S.

Proof :

By definition,

$$(K, A \times B \times C) = (F,A) * (G,B) * (H,C),$$

where $K(a,b,c) = F(a)G(b)H(c)$ for all $(a,b,c) \in A \times B \times C$.

we show that $(F,A) * (G,B) * (H,C)$ is a soft ternary semigroup over S. It is enough to show that $K(a,b,c) = F(a)G(b)H(c)$ is a ternary subsemigroup of S. Since S is commutative, so

$\big(K(a,b,c)K(a,b,c)K(a,b,c)\big)$

$$= ((F(a)G(b)H(c))(F(a)G(b)H(c))(F(a)G(b)H(c)))$$

$$= (\big(F(a)G(b)H(c)\big)F(a)G(b)H(c)\big(F(a)\big)G(b)H(c))$$

$$= (F(a)(G(b)H(c)F(a))(F(a)H(c)G(b)G(b)H(c))$$

$$= (F(a)F(a)\big(H(c)G(b)F(a)\big)\big(H(c)G(b)G(b)\big)H(c))$$

$$= ((F(a)F(a)F(a))G(b)(H(c)H(c)G(b)G(b)H(c))$$

$$= (\big(F(a)F(a)F(a)\big)G(b)G(b)\big(H(c)H(c)G(b)\big)H(c))$$

$$= ((F(a)F(a)F(a))(G(b)G(b)G(b))(H(c)H(c)H(c)))$$

$$\subseteq (F(a)G(b)H(c))$$

$$= K(a, b, c)$$

because $F(a), G(b)$ and $H(c)$ are ternary subsemigroups. This implies that

$$K(a, b, c)K(a, b, c)K(a, b, c) \subseteq K(a, b, c)$$

Thus $K(a, b, c)$ is a ternary subsemigroup of S. Hence $(F, A) * (G, B) * (H, C)$ is a soft ternary semigroup over$S$.

Theorem : 4.16

Let (F, A) and (G, B) be any two soft ideals over a ternary semigroups over S, with $A \cap B \neq \emptyset$. Then $(F, A) \cap_R (G, B)$ is a soft ideal over S contained in both (F, A) and (G, B).

Theorem : 4.17

Let (F, A) and (G, B) be any two soft ideals over a ternary semigroups over S. Then $(F, A) \cup_E (G, B)$ is a soft ideal over S containing both (F, A) and (G, B).

Proof :

By definition,

$$(H, C) = (F, A) \cup_E (G, B),$$

where $C = A \cup B$ for all $c \in C$, either $c \in A - B$ or $c \in B - A$ or $c \in A \cap B$. If $c \in A - B$, then $H(c) = F(c)$, if $c \in B - A$, then $H(c) = G(c)$, and if $c \in A \cap B$, then $H(c) = F(c) \cup G(c)$, in all the case $H(c)$ is an ideal of S. Hence (H, C) is soft ideal over S. As $A \subseteq A \cup B$, $B \subseteq A \cup B$, and $F(c) \subseteq H(c), G(c) \subseteq H(c)$ for all $c \in C$.

Therefore, by definition of soft subsets $(F, A) \subseteq (H, C)$ and $(G, B) \subseteq (H, C)$.

Theorem : 4.18

i) Let (F, A) and (G, B) be any two soft ideals over a ternary semigroups over S. Then $(F, A) \wedge (G, B)$ is a soft ideal over S.

ii) Let (F, A) and (G, B) be any two soft ideals over a ternary semigroups over S. Then $(F, A) \vee (G, B)$ is a soft ideal over S.

Definition : 4.19

Let (G, B) be a soft subset of a ternary semigroup (F, A) over S. Then (G, B) is called a **soft ternary subsemigroup** of (F, A) if $G(b)$ is a ternary subsemigroup of $F(b)$ for all $b \in B$.

Theorem : 4.20

Let (F, A) be a soft ternary semigroup over S and $\{(H_i, B_i) : i \in I\}$ be a nonempty family of soft ternary subsemigroup of (F, A). Then

i) $\cap_{R\, i \in I}(H_i, B_i)$ is a soft ternary subsemigroup of (F, A).

ii) $\wedge_{i \in I}(H_i, B_i)$ is a soft ternary subsemigroup of $\wedge_{i \in I}(F, A)$.

iii) If $B_i \cap B_j = \emptyset$ for all different $i, j \in I$, then $\cup_{E\, i \in I}(H_i, B_i)$ is a soft ternary subsemigroup of (F, A).

Theorem : 4.21

Let (F, A) be a soft ternary semigroup over S and $\{(H_i, B_i) : i \in I\}$ be a nonempty family of soft ideals of (F, A). Then

i) $\cap_{R\, i \in I}(H_i, B_i)$ is a soft ideal of (F, A).

ii) $\wedge_{i \in I}(H_i, B_i)$ is a soft ideal of $\wedge_{i \in I}(F, A)$.

iii) $\cup_{R\, i\in I}(H_i, B_i)$ is a soft ideal of (F, A).

iv) $\vee_{i\in I}(H_i, B_i)$ is a soft ideal of $\vee_{i\in I}(F, A)$.

Soft Quasi-Ideals over Ternary Semigroups

Definition : 4.22

A soft set (F, A) over a ternary semigroup S is called a **soft quasi-ideal** over S if

i) $(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(F, A)\,\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A)$

ii) $(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A)\,\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A)$

$\subseteq (F, A)$

where (S, E) is the absolute soft set over S.

Theorem : 4.23

A soft set (F, A) over a ternary semigroup S is a soft quasi-ideal over S if and only if for all $a \in A$, $F(a) \neq \emptyset$ is a quasi-ideal of S.

Proof :

Suppose that a soft set (F, A) over S is a soft quasi ideal over S. We have to prove that $F(a)$ is a quasi-ideal of S. By definition of restricted product,

$$(F, A)\hat{o}(S, E)\hat{o}(S, E) = (G, A \cap E \cap E) = (G, A) \qquad \rightarrow (1)$$

$$(S, E)\hat{o}(F, A)\hat{o}(S, E) = (H, E \cap A \cap E) = (H, A) \qquad \rightarrow (2)$$

$$(S, E)\hat{o}(S, E)\hat{o}(F, A) = (I, E \cap E \cap A) = (I, A) \qquad \rightarrow (3)$$

$$(S, E)\hat{o}(S, E)\hat{o}(F, A)\hat{o}(S, E)\hat{o}(S, E) = (J, E \cap E \cap A \cap E \cap E) = (J, A) \qquad \rightarrow (4)$$

From $(1), (2)$ and (3) we have,

$$(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(F, A)\ \hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A)$$

$$= (G, A) \cap_R (H, A) \cap_R (I, A) \qquad \rightarrow (5)$$

$$(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(F, A)\ \hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (K, A) \rightarrow (6)$$

But $(F, A)$is a soft quasi-ideal over S. Thus,

$$(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(F, A)\ \hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A).$$

From (6), $(K, A) \subseteq (F, A)$, that is, $K(a) \subseteq F(a)$ for all $a \in A$, again from (6), we have

$F(a)S(a)S(a) \cap S(a)F(a)S(a) \cap S(a)S(a)F(a) = K(a)$ for all $a \in A$.

This implies that,

$$F(a)S(a)S(a) \cap S(a)F(a)S(a) \cap S(a)S(a)F(a) \subseteq F(a) \qquad \rightarrow (7)$$

Similarly, from $(1), (3)$ and (4),

$$(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A)\ \hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A)$$

$$= (P, A) \qquad \rightarrow (8)$$

As $(F, A)$is a soft quasi-ideal over S, then

$$(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A)\ \hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A)$$

$$\subseteq (F, A)$$

Therefore $(P, A) \subseteq (F, A)$, that is $P(a) \subseteq F(a)$ for all $a \in A$. From (8),

$F(a)S(a)S(a) \cap S(a)S(a)F(a)S(a)S(a) \cap S(a)S(a)F(a) = P(a)$ for all $a \in A$. This implies that,

$$F(a)S(a)S(a) \cap S(a)S(a)F(a)S(a)S(a) \cap S(a)S(a)F(a) \subseteq F(a) \qquad \rightarrow (9)$$

From (7) and (9), it is clear that $\mathrm{F(a)}$ is a quasi-ideal over S.

Conversely, let $\mathrm{F(a)} \neq \emptyset$ be a quasi-ideal of S for all $\mathrm{a} \in \mathrm{A}$. we have to prove that $(\mathrm{F,A})$ is quasi-ideal over S. From $(1), (2)$ and (3), we have

$$(\mathrm{F,A})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{S,E}) \cap_{\mathrm{R}} (\mathrm{S,E})\hat{o}(\mathrm{F,A})\ \hat{o}(\mathrm{S,E}) \cap_{\mathrm{R}} (\mathrm{S,E})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{F,A})$$

$$= (\mathrm{G,A}) \cap_{\mathrm{R}} (\mathrm{H,A}) \cap_{\mathrm{R}} (\mathrm{I,A})$$

$$= (\mathrm{K,A})$$

By definition,

$\mathrm{F(a)S(a)S(a)} \cap \mathrm{F(a)S(a)} \cap \mathrm{S(a)S(a)F(a)} = \mathrm{K(a)}$ for all $\mathrm{a} \in \mathrm{A}$. But $\mathrm{F(a)}$ is a quasi-ideal over S. Therefore,

$$\mathrm{K(a)} = \mathrm{F(a)S(a)S(a)} \cap \mathrm{S(a)F(a)S(a)} \cap \mathrm{S(a)S(a)F(a)} \subseteq \mathrm{F(a)},$$

And so $(\mathrm{K,A}) \subseteq (\mathrm{F,A})$. Thus,

$$(\mathrm{F,A})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{S,E}) \cap_{\mathrm{R}} (\mathrm{S,E})\hat{o}(\mathrm{F,A})\ \hat{o}(\mathrm{S,E}) \cap_{\mathrm{R}} (\mathrm{S,E})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{F,A}) \subseteq (\mathrm{F,A})$$

$$\rightarrow (10)$$

Now from $(1), (3)$ and (4), we have

$$(\mathrm{F,A})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{S,E}) \cap_{\mathrm{R}}(\mathrm{S,E})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{F,A})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{S,E}) \cap_{\mathrm{R}} (\mathrm{S,E})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{F,A})$$

$$= (\mathrm{G,A}) \cap_{\mathrm{R}} (\mathrm{I,A}) \cap_{\mathrm{R}} (\mathrm{J,A}) = (\mathrm{P,A}),$$

$$(\mathrm{F,A})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{S,E}) \cap_{\mathrm{R}}(\mathrm{S,E})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{F,A})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{S,E}) \cap_{\mathrm{R}}(\mathrm{S,E})\hat{o}(\mathrm{S,E})\hat{o}(\mathrm{F,A})$$

$$= (\mathrm{P,A}),$$

$\mathrm{F(a)S(a)S(a)} \cap \mathrm{S(a)S(a)F(a)S(a)S(a)} \cap \mathrm{S(a)S(a)F(a)} = \mathrm{P(a)}$ for all $\mathrm{a} \in \mathrm{A}$.

But $\mathrm{F(a)}$ is a quasi-ideal over S. Then

$$P(a) = F(a)S(a)S(a) \cap S(a)S(a)F(a)S(a)S(a) \cap S(a)S(a)F(a)$$

Therefore $(P, A) \subseteq (F, A)$.

Hence,

$$(F, A)\hat{o}(S, E)\hat{o}(S, E)_R(S, E)\hat{o}(S, E)\hat{o}(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R(S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A) \qquad \rightarrow (11)$$

From (10) and (11) (F, A) is a soft quasi-ideal over S.

Theorem : 4.24

Let $(R, A), (L, B)$ and (M, C) be soft right, soft left and soft lateral ideals over S. Then $(R, A) \cap_R (L, B) \cap_R (M, C)$ is a soft quasi-ideal over S.

Theorem : 4.25

Let $(R, A), (L, B)$ and (M, C) be the soft right, soft left and soft lateral ideals over S, such that $A \cap B \cap C = \emptyset$. Then $(R, A) \cap_E (L, B) \cap_E (M, C)$ is a soft quasi-ideal over S.

Proof :

By definition

$$(H, D) = (R, A) \cap_E (L, B) \cap_E (M, C)$$

where $D = A \cup B \cup C$, $A \cap B \cap C = \emptyset$ and

$$H(d) = \begin{cases} R(d) & \text{if } d \in A - B \cap C \\ M(d) & \text{if } d \in C - A \cap B \\ L(d) & \text{if } d \in B - A \cap C \end{cases}$$

for any $d \in D$. In each case $H(d)$ is a quasi-ideal of S. As every left, right and a lateral ideal of a ternary semigroup S is a quasi-ideal of S, then by definition,

$(H, D) = (R, A) \cap_E (L, B) \cap_E (M, C)$ is a soft quasi-ideal over S.

Theorem : 4.26

Every soft left (right, lateral) ideal over a ternary semigroup S is a soft quasi-ideal over S.

Proof :

Let (L, A) be a soft left ideal over S. Then $L(a)$ is a left ideal of S. As each left deal of S is a quasi-ideal of S, therefore $L(a)$ is a quasi-ideal of S. Hence (L, A) is a soft quasi-ideal over S.

Remark : 4.27

The converse of theorem.4.26 is not true in general as seen in the following example.

Example : 4.28

Let

$$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be the ternary semigroup under matrix multiplication defined as

$$ABC = (AB)C = A(BC)$$

for all $A, B, C \in S$. Let $B = \{\alpha\}$ and $G : B \to P(S)$ defined by

$$G(\alpha\} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

Then (G, B) is a soft quasi-ideal over S. But it is not a soft left, soft right and a soft lateral ideal over S.

Theorem : 4.29

i) Every soft left (right, lateral) ideal over S is a soft ternary semi-group over S.

ii) Every soft quasi-ideal is a soft ternary semigroup over S.

Theorem : 4.30

Let $(R, A), (L, B)$ and (M, C) be soft right, soft left and soft lateral ideals over S. Then $(R, A) \wedge (L, B) \wedge (M, C)$ is a soft quasi-ideal over S.

Proof :

By definition $(H, D) = (R, A) \wedge (L, B) \wedge (M, C)$ where $D = A \times C \times B$, and for any $(a, c, b) \in A \times C \times B$, $H\,(a, c, b) = R(a) \cap M\,(c) \cap L(b)$ is a quasi-ideal of S. Since the intersection of a left, right and a lateral ideal is a quasi-ideal of S, thus, $(R, A) \wedge (L, B) \wedge (M, C)$ is a soft quasi-ideal over S.

Theorem : 4.31

Let (F, A) and (G, B) be two soft quasi-ideals over a ternary semigroup S. Then the following statements hold.

i) $(F, A) \cap_R (G, B)$is a soft quasi-ideal over S.

ii) $(F, A) \cap_E (G, B)$ is a soft quasi-ideal over S.

iii) $(F, A) \wedge (G, B)$is a soft quasi-ideal over S.

iv)$(F, A) \cup_E (G, B)$ is a soft quasi-ideal over S, whenever $A \cap B = \emptyset$.

Theorem : 4.32

Let (F, A) be a soft quasi-ideal and (G, B) a soft ternary semi-group over S. Then $(F, A) \cap_R (G, B)$ is a soft quasi-ideal of (G, B).

Proof :

By definition,

$$(H, C) = (F, A) \cap_R (G, B),$$

where $C = A \cap B \neq \emptyset$ and

$$H(c) = F(c)G(c) \text{ for all } c \in C, \text{ as } H(c) \subseteq F(c) \text{and } H(c) \subseteq G(c).$$

we have to prove that $H(c)$ is a quasi-ideal of $G(c)$. Since $H(c) \subseteq G(c)$,

$$\begin{aligned} &H(c)G(c)G(c) \cap G(c)H(c)G(c) \cap G(c)G(c)H(c) \\ &\subseteq G(c)G(c)G(c) \cap G(c)G(c)G(c) \cap G(c)G(c)G(c) \\ &\subseteq G(c)G(c)G(c) \\ &\subseteq G(c) \end{aligned}$$

because $G(c)$ is a ternary subsemigroup of S. This implies that

$$H(c)G(c)G(c) \cap G(c)H(c)G(c) \cap G(c)G(c)H(c) \subseteq G(c) \qquad \rightarrow (1)$$

Also $H(c) \subseteq G(c)$. Then

$$\begin{aligned} &H(c)G(c)G(c) \cap G(c)H(c)G(c) \cap G(c)G(c)H(c) \\ &\subseteq F(c)G(c)G(c) \cap G(c)F(c)G(c) \cap G(c)G(c)F(c) \\ &\subseteq F(c)S(c)S(c) \cap S(c)F(c)S(c) \cap S(c)S(c)F(c) \subseteq F(c) \end{aligned}$$

because $F(c)$ is a quasi-ideal of S. Thus,

$$H(c)G(c)G(c) \cap G(c)H(c)G(c) \cap G(c)G(c)H(c) \subseteq F(c) \qquad \rightarrow (2)$$

From eqn (1) and (2), we have

$$H(c)G(c)G(c) \cap G(c)H(c)G(c) \cap G(c)G(c)H(c) \subseteq F(c) \cap G(c) \subseteq H(c) \quad \rightarrow (3)$$

Similarly, we can prove that

$$H(c)G(c)G(c) \cap G(c)G(c)H(c)G(c)G(c) \cap G(c)G(c)H(c) \subseteq H(c) \qquad \rightarrow (4)$$

From (3) and (4), $H(c)$ is a quasi-ideal of $G(c)$. Thus $(F, A) \cap_R (G, B)$ is a soft quasi-ideal of (G, B).