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(i, j)-$\psi^*\alpha$-Continuous Maps in Bitopological Spaces

7.1 Introduction

The concept of continuity in topological spaces was extended to bitpological spaces by Pervin (1967). In this chapter, (i, j)-$\psi^*\alpha$-σ_k-continuous maps, quasi (i, j)-$\psi^*\alpha$-continuous maps, perfectly (i, j)-$\psi^*\alpha$-continuous maps, totally (i, j)-$\psi^*\alpha$-σ_k-continuous maps, strongly (i, j)-$\psi^*\alpha$-σ_k-continuous maps, contra (i, j)-$\psi^*\alpha$-σ_k-continuous maps, (i, j)-$\psi^*\alpha$-irresolute maps and contra (i, j)-$\psi^*\alpha$-irresolute maps are introduced and their properties are discussed.

7.2 (i, j)-$\psi^*\alpha$-σ_k-continuous maps

In this section, (i, j)-$\psi^*\alpha$-σ_k-continuous maps using (i, j)-$\psi^*\alpha$-closed sets in bitopological spaces are introduced and some of their basic properties are studied.

Definition 7.2.1 A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called **(i, j)-$\psi^*\alpha$-σ_k-continuous** if $f^{-1}(V)$ is an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) for every σ_k-closed set V in (Y, σ_1, σ_2), where i, j, k = 1, 2 and $i \neq j$.

Example 7.2.2 Let X = Y = {a, b, c}, $\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\sigma_1 = \{\phi, \{a, b\}, Y\}$ and $\sigma_2 = \{\phi, \{a\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity map. Then f is (1, 2)-$\psi^*\alpha$-σ_1-continuous.

Remark 7.2.3 If $\tau_1 = \tau_2 = \tau$ and $\sigma_1 = \sigma_2 = \sigma$ then (i, j)-$\psi^*\alpha$-σ_k-continuous map coincides with $\psi^*\alpha$-continuous map.

Proposition 7.2.4

(i) Every τ_j-σ_k-continuous map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an (i, j)-$\psi^*\alpha$-σ_k-continuous map.

(ii) Every τ_j-α-σ_k-continuous map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an (i, j)-$\psi^*\alpha$-σ_k-continuous map.

Proof: Since every τ_j -closed set and τ_j-α-closed set is an (i, j)-$\psi^*\alpha$-closed set, the result follows.

The converse of the statements in the above proposition need not be true as can be seen from the following example.

Example 7.2.5 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a, b}, X}, τ_2 = {ϕ, {a}, {a, b}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) = a, f(c) = c. Then f is (1, 2)-$\psi^*\alpha$-σ_1-continuous but not τ_2-σ_1-continuous and not τ_2-α-σ_1-continuous, since {b, c} is σ_1-closed in (Y, σ_1, σ_2) but f^{-1}({b, c}) = {a, c} is not τ_2-closed and not τ_2-α-closed in (X, τ_1, τ_2).

Remark 7.2.6 The converse of the statements in **Proposition 7.2.4** is true if (X, τ_1, τ_2) is, respectively,

(i) an (i, j)-${}_{\psi^*\alpha}T_c$-space,

(ii) an (i, j)-${}_{\psi^*\alpha}T_\alpha$-space.

Proposition 7.2.7 Every (i, j)-$\psi^*\alpha$-σ_k-continuous map is an (i, j)-gp-σ_k-continuous map but not conversely.

Proof: Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be an (i, j)-$\psi^*\alpha$-σ_k-continuous map. Let V be a σ_k-closed set in (Y, σ_1, σ_2). Then f^{-1}(V) is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Since every (i, j)-$\psi^*\alpha$-closed set is (i, j)-gp-closed, f^{-1}(V) is (i, j)-gp-closed and hence f is an (i, j)-gp-σ_k-continuous map.

Example 7.2.8 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, X}, τ_2 = {ϕ, {a}, {b}, {a, b},{a, c}, X}, σ_1 = {ϕ, {c},{a, b}, Y} and σ_2 = {ϕ, {a}, {b}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be the identity map. Then f is (1, 2)-gp-σ_1-continuous but not (1, 2)-$\psi^*\alpha$-σ_1-continuous, since {a, b} is σ_1-closed in (Y, σ_1, σ_2) but f^{-1}({a, b}) = {a, b} is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Proposition 7.2.9 Every (i, j)-$\psi^*\alpha$-σ_k-continuous map is an (i, j)-gpr-σ_k-continuous map but not conversely.

Proof: Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be an (i, j)-$\psi^*\alpha$-σ_k-continuous map. Let V be a σ_k-closed set in (Y, σ_1, σ_2). Then $f^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Since every (i, j)-$\psi^*\alpha$-closed set is (i, j)-gpr-closed, $f^{-1}(V)$ is (i, j)-gpr-closed and hence f is an (i, j)-gpr-σ_k-continuous map .

Example 7.2.10 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, X}, τ_2 = {ϕ, {a}, {a, b}, {a, c}, X}, σ_1 = {ϕ, {c}, {a, b}, Y} and σ_2 = {ϕ, {a}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) = c, f(c) = a. Then f is (1, 2)-gpr-σ_1-continuous but not (1, 2)-$\psi^*\alpha$-σ_1-continuous, since {a, b} is σ_1-closed in (Y, σ_1, σ_2) but f^{-1}({a, b}) = {a, c} is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Proposition 7.2.11 Every (i, j)-$\psi^*\alpha$-σ_k-continuous map is an (i, j)- $\tilde{g}_\alpha$-σ_k-continuous map but not conversely.

Proof: Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be an (i, j)-$\psi^*\alpha$-σ_k-continuous map . Let V be a σ_k-closed set in (Y, σ_1, σ_2). Then $f^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Since every (i, j)-$\psi^*\alpha$-closed set is (i, j)- $\tilde{g}_\alpha$-closed, $f^{-1}(V)$ is (i, j)- $\tilde{g}_\alpha$-closed and hence f is an (i, j)- $\tilde{g}_\alpha$-σ_k-continuous map.

Example 7.2.12 Let X = Y = {a, b, c, d}, τ_1 = {ϕ, {a}, {a, b}, X}, τ_2 = {ϕ, {d}, {a, b}, {a, b, d}, X}, σ_1 = {ϕ, {a, b}, Y} and σ_2 = {ϕ, {a}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) = c, f(c) = d, f(d) = a. Then f is (1, 2)- $\tilde{g}_\alpha$-σ_1-continuous but not (1, 2)-$\psi^*\alpha$-σ_1-continuous, since {c, d} is σ_1-closed in (Y, σ_1, σ_2) but f^{-1}({c, d}) = {b, c} is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Proposition 7.2.13 Every (i, j)-$\psi^*\alpha$-σ_k-continuous map is an (i, j)-gα-σ_k-continuous map but not conversely.

Proof: Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be an (i, j)-$\psi^*\alpha$-σ_k-continuous map. Let V be a σ_k-closed set in (Y, σ_1, σ_2). Then $f^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Since every (i, j)-$\psi^*\alpha$-closed set is (i, j)-gα-closed, $f^{-1}(V)$ is (i, j)-gα-closed and hence f is an (i, j)-gα-σ_k-continuous map.

Example 7.2.14 Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}$, $\sigma_1 = \{\phi, \{a\}, \{a, b\}, Y\}$ and $\sigma_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a map defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is (1, 2)-gα-σ_1-continuous but not (1, 2)-$\psi^*\alpha$-σ_1-continuous, since {b, c} is σ_1-closed in (Y, σ_1, σ_2) but $f^{-1}(\{b, c\}) = \{a, c\}$ is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Proposition 7.2.15 Every (i, j)-$\psi^*\alpha$-σ_k-continuous map is an (i, j)-αg-σ_k-continuous map but not conversely.

Proof: Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an (i, j)-$\psi^*\alpha$-σ_k-continuous map. Let V be a σ_k-closed set in (Y, σ_1, σ_2). Then $f^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Since every (i, j)-$\psi^*\alpha$-closed set is (i, j)-αg-closed, $f^{-1}(V)$ is (i, j)-αg-closed and hence f is an (i, j)-αg-σ_k-continuous map.

Example 7.2.16 Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\sigma_1 = \{\phi, \{b\}, Y\}$ and $\sigma_2 = \{\phi, \{a, b\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be the identity map. Then f is (1, 2)-αg-σ_1-continuous but not (1, 2)-$\psi^*\alpha$-σ_1-continuous, since {a, c} is σ_1-closed in (Y, σ_1, σ_2) but $f^{-1}(\{a, c\}) = \{a, c\}$ is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Proposition 7.2.17 Every (i, j)-$\psi^*\alpha$-σ_k-continuous map is an (i, j)-ψg-σ_k-continuous map but not conversely.

Proof: Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an (i, j)-$\psi^*\alpha$-σ_k-continuous map. Let V be a σ_k-closed set in (Y, σ_1, σ_2). Then $f^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Since every (i, j)-$\psi^*\alpha$-closed set is (i, j)-ψg-closed, $f^{-1}(V)$ is (i, j)-ψg-closed and hence f is an (i, j)-ψg-σ_k-continuous map.

Example 7.2.18 Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma_1 = \{\phi, \{a\}, \{b, c\}, Y\}$ and $\sigma_2 = \{\phi, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a map defined by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is (1, 2)-ψg-σ_1-continuous but not (1, 2)-$\psi^*\alpha$-σ_1-continuous, since {a} is σ_1-closed in (Y, σ_1, σ_2) but $f^{-1}(\{a\}) = \{b\}$ is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Remark 7.2.19 The following examples show that (i, j)-g-σ_k-continuous maps and (i, j)-$\psi^*\alpha$-σ_k-continuous maps are independent.

Example 7.2.20 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, X},τ_2 = {ϕ, {a}, {a, b}, X}, σ_1 = {ϕ, {b}, {a, b}, {b, c}, Y} and σ_2 = {ϕ, {b}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) = c, f(c) = a. Then f is (1, 2)-$\psi^*\alpha$-σ_1-continuous but not (1, 2)-g-σ_1-continuous, since {c} is σ_1-closed in (Y, σ_1, σ_2) but f^{-1}({c}) = {b} is not (1, 2)-g-closed in (X, τ_1, τ_2).

Example 7.2.21 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, X}, τ_2 = {ϕ, {a}, {a, b}, {a, c}, X}, σ_1 = {ϕ, {c}, {a, b}, Y} and σ_2 = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be the identity map. Then f is (1, 2)-g-σ_1-continuous but not (1, 2)-$\psi^*\alpha$-σ_1-continuous, since {a, b} is σ_1-closed in (Y, σ_1, σ_2) but f^{-1}({a, b}) = {a, b} is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Remark 7.2.22 The following examples show that (i, j)-g^*-σ_k-continuous maps and (i, j)-$\psi^*\alpha$-σ_k-continuous maps are independent.

Example 7.2.23 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, X}, τ_2 ={ϕ, {a}, {a, b}, X}, σ_1 = {ϕ, {b}, {a, b}, {b, c}, Y} and σ_2 = {ϕ, {b}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) = c, f(c) = a. Then f is (1, 2)-$\psi^*\alpha$-σ_1-continuous but not (1, 2)-g^*-σ_1-continuous, since {c} is σ_1-closed in (Y, σ_1, σ_2) but f^{-1}({c}) = {b} is not (1, 2)-g^*-closed in (X, τ_1, τ_2).

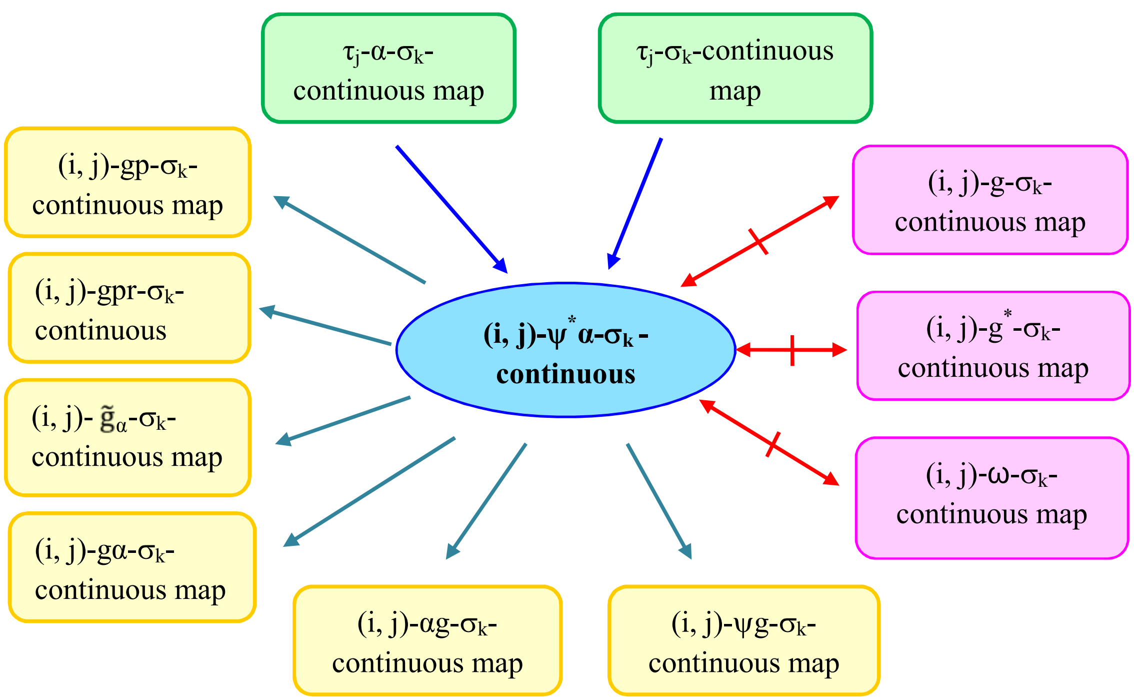
Example 7.2.24 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, X}, τ_2 = {ϕ, {a}, {a, b}, X}, σ_1 = {ϕ, {c}, {b, c}, Y} and σ_2 = {ϕ, {c}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) = c, f(c) = a. Then f is (1, 2)-g^*-σ_1-continuous but not (1, 2)-$\psi^*\alpha$-σ_1-continuous, since {a, b} is σ_1-closed in (Y, σ_1, σ_2) but f^{-1}({a, b}) = {a, c} is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Remark 7.2.25 The following examples show that (i, j)-ω-σ_k-continuous maps and (i, j)-$\psi^*\alpha$-σ_k-continuous maps are independent.

Example 7.2.26 Let X = Y = {a, b, c, d}, τ_1 = {ϕ, {d}, {a, b}, {a, b, d}, X}, τ_2 = {ϕ, {a}, X}, σ_1 = {ϕ, {a, b}, Y} and σ_2 = {ϕ, {a}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) = c, f(c) = d, f(d) = a. Then f is (1, 2)-$\psi^*\alpha$-σ_1-continuous but not (1, 2)-ω-σ_1-continuous, since {c, d} is σ_1-closed in (Y, σ_1, σ_2) but f^{-1}({c, d}) = {b, c} is not (1, 2)-ω-closed in (X, τ_1, τ_2).

Example 7.2.27 Let X = Y = {a, b, c, d}, τ_1 = {ϕ, {d}, {a, b}, {a, b, d}, X}, τ_2 = {ϕ, {a}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {b}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) = a, f(c) = c, f(d) = d. Then f is (1, 2)-ω-σ_1-continuous but not (1, 2)-$\psi^*\alpha$-σ_1-continuous, since {b, c, d} is σ_1-closed in (Y, σ_1, σ_2) but f^{-1}({b, c, d}) = {a, c, d} is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Remark 7.2.28 The above observations are depicted in the following diagram.



Theorem 7.2.29 Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map. Then the following statements are equivalent.

(i) f is an (i, j)-$\psi^*\alpha$-σ_k-continuous.

(ii) f^{-1}(V) is an (i, j)-$\psi^*\alpha$-open for each σ_k-open set V in (Y, σ_1, σ_2), where i, j, k = 1, 2 and $i \neq j$.

Proposition 7.2.30 Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i, j)-$\psi^*\alpha$-σ_k-continuous map. Then for every subset U of (X, τ_1, τ_2), $f[(i, j)\text{-}\psi^*\alpha cl(U)] \subseteq \sigma_k\text{-}cl(f(U))$.

Proof: Let U be any subset of (X, τ_1, τ_2). Then $\sigma_k\text{-}cl(f(U))$ is a σ_k-closed set in (Y, σ_1, σ_2). Since f is an (i, j)-$\psi^*\alpha$-σ_k-continuous map, $f^{-1}[\sigma_k\text{-}cl(f(U))]$ is an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2). Since $f(U) \subseteq \sigma_k\text{-}cl(f(U))$, $U \subseteq f^{-1}[f(U)] \subseteq f^{-1}[\sigma_k\text{-}cl(f(U))]$ and hence $f^{-1}[\sigma_k\text{-}cl(f(U))]$ is an (i, j)-$\psi^*\alpha$-closed set containing U. Therefore $(i, j)\text{-}\psi^*\alpha cl(U) \subseteq f^{-1}[\sigma_k\text{-}cl(f(U))]$ which implies that $f[(i, j)\text{-}\psi^*\alpha cl(U)] \subseteq \sigma_k\text{-}cl(f(U))$.

Corollary 7.2.31 Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a τ_j-σ_k-continuous map Then for every subset U of (X, τ_1, τ_2), $f[(i, j)\text{-}\psi^*\alpha cl(U)] \subseteq \sigma_k\text{-}cl(f(U))$.

Proof: Follows from the fact that every τ_j-σ_k-continuous map is an (i, j)-$\psi^*\alpha$-σ_k-continuous map and from **Proposition 7.2.30.**

Theorem 7.2.32 Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a map. Then the following statements are equivalent.

(i) For every subset A of (X, τ_1, τ_2), $f [(i, j)\text{-}\psi^*\alpha cl(A)] \subseteq \sigma_k\text{-}cl(f(A))$.

(ii) For every subset B of (Y, σ_1, σ_2), $[(i, j)\text{-}\psi^*\alpha cl(f^{-1}(B))] \subseteq f^{-1} [\sigma_k\text{-}cl(B)]$.

Proof: Suppose that (i) holds and let B be any subset of (Y, σ_1, σ_2). Replacing A by $f^{-1}(B)$, $f [(i, j)\text{-}\psi^*\alpha cl(f^{-1}(B))] \subseteq \sigma_k\text{-}cl(f[f^{-1}(B)]) \subseteq \sigma_k\text{-}cl(B)$. Hence $[(i, j)\text{-}\psi^*\alpha cl(f^{-1}(B))] \subseteq f^{-1} [\sigma_k\text{-}cl(B)]$.

Conversely, suppose that (ii) holds and let B = f(A) where A is a subset of (X, τ_1, τ_2), Then $[(i, j)\text{-}\psi^*\alpha cl(f^{-1}(f(A))] \subseteq f^{-1} [\sigma_k\text{-}cl(f(A))]$. Therefore $f [(i, j)\text{-}\psi^*\alpha cl(A)] \subseteq [\sigma_k\text{-}cl(f(A))]$.

Proposition 7.2.33 If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an (i, j)-$\psi^*\alpha$-σ_j-continuous map and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ is a σ_j-η_k-continuous map, then $g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is an (i, j)-$\psi^*\alpha$-η_k-continuous map, where i, j, k = 1, 2 and $i \neq j$.

Proof: Let V be a η_k-closed set in (Z, η_1, η_2). Since g is σ_j-η_k-continuous, $g^{-1}(V)$ is σ_j-closed in (Y, σ_1, σ_2). Since f is an (i, j)-$\psi^*\alpha$-σ_j-continuous map, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2). Therefore g∘f is an (i, j)-$\psi^*\alpha$-η_k-continuous map.

Proposition 7.2.34 If f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is τ_j-σ_k-continuous (resp. τ_j-α-σ_k-continuous) and g : (Y, σ_1, σ_2) → (Z, η_1, η_2) is σ_k-η_i-continuous, then g∘f : (X, τ_1, τ_2) → (Z, η_1, η_2) is an (i, j)-$\psi^*\alpha$-η_i-continuous map, where i, j, k = 1, 2 and $i \neq j$.

Proof: Let V be a η_i-closed set in (Z, η_1, η_2). Since g is σ_k-η_i-continuous, $g^{-1}(V)$ is σ_k-closed in (Y, σ_1, σ_2). Since f is τ_j-σ_k-continuous (resp. τ_j-α-σ_k-continuous), $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is τ_j-closed (resp. τ_j-α-closed) in (X, τ_1, τ_2). Since every τ_j-closed (resp. τ_j-α-closed) set is (i, j)-$\psi^*\alpha$-closed, $(g\circ f)^{-1}(V)$ is an (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Therefore g∘f is an (i, j)-$\psi^*\alpha$-η_i-continuous map.

Remark 7.2.35 The composition of two (i, j)-$\psi^*\alpha$-σ_k-continuous maps need not be an (i, j)-$\psi^*\alpha$-σ_k-continuous map as seen from the following example.

Example 7.2.36 Let X = Y = Z = {a, b, c}, τ_1 = {ϕ, {a, b}, X}, τ_2 = {ϕ, {a}, X}, σ_1 = {ϕ, {a}, Y}, σ_2 = {ϕ, {a}, {b, c}, Y}, η_1 = {ϕ, {a}, {a, b}, Z} and η_2 = {ϕ, {a}, {b}, {a, b}, Z}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = c, f(b) = a, f(c) = b and g : (Y, σ_1, σ_2) → (Z, η_1, η_2) be a map defined by g(a) = b, g(b) = a, g(c) = c. Then the maps f is (1, 2)-$\psi^*\alpha$-σ_2-continuous and g is (1, 2)-$\psi^*\alpha$-η_2-continuous but their composition g∘f : (X, τ_1, τ_2) → (Z, η_1, η_2) is not a (1, 2)-$\psi^*\alpha$-η_2-continuous map, since {b, c} is η_2-closed in (Z, η_1, η_2) but $(g\circ f)^{-1}$({b, c}) = {a, b} is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Proposition 7.2.37 Let (Y, σ_1, σ_2) be an (i, j)-${}_{\psi^*\alpha}T_c$ - space. If f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is an (i, j)-$\psi^*\alpha$-σ_j-continuous map and g : (Y, σ_1, σ_2) → (Z, η_1, η_2) is an (i, j)-$\psi^*\alpha$-η_k-continuous map then g∘f : (X, τ_1, τ_2) → (Z, η_1, η_2) is also an (i, j)-$\psi^*\alpha$-η_k-continuous map, where i, j, k = 1, 2 and $i \neq j$.

Proof: Let V be a η_k-closed set in (Z, η_1, η_2). Since g is (i, j)-$\psi^*\alpha$-η_k-continuous, $g^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since (Y, σ_1, σ_2) be an (i, j)-${}_{\psi^*\alpha}T_c$-space, $g^{-1}(V)$ is σ_j-closed.

Since f is (i, j)-$\psi^*\alpha$-σ_j -continuous, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence $g\circ f$ is an (i, j)-$\psi^*\alpha$-η_k-continuous map .

7.3 Quasi (i, j)-$\psi^*\alpha$-continuous maps and perfectly (i, j)-$\psi^*\alpha$-continuous maps

In this section, quasi (i, j)-$\psi^*\alpha$-continuous maps and perfectly (i, j)-$\psi^*\alpha$-continuous maps are defined and their properties are studied.

Definition 7.3.1 A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called **quasi (i, j)-$\psi^*\alpha$-continuous** if $f^{-1}(V)$ is τ_j-closed in (X, τ_1, τ_2) for every (i, j)-$\psi^*\alpha$-closed set V in (Y, σ_1, σ_2), where i, j = 1, 2 and $i \neq j$.

Example 7.3.2 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, X}, τ_2 = {ϕ, {a}, {b}, {a, b}, {a, c}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {a}, {a, b}, {a, c}, Y}. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a map defined by f(a) = a, f(b) = b, f(c) = c. Then f is quasi (1, 2)-$\psi^*\alpha$-continuous.

Theorem 7.3.3 A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is quasi (i, j)-$\psi^*\alpha$-continuous if and only if the inverse image of every (i, j)-$\psi^*\alpha$-open set in (Y, σ_1, σ_2) is τ_j-open in (X, τ_1, τ_2), where i, j = 1, 2 and $i \neq j$.

Proof: (Necessity) Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a quasi (i, j)-$\psi^*\alpha$-continuous map and V be any (i, j)-$\psi^*\alpha$-open set in (Y, σ_1, σ_2). Then $Y - V$ is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since f is quasi (i, j)-$\psi^*\alpha$-continuous, $f^{-1}(Y - V) = X - f^{-1}(V)$ is τ_j-closed in (X, τ_1, τ_2). Hence $f^{-1}(V)$ is τ_j-open in (X, τ_1, τ_2).

(Sufficiency): Let F be any (i, j)-$\psi^*\alpha$-closed set in (Y, σ_1, σ_2). Then $Y - F$ is (i, j)-$\psi^*\alpha$-open in (Y, σ_1, σ_2). By assumption, $f^{-1}(Y - F) = X - f^{-1}(F)$ is τ_j-open in (X, τ_1, τ_2) which implies that $f^{-1}(F)$ is τ_j-closed in (X, τ_1, τ_2). Hence f is a quasi (i, j)-$\psi^*\alpha$-continuous map.

Definition 7.3.4 A map $f :(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called **perfectly (i, j)-$\psi^*\alpha$-continuous** if $f^{-1}(V)$ is τ_j-clopen in (X, τ_1, τ_2) for every (i, j)-$\psi^*\alpha$-closed set V in (Y, σ_1, σ_2), where i, j = 1, 2 and $i \neq j$.

Example 7.3.5 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, X}, τ_2 = {ϕ, {a}, {b, c}, X}, σ_1 = {ϕ, {a}, {b}, {a, b}, Y} and σ_2 = {ϕ, {a}, {b, c}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = a, f(b) = b, f(c) = c. Then f is perfectly (1, 2)-$\psi^*\alpha$-continuous.

Theorem 7.3.6 A map f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is perfectly (i, j)-$\psi^*\alpha$-continuous if and only if the inverse image of every (i, j)-$\psi^*\alpha$-open set in (Y, σ_1, σ_2) is τ_j-clopen in (X, τ_1, τ_2), where i, j = 1, 2 and i ≠ j.

Proof: Similar to **Theorem 7.3.3.**

Proposition 7.3.7 If f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is a perfectly (i, j)-$\psi^*\alpha$-continuous map, then it is a quasi (i, j)-$\psi^*\alpha$-continuous map but not conversely.

Proof: Let V be an (i, j)-$\psi^*\alpha$-closed set in (Y, σ_1, σ_2). Since f is perfectly (i, j)-$\psi^*\alpha$-continuous, $f^{-1}(V)$ is τ_j-clopen in (X, τ_1, τ_2) and hence $f^{-1}(V)$ is τ_j-closed in (X, τ_1, τ_2). Therefore f is a quasi (i, j)-$\psi^*\alpha$-continuous map.

Example 7.3.8 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, X}, τ_2 = {ϕ, {a}, {b}, {a, b}, {a, c}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {a}, {a, b}, {a, c}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be the identity map. Then f is quasi (1, 2)-$\psi^*\alpha$-continuous but not perfectly (1, 2)-$\psi^*\alpha$-continuous, since {c} is (1, 2)-$\psi^*\alpha$-closed set in (Y, σ_1, σ_2) but f^{-1}({c}) = {c} is not τ_2-clopen in (X, τ_1, τ_2).

Proposition 7.3.9 If f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is a quasi (i, j)-$\psi^*\alpha$-continuous map, then it is an (i, j)-$\psi^*\alpha$-σ_k-continuous map but not conversely.

Proof: Let V be a σ_k-closed set in (Y, σ_1, σ_2). Since every σ_k-closed set is (i, j)-$\psi^*\alpha$-closed, V is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since f is quasi (i, j)-$\psi^*\alpha$-continuous, $f^{-1}(V)$ is τ_j-closed in (X, τ_1, τ_2). Since every τ_j-closed set is (i, j)-$\psi^*\alpha$-closed, $f^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence f is an (i, j)-$\psi^*\alpha$-σ_k-continuous map.

Example 7.3.10 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, X}, τ_2 = {ϕ, {a},{a, b}, {a, c}, X}, σ_1 = {ϕ, {a, b}, Y} and σ_2 = {ϕ, {a}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be the identity map. Then f is (1, 2)-$\psi^*\alpha$-σ_2-continuous but not quasi (1, 2)-$\psi^*\alpha$-continuous, since {a, c} is (1, 2)-$\psi^*\alpha$-closed set in (Y, σ_1, σ_2) but f^{-1}({a, c}) = {a, c} is not τ_2-closed in (X, τ_1, τ_2).

Proposition 7.3.11 If f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is a quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous) map and g : (Y, σ_1, σ_2) → (Z, η_1, η_2) is a perfectly (i, j)-$\psi^*\alpha$-continuous (resp. quasi (i, j)-$\psi^*\alpha$-continuous) then g∘f : (X, τ_1, τ_2) → (Z, η_1, η_2) is a quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous) map.

Proof: Let V be an (i, j)-$\psi^*\alpha$-closed set in (Z, η_1, η_2). Then g^{-1}(V) is σ_j-clopen (resp. σ_j-closed) in (Y, σ_1, σ_2) as g is perfectly (i, j)-$\psi^*\alpha$-continuous (resp. quasi (i, j)-$\psi^*\alpha$-continuous). Since every σ_j-closed set is (i, j)-$\psi^*\alpha$-closed, g^{-1}(V) is an (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since f is quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous), $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is τ_j-closed (resp. τ_j-clopen) in (X, τ_1, τ_2) and hence g∘f is a quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous) map.

Proposition 7.3.12 If f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) and g : (Y, σ_1, σ_2) → (Z, η_1, η_2) are quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous) maps, then g∘f : (X, τ_1, τ_2) → (Z, η_1, η_2) is also a quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous) map.

Proof: Let V be an (i, j)-$\psi^*\alpha$-closed set in (Z, η_1, η_2). Then g^{-1}(V) is σ_j-closed (resp. σ_j-clopen) in (Y, σ_1, σ_2) as g is quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous). Since every σ_j-closed set is (i, j)-$\psi^*\alpha$- closed, g^{-1}(V) is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since f is quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$- continuous), $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is τ_j-closed (resp. τ_j-clopen) in (X, τ_1, τ_2) and hence g∘f is a quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous) map.

7.4 Totally (i, j)-$\psi^*\alpha$-σ_k-continuous maps and strongly (i, j)-$\psi^*\alpha$-σ_k-continuous maps

In this section, totally (i, j)-$\psi^*\alpha$-σ_k-continuous maps and strongly (i, j)-$\psi^*\alpha$-σ_k-continuous maps are introduced and some of their properties are discussed.

Definition 7.4.1 A subset A of (X, τ_1, τ_2) is called **(i, j)-$\psi^*\alpha$-clopen** if it is both (i, j)-$\psi^*\alpha$-open and (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Definition 7.4.2 A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called **totally (i, j)-$\psi^*\alpha$-σ_k-continuous** if $f^{-1}(V)$ is an (i, j)-$\psi^*\alpha$-clopen set in (X, τ_1, τ_2) for every σ_k-open set V in (Y, σ_1, σ_2), where i, j, k = 1, 2 and $i \neq j$.

Example 7.4.3 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, X}, τ_2 = {ϕ, {a}, {b, c}, X}, σ_1 = {ϕ, {a}, {b, c}, Y} and σ_2 = {ϕ, {b, c}, Y}. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be the identity map. Then f is totally (1, 2)-$\psi^*\alpha$-σ_2-continuous.

Theorem 7.4.4 A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is totally (i, j)-$\psi^*\alpha$-σ_k-continuous if and only if the inverse image of every σ_k-closed subset of (Y, σ_1, σ_2) is an (i, j)-$\psi^*\alpha$-clopen subset of (X, τ_1, τ_2).

Definition 7.4.5 A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called **strongly (i, j)-$\psi^*\alpha$-σ_k -continuous** if $f^{-1}(V)$ is an (i, j)-$\psi^*\alpha$-clopen subset in (X, τ_1, τ_2) for every σ_k-subset V in (Y, σ_1, σ_2), where i, j, k= 1, 2 and $i \neq j$.

Example 7.4.6 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, X}, τ_2 = {ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c} X}, σ_1 = {ϕ, {a}, {b, c}, Y} and σ_2 = {ϕ, {a}, {b},{a, b}, Y}. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a map defined by f(a) = c, f(b) = a, f(c) = b. Then f is strongly (1, 2)-$\psi^*\alpha$-σ_2-continuous.

Proposition 7.4.7 Every strongly (i, j)-$\psi^*\alpha$-σ_k-continuous map is a totally (i, j)-$\psi^*\alpha$-σ_k-continuous map but not conversely.

Proof : The proof follows from the **Definition 7.4.2** and **Definition 7.4.5.**

Example 7.4.8 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, X}, τ_2 = {ϕ, {a}, {b, c}, X}, σ_1 = {ϕ, {a}, {b, c}, Y} and σ_2 = {ϕ, {b, c}, Y}. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be the identity map. Then f is totally (1, 2)-$\psi^*\alpha$-σ_2-continuous but not strongly (1, 2)-$\psi^*\alpha$-σ_2-continuous, since {b} is σ_2-subset in (Y, σ_1, σ_2) but f^{-1}({b}) = {b} is not (1, 2)-$\psi^*\alpha$-clopen in (X, τ_1, τ_2).

Proposition 7.4.9 Every totally (i, j)-$\psi^*\alpha$-σ_k-continuous map is an (i, j)-$\psi^*\alpha$-σ_k-continuous map but not conversely.

Proof: Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a totally (i, j)-$\psi^*\alpha$-σ_k-continuous map. Let V be a σ_k-closed set in (Y, σ_1, σ_2). Since f is totally (i, j)-$\psi^*\alpha$-σ_k-continuous, $f^{-1}(V)$ is (i, j)-$\psi^*\alpha$-clopen in (X, τ_1, τ_2). Hence f is an (i, j)-$\psi^*\alpha$-σ_k-continuous map.

Example 7.4.10 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a, b}, X}, τ_2 = {ϕ, {a}, {a, b}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) = a, f(c) = c. Then f is (1, 2)-$\psi^*\alpha$-σ_2-continuous but not totally (1, 2)-$\psi^*\alpha$-σ_2-continuous, since {c} is σ_2-closed in (Y, σ_1, σ_2) but f^{-1}({c}) = {c} is not (1, 2)-$\psi^*\alpha$-clopen in (X, τ_1, τ_2).

7.5 Contra (i, j)-$\psi^*\alpha$-σ_k-continuous maps

In this section, contra (i, j)-$\psi^*\alpha$-σ_k-continuous maps are introduced and some of their properties are discussed.

Definition 7.5.1 A map f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is called **contra (i, j)-$\psi^*\alpha$-σ_k-continuous** if $f^{-1}(V)$ is an (i, j)-$\psi^*\alpha$-open set in (X, τ_1, τ_2) for every σ_k-closed set V in (Y, σ_1, σ_2), where i, j, k= 1, 2 and i ≠ j.

Example 7.5.2 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, {a, c}, X}, τ_2 = {ϕ, {a}, {b, c}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = c, f(b) = b, f(c) = a. Then f is contra (1, 2)-$\psi^*\alpha$-σ_2-continuous.

Theorem 7.5.3 A map f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is contra (i, j)-$\psi^*\alpha$-σ_k-continuous if and only if the inverse image of every σ_k-open set in (Y, σ_1, σ_2) is an (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2), where i, j, k = 1, 2 and i ≠ j.

Proof : Follows from the **Definition 7.5.1.**

Theorem 7.5.4 Every totally (i, j)-$\psi^*\alpha$-σ_k-continuous map is a contra (i, j)-$\psi^*\alpha$-σ_k-continuous map but not conversely.

Proof: Follows from the **Definition 7.4.2** and **Definition 7.5.1**

Example 7.5.5 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, {a, c}, X}, τ_2 = {ϕ, {a}, {b, c}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = c, f(b) = b, f(c) = a. Then f is contra (1, 2)-$\psi^*\alpha$- σ_2 -continuous but not totally (1, 2)-$\psi^*\alpha$-σ_2-continuous, since {a} and {a, b} are σ_2-open sets in (Y, σ_1, σ_2) but f^{-1}({a}) = {c} and f^{-1}({a, b}) = {b, c} are not (1, 2)-$\psi^*\alpha$-open in (X, τ_1, τ_2).

Theorem 7.5.6 Every strongly (i, j)-$\psi^*\alpha$-σ_k-continuous map is a contra (i, j)-$\psi^*\alpha$-σ_k-continuous map but not conversely.

Proof: Follows from the **Definition 7.4.5** and **Definition 7.5.1**

Example 7.5.7 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, {a, c}, X}, τ_2 = {ϕ, {a}, {b, c}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = c, f(b) = b, f(c) = a. Then f is contra (1, 2)-$\psi^*\alpha$-σ_2-continuous but not strongly (1, 2)-$\psi^*\alpha$-σ_2-continuous, since {a, b} is σ_2-subset in (Y, σ_1, σ_2) but f^{-1}({a, b}) = {b, c} is not (1, 2)-$\psi^*\alpha$-open in (X, τ_1, τ_2).

Remark 7.5.8 The following examples show that the (i, j)-$\psi^*\alpha$-σ_k-continuous maps and contra (i, j)-$\psi^*\alpha$-σ_k-continuous maps are independent.

Example 7.5.9 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a, b}, X}, τ_2 = {ϕ, {a}, X}, σ_1 = {ϕ, {a, c}, Y} and σ_2 = {ϕ, {a}, {a, c}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) = c, f(c) = a. Then f is contra (1, 2)-$\psi^*\alpha$-σ_2-continuous but not (1, 2)-$\psi^*\alpha$-σ_2-continuous, since {b} and {b, c} are σ_2-closed sets in (Y, σ_1, σ_2) but f^{-1}({b}) = {a} and f^{-1}({b, c}) = {a, b} are not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Example 7.5.10 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, X}, τ_2 = {ϕ, {a}, {a, b}, {a, c}, X}, σ_1 = {ϕ, {a, b}, Y} and σ_2 = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be the identity map. Then f is (1, 2)-$\psi^*\alpha$-σ_2-continuous but not contra (1, 2)-$\psi^*\alpha$-σ_2-continuous, since {c} and {b, c} are σ_2-closed sets in (Y, σ_1, σ_2) but f^{-1}({c}) = {c} and f^{-1}({b, c}) = {b, c} are not (1, 2)-$\psi^*\alpha$-open in (X, τ_1, τ_2).

Remark 7.5.11 The following examples show that quasi (i, j)-$\psi^*\alpha$-continuous maps and contra (i, j)-$\psi^*\alpha$-σ_k-continuous maps are independent.

Example 7.5.12 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, {a, c}, X}, τ_2 = {ϕ, {a}, {b, c}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = c, f(b) = b, f(c) = a. Then f is contra (1, 2)-$\psi^*\alpha$-σ_2-continuous but not quasi (1, 2)-$\psi^*\alpha$-continuous since {b} and {b, c} are (1, 2)-$\psi^*\alpha$-closed sets in (Y, σ_1, σ_2) but f^{-1}({b}) = {b} and f^{-1}({b, c}) = {a, b} are not τ_2-closed sets in (X, τ_1, τ_2).

Example 7.5.13 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, X}, τ_2 = {ϕ, {a}, {b}, {a, b}, {a, c}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {a}, {a, b}, {a, c}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be the identity map. Then f is quasi (1, 2)-$\psi^*\alpha$-continuous but not contra (1, 2)-$\psi^*\alpha$-σ_2-continuous, since {c} and {b, c} are σ_2-closed sets in (Y, σ_1, σ_2) but f^{-1}({c}) = {c} and f^{-1}({b, c}) = {b, c} are not (1, 2)-$\psi^*\alpha$-open in (X, τ_1, τ_2).

Remark 7.5.14 The composition of two contra (i, j)-$\psi^*\alpha$-σ_k-continuous maps need not be a contra (i, j)-$\psi^*\alpha$-σ_k-continuous map as seen from the following example.

Example 7.5.15 Let X = Y = Z = {a, b, c}, τ_1 = {ϕ, {a},{a, b}, {a, c}, X}, τ_2 = {ϕ, {a}, {b, c}, X}, σ_1 = {ϕ, {a}, Y}, σ_2 = {ϕ, {a}, {a, b}, Y}, η_1 = {ϕ, {b}, {a, b}, Z} and η_2 = {ϕ, {a, b}, Z}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = c, f(b) = b, f(c) = a and g : (Y, σ_1, σ_2) → (Z, η_1, η_2) be a map defined by g(a) = c, g(b) = a, g(c) = b. Then the map f is contra (1, 2)-$\psi^*\alpha$-σ_2-continuous and g is contra (1, 2)-$\psi^*\alpha$-η_2-continuous but their composition g∘f : (X, τ_1, τ_2) → (Z, η_1, η_2) is not a contra (1, 2) $\psi^*\alpha$-η_2-continuous map, since {c} is η_2-closed in (Z, η_1, η_2) but $(g\circ f)^{-1}$({c}) = {c} is not (1, 2)-$\psi^*\alpha$-open in (X, τ_1, τ_2).

Proposition 7.5.16 Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a perfectly (i, j)-$\psi^*\alpha$-continuous map and g : (Y, σ_1, σ_2) → (Z, η_1, η_2) be a contra (i, j)-$\psi^*\alpha$-η_k-continuous map. Then g∘f : (X, τ_1, τ_2) → (Z, η_1, η_2) is a totally (i, j)-$\psi^*\alpha$-η_k-continuous map.

Proof: Let U be η_k-open set in (Z, η_1, η_2). Since g is contra (i, j)-$\psi^*\alpha$-η_k-continuous, g^{-1}(U) is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since f is perfectly (i, j)-$\psi^*\alpha$-continuous,

$(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is τ_j-clopen in (X, τ_1, τ_2). Since every τ_j-closed set is (i, j)-$\psi^*\alpha$-closed and τ_j-open set is (i, j)-$\psi^*\alpha$-open, $g\circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is a totally (i, j)-$\psi^*\alpha$-η_k-continuous map.

7.6 (i, j)-$\psi^*\alpha$-irresolute maps

In this section (i, j)-$\psi^*\alpha$-irresolute maps are introduced and some of their properties are analyzed.

Definition 7.6.1 A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called **(i, j)-$\psi^*\alpha$-irresolute** if $f^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2) for every (i, j)-$\psi^*\alpha$-closed set V in (Y, σ_1, σ_2), where i, j = 1, 2 and $i \neq j$.

Example 7.6.2 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a, b}, X}, τ_2 = {ϕ, {a}, {a, b}, X}, σ_1 = {ϕ, {a}, {a, b}, Y} and σ_2 = {ϕ, {a}, {a, b}, {a, c}, Y}. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity map. Then f is (1, 2)-$\psi^*\alpha$-irresolute.

Theorem 7.6.3 A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an (i, j)-$\psi^*\alpha$-irresolute if and only if the inverse image of every (i, j)-$\psi^*\alpha$-open set in (Y, σ_1, σ_2) is an (i, j)-$\psi^*\alpha$-open in (X, τ_1, τ_2), where i, j = 1, 2 and $i \neq j$.

Proof: Follows from **Definition 7.6.1**

Proposition 7.6.4 Every quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous) map is an (i, j)-$\psi^*\alpha$- irresolute map but not conversely.

Proof: Assume that $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous). Let V be an (i, j)-$\psi^*\alpha$-closed set in (Y, σ_1, σ_2). Then $f^{-1}(V)$ is τ_j-closed (resp. τ_j-clopen) in (X, τ_1, τ_2). Since every τ_j-closed set is (i, j)-$\psi^*\alpha$-closed, $f^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence f is an (i, j)-$\psi^*\alpha$-irresolute map.

Example 7.6.5 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, {a, c}, X}, τ_2 = {ϕ, {a}, X}, σ_1 = {ϕ, {a}, {a, b}, Y} and σ_2 = {ϕ, {a}, {a, b}, {a, c}, Y}. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity map. Then f is (1, 2)-$\psi^*\alpha$-irresolute but not quasi (1, 2)-$\psi^*\alpha$-continuous and not perfectly (1, 2)-$\psi^*\alpha$-continuous, since {b} and {c} are (1, 2)-$\psi^*\alpha$-closed sets in (Y, σ_1, σ_2) but f^{-1}({b}) = {b} and f^{-1}({c}) = {c} are not τ_2 -closed in (X, τ_1, τ_2).

Proposition 7.6.6 Every (i, j)-$\psi^*\alpha$-irresolute map is an (i, j)-$\psi^*\alpha$-σ_k-continuous map but not conversely.

Proof: Assume that f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is an (i, j)-$\psi^*\alpha$-irresolute map. Let V be a σ_k-closed set in (Y, σ_1, σ_2). Since every σ_k-closed set is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2) and f is (i, j)-$\psi^*\alpha$-irresolute, f^{-1}(V) is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence f is a (i, j)-$\psi^*\alpha$-σ_k-continuous map.

Example 7.6.7 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, X}, τ_2 = {ϕ, {a},{a, b}, {a, c}, X}, σ_1 = {ϕ, {a, b}, Y} and σ_2 = {ϕ, {a}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be the identity map. Then f is (1, 2)-$\psi^*\alpha$-σ_2-continuous but not (1, 2)-$\psi^*\alpha$-irresolute, since {a, c} is (1, 2)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2) but f^{-1}({a, c}) = {a, c}is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2)

Proposition 7.6.8 The following examples show that (i, j)-$\psi^*\alpha$-irresolute maps and contra (i, j)-$\psi^*\alpha$-σ_k-continuous maps are independent.

Example 7.6.9 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {b}, {a, b}, {a, c}, X}, τ_2 = {ϕ, {a}, X}, σ_1 = {ϕ, {a}, {a, b}, Y} and σ_2 = {ϕ, {a}, {a, b}, {a, c}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = a, f(b) = b, f(c) = c. Then f is (1, 2)-$\psi^*\alpha$-irresolute but not contra (1, 2)-$\psi^*\alpha$-σ_2-continuous, since {b},{c} and {b, c}are σ_2-closed in (Y, σ_1, σ_2) but f^{-1}({b}) = {b}, f^{-1}({c}) = {c}and f^{-1}({b, c}) = {b, c}are not (1, 2)-$\psi^*\alpha$-open in (X, τ_1, τ_2).

Example 7.6.10 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, {a, c}, X}, τ_2 = {ϕ, {a}, {b, c}, X}, σ_1 = {ϕ, {a}, Y} and σ_2 = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = c, f(b) = b, f(c) = a. Then f is contra (1, 2)-$\psi^*\alpha$-σ_2-continuous but not (1, 2)-$\psi^*\alpha$-irresolute, since {c} and {b, c} are (1, 2)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2) but f^{-1}({c}) = {a} and f^{-1}({b, c}) = {a, b}are not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Proposition 7.6.11 If f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is (i, j)-$\psi^*\alpha$-irresolute then for every subset A of (X, τ_1, τ_2) such that f(A) is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2), f[(i, j)-$\psi^*\alpha$cl(f(A))] ⊆ (i, j)-$\psi^*\alpha$cl(f(A)).

Proof: Let A be a subset of (X, τ_1, τ_2) such that f(A) is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since f is (i, j)-$\psi^*\alpha$-irresolute, f^{-1}[(i, j)-$\psi^*\alpha$cl(f(A))] is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Now $A \subseteq f^{-1}[f(A)] \subseteq f^{-1}$[(i, j)-$\psi^*\alpha$cl(f(A))]. Therefore (i, j)-$\psi^*\alpha$cl(A) $\subseteq f^{-1}$[(i, j)-$\psi^*\alpha$cl(f(A))] and hence f[(i, j)-$\psi^*\alpha$cl(A)] $\subseteq$ f[f^{-1}((i, j)-$\psi^*\alpha$cl(f(A)))] = (i, j)-$\psi^*\alpha$cl(f(A)).

Proposition 7.6.12 If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-$\psi^*\alpha$-irresolute then for every (i, j)-$\psi^*\alpha$-closed set B in (Y, σ_1, σ_2), (i, j)-$\psi^*\alpha$cl(f^{-1}(B)) $\subseteq f^{-1}$[(i, j)-$\psi^*\alpha$cl(B)].

Proof: Let B be an (i, j)-$\psi^*\alpha$-closed set in (Y, σ_1, σ_2). Then (i, j)-$\psi^*\alpha$cl(B) is an (i, j)-$\psi^*\alpha$-closed set in (Y, σ_1, σ_2). Since f is (i, j)-$\psi^*\alpha$- irresolute, f^{-1}[(i, j)-$\psi^*\alpha$cl(B)] is an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2). Since B $\subseteq$ (i, j)-$\psi^*\alpha$cl(B), $f^{-1}(B) \subseteq f^{-1}$[(i, j)-$\psi^*\alpha$cl(B)]. Therefore (i, j)-$\psi^*\alpha$cl(f^{-1}(B)) $\subseteq f^{-1}$((i, j)-$\psi^*\alpha$cl(B)).

Proposition 7.6.13 If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is an (i, j)-$\psi^*\alpha$-irresolute map and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ is an (i, j)-$\psi^*\alpha$-η_k-continuous map, then $g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is an (i, j)-$\psi^*\alpha$-η_k-continuous map.

Proof: Let V be any η_k-closed set in (Z, η_1, η_2). Since g is (i, j)-$\psi^*\alpha$-η_k-continuous, $g^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since f is (i, j)-$\psi^*\alpha$-irresolute, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence $g \circ f$ is an (i, j)-$\psi^*\alpha$-η_k-continuous map.

Proposition 7.6.14 Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be (i, j)-$\psi^*\alpha$-irresolute and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ be quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous). Then $g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is an (i, j)-$\psi^*\alpha$-irresolute map.

Proof: Let V be any (i, j)-$\psi^*\alpha$-closed set in (Z, η_1, η_2). Since g is quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous), $g^{-1}(V)$ is σ_j-closed (resp. σ_j-clopen) in (Y, σ_1, σ_2). Since every σ_j-closed set is (i, j)-$\psi^*\alpha$-closed, $g^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed. Since f is (i, j)-$\psi^*\alpha$-irresolute, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence $g \circ f$ is an (i, j)-$\psi^*\alpha$-irresolute map.

Proposition 7.6.15 Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an (i, j)-$\psi^*\alpha$-irresolute map and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ be a σ_j-η_k-continuous (resp. σ_j -α-η_k-continuous) map. Then $g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is an (i, j)-$\psi^*\alpha$-η_k-continuous map.

Proof: Let V be any η_k-closed set in (Z, η_1, η_2). Since g is σ_j-η_k-continuous (resp. σ_j -α-η_k-continuous), $g^{-1}(V)$ is σ_j-closed (resp. σ_j-α-closed) in (Y, σ_1, σ_2). Since every σ_j-closed (resp. σ_j-α-closed) set is (i, j)-$\psi^*\alpha$-closed, $g^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since f is (i, j)-$\psi^*\alpha$-irresolute, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is an (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence $g \circ f$ is (i, j)-$\psi^*\alpha$-η_k-continuous map.

Proposition 7.6.16 If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ are (i, j)-$\psi^*\alpha$-irresolute maps, then $g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is also an (i, j)-$\psi^*\alpha$-irresolute map .

Proof: Let V be any (i, j)-$\psi^*\alpha$-closed set in (Z, η_1, η_2). Since g is (i, j)-$\psi^*\alpha$-irresolute, $g^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since f is (i, j)-$\psi^*\alpha$-irresolute, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence $g \circ f$ is an (i, j)-$\psi^*\alpha$-irresolute map.

7.7 Contra (i, j)-$\psi^*\alpha$-irresolute maps

In this section, contra (i, j)-$\psi^*\alpha$-irresolute maps are introduced and their basic properties are studied.

Definition 7.7.1 A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called **contra (i, j)-$\psi^*\alpha$-irresolute** if $f^{-1}(V)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2) for every (i, j)-$\psi^*\alpha$-open set V in (Y, σ_1, σ_2), where i, j = 1, 2 and $i \neq j$.

Example 7.7.2 Let X = Y = {a, b, c}, $\tau_1 = \{\phi, \{a, b\}, X\}$, $\tau_2 = \{\phi, \{a\}, \{a, b\}, X\}$, $\sigma_1 = \{\phi, \{a\},\{b\}, \{a, b\}, Y\}$ and $\sigma_2 = \{\phi, \{a\}, \{b, c\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a map defined by f(a) = c, f(b) = a, f(c) = b. Then f is contra (1, 2)-$\psi^*\alpha$-irresolute.

Theorem 7.7.3 Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a map. Then the following statements are equivalent.

(i) f is contra (i, j)-$\psi^*\alpha$-irresolute.

(ii) The inverse image of every (i, j)-$\psi^*\alpha$-closed set in (Y, σ_1, σ_2) is an (i, j)-$\psi^*\alpha$-open in (X, τ_1, τ_2), where i, j = 1, 2 and i ≠ j.

Proof : Follows from the **Definition 7.7.1**

Proposition 7.7.4 Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a perfectly (i, j)-$\psi^*\alpha$-continuous map, then f is a contra (i, j)-$\psi^*\alpha$-irresolute map but not conversely.

Proof: Let U be an (i, j)-$\psi^*\alpha$-open set in (Y, σ_1, σ_2). Since f is perfectly (i, j)-$\psi^*\alpha$-continuous, $f^{-1}(U)$ is τ_j-clopen in (X, τ_1, τ_2). Since every τ_j-closed set is (i, j)-$\psi^*\alpha$-closed, $f^{-1}(U)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence f is an contra (i, j)-$\psi^*\alpha$-irresolute map.

Example 7.7.5 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a, b}, X}, τ_2 = {ϕ, {a}, {a, b}, X}, σ_1 = {ϕ, {a},{b}, {a, b}, Y} and σ_2 = {ϕ, {a}, {b, c}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = c, f(b) = a, f(c) = b. Then f is contra (1, 2)-$\psi^*\alpha$-irresolute but not perfectly (1, 2)-$\psi^*\alpha$-continuous, since {a} is (1, 2)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2) but f^{-1}({a}) = {b} is not τ_2-clopen in (X, τ_1, τ_2).

Proposition 7.7.6 If f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) is a strongly (i, j)-$\psi^*\alpha$-σ_k-continuous map, then f is a contra (i, j)-$\psi^*\alpha$-irresolute map but not conversely.

Proof: Let U be any (i, j)-$\psi^*\alpha$-open set in (Y, σ_1, σ_2). Since f is strongly (i, j)-$\psi^*\alpha$-σ_k-continuous, $f^{-1}(U)$ is (i, j)-$\psi^*\alpha$-clopen in (X, τ_1, τ_2). Therefore $f^{-1}(U)$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence f is a contra (i, j)-$\psi^*\alpha$-irresolute map.

Proposition 7.7.7 The following examples show that (i, j)-$\psi^*\alpha$-irresolute maps and contra (i, j)-$\psi^*\alpha$-irresolute maps are independent.

Example 7.7.8 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a, b}, X}, τ_2 = {ϕ, {a}, {a, b}, X}, σ_1 = {ϕ, {a}, {a, b}, Y} and σ_2 = {ϕ, {a}, {a, b}, {a, c}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be the identity map. Then f is (1, 2)-$\psi^*\alpha$-irresolute but not contra (1, 2)-$\psi^*\alpha$-irresolute, since {a} and {a, b} are (1, 2)-$\psi^*\alpha$-open in (Y, σ_1, σ_2) but f^{-1}({a}) = {a} and f^{-1}({a b}) = {a, b}are not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Example 7.7.9 Let X = Y = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, X}, τ_2 = {ϕ, {a},{b}, {a, b}, X}, σ_1 = {ϕ, {a}, {a, b},{a, c}, Y} and σ_2 = {ϕ, {a}, {b, c}, Y}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = b, f(b) =c, f(c) =a. Then f is contra (1, 2)-$\psi^*\alpha$-irresolute but not (1, 2)-$\psi^*\alpha$-irresolute, since {b, c} is (1, 2)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2) but f^{-1}({b, c}) = {a, b} is not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Proposition 7.7.10 Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be an (i, j)-$\psi^*\alpha$-irresolute map and g : (Y, σ_1, σ_2) → (Z, η_1, η_2) be a contra (i, j)-$\psi^*\alpha$-irresolute map. Then g∘f : (X, τ_1, τ_2) → (Z, η_1, η_2) is a contra (i, j)-$\psi^*\alpha$-irresolute map.

Proof: Let U be any (i, j)-$\psi^*\alpha$-open set in (Z, η_1, η_2). Since g is contra (i, j)-$\psi^*\alpha$-irresolute, g^{-1}(U) is (i, j)-$\psi^*\alpha$-closed in (Y, σ_1, σ_2). Since f is (i, j)-$\psi^*\alpha$-irresolute, $(g∘f)^{-1}$(U) = $f^{-1}(g^{-1}(U))$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence g∘f is a contra (i, j)-$\psi^*\alpha$-irresolute map.

Proposition 7.7.11 Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be contra (i, j)-$\psi^*\alpha$-irresolute g : (Y, σ_1, σ_2) → (Z, η_1, η_2) be quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous). Then g∘f : (X, τ_1, τ_2) → (Z, η_1, η_2) is contra (i, j)-$\psi^*\alpha$-irresolute.

Proof: Let U be any (i, j)-$\psi^*\alpha$-open set in (Z, η_1, η_2). Since g is quasi (i, j)-$\psi^*\alpha$-continuous (resp. perfectly (i, j)-$\psi^*\alpha$-continuous), g^{-1}(U) is σ_j-open (resp. σ_j-clopen) in (Y, σ_1, σ_2). Since every σ_j-open set is (i, j)-$\psi^*\alpha$-open, g^{-1}(U) is (i, j)-$\psi^*\alpha$-open. Since f is contra (i, j)-$\psi^*\alpha$-irresolute, $(g∘f)^{-1}$(U) = $f^{-1}(g^{-1}(U))$ is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Hence g∘f is contra (i, j)-$\psi^*\alpha$-irresolute.

Remark 7.7.12 The composition of two contra (i, j)-$\psi^*\alpha$-irresolute maps need not be a contra (i, j)-$\psi^*\alpha$-irresolute map as seen from the following example.

Example 7.7.13 Let X = Y = Z = {a, b, c}, τ_1 = {ϕ, {a},{a, b}, X}, τ_2 = {ϕ, {a}, {a, b}, X}, σ_1 = {ϕ, {a}, Y}, σ_2 = {ϕ, {a}, {a, b}, {a, c}, Y}, η_1 = {ϕ, {a}, {a, b}, Z} and η_2 = {ϕ, {a}, {b}, {a, b}, Z}. Let f : (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a map defined by f(a) = c, f(b) = b, f(c) = a and g : (Y, σ_1, σ_2) → (Z, η_1, η_2) be a map defined by g(a) = c, g(b) = b, g(c) = a. Then the maps f and g are contra (1, 2)-$\psi^*\alpha$-irresolute maps but their composition g∘f : (X, τ_1, τ_2) → (Z, η_1, η_2) is not a contra (1, 2) $\psi^*\alpha$-irresolute map, since {a} and {a, b}are (1, 2)-$\psi^*\alpha$-open in (Z, η_1, η_2) but $(g\circ f)^{-1}(\{a\}) = \{a\}$and $(g\circ f)^{-1}(\{a, b\}) = \{a, b\}$ are not (1, 2)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).

Remark 7.7.12 The above observations are depicted in the following diagram.