Chapter 3

Other Notions Related to λ_g^δ-Closed Sets

3.1 Introduction

The aim of this chapter is to introduce a new class of neighborhoods using λ_g^δ-open sets namely λ_g^δ-neighborhoods in topological spaces. Further, we discuss some vital relations and interesting characterizations of λ_g^δ-neighborhood. This new notion can be applied in the Geographic Information System (GIS), where the concept of nearness is studied in terms of the relation between objects rather than the distance between them. In a consequent manner, concepts of λ_g^δ-Frontier, λ_g^δ-Boundary, λ_g^δ-Exterior and λ_g^δ-Saturated Set are studied.

The idea of grills was developed by Choquet in 1974. Grill is a powerful tool in dealing with proximity spaces, closure spaces and in the theory of compactification. It is efficient in dealing with many topological situations. With this motivation, λ_g^δ-open sets are analyzed through grills.

3.2 λ_g^δ-Neighborhood

Definition 3.2.1. A subset N of a topological space (X, τ) is called a ***λ_g^δ-neighborhood*** of $x \in X$ if there exists a λ_g^δ-open set Q such that $x \in Q \subseteq N$.

The set of all λ_g^δ-neighborhood of x is denoted by $\lambda_g^\delta N(x)$.

Example 3.2.2. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then $\lambda_g^\delta O(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}\}$. Now, $\{a, c, d\}$ is a λ_g^δ-neighborhood of a, as there exists a λ_g^δ-open set $\{a, c\}$ such that $a \in \{a, c\} \subseteq \{a, c, d\}$.

Definition 3.2.3. A subset N of $A \subseteq X$ is called a ***λ_g^δ-neighborhood of A*** if there exists a λ_g^δ-open set Q such that $A \subseteq Q \subseteq N$.

Example 3.2.4. Let X and τ be defined as in Example 3.2.2. Now, $\{a, c, d\}$ is a λ_g^δ-neighborhood of $\{a\}$ as there exists a λ_g^δ-open set $\{a, c\}$ such that $\{a\} \subseteq \{a, c\} \subseteq \{a, c, d\}$.

Theorem 3.2.5. *If a subset N of a topological space X is λ_g^δ-open then N is a λ_g^δ-neighborhood of each of its points.*

Proof. Let N be a λ_g^δ-open set and $x \in N$. Then $x \in N \subseteq N$. Since x is an arbitrary point in N, N is a λ_g^δ-neighborhood of each of its points. □

The converse of the above theorem need not be true as seen from Example 3.2.2.

Remark 3.2.6. A λ_g^δ-neighborhood need not be a λ_g^δ-open set as observed from the following example.

Example 3.2.7. Let $X = \{a, b, c\}$ and $\tau = \{X, \tau, \{a\}, \{b\}, \{a, b\}\}$ then $\lambda_g^\delta O(X, \tau) = \tau$. The set $\{b, c\}$ is a λ_g^δ-neighborhood of the point b, as $b \in \{b\} \subseteq \{b, c\}$ but $\{b, c\}$ is not a λ_g^δ-open set of (X, τ).

Theorem 3.2.8. *If F is a λ_g^δ-closed subset of a topological space and $x \in X \setminus F$ then there exists a λ_g^δ-neighborhood N of x such that $N \cap F = \phi$.*

Proof. Let F be a λ_g^δ-closed subset of X then $X \setminus F$ is λ_g^δ-open in X. By Theorem 3.2.5 , $X \setminus F$ contains a λ_g^δ-neighborhood of each of its points. Therefore there exists a λ_g^δ-open set N of x such that $N \subseteq (X \setminus F)$ which in turn implies that $N \cap F = \phi$. □

Theorem 3.2.9. *In a topological space (X, τ) with $x \in X$, the following results are* *true.*

(i) $\lambda_g^\delta N(x) \neq \phi$.

(ii) If $N \in \lambda_g^\delta N(x)$ *then* $x \in N$.

(iii) If $N \in \lambda_g^\delta N(x)$ *and* $N \subseteq M$ *then* $M \in \lambda_g^\delta N(x)$.

(iv) If $N \in \lambda_g^\delta N(x)$ *and* $M \in \lambda_g^\delta N(x)$ *then* $N \cap M \in \lambda_g^\delta N(x)$.

(v) If $N \in \lambda_g^\delta N(x)$ *then there exists* $M \in \lambda_g^\delta N(x)$ *such that* $M \subseteq N$ *and* $M \in \lambda_g^\delta N(y)$*, for all* $y \in M$.

Proof. (i) Since X is a λ_g^δ-open set irrespective of the topology, it is a λ_g^δ-neighborhood for every $x \in X$. That is $X \in \lambda_g^\delta N(x)$, for all $x \in X$. Hence $\lambda_g^\delta N(x) \neq \phi$, for all $x \in X$.

(ii) Let $N \in \lambda_g^\delta N(x)$ then N is a neighborhood of x. Therefore $x \in N$ follows from the definition of λ_g^δ-neighborhood.

(iii) Let $N \in \lambda_g^\delta N(x)$ and $N \subseteq M$. Since $N \in \lambda_g^\delta N(x)$ there exists a λ_g^δ-open set Q such that $x \in Q \subseteq N \subseteq M$ and hence M is a λ_g^δ-neighborhood of x. Hence $M \in \lambda_g^\delta N(x)$.

(iv) Let $N \in \lambda_g^\delta N(x)$ and $M \in \lambda_g^\delta N(x)$ then there exists λ_g^δ-open sets Q_1 and Q_2 such that $x \in Q_1 \subseteq N$ and $x \in Q_2 \subseteq M$. This implies $x \in Q_1 \cap Q_2 \subseteq N \cap M$. Now in order to prove $N \cap M$ is a λ_g^δ-neighborhood of x, it suffices to prove that $Q_1 \cap Q_2$ is λ_g^δ-open. Since arbitrary intersection of λ_g^δ-open sets is λ_g^δ-open, $Q_1 \cap Q_2$ is λ_g^δ-open and hence $N \cap M$ is a λ_g^δ-neighborhood of x. Therefore $N \cap M \in \lambda_g^\delta N(x)$.

(v) Let $N \in \lambda_g^\delta N(x)$ then there exists a λ_g^δ-open set M such that $x \in M \subseteq N$. Since M is λ_g^δ-open, it is a λ_g^δ-neighborhood of each of its points (by Theorem 3.2.5). Thus $M \in \lambda_g^\delta N(y)$, for all $y \in M$.

□

Lemma 3.2.10. Let (X, τ) be a topological space and $x \in X$. Suppose that a collection $\mathscr{A}_x$ satisfies

(i) $N \in \mathscr{A}_x$ such that $x \in N$

(ii) $N, M \in \mathscr{A}_x \Rightarrow N \cap M \in \mathscr{A}_x$

then $\mathscr{B}$ forms a basis for a topology where $\mathscr{B} = \{\phi\} \cup \{G \subseteq X | x \in G \Rightarrow$ there exists $N \in \mathscr{A}_x$ such that $x \in N \subseteq G\}$.

Proof. (a) $\phi \in \mathscr{B}$, by definition of $\mathscr{B}$. For all $x \in X$, there exists $N \in \mathscr{A}_x$ such that $x \in N \subseteq \mathscr{A}_x$, by hypothesis (a). Thus $X \in \mathscr{B}$.

(b) Let $G_1, G_2 \in \mathscr{B}$ and $x \in G_1 \cap G_2 \Rightarrow x \in G_1$ and $x \in G_2$. $x \in G_1 \Rightarrow$ there exists $N \in \mathscr{A}_x$ such that $x \in N \subseteq G_1$ and $x \in G_2 \Rightarrow$ there exists $M \in \mathscr{A}_x$ such that $x \in M \subseteq G_2$. Now $x \in N \cap M \subseteq G_1 \cap G_2$. By (b), $N \cap M \subseteq \mathscr{A}_x$. Therefore for all $x \in G_1 \cap G_2$, there exists $N \cap M = N'$ such that $x \in= N' \subseteq G_1 \cap G_2$. Thus $G_1 \cap G_2 \in \mathscr{B}$.

□

Corollary 3.2.11. Suppose that $A_x = \lambda_g^\delta N(x)$ in Lemma 3.2.10. Then $\mathscr{B} = \{\phi\} \cup \{G \subseteq X | x \in G \Rightarrow \exists\ N \in \lambda_g^\delta N(x) \ni x \in N \subseteq G\}$ forms a basis for a topology.

Definition 3.2.12. Let $x \in A \subseteq X$. Then x is a ***λ_g^δ-limit point*** of A if every λ_g^δ-open containing x contains at least one point of A other than x. That is, $N \cap (A \setminus \{x\}) \neq \phi$, for all $N \in \lambda_g^\delta O(x)$. The set of all λ_g^δ-limit points of $A \subseteq X$ is called a ***λ_g^δ-derived set*** and denoted by $\lambda_g^\delta D(A)$.

Example 3.2.13. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then $\lambda_g^\delta O(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. d is a λ_g^δ-limit point of $\{a, d\}$ as every λ_g^δ-open set containing d (only X in this case) contains at least one of $\{a, d\}$ other than d. Further, $\lambda_g^\delta D(A) = \{a, b, d\}$.

Proposition 3.2.14. Every λ_g^δ-limit point is a δ-limit point.

Proof. Follows from the fact that $\delta O(X, \tau) \subseteq \lambda_g^\delta O(X, \tau)$ and Theorem 3.2.5. □

Theorem 3.2.15. *Let $A, B \subseteq X$. Then the following statements are valid.*

(i) $\lambda_g^\delta D(\phi) = \phi$.

(ii) $\lambda_g^\delta D(A) \subseteq D_\delta(A)$.

(iii) *If* $A \subseteq B$ *then* $\lambda_g^\delta D(A) \subseteq \lambda_g^\delta D(B)$.

(iv) $\lambda_g^\delta D(A \cup B) = \lambda_g^\delta D(A) \cup \lambda_g^\delta D(B)$.

(v) $\lambda_g^\delta D(A \cap B) \subseteq \lambda_g^\delta D(A) \cap \lambda_g^\delta D(B)$.

(vi) $\lambda_g^\delta D(\lambda_g^\delta D(A)) \setminus A \subseteq \lambda_g^\delta D(A)$.

(vii) $\lambda_g^\delta D(A \cup \lambda_g^\delta D(A)) \subseteq A \cup \lambda_g^\delta D(A)$.

Proof. (i) Trivial

(ii) Follows from Proposition 3.2.14.

(iii) Let $x \in \lambda_g^\delta D(A)$ then $N \cap (A \setminus \{x\}) \neq \phi$, for each λ_g^δ-open set N of x. Since $A \subseteq B$, $N \cap (A \setminus \{x\}) \subseteq N \cap (B \setminus \{x\}) \neq \phi \Rightarrow x \in \lambda_g^\delta D(B)$. Therefore $\lambda_g^\delta D(A) \subseteq \lambda_g^\delta D(B)$.

(iv) Since $A \subseteq A \cup B$, $\lambda_g^\delta D(A) \subseteq \lambda_g^\delta D(A \cup B)$. Similarly $B \subseteq A \cup B \Rightarrow \lambda_g^\delta D(B) \subseteq \lambda_g^\delta D(A \cup B)$. Therefore $\lambda_g^\delta D(A) \cup \lambda_g^\delta D(B) \subseteq \lambda_g^\delta D(A \cup B)$. Suppose if $x \notin \lambda_g^\delta D(A) \cup \lambda_g^\delta D(B)$ then $x \notin \lambda_g^\delta D(A)$ and $x \notin \lambda_g^\delta D(B)$ so that x is neither a limit point of A nor B. Therefore there exist λ_g^δ-open sets N_1 and N_2 of x such that $N_1 \cap (A \setminus \{x\}) = \phi$ and $N_2 \cap (B \setminus \{x\}) = \phi$. Since $N_1 \cap N_2$ is a λ_g^δ-open set containing x, $(N_1 \cap N_2) \cap [(A \cup B) \setminus \{x\}] = \phi \Rightarrow x \notin \lambda_g^\delta D(A \cup B)$ giving $\lambda_g^\delta D(A \cup B) \subseteq \lambda_g^\delta D(A) \cup \lambda_g^\delta D(B)$.

(v) Follows from (iii) as $A \cap B \subseteq A, B$.

(vi) Let $x \in \lambda_g^\delta D(\lambda_g^\delta D(A)) \setminus A$ and U be a λ_g^δ-open set containing x. Then $x \in \lambda_g^\delta D(\lambda_g^\delta D(A)) \Rightarrow U \cap [\lambda_g^\delta D(A) \setminus \{x\}] \neq \phi$. Now let $y \in U \cap [\lambda_g^\delta D(A) \setminus \{x\}] \Rightarrow y \in U$ and $y \in \lambda_g^\delta D(A)$ so that $U \cap [A \setminus \{y\}] \neq \phi$. Let $z \in U \cap [A \setminus \{y\}]$. Then $z \neq x$ as $z \in A$ and $x \notin A$. Therefore $U \cap [A \setminus \{x\}] \neq \phi$. Hence $x \in \lambda_g^\delta D(A)$.

(vii) Let $x \in \lambda_g^\delta D(A \cup \lambda_g^\delta D(A))$. If $x \in A$, the result is obvious. Suppose if $x \in \lambda_g^\delta D(A \cup \lambda_g^\delta D(A)) \setminus A$ and let U be a λ_g^δ-open set containing x. Then $U \cap (A \cup \lambda_g^\delta D(A) \setminus \{x\}) \neq \phi \Rightarrow U \cap (A \setminus \{x\}) \neq \phi$ or $U \cap (\lambda_g^\delta D(A) \setminus \{x\}) \neq \phi$. Now let $y \in U \cap [\lambda_g^\delta D(A) \setminus \{x\}] \Rightarrow$

$y \in U$ and $y \in \lambda_g^\delta D(A)$ so that $U \cap [A \setminus \{y\}] \neq \phi$. Let $z \in U \cap [A \setminus \{y\}]$. Then $z \neq x$ as $z \in A$ and $x \notin A$. Therefore $U \cap [A \setminus \{x\}] \neq \phi$. Hence $x \in \lambda_g^\delta D(A)$ and thus $x \in A \cup \lambda_g^\delta D(A)$.

□

Theorem 3.2.16. *Let $A \subseteq X$. If A is λ_g^δ-closed then $\lambda_g^\delta D(A) \subseteq A$.*

Proof. As A is λ_g^δ-closed, $X \setminus A$ is λ_g^δ-open. For each $x \in X \setminus A$, there exists a λ_g^δ-open set N of x such that $x \in N \subseteq X \setminus A$. Since $A \cap (X \setminus A) = \phi$, the λ_g^δ-open set N contains no point of A and hence x is not a λ_g^δ-limit point of A. Therefore no point of $X \setminus A$ can be a λ_g^δ-limit point of A which means A contains all its limit points. Hence $\lambda_g^\delta D(A) \subseteq A$. □

3.3 λ_g^δ - Frontier

Remark 3.3.1. For a subset A of a topological space (X, τ), the following are true.

1. $\lambda_g^\delta cl(A) \setminus \lambda_g^\delta cl(B) \neq \lambda_g^\delta cl(A \setminus B)$.
2. $\lambda_g^\delta int(A) \setminus \lambda_g^\delta int(B) \neq \lambda_g^\delta int(A \setminus B)$.

Example 3.3.2. (i) Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $\lambda_g^\delta C(X, \tau) = \{X, \phi, \{c\}, \{a, c\}, \{b, c\}\}$ and $\lambda_g^\delta O(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{b, c\}$ and $B = \{b\}$. Now, $\lambda_g^\delta cl(A) \setminus \lambda_g^\delta cl(B) = \{b, c\} \setminus \{b, c\} = \phi$ but $\lambda_g^\delta cl(A \setminus B) = \{c\}$. Hence $\lambda_g^\delta cl(A) \setminus \lambda_g^\delta cl(B) \neq \lambda_g^\delta cl(A \setminus B)$.

(ii) Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then $\lambda_g^\delta C(X, \tau) = \{X, \phi, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\lambda_g^\delta O(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$. Let $A = \{b, d\}$ and $B = \{c, d\}$. Now, $\lambda_g^\delta int(A) \setminus \lambda_g^\delta int(B) = \{b\} \setminus \{c\} = \phi$ but $\lambda_g^\delta int(A \setminus B) = \{b\}$. Hence $\lambda_g^\delta int(A) \setminus \lambda_g^\delta int(B) \neq \lambda_g^\delta int(A \setminus B)$.

Definition 3.3.3. For a subset A of a topological space (X, τ), ***λ_g^δ-frontier*** of A is denoted by $\lambda_g^\delta F(A)$ and defined as $\lambda_g^\delta F(A) = \lambda_g^\delta cl(A) \setminus \lambda_g^\delta int(A)$.

Example 3.3.4. Let X and τ be defined as in Example 3.3.2(i). Then for $A = \{a, c\}$, $\lambda_g^\delta F(\{a, c\}) = \lambda_g^\delta cl(\{a, c\}) \setminus \lambda_g^\delta int(\{a, c\}) = \{a, c\} \setminus \{a\} = \{c\}$.

Proposition 3.3.5. For a subset A of a topological space (X, τ), the following results are true.

(i) $\lambda_g^\delta cl(A) = \lambda_g^\delta int(A) \cup \lambda_g^\delta F(A)$.

(ii) $\lambda_g^\delta int(A) \cap \lambda_g^\delta F(A) = \phi$.

(iii) $\lambda_g^\delta F(A) = \lambda_g^\delta cl(A) \cap \lambda_g^\delta cl(X \setminus A)$.

(iv) $\lambda_g^\delta F(A)$ is δ-closed.

(v) $\lambda_g^\delta F(A) = \lambda_g^\delta F(X \setminus A)$.

(vi) $Fr_\delta(\lambda_g^\delta F(A)) \subseteq \lambda_g^\delta F(A)$.

Proof. (i) $\lambda_g^\delta int(A) \cup \lambda_g^\delta F(A) = \lambda_g^\delta int(A) \cup [\lambda_g^\delta cl(A) \setminus \lambda_g^\delta int(A)] = \lambda_g^\delta cl(A)$.

(ii) $\lambda_g^\delta int(A) \cap \lambda_g^\delta F(A) = \lambda_g^\delta int(A) \cap [\lambda_g^\delta cl(A) \setminus \lambda_g^\delta int(A)] = \phi$.

(iii) $\lambda_g^\delta cl(A) \cap \lambda_g^\delta cl(X \setminus A) = \lambda_g^\delta cl(A) \setminus \lambda_g^\delta int(A) = \lambda_g^\delta F(A)$.

(iv) $cl_\delta(\lambda_g^\delta F(A)) = cl_\delta(\lambda_g^\delta cl(A) \cap \lambda_g^\delta cl(X \setminus A)) \subseteq cl_\delta(\lambda_g^\delta cl(A)) \cap cl_\delta(\lambda_g^\delta cl(X \setminus A)) \subseteq cl_\delta(cl_\delta(A)) \cap cl_\delta(cl_\delta(X \setminus A)) = cl_\delta(A) \cap cl_\delta(X \setminus A) = Fr_\delta(A)$, which is δ-closed.

(v) $\lambda_g^\delta F(X \setminus A) = \lambda_g^\delta cl(X \setminus A) \setminus \lambda_g^\delta int(X \setminus A) = [X \setminus \lambda_g^\delta int(A)] \setminus [X \setminus \lambda_g^\delta cl(A)] = [X \setminus \lambda_g^\delta int(A)] \cap \lambda_g^\delta cl(A) = \lambda_g^\delta cl(A) \cap [X \setminus \lambda_g^\delta int(A)] = \lambda_g^\delta cl(A) \setminus \lambda_g^\delta int(A) = \lambda_g^\delta F(A)$.

(vi) $Fr_\delta(\lambda_g^\delta F(A)) = cl_\delta(\lambda_g^\delta F(A)) \cap cl_\delta(X \setminus \lambda_g^\delta F(A)) \subseteq cl_\delta(\lambda_g^\delta F(A)) = \lambda_g^\delta F(A)$, as $\lambda_g^\delta F(A)$ is δ-closed.

□

Proposition 3.3.6. Let $A \subseteq B$ and $\lambda_g^\delta int(B) = \phi$ then $\lambda_g^\delta F(A) \subseteq \lambda_g^\delta F(B)$.

Proof. Let $A \subseteq B$ and $\lambda_g^\delta int(B) = \phi$. Let $x \in \lambda_g^\delta F(A) = \lambda_g^\delta cl(A) \setminus \lambda_g^\delta int(B)$. Then $x \in \lambda_g^\delta cl(A) \subseteq \lambda_g^\delta cl(B) = \lambda_g^\delta cl(B) \setminus \lambda_g^\delta int(B)$, as $\lambda_g^\delta int(B) = \phi$. Hence $x \in \lambda_g^\delta F(B)$. □

3.4 λ_g^δ-Boundary, λ_g^δ-Exterior, λ_g^δ-Saturated Set

Definition 3.4.1. For a subset A of a topological space (X, τ), ***λ_g^δ-boundary*** of A is denoted by $\lambda_g^\delta B(A)$ and defined as $\lambda_g^\delta B(A) = A \setminus \lambda_g^\delta int(A)$.

Example 3.4.2. Let$X$, τ and A be defined as in Example 3.3.4. Then $\lambda_g^\delta B(\{a, c\}) = \{a, c\} \setminus \{a\} = \{c\}$.

Theorem 3.4.3. *For a subset A of a topological space (X, τ), the following results are true.*

(i) $\lambda_g^\delta B(\phi) = \phi$.

(ii) $\lambda_g^\delta B(X) = \phi$.

(iii) $A = \lambda_g^\delta int(A) \cup \lambda_g^\delta B(A)$.

(iv) If A is $\lambda_g^\delta -$ open then $\lambda_g^\delta B(A) = \phi$.

(v) $\lambda_g^\delta int(A) \cap \lambda_g^\delta B(A) = \phi$.

(vi) $\lambda_g^\delta B(\lambda_g^\delta int(A)) = \phi$.

Proof. (i) $\lambda_g^\delta B(\phi) = \phi \setminus \lambda_g^\delta int(\phi) = \phi$.

(ii) $\lambda_g^\delta B(X) = X \setminus \lambda_g^\delta int(X) = \phi$.

(iii) $\lambda_g^\delta int(A) \cup \lambda_g^\delta B(A) = \lambda_g^\delta int(A) \cup [A \setminus \lambda_g^\delta int(A)] = A$.

(iv) Let A be λ_g^δ-open then $\lambda_g^\delta int(A) = A$. Now, $\lambda_g^\delta B(A) = A \setminus \lambda_g^\delta int(A) = A \setminus A = \phi$.

(v) $\lambda_g^\delta int(A) \cap \lambda_g^\delta B(A) = \lambda_g^\delta int(A) \cap [A \setminus \lambda_g^\delta int(A)] = \phi$.

(vi) $\lambda_g^\delta B(\lambda_g^\delta int(A)) = \lambda_g^\delta int(A) \setminus \lambda_g^\delta int(\lambda_g^\delta int(A)) = \lambda_g^\delta int(A) \setminus \lambda_g^\delta int(A) = \phi$.

□

Remark 3.4.4. In the previous Theorem, converse of (iv) is not true as seen from the following Example.

Example 3.4.5. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ then for $A = \{c, d\}$, $\lambda_g^\delta int(A) = A$. Therefore $\lambda_g^\delta B(A) = \phi$ but A is not a λ_g^δ-open set.

Definition 3.4.6. For a subset A of a topological space (X, τ), ***λ_g^δ-exterior*** of A is denoted by $\lambda_g^\delta E(A)$ and defined as $\lambda_g^\delta E(A) = X \setminus \lambda_g^\delta cl(A)$.

Example 3.4.7. Let$X$, τ and A be defined as in Example 3.3.4. Then $\lambda_g^\delta E(\{a, c\}) = X \setminus \lambda_g^\delta cl(\{a, c\}) = X \setminus \{a, c\} = \{b\}$

Theorem 3.4.8. *For a subset A of a topological space (X, τ), the following results are true.*

(i) $Ext_\delta(A) \subseteq \lambda_g^\delta E(A)$.

(ii) $\lambda_g^\delta E(X) = \phi$.

(iii) $\lambda_g^\delta E(\phi) = X$.

(iv) $\lambda_g^\delta E(A) = \lambda_g^\delta int(X \setminus A)$.

(v) *If* $A \subseteq B$ *then* $\lambda_g^\delta E(A) \supseteq \lambda_g^\delta E(B)$.

(vi) $\lambda_g^\delta E(A \cup B) \subseteq \lambda_g^\delta E(A) \cup \lambda_g^\delta E(B)$.

(vii) $\lambda_g^\delta E(A \cap B) \supseteq \lambda_g^\delta E(A) \cap \lambda_g^\delta E(B)$.

(viii) $\lambda_g^\delta E(\lambda_g^\delta E(A)) = \lambda_g^\delta int(\lambda_g^\delta cl(A))$.

(ix) $\lambda_g^\delta E(A) = \lambda_g^\delta E(X \setminus \lambda_g^\delta E(A))$.

(x) $X = \lambda_g^\delta int(A) \cup \lambda_g^\delta E(A) \cup \lambda_g^\delta F(A)$.

(xi) $\lambda_g^\delta int(A) \subseteq \lambda_g^\delta E(\lambda_g^\delta E(A))$.

Proof. (i) Let $x \in Ext_\delta(A)$. Then $x \in X \setminus cl_\delta(A) \Rightarrow x \notin cl_\delta(A) \Rightarrow x \notin \lambda_g^\delta cl(A)$, as $\lambda_g^\delta cl(A) \subseteq cl_\delta(A)$. That is, $x \in X \setminus \lambda_g^\delta cl(A) \Rightarrow x \in \lambda_g^\delta E(A)$.

(ii) $\lambda_g^\delta E(X) = X \setminus \lambda_g^\delta cl(X) = \phi$.

(iii) $\lambda_g^\delta E(\phi) = X \setminus \lambda_g^\delta cl(\phi) = X$.

(iv) $\lambda_g^\delta E(A) = X \setminus \lambda_g^\delta cl(A) = \lambda_g^\delta int(X \setminus A)$, by Lemma 2.5.14.

(v) Let $A \subseteq B$ then $\lambda_g^\delta E(A) = \lambda_g^\delta int(X \setminus A) \supseteq \lambda_g^\delta int(X \setminus B) = \lambda_g^\delta E(B)$, by Proposition 2.5.11(ii).

(vi) We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$. By (iv), $\lambda_g^\delta E(A) \supseteq \lambda_g^\delta E(A \cup B)$ and $\lambda_g^\delta E(B) \supseteq \lambda_g^\delta E(A \cup B)$ which implies $\lambda_g^\delta E(A) \cup \lambda_g^\delta E(B) \supseteq E(A \cup B)$.

(vii) We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. By (iv), $\lambda_g^\delta E(A \cap B) \supseteq \lambda_g^\delta E(A)$ and $\lambda_g^\delta E(A \cap B) \supseteq \lambda_g^\delta E(B)$ which implies $\lambda_g^\delta E(A \cap B) \supseteq \lambda_g^\delta E(A) \cap \lambda_g^\delta E(B)$.

(viii) $\lambda_g^\delta E(\lambda_g^\delta E(A)) = \lambda_g^\delta E(X \setminus \lambda_g^\delta cl(A)) = \lambda_g^\delta int[\lambda_g^\delta cl(A)]$.

(ix) $\lambda_g^\delta E(X \setminus \lambda_g^\delta E(A)) = \lambda_g^\delta E(X \setminus \lambda_g^\delta int(X \setminus A)]) = \lambda_g^\delta E(\lambda_g^\delta cl(A)) = \lambda_g^\delta int[X \setminus \lambda_g^\delta cl(A)] = \lambda_g^\delta int[\lambda_g^\delta int(X \setminus A)] = \lambda_g^\delta int(X \setminus A) = \lambda_g^\delta E(A)$, by (iii).

(x) $\lambda_g^\delta int(A) \cup \lambda_g^\delta E(A) \cup \lambda_g^\delta F(A) = \lambda_g^\delta int(A) \cup \lambda_g^\delta int(X \setminus A) \cup [\lambda_g^\delta cl(A) \setminus \lambda_g^\delta int(A)] = \lambda_g^\delta cl(A) \cup [X \setminus \lambda_g^\delta cl(A)] = X$.

(xi) $\lambda_g^\delta int(A) = \lambda_g^\delta int(\lambda_g^\delta cl(A)) = \lambda_g^\delta int(X \setminus \lambda_g^\delta int(X \setminus A)) = \lambda_g^\delta int(X \setminus \lambda_g^\delta E(A)) = \lambda_g^\delta E(\lambda_g^\delta E(A))$.

□

Remark 3.4.9. The reverse implications (vi) and (vii) are not true in general as seen from the following Example.

Example 3.4.10. (i) Let X and τ be defined as in Example 3.3.4. Then for $A = \{a\}$ and $B = \{b\}, \lambda_g^\delta E(\{a\} \cup \{b\}) = \lambda_g^\delta E(\{a, b\}) = \{c\}$ but $\lambda_g^\delta E(\{a\}) \cup \lambda_g^\delta E(\{b\}) = \{b, c\} \cup \{a, c\} = \{a, b, c\}$.

(ii) Let X and τ be defined as in Example 3.3.2(ii). Then for $A = \{a\}$ and $B = \{a, c\}, \lambda_g^\delta E(\{a\} \cap \{a, c\}) = \lambda_g^\delta E(\{a\}) = \{b, c, d\}$ but $\lambda_g^\delta E(\{a\}) \cap \lambda_g^\delta E(\{a, b\}) = \{b, c, d\} \cap \{c\} = \{c\}$.

Corollary 3.4.11. $Ext_\delta(A) \cup Ext_\delta(B) \subseteq \lambda_g^\delta E(A \cap B)$.

Definition 3.4.12. A subset A of a topological space (X, τ) is said to be ***λ_g^δ-saturated*** if $\lambda_g^\delta cl(\{x\}) \subseteq A$ for every $x \in A$. The set of all λ_g^δ-saturated sets in (X, τ) is denoted by $\lambda_g^\delta Sat(X)$.

Remark 3.4.13. In the usual topological spaces, for any subset $A \in \lambda_g^\delta Sat(X) \Rightarrow A^c \in \lambda_g^\delta Sat(X)$. But in the case of λ_g^δ-closed sets, the result fails as seen from the following Example.

Example 3.4.14. Let X and τ be defined as in Example 3.3.4. Take $A = \{d\}$ then $A \in \lambda_g^\delta Sat(X)$ but $X \setminus A = \{a, b, c\} \notin \lambda_g^\delta Sat(X)$.

Theorem 3.4.15. *Every λ_g^δ-closed set is a λ_g^δ-saturated set but not conversely.*

Proof. Let A be a λ_g^δ-closed set in (X, τ) with $x \in A$. Then $\{x\} \subseteq A$. Taking λ_g^δ-closure on both sides, $\lambda_g^\delta cl\{x\} \subseteq \lambda_g^\delta cl(A) = A$, as A is λ_g^δ-closed. Therefore A is a λ_g^δ-saturated set. □

Example 3.4.16. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Take $A = \{a, b\}$. Then A is a λ_g^δ-saturated set but not λ_g^δ-closed set.

3.5 λ_g^δ-open sets via grill

Definition 3.5.1. Let $(X, \tau^{\lambda_g^\delta}, \mathcal{G})$ be a grill λ_g^δ-space. We define a function $\psi_{\lambda_g^\delta \mathcal{G}} : P(X) \to P(X)$, denoted by $\psi_{\lambda_g^\delta \mathcal{G}}(A, \tau^{\lambda_g^\delta})$ or $\psi_{\lambda_g^\delta \mathcal{G}}(A)$ (for $A \in P(X)$), called the ***λ_g^δ-operator*** associated with the grill $\mathcal{G}$ and $\tau^{\lambda_g^\delta}$, and is defined by $\psi_{\lambda_g^\delta \mathcal{G}}(A) = \{x \in X \mid U \cap A \in \mathcal{G}$, for all λ_g^δ-open set U containing $x\}$.

Example 3.5.2. Let X, τ be defined as in Example 3.3.4 and $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Then $\psi_{\lambda_g^\delta \mathcal{G}}(\{a\}) = \{a, c\}$.

Theorem 3.5.3. *Let $(X, \tau^{\lambda_g^\delta})$ be a λ_g^δ-space and $\mathcal{G}$, $\mathcal{H}$ be two grills on X. Then for a subset A of X, the following conditions are valid:*

(i) $\mathcal{G} \subseteq \mathcal{H} \Rightarrow \psi_{\lambda_g^\delta \mathcal{G}}(A) \subseteq \psi_{\lambda_g^\delta \mathcal{H}}(A)$.

(ii) $\psi_{\lambda_g^\delta(\mathcal{G}\cup\mathcal{H})}(A) = \psi_{\lambda_g^\delta\mathcal{G}}(A) \cup \psi_{\lambda_g^\delta\mathcal{H}}(A)$.

Proof. (i) Let $x \in \psi_{\lambda_g^\delta\mathcal{G}}(A)$. Then $U \cap A \in \mathcal{G}$, for every λ_g^δ-open set containing x. Since $\mathcal{G} \subseteq \mathcal{H}$, $U \cap A \in \mathcal{H}$, for every λ_g^δ-open set containing x. Therefore $x \in \psi_{\lambda_g^\delta\mathcal{H}}(A)$ and hence $\psi_{\lambda_g^\delta\mathcal{G}}(A) \subseteq \psi_{\lambda_g^\delta\mathcal{H}}(A)$.

(ii) $\psi_{\lambda_g^\delta\mathcal{G}}(A) \cup \psi_{\lambda_g^\delta\mathcal{H}}(A) \subseteq \psi_{\lambda_g^\delta(\mathcal{G}\cup\mathcal{H})}(A)$ is obvious. Suppose $x \notin \psi_{\lambda_g^\delta\mathcal{G}}(A) \cup \psi_{\lambda_g^\delta\mathcal{H}}(A)$. Then $x \notin \psi_{\lambda_g^\delta\mathcal{G}}(A)$ and $x \notin \psi_{\lambda_g^\delta\mathcal{H}}(A)$. Therefore there exist λ_g^δ-open sets U_1, U_2 containing x such that $(U_1 \cap A) \notin \mathcal{G}$ and $(U_2 \cap A) \notin \mathcal{H}$. Now let $U = U_1 \cap U_2$ then U is an λ_g^δ-open set containing x such that $(U \cap A) \notin \mathcal{G}$ and $(U \cap A) \notin \mathcal{H}$. This implies $(U \cap A) \notin \mathcal{G} \cup \mathcal{H}$ so that $x \notin \psi_{\lambda_g^\delta(\mathcal{G}\cup\mathcal{H})}(A)$. Thus $\psi_{\lambda_g^\delta(\mathcal{G}\cup\mathcal{H})}(A) \subseteq \psi_{\lambda_g^\delta\mathcal{G}}(A) \cup \psi_{\lambda_g^\delta\mathcal{H}}(A)$. □

Theorem 3.5.4. *Let $(X, \tau^{\lambda_g^\delta}, \mathcal{G})$ be a grill λ_g^δ-space. Then for any two subsets A and B of X, the following conditions hold:*

(i) $\psi_{\lambda_g^\delta\mathcal{G}}(A) \subseteq \psi_{\lambda_g^\delta\mathcal{G}}(B)$, *if* $A \subseteq B$.

(ii) $\psi_{\lambda_g^\delta\mathcal{G}}(A \cup B) = \psi_{\lambda_g^\delta\mathcal{G}}(A) \cup \psi_{\lambda_g^\delta\mathcal{G}}(B)$.

(iii) $\psi_{\lambda_g^\delta\mathcal{G}}(A \cap B) \subseteq \psi_{\lambda_g^\delta\mathcal{G}}(A) \cap \psi_{\lambda_g^\delta\mathcal{G}}(B)$.

(iv) $\psi_{\lambda_g^\delta\mathcal{G}}(A) = \phi$, *if* $A \notin \mathcal{G}$.

(v) $\psi_{\lambda_g^\delta\mathcal{G}}(A) \subseteq \psi_{\delta\mathcal{G}}(A)$.

(vi) $\psi_{\lambda_g^\delta\mathcal{G}}(A) \subseteq \lambda_g^\delta cl(A) \subseteq cl_\delta(A)$.

Proof. (i) Suppose $x \notin \psi_{\lambda_g^\delta\mathcal{G}}(B)$. Then $U \cap B \notin \mathcal{G}$, for every λ_g^δ-open set U containing x. This implies $U \cap A \notin \mathcal{G}$, for every λ_g^δ-open set U containing x. Thus $x \notin \psi_{\lambda_g^\delta\mathcal{G}}(A)$. Hence $\psi_{\lambda_g^\delta\mathcal{G}}(A) \subseteq \psi_{\lambda_g^\delta\mathcal{G}}(B)$.

(ii) $\psi_{\lambda_g^\delta\mathcal{G}}(A) \cup \psi_{\lambda_g^\delta\mathcal{G}}(B) \subseteq \psi_{\lambda_g^\delta\mathcal{G}}(A \cup B)$ is obvious. Suppose $x \notin \psi_{\lambda_g^\delta\mathcal{G}}(A) \cup \psi_{\lambda_g^\delta\mathcal{G}}(B)$ then $x \notin \psi_{\lambda_g^\delta\mathcal{G}}(A)$ and $x \notin \psi_{\lambda_g^\delta\mathcal{G}}(B)$. That is, there exists λ_g^δ-open sets U_1, U_2 containing x such that $(U_1 \cap A) \notin \mathcal{G}$ and $(U_2 \cap B) \notin \mathcal{G}$. Therefore $(A \cap U_1) \cup (B \cap U_2) \notin \mathcal{G}$. Now $(U_1 \cap U_2)$ is a λ_g^δ-open set containing x and $(A \cup B) \cap (U_1 \cap U_2) \notin \mathcal{G}$. Hence $x \notin \psi_{\lambda_g^\delta\mathcal{G}}(A \cup B)$. Thus $\psi_{\lambda_g^\delta\mathcal{G}}(A \cup B) \subseteq \psi_{\lambda_g^\delta\mathcal{G}}(A) \cup \psi_{\lambda_g^\delta\mathcal{G}}(B)$.

(iii) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, it follows from(i).

(iv) Suppose there exists $x \in \psi_{\lambda_g^\delta \mathcal{G}}(A)$ then for all λ_g^δ-open set U containing x, $U \cap A \in \mathcal{G} \Rightarrow A \in \mathcal{G}$, which is a contraction to $A \notin \mathcal{G}$.

(v) Let $x \in \psi_{\lambda_g^\delta \mathcal{G}}(A)$. Then for every λ_g^δ-open set U containing x, $U \cap A \in \mathcal{G}$. Let U' be a δ-open set then it is λ_g^δ-open which implies $U' \cap A \in \mathcal{G}$. Therefore $x \in \psi_{\delta\mathcal{G}}(A)$.

(vi) Let $x \in \psi_{\lambda_g^\delta \mathcal{G}}(A)$ then for all λ_g^δ-open set U containing x, $U \cap A \in \mathcal{G} \Rightarrow A \in \mathcal{G}$. Since $\phi \in \mathcal{G}$, $U \cap A \neq \phi \Rightarrow x \in \lambda_g^\delta cl(A) \subseteq cl_\delta(A)$.

□

Theorem 3.5.5. *Let $(X, \tau^{\lambda_g^\delta}, \mathcal{G})$ be a grill λ_g^δ-space. Then for any two subsets A and B of X, the following conditions hold:*

(i) $\psi_{\lambda_g^\delta \mathcal{G}}(A) \setminus \psi_{\lambda_g^\delta \mathcal{G}}(B) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B)$.

(ii) If $B \notin \mathcal{G}, \psi_{\lambda_g^\delta \mathcal{G}}(A \cup B) = \psi_{\lambda_g^\delta \mathcal{G}}(A) = \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B)$.

(iii) If $(A \setminus B) \cup (B \setminus A) \notin \mathcal{G}$ *then* $\psi_{\lambda_g^\delta \mathcal{G}}(A) = \psi_{\lambda_g^\delta \mathcal{G}}(B)$.

Proof. (i) Let A and B be subsets of X and $A = (A \setminus B) \cup (A \cap B)$. By (ii) of Theorem 3.5.4, $\psi_{\lambda_g^\delta \mathcal{G}}(A) = \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B) \cup \psi_{\lambda_g^\delta \mathcal{G}}(A \cap B)$. Further by (iii) of Theorem 3.5.4, $\psi_{\lambda_g^\delta \mathcal{G}}(A \cap B) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(B)$. Therefore $\psi_{\lambda_g^\delta \mathcal{G}}(A) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B) \cup \psi_{\lambda_g^\delta \mathcal{G}}(B)$. Now, $\psi_{\lambda_g^\delta \mathcal{G}}(A) \setminus \psi_{\lambda_g^\delta \mathcal{G}}(B) \subseteq [\psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B) \cup \psi_{\lambda_g^\delta \mathcal{G}}(B)] \setminus \psi_{\lambda_g^\delta \mathcal{G}}(B) = \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B)$. Hence $\psi_{\lambda_g^\delta \mathcal{G}}(A) \setminus \psi_{\lambda_g^\delta \mathcal{G}}(B) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B)$.

(ii) Let $B \notin \mathcal{G}$. Then $\psi_{\lambda_g^\delta \mathcal{G}}(A \cup B) = \psi_{\lambda_g^\delta \mathcal{G}}(A) \cup \psi_{\lambda_g^\delta \mathcal{G}}(B) = \psi_{\lambda_g^\delta \mathcal{G}}(A)$, as $B \notin \mathcal{G} \Rightarrow \psi_{\lambda_g^\delta \mathcal{G}}(B) = \phi$. In general, $A \setminus B \subseteq A \cup B$. By (i) of Theorem 3.5.4, $\psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(A \cup B) = \psi_{\lambda_g^\delta \mathcal{G}}(A)$. Further by hypothesis, $\psi_{\lambda_g^\delta \mathcal{G}}(A) \setminus \psi_{\lambda_g^\delta \mathcal{G}}(B) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B) \Rightarrow \psi_{\lambda_g^\delta \mathcal{G}}(A) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B)$. Therefore $\psi_{\lambda_g^\delta \mathcal{G}}(A \cup B) = \psi_{\lambda_g^\delta \mathcal{G}}(A) = \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B)$.

(iii) Let $V = (A \setminus B) \cup (B \setminus A) \notin \mathcal{G}$. Then $A = (V \setminus B) \cup (B \setminus V)$. By (ii) of Theorem 3.5.4, $\psi_{\lambda_g^\delta \mathcal{G}}(A) = \psi_{\lambda_g^\delta \mathcal{G}}(V \setminus B) \cup \psi_{\lambda_g^\delta \mathcal{G}}(B \setminus V)$. By (ii), $\psi_{\lambda_g^\delta \mathcal{G}}(A) = \psi_{\lambda_g^\delta \mathcal{G}}((A \setminus B) \cup (B \setminus A)) = \psi_{\lambda_g^\delta \mathcal{G}}(V \setminus B) \cup \psi_{\lambda_g^\delta \mathcal{G}}(B) = \psi_{\lambda_g^\delta \mathcal{G}}(V \cup B) = \psi_{\lambda_g^\delta \mathcal{G}}(B)$, again by (ii)

□

Corollary 3.5.6. Let $(X, \tau^{\lambda_g^\delta}, \mathcal{G})$ be a grill λ_g^δ-space and $A \subseteq X$. Then

(i) $\phi \notin \mathcal{G} \Rightarrow \psi_{\lambda_g^\delta \mathcal{G}}(\phi) = \phi$.

(ii) $\psi_{\lambda_g^\delta \mathcal{G}}(A) \setminus \psi_{\lambda_g^\delta \mathcal{G}}(\psi_{\lambda_g^\delta \mathcal{G}}(A)) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus \psi_{\lambda_g^\delta \mathcal{G}}(A))$.

(iii) $\psi_{\lambda_g^\delta \mathcal{G}}(A) \setminus \psi_{\lambda_g^\delta \mathcal{G}}(B) = \psi_{\lambda_g^\delta \mathcal{G}}(A \setminus B) \setminus \psi_{\lambda_g^\delta \mathcal{G}}(B)$.

(iv) $A \cap \psi_{\lambda_g^\delta \mathcal{G}}(X) = \phi$, for every λ_g^δ-open set A with $A \notin \mathcal{G}$.

Lemma 3.5.7. Let $(X, \tau^{\lambda_g^\delta}, \mathcal{G})$ be a grill λ_g^δ-space and $A \subseteq X$ be λ_g^δ-closed. Then $\psi_{\lambda_g^\delta \mathcal{G}}(A) \subseteq A$.

Proof. Suppose $x \notin A$, where A is λ_g^δ-closed. Then $X \setminus A$ is a λ_g^δ-open set containing x and $(X \setminus A) \cap A = \phi \notin \mathcal{G}$. Therefore $x \notin \psi_{\lambda_g^\delta \mathcal{G}}(A)$ and hence $\psi_{\lambda_g^\delta \mathcal{G}}(A) \subseteq A$. □

Corollary 3.5.8. Let $(X, \tau^{\lambda_g^\delta}, \mathcal{G})$ be a grill λ_g^δ-space and $A \subseteq X$ be λ_g^δ-closed. Then $\psi_{\lambda_g^\delta \mathcal{G}}(\psi_{\lambda_g^\delta \mathcal{G}}(A)) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(A)$.

Proof. Follows from Theorem 3.5.4(i). □

Theorem 3.5.9. *Let $(X, \tau^{\lambda_g^\delta}, \mathcal{G})$ be a grill λ_g^δ-space and $A \subseteq X$. If V is λ_g^δ-open set containing x then $\psi_{\lambda_g^\delta \mathcal{G}}(A) = \psi_{\lambda_g^\delta \mathcal{G}}(V \cap A)$.*

Proof. Since $V \cap A \subseteq A$, by (i) of Theorem 3.5.4, we have $\psi_{\lambda_g^\delta \mathcal{G}}(V \cap A) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(A)$. Suppose $x \notin \psi_{\lambda_g^\delta \mathcal{G}}(V \cap A)$, then there exists λ_g^δ-open set U containing x such that $U \cap (V \cap A) \notin \mathcal{G}$. Now, $W = U \cap V$ is a λ_g^δ-open set containing x such that $W \cap A \notin \mathcal{G}$. Therefore $x \notin \psi_{\lambda_g^\delta \mathcal{G}}(A)$. Hence $\psi_{\lambda_g^\delta \mathcal{G}}(A) \subseteq \psi_{\lambda_g^\delta \mathcal{G}}(V \cap A)$. □

Theorem 3.5.10. *Let $(X, \tau^{\lambda_g^\delta}, \mathcal{G})$ be a grill λ_g^δ-space. Then a function $\zeta_{\lambda_g^\delta \mathcal{G}} : P(X) \to P(X)$ defined by $\zeta_{\lambda_g^\delta \mathcal{G}}(A) = A \cup \psi_{\lambda_g^\delta \mathcal{G}}(A)$, satisfies Kuratowski closure axioms, for every λ_g^δ-open set A.*

Proof. (i) $\zeta_{\lambda_g^\delta \mathcal{G}}(\phi) = \phi \cup \psi_{\lambda_g^\delta \mathcal{G}}(\phi)$.

(ii) Let $A \subseteq X$ and $x \notin \zeta_{\lambda_g^\delta \mathcal{G}}(A)$. Then $x \notin A$ and $x \notin \psi_{\lambda_g^\delta \mathcal{G}}(A)$. Therefore $A \subseteq \zeta_{\lambda_g^\delta \mathcal{G}}(A)$ and $\psi_{\lambda_g^\delta \mathcal{G}}(A) \subseteq \zeta_{\lambda_g^\delta \mathcal{G}}(A)$.

(iii) $\zeta_{\lambda_g^\delta \mathcal{G}}(A \cup B) = (A \cup B) \cup \psi_{\lambda_g^\delta \mathcal{G}}(A \cup B) = (A \cup B) \cup \psi_{\lambda_g^\delta \mathcal{G}}(A) \cup \psi_{\lambda_g^\delta \mathcal{G}}(B) = (A \cup \psi_{\lambda_g^\delta \mathcal{G}}(A)) \cup (B \cup \psi_{\lambda_g^\delta \mathcal{G}}(B)) = \zeta_{\lambda_g^\delta \mathcal{G}}(A) \cup \zeta_{\lambda_g^\delta \mathcal{G}}(B)$.

(iv) $\zeta_{\lambda_g^\delta \mathcal{G}}(\zeta_{\lambda_g^\delta \mathcal{G}}(A)) = \zeta_{\lambda_g^\delta \mathcal{G}}(A \cup \psi_{\lambda_g^\delta \mathcal{G}}(A)) = (A \cup \psi_{\lambda_g^\delta \mathcal{G}}(A)) \cup \psi_{\lambda_g^\delta \mathcal{G}}(A \cup \psi_{\lambda_g^\delta \mathcal{G}}(A)) = A \cup \psi_{\lambda_g^\delta \mathcal{G}}(A) = \zeta_{\lambda_g^\delta \mathcal{G}}(A)$.

□