Chromatic Polynomial for Lexicographic Product of Some Graphs

Uma, M

(14PMA017)

Thesis Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women

Coimbatore-641 043

In Partial Fulfilment of the Requirements for the

Degree of Master of Science in Mathematics

April 2016

Chromatic Polynomial for Lexicographic Product of Some Graphs

Uma, M

(14PMA017)

Thesis submitted to

Avinashilingam Institute for Home Science and Higher Education for Women

Coimbatore-641 043

In Partial Fulfilment of the Requirements for the

Degree of Master of Science in Mathematics

April 2016

Signature of the Head of the Department

Signature of the Supervisor

Acknowledgement

ACKNOWLEDGEMENT

First and foremost, I extremely thankful to the **LORD ALMIGHTY** for giving me the power to believe in myself and pursue my dreams.

I takes immense pleasure in pleasure in rendering my heartfelt to **Dr.(Thiru) P.R.KRISHNAKUMAR,** Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for providing the conducive infrastructure for the conduct of the research study.

I would like to thank **Dr. (Tmt.) PREMAVATHY VIJAYAN**, Vice Chancellor (i/c), Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the encouragement and for providing the opportunity to develop and establish my skills.

I extends her heartfelt thanks to **Dr. (Tmt.)A.VENMATHI,** Registrar (i/c) and **Dr.(Tmt.) GOWRI RAMAKRISHNAN**, Former registrar, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the encouragement given by her during the investigation.

I would like to express my thank **Hony. Col. Dr. (Tmt.) SAROJA PRABHAKARAN,** Former Vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, and The Director, Hall of Residence, Avinashilingam Education trust institutions- Hostel for Women, Coimbatore, for all the necessary support and guidance towards the completion of the study.

I express my heartfelt thanks to **Dr. (Tmt.) A.PARVATHI,** Professor, Dean, Faculty of Science, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her excellent support, unflinching encouragement and guidance during the course of the investigation.

I express my heartfelt thanks to **Dr. (Tmt.) K.UDAYA CHANDRIKA,** professor, Head of the department of mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her support and guidance during the course of the investigation.

I deeply indebted to my thesis supervisor (**MISS.)R.SANGEETHA,** Teaching Assistant, Department of mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her inspiring guidance, innovative ideas, critical suggestions and constant encouragement throughout the completion of this work.

I would like to express sincere thanks to all the **STAFF MEMBERS OF THE DEPARTMENT OF MATHEMATICS** who were responsible for the good finish of this dissertation.

Words fail to express her deep indebtedness to her **BELOVED PARENTS, LOVING BROTHER, FRIENDS AND ALSO THE GRACEFUL RELATIVES** to their belief in the importance of education. They inspired her with the importance of learning and the clarity of understanding from childhood.

Content

CONTENT

Introduction

Introduction

In graph theory, as in discrete mathematics in general, not only the existence, but also the counting of objects with some given properties is of main interest. To count and to encode the number of structures with given properties, generating functions, formally written as polynomials, are widely used.

Graph polynomials are an important topic of interest to many combinatorialists. The coefficients of a graph polynomial encode various combinatorial properties of a graph, such as the number of independent sets or the number of matchings. There are several graph polynomials that are active areas of combinatorial research such as chromatic polynomials, matchings polynomials, reliability polynomials, Tutte polynomials, clique polynomials and independence polynomials.

Given an arbitrary graph G on n vertices, we can compute a graph polynomial by enumerating the number of occurrences of a particular property. Every graph polynomial is wall defined. A graph cannot have two different polynomials. Here we are interested in chromatic polynomial.

As the name suggests, the chromatic polynomial is originally defined in terms of colorings and therefore by a coloring representation.

In this paper, we investigate the chromatic polynomial of circulant graphs and lexicographic product of two circulant graph.

In chapter I, we provide the necessary background to work on with chromatic polynomial of various graphs. The basic graph theory definitions are defined illustrating examples for each.

In chapter II, chromatic polynomials of graphs such as Complete graph, Path graph, Tree , Cycle graph, Star graph, Wheel graph, Pan graph, Sunlet graph, Sun graph, and Petersen graph are computed using elementary combinatorial techniques. An example has been illustrated for each graph briefly.

Chapter III, the main part of this paper focuses on the chromatic polynomial of focuses on the chromatic polynomial for lexicographic product of graphs such as Ladder graph, Book graph. The lexicographic product of two circulant graphs is a circulant graph. The chromatic polynomial of the so obtained circulant graph is calculated using the basic combinatorial technique.

Review of literature

REVIEW OF LITERATURE

A research on the field of chromatic polynomial for lexicographic product of some graphs was developed by many authors in the last 15 years.

Since the advent of these notions, several authors contributed to the study of these concepts and severed worthwhile research papers have been published. We proved a brief review of literature in some of the important articles published that are related to this topic.

The invitation of the study of “On the Theory of the Matching Polynomial” was done by C.D.Godsil, I.Gutman in the year of 1981.

The study of “Real Graph Polynomial, Progress in Graph Theory” was done by C.D.Gobsil in the year of 1984.

The invitation of the study of “Some Problems on Chromatic Polynomial” was done by G.L.Chia in the year of 1997.

In this paper G.L.Chia has described some unsolved problems, on chromatic polynomials along with a brief account of their progresses. Most of these problems are concerned with graphs that are uniquely determined by their chromatic polynomial. It is hoped that the solutions to these problems will uncover new methods and facts concerning chromatic polynomial.

The study of “Algebraic Methods of Chromatic Polynomial” was done by N.L.Biggs, M.H.Klin, and P.Reinfeld in the year of 2004.

In this paper they have discussed the chromatic polynomial of a 'bracelet', when the base graph is a complete graph K_b and arbitray links L between the consecutive copies are allowed. If there are n copies of the base graph the resulting graph will be denoted by $L_n(b)$, we show that the chromatic polynomial of $L_n(b)$ can be written in the form

$$P(L_n(b); K) = \sum_{e=0} \sum_{\pi \vdash l} m(k)\, tr\, (N_L^{\pi})^n$$

Here the notation $\pi \vdash l$ means that π is a partition of l, and $m_{\pi}(k)$ is a polynomial that doesnot depend on the square matrix N_L^{π} has size $\binom{b}{l} n_{\pi}$, where n_{π} is the degree of the representation R^n of syme associated with π.

They have driven an explicit formula for $m_{\pi}(k)$ and described a method for calculating the matrices N_L^{π}. Examples are given; finally, the paper has discussed the application of the theses results to the problem of locating the chromatic zeros.

Chapter - 1

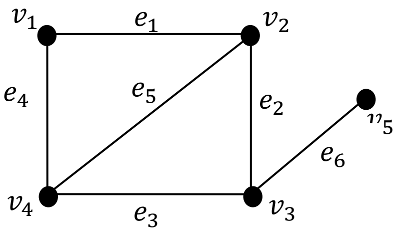
CHAPTER – I

Preliminaries

Definition: 1.1 [4]

A linear *graph* (or simply a graph) $G = < V, E >$ consist a set of objects $V = \{v_1, v_{2,\ldots\ldots} v_n\}$ called vertices and another set $E = \{e_1, e_2, \ldots \ldots \ldots e_n\}$ whose element are called edges such that each e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i,v_j associated with edge e_k are called tha end vertices of e_k .

Example: 1.2

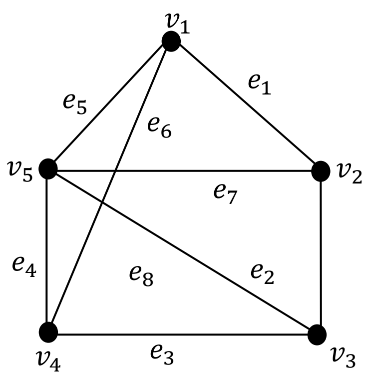


Simple graph with 5

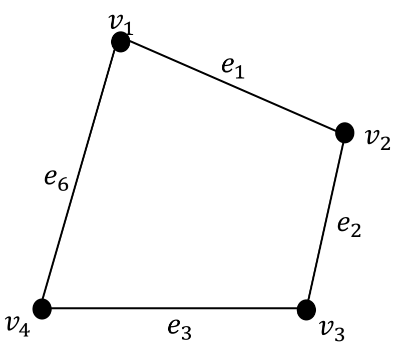
Definition: 1.3 [4]

A graph $G^{'}$ is said to be a *subgraph* of graph G.If all the vertices and all the edges of $G^{'}$are in G and each edge of $G^{'}$ has the same end vertices in $G^{'}$ as in G.

Example: 1.4



G graph with 5 vertices

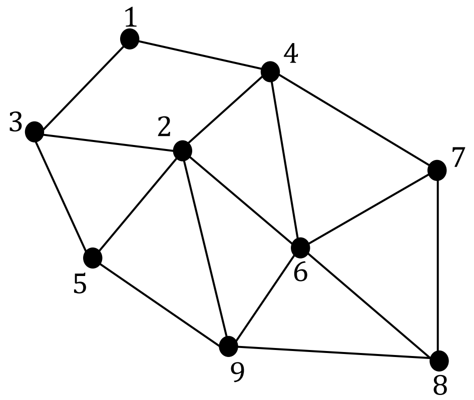


$G^{'}$subgraph of G

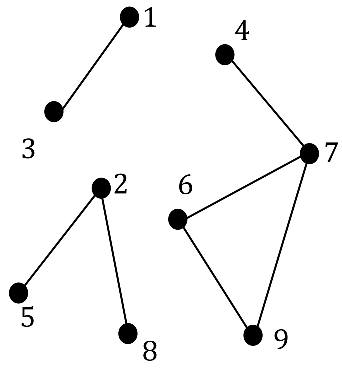
Definition: 1.5 [4]

H is an induced subgraph of G when every vertex of H is a vertex in G and every edge in G with both end points in H is an edge in H.

Example: 1.6



G-graph

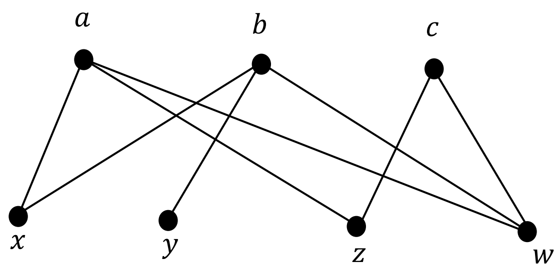


H-induced subgraphs of G

Definition: 1.7 [4]

A bipartite graph, also called a *bigraph* is a set of graph vertices decomposed into two disjoint sets such that no two vertices with in the same set are adjacent.

Example: 1.8

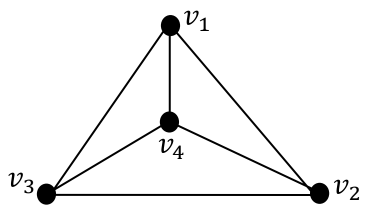


Bipartite graph with 8 vertices

Definition: 1.9 [1]

A graph in which any two vertices are adjacent is called a *complete graph* a complete graph. Any complete graph with n vertices is called by k_n.

Example: 1.10

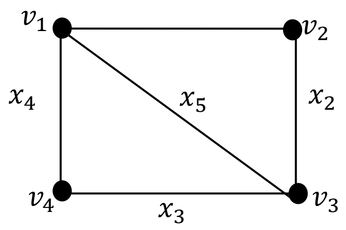


Complete graph with 4 vertices

Definition: 1.11 [4]

A *walk* of graph G is an alternating sequence of vertex and edge $v_0 x_1 v_1 x_{2,\ldots\ldots} v_{n-1} x_n v_n$ begining ending with vertex such that each edge x_i is incident with v_{i-1} and v_i.

Example: 1.12

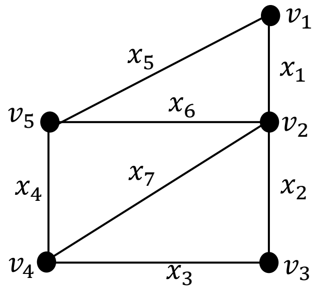


Walk-$v_1 x_1 v_2 x_2 v_3 x_5 v_1 x_4 v_4 x_3 v_3$

Definition: 1.13 [4]

A walk is called a *path* if all its vertices are distinct.

Example: 1.14

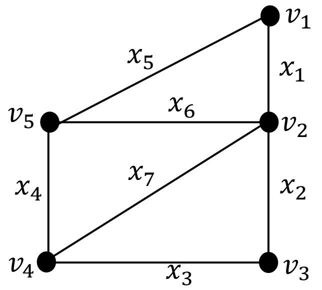


Path- $v_1 x_1 v_2 x_7 v_4 x_4 v_5$

Definition: 1.15 [4]

A walk is called a *trail* if all its edges are distinct.

Example: 1.16

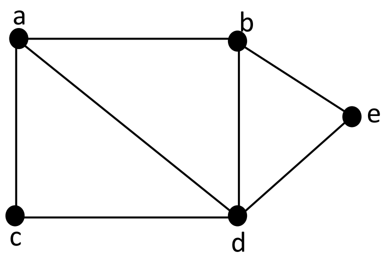


Trail-$v_1 x_1 v_2 x_7 v_4 x_3 v_3 x_2 v_2 x_6 v_5$

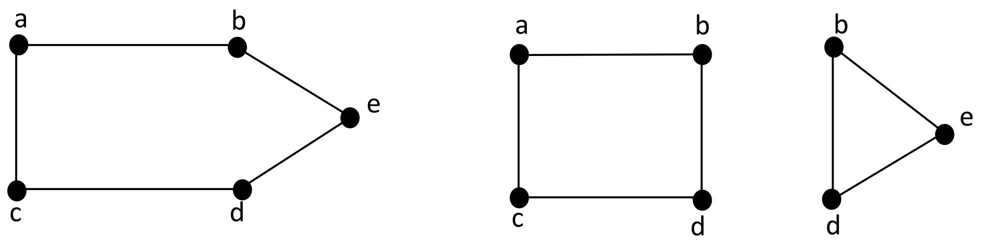
Definition: 1.17 [8]

A cycle is defined as a closed walk with no repetitions of vertices and edges other than the repetition of the starting and ending vertex. A closed path is called a *cycle*.

Example: 1.18



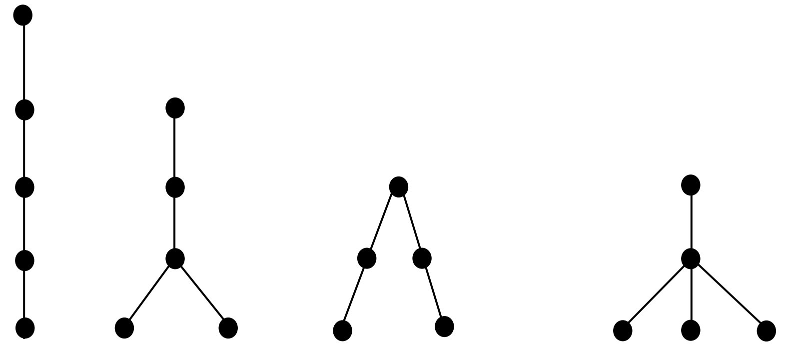
In the above graph abdca, abedca, bcd are cycles.



Definition: 1.19 [8]

An acyclic graph is one that contains no cycle. A *tree* is a connected acyclic graph.

Example: 1.20

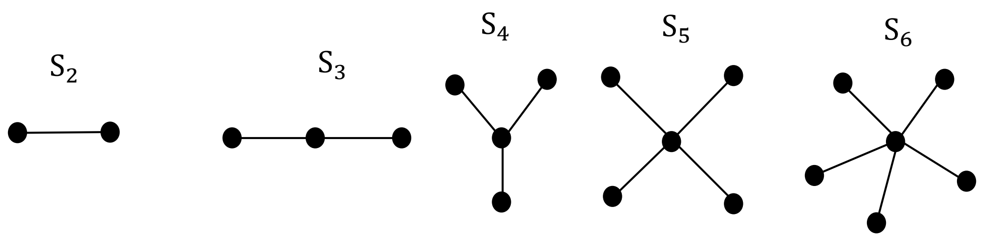


Tree on 5 vertices

Definition: 1.21 [8]

Suppose we start with n vertices, choose one special vertex and then draw edges from the special vertex to every other vertex. The graph we would obtain is called the *star* on n vertices S_n.

Example: 1.22

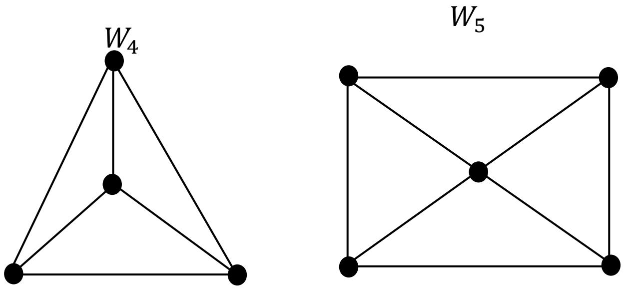


Star graph with 2, 3,4,5,6 vertices

Definition: 1.23 [8]

A *wheel graph* W_n of order *n*, sometime is a graph that contains a cycle of order *n*-1 and for which every graph vertex in the cycle is connected to one other graph vertex (which is known as the hub). The edges of a wheel which include the hub are called spokes. The wheel W_n can be defined as the graph $K_1 + C_{n-1}$, where K_1 is the singleton graph and $C_n$is the cycle graph. A wheel graph $W_n$is simply called as a *n*-wheel.

Example: 1.24 [8]

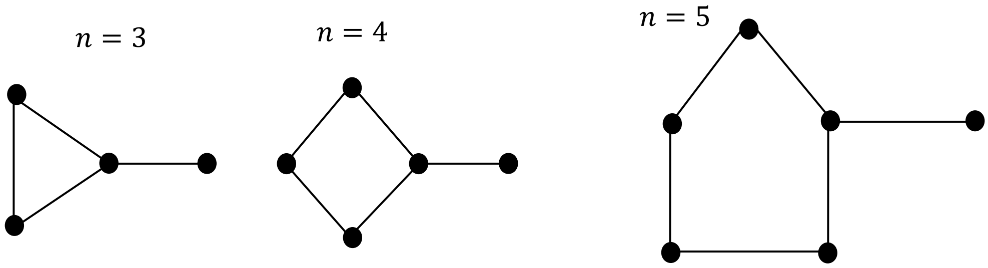


Wheel graph with 4, 5 vertices

Definition: 1.25 [8]

The *n-pan graph* is the graph obtained by joining a cycle graph C_n to a single ton graph K_1 with a bridge.

Example: 1.26

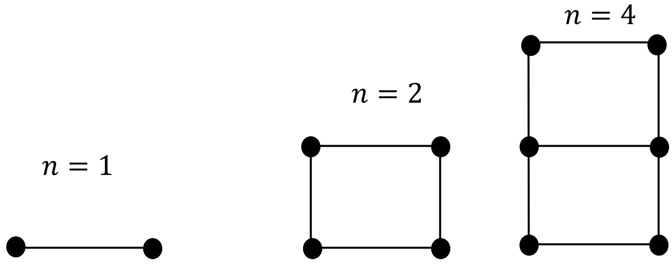


Pan graph with 3, 4, 5 vertices

Definition: 1.27 [8]

The *n- ladder graph* can be defined as $P_2 \circ P_n$, where P_n is a path graph.

Example: 1.28

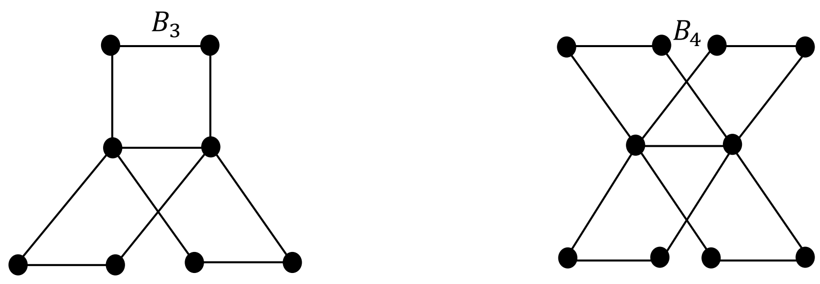


Ladder graph with 2, 4, 6 vertices

Definition: 1.29 [8]

The *m-book graph* is defined as the graph Cartesian product $S_{m+1} \times P_2$, where S_m is a star graph and P_2 is the path graph on two nodes.

Example: 1.30

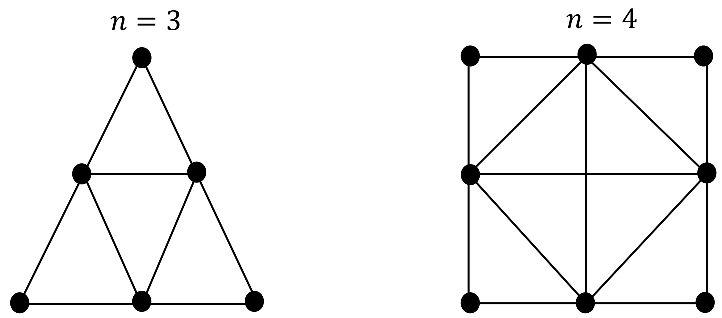


Book graph with 8, 10 vertices

Definition: 1.31 [8]

The *n-sungraph* as a graph on 2n vertices consisting of a central complete graph K_n with an outer ring of n vertices , each of which is joined to both endpoints of the closed outer edge of the central core.

Example: 1.32

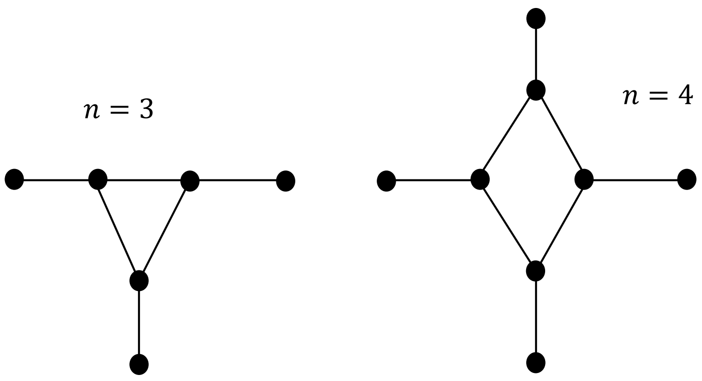


Sun graph with 6, 8 vertices

Definition: 1.33 [8]

The *n-sunlet graph* is the graph on 2n vertices obtained by attaching n pendant edges to a cycle graph C_n.

Example: 1.34



Sunlet graph with 6,8 vertices

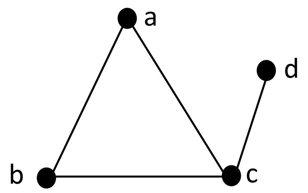
Definition: 1.35 [7]

The *falling factorial* $(x)_n$ is defined dy $(x)_n = x(x-1)\ldots\ldots.(x-(n-1))$ for $n \geq 0$. It is also known as the lower factorial falling factorial power or factorial power.

Definition: 1.36 [7]

An edge of a graph is said to be *pendant* if one of its vertices is a pendant vertex.

Example: 1.37

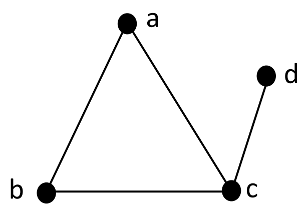


cd pendant edge

Definition: 1.38 [7]

A vertex of a graph is said to be *pendant* if its neighbourhood contains exactly one vertex.

Example: 1.39

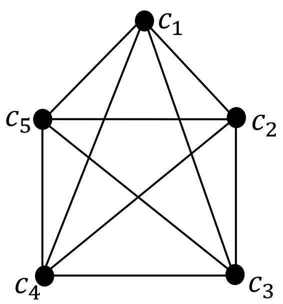


d pendant vertex

Definition: 1.40 [8]

A *proper coloring* of a graph is an assignment of colors to the vertices of the graph so that no two adjacent vertices have the same color.

Example: 1.41



c_1-color 1, c_2-color 2,

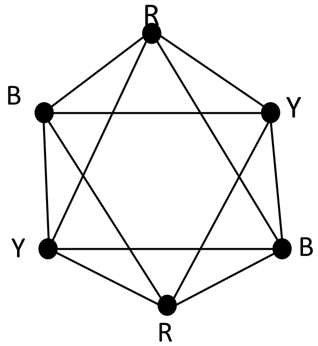
c_3-color 3, c_4-color 4,

c_5-color 5.

Definition: 1.42 [8]

A *vertex coloring* of a graph G is coloring of vertices of G such that adjacent vertices receive different colors.

Example: 1.43



R- red ,

B- blue ,

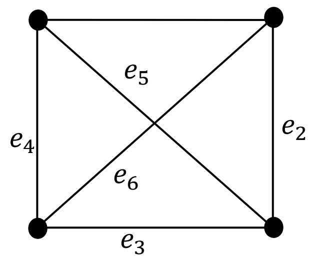
Y- yellow

Vertex coloring of a graph with 6 vertices

Definition: 1.44 [8]

An *edge coloring* of graph G is coloring of edges of G such that adjacent edges receive different colors.

Example: 1.45



e_1-color 1, e_2-color 2,

e_3-color 3, e_4-color 4,

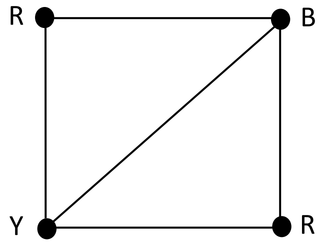
e_5-color 5, e_6-color 6.

Edge coloring of a graph with 4 vertices

Definition: 1.46 [8]

The least number of colors required to color the vertices of a graph so that the adjacent vertices do not have the same color is called the *chromatic number*. The chromatic number of a graph is denoted by the symbol λ.

Example: 1.47



R- red ,

B- blue ,

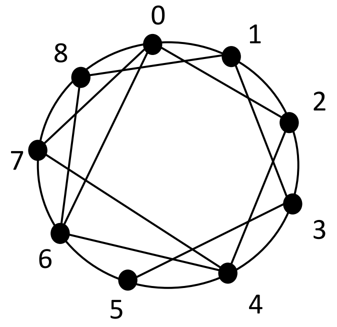
Y- yellow

Chromatic number of the graph $G = \lambda(G)$ =3

Definition: 1.48 [8]

Given a set S$\subseteq \left\{1,2, \dots \dots \dots \left[\frac{n}{2}\right]\right\}$ the *circulant graph* $C_{n,s}$is the graph with vertex set $V(G) = Z_n$and edge set $E(G) = \{uv: |u - v|_n \in s\}$. Where $\{x / n = min(|x|, n - |x|)\}$ is the circular distance modulo n.

Example: 1.49

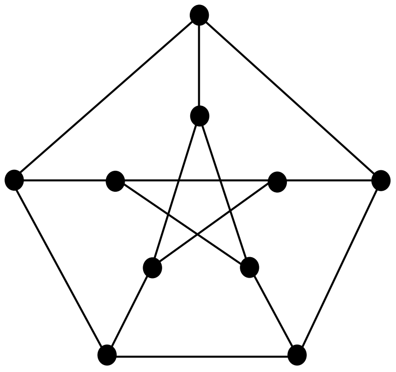


Circulant graph of 9 vertices

Definition: 1.50 [8]

The Petersen graph is a graph with 10 vertices and 15 edges. All of whose notes have degree three.

Example: 1.51



Petersen graph with 10 vertices

Chapter - 2

CHAPTER – II

On the Chromatic Polynomial of Some Graphs

Section 2.1

Preliminaries

Definition: 2.1.1 [3]

The *chromatic polynomial* $P_n(\lambda)$ of a graph with n vertices given the number of ways of properly coloring the graph using λ or fewer colors.

Let c_i be the different ways of properly coloring G using exactly i different colors. Since i colors can be chosen out of λ colors in $\binom{\lambda}{i}$ different ways there are $c_i\binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors.

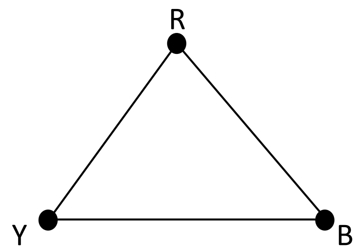
Since i can be any positive integer from 1 to n (it is not possible to use than n colors on n vertices) the chromatic polynomial is a sum of these terms that is

$$P_n(\lambda) = \sum_{i}^{n} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c\frac{\lambda(\lambda-1)}{2!} + \ \dots\dots. + c_n \frac{\lambda(\lambda-1)(\lambda-2)\dots\big(\lambda-(n-1)\big)}{n!}$$

Each c_i has to be evaluated individually for the given graph.

Example: 2.1.2



R- red ,

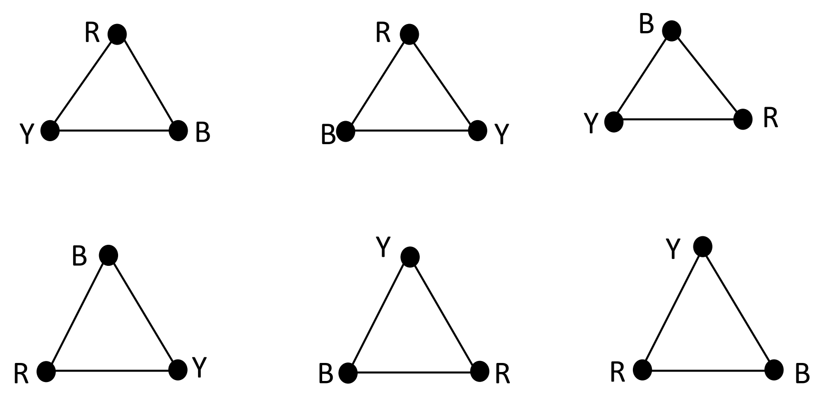
B- blue ,

Y- yellow

c_i is the number of ways of coloring the given graph using *i* colors

Let $c_1 = 0, c_2 = 0$, because given graph is not a proper coloring.

c_3-complete graph of 3 vertices



Keeping R fixed for one vertices and coloring with the remaining two vertices in all possible combination, we obtain 2 as the number of ways of proper coloring, similarly Y, B are fixed also.

Therefore total number of proper coloring with 3 colors using = 3.2=6

$\therefore c_1 = 0, c_2 = 0, c_3 = 3!, n = 3.$

$$P_3(\lambda) = \sum_{1}^{3} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$= (0)\frac{\lambda}{1!} + (0)\frac{\lambda(\lambda-1)}{2!} + (3!)\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$= \lambda(\lambda-1)(\lambda-2)$$

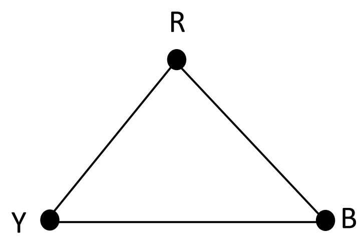
$$= \lambda(\lambda^2 - 3\lambda + 2)$$

$$P_3(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda$$

Section 2.2

Chromatic Polynomial of Some Graphs

Chromatic Polynomial of Complete Graph: 2.2.1



R- red ,

B- blue ,

Y- yellow

K_3-Complete graph of 3 vertices

c_i is the number of ways of coloring the given graph using *i* colors

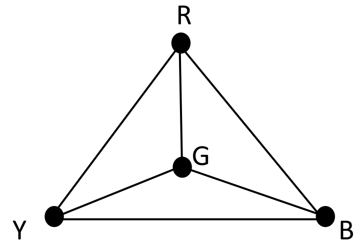
Here $c_1 = 0, c = 0, c_3 = 3!, n = 3$

$$P_3(\lambda) = \sum_{1}^{3} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$= 3! \frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$P_3(\lambda) = \lambda(\lambda-1)(\lambda-2)$$



R- red ,B- blue ,

Y- yellow, R-green

K_4-complete graph of 4 vertices

c_i is the number of ways of coloring the given graph using *i* colors

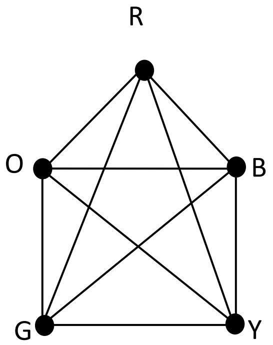
Here $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 4!, n = 4$

$$P_4(\lambda) = \sum_{1}^{4} c_i \binom{\lambda}{i}$$

$$P_4(\lambda) = c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= (0)\frac{\lambda}{1!} + (0)\frac{\lambda(\lambda-1)}{2!} + (0)\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + (4!)\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$P_4(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$



R- red ,B- blue ,
Y- yellow, R-green,
O-orange .

K_5-complete graph of 5 vertices

c_i is the number of ways of coloring the given graph using i colors

Here $c_1 = 0, c = 0, c_3 = 0, c_4 = 0, c_5 = 5!, n = 4.$

$$P_5(\lambda) = \sum_{1}^{5} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

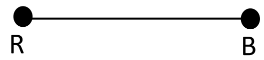
$$= (0)\frac{\lambda}{1!} + (0)\frac{\lambda(\lambda-1)}{2!} + (0)\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + (0)\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 5!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$P_5(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$

Similarly we can find the chromatic polynomial for K_6, K_7 and so

Therefore the general form of complete graph K_n is $\lambda(\lambda-1)(\lambda-2) \dots\dots\dots\dots\dots\dots.(\lambda-(n-1))$.

Chromatic Polynomial of Path Graph: 2.2.2



R- red ,B- blue

P_2-Path of 2 vertices

c_i is the number of ways of coloring the given graph using *i* colors

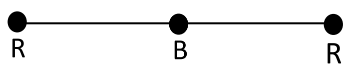
Here $c_1 = 0, c_2 = 2, n = 2.$

$$P_2(\lambda) = \sum_{1}^{2} c_i \binom{\lambda}{i}$$

$$P_2(\lambda) = c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!}$$

$$= (0)\frac{\lambda}{1!} + 2!\frac{\lambda(\lambda-1)}{2!}$$

$P_2(\lambda) = \lambda(\lambda-1)$



R- red ,B- blue

P_3-Path of 3 vertices

c_i is the number of ways of coloring the given graph using *i* colors

Here $c_1 = 0, c_2 = 2!, c_2 = 3!, n = 3.$

$$P_3(\lambda) = \sum_{1}^{3} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

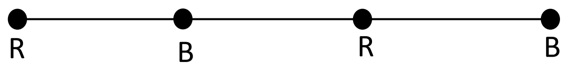
$$= (0)\frac{\lambda}{1!} + 2!\frac{\lambda(\lambda-1)}{2!} + 3!\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$= \lambda(\lambda-1) + \lambda(\lambda-1)(\lambda-2)$$

$$= \lambda(\lambda-1)(1+\lambda-2)$$

$$= \lambda(\lambda-1)(\lambda-1)$$

$$P_3(\lambda) = \lambda(\lambda-1)^2$$



R- red ,B- blue

P_4-Path of 4 vertices

c_i is the number of ways of coloring the given graph using *i* colors

Here $c_1 = 0, c_2 = 2, c_3 = 18,\ c_4 = 4!, n = 4.$

$$P_4(\lambda) = \sum_{1}^{4} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= (0)\frac{\lambda}{1!} + 2\frac{\lambda(\lambda-1)}{2!} + 18\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 4!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= \lambda(\lambda-1) + 3\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

$$= \lambda(\lambda-1)[1 + 3\lambda - 6 + (\lambda-2)(\lambda-3)]$$

$$= \lambda(\lambda-1)[3\lambda - 5 + \lambda^2 - 5\lambda + 6]$$

$$= \lambda(\lambda-1)[\lambda^2 - 2\lambda + 1]$$

$$= \lambda(\lambda - 1)(\lambda - 1)^2$$

$$P_4(\lambda) = \lambda(\lambda - 1)^3$$

Similarly we can find the chromatic polynomial for P_6, P_7 and so on.

Therefore the general form of Path graph P_n is given by $\lambda(\lambda - 1)^{n-1}$.

Chromatic Polynomial of Tree: 2.2.3

R-red, B-blue.

T_2-Tree for 2 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

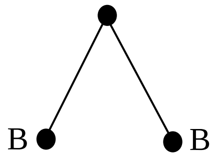
Here $c_1 = 0,\ c_2 = 2, n = 2$.

$$P_2(\lambda) = \sum_1^2 c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda - 1)}{2!}$$

$$= (0)\frac{\lambda}{1!} + 2! \frac{\lambda(\lambda - 1)}{2!}$$

$$P_2(\lambda) = \lambda(\lambda - 1)$$



R-red, B-blue.

T_3-Tree for 3 vertices

c_i is the number of ways of coloring the given graph using i colors.

Here $c_1 = 0, c_2 = 2!, c_3 = 3!, n = 3.$

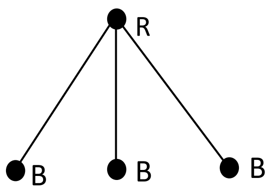
$$P_3(\lambda) = \sum_{1}^{3} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$= (0)\frac{\lambda}{1!} + 2!\frac{\lambda(\lambda-1)}{2!} + 3!\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$= \lambda(\lambda-1)[1+\lambda-2]$$

$$P_3(\lambda) = \lambda(\lambda-1)^2$$



R-red, B-blue.

T_4-Tree for 4 vertices

c_i is the number of ways of coloring the given graph using i colors.

Here $c_1 = 0, c_2 = 2, c = 18, c_4 = 4!, n = 4,$

$$P_4(\lambda) = \sum_{1}^{4} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= (0)\frac{\lambda}{1!} + 2\frac{\lambda(\lambda-1)}{2!} + 18\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 4!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= \lambda(\lambda-1) + 3\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

$$= \lambda(\lambda-1)[1 + 3\lambda - 6 + (\lambda-2)(\lambda-3)]$$

$$= \lambda(\lambda-1)[3\lambda - 5 + \lambda^2 - 5\lambda + 6]$$

$= \lambda(\lambda - 1)[\lambda^2 - 2\lambda + 1]$

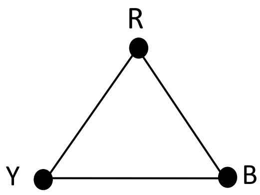
$= \lambda(\lambda - 1)(\lambda - 1)^2$

$P_4(\lambda) = \lambda(\lambda - 1)^3$

Similarly we can find the chromatic polynomial for T_6, T_7 and so on.

Therefore the general form of Tree graph T_n is given by $\lambda(\lambda - 1)^{n-1}$.

Chromatic Polynomial of Cycle Graph: 2.2.4



R-red, B-blue, Y-yellow

C_3- Cycle graph for 3 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 0, c_3 = 3!, n = 3$.

$$P_3(\lambda) = \sum_{1}^{3} c_i \binom{\lambda}{i}$$

$$P_3(\lambda) = c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda - 1)}{2!} + c_3 \frac{\lambda(\lambda - 1)(\lambda - 2)}{3!}$$

$$= (0)\frac{\lambda}{1!} + (0)\frac{\lambda(\lambda - 1)}{2!} 3! \frac{\lambda(\lambda - 1)(\lambda - 2)}{3!}$$

$= \lambda(\lambda - 1)(\lambda - 2)$

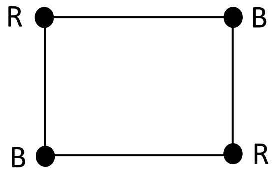
$= \lambda(\lambda^2 - 3\lambda + 2)$

$= \lambda^3 - 3\lambda^2 + 2\lambda$

$= \lambda^3 - 3\lambda^2 + 2\lambda + \lambda - \lambda + 1 - 1$

$= \lambda^3 - 3\lambda^2 + 3\lambda - 1 + (1 - \lambda)$

$P_3(\lambda) = (\lambda - 1)^3 + (-1)^3(\lambda - 1)$



R-red,B-bule.

C_4- Cycle graph for 4 vertices

c_i is the number of ways of coloring the given graph using i colors

Here $c_1 = 0, c_2 = 2, c_3 = 12, c_4 = 4!, n = 4.$

$$P_4(\lambda) = \sum_{1}^{4} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= (0)\frac{\lambda}{1!} + 2\frac{\lambda(\lambda-1)}{2!} + 12\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 4!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= (\lambda-1) + 2[\lambda(\lambda-1)(\lambda-2)] + \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

$$= \lambda(\lambda-1)[1 + 2(\lambda-2) + (\lambda-2)(\lambda-3)]$$

$$= (\lambda^2-\lambda)[1 + 2\lambda - 4 + \lambda^2 - 5\lambda + 6]$$

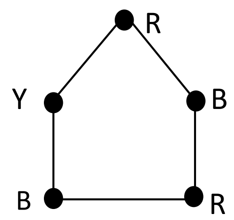
$$= (\lambda^2-\lambda)[\lambda^2 - 3\lambda + 3]$$

$$= \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda^3 + 3\lambda^2 - 3\lambda$$

$$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda + \lambda - \lambda + 1 - 1$$

$$= (\lambda-1)^4 + (\lambda-1)$$

$$P_4(\lambda) = (\lambda-1)^4 + (-1)^4(\lambda-1)$$



R-red, B-blue,
Y-yellow

C_5- Cycle graph for 5vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 0, c_3 = 30, c_4 = 120, c_5 = 5!, n = 5.$

$$P_5(\lambda) = \sum_{1}^{5} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$= (0)\frac{\lambda}{1!} + (0)\frac{\lambda(\lambda-1)}{2!} + 30\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 120\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 5!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$= [5\lambda(\lambda-1)(\lambda-2) + 5\lambda(\lambda-1)(\lambda-2)(\lambda-3) \ + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

$$= \lambda(\lambda-1)(\lambda-2)[5 + 5(\lambda-3) + (\lambda-3)(\lambda-4)$$

$$= (\lambda^2-\lambda)(\lambda-2)[5 + 5\lambda - 15 + \lambda^2 - 7\lambda + 12]$$

$$= (\lambda^3 - 2\lambda^2 - \lambda^2 + 2\lambda)[\lambda^2 - 2\lambda + 2]$$

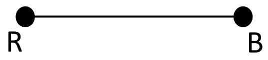
$$= \lambda^5 - 5\lambda^4 + 10\lambda^3 - 10\lambda^2 + 4\lambda + \lambda - \lambda + 1 - 1$$

$$P_6(\lambda) = (\lambda-1)^5 + (-1)^5(\lambda-1)$$

Similarly we can find the chromatic polynomial for C_6, C_7 and so on

Therefore the general form of Cycle graph C_n is given by $(\lambda-1)^n + (-1)^n(\lambda-1)$.

Chromatic Polynomial of Star Graph: 2.2.5



R-red, B-blue

S_2- Star graph of 2 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

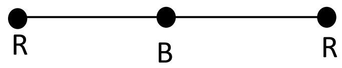
Here $c_1 = 0, c_2 = 2, n = 2.$

$$P_2(\lambda) = \sum_{1}^{2} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda - 1)}{2!}$$

$$= (0)\frac{\lambda}{1!} + 2!\frac{\lambda(\lambda - 1)}{2!}$$

$$P_2(\lambda) = \lambda(\lambda - 1)$$



R-red, B-blue

S_3- Star graph of 3 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 2!, c_3 = 3!, n = 3.$

$$P_3(\lambda) = \sum_{1}^{3} c_i \binom{\lambda}{i}$$

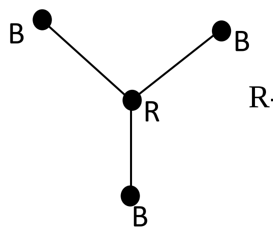
$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda - 1)}{2!} + c_3 \frac{\lambda(\lambda - 1)(\lambda - 2)}{3!}$$

$$= (0)\frac{\lambda}{1!} + 2!\frac{\lambda(\lambda - 1)}{2!} + 3!\frac{\lambda(\lambda - 1)(\lambda - 2)}{3!}$$

$$= \lambda(\lambda - 1)[1 + \lambda - 2]$$

$$= \lambda(\lambda - 1)(\lambda - 1)$$

$$P_3(\lambda) = \lambda(\lambda - 1)^2$$



R-red, B-blue

S_3- Star graph of 3 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 2, c_3 = 18, c_4 = 4!, n = 4.$

$$P_4(\lambda) = \sum_{1}^{4} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda - 1)}{2!} + c_3 \frac{\lambda(\lambda - 1)(\lambda - 2)}{3!} + c_4 \frac{\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)}{4!}$$

$$= (0) \frac{\lambda}{1!} + 2 \frac{\lambda(\lambda - 1)}{2!} + 18 \frac{\lambda(\lambda - 1)(\lambda - 2)}{3!} + 4! \frac{\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)}{4!}$$

$$= \lambda(\lambda - 1)[1 + 3(\lambda - 2) + (\lambda - 2)(\lambda - 3)]$$

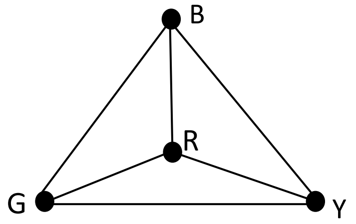
$$= \lambda(\lambda - 1)[1 + 3(\lambda - 2) + (\lambda^2 - 5\lambda + 6)]$$

$$= \lambda(\lambda - 1)[\lambda^2 - 2\lambda + 1]$$

$$P_3(\lambda) = \lambda(\lambda - 1)^3$$

Similarly we can find the chromatic polynomial for S_6, S_7 and so on. Therefore the general form of Star graph $S_n$is given by $\lambda(\lambda - 1)^n$.

Chromatic Polynomial of Wheel Graph: 2.2.6



R-red, B-blue,
Y-yellow, G-green

W_4- Wheel graph of 4 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 4!, n = 4.$

$$P_4(\lambda) = \sum_{1}^{4} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= (0)\frac{\lambda}{1!} + (0)\frac{\lambda(\lambda-1)}{2!} + (0)\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 4!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

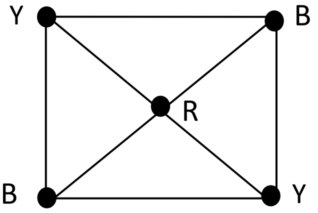
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

$$= \lambda(\lambda-1)(\lambda^2-5\lambda+6)$$

$$= \lambda[\lambda^3 - 5\lambda^2 + 6\lambda - \lambda^2 + 5\lambda - 6]$$

$$= \lambda[\lambda^3 - 5\lambda^2 + 6\lambda - \lambda^2 + 5\lambda - 6 + \lambda - \lambda + 2 - 2]$$

$$P_4(\lambda) = \lambda[(\lambda-2)^3 + (-1)^3(\lambda-2)]$$



R-red, B-blue, Y-yellow

W_5- Wheel graph of 5 vertices

c_i is the number of ways of coloring the given graph using i colors

Here $c_1 = 0, c_2 = 0, c_3 = 6, c_4 = 48, c_5 = 5!, n = 5.$

$$P_5(\lambda) = \sum_1^5 c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$= (0)\frac{\lambda}{1!} + (0)\frac{\lambda(\lambda-1)}{2!} 6 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 48 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 5! \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

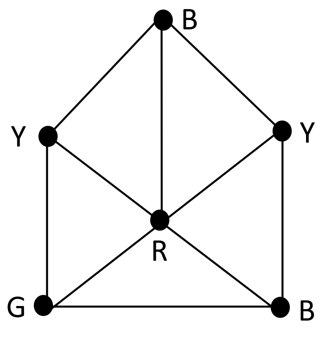
$$= \lambda(\lambda-1)(\lambda-2)[1 + 2(\lambda-3) + (\lambda-3)(\lambda-4)]$$

$$= \lambda(\lambda-1)(\lambda-2)[\lambda^2 - 5\lambda + 7]$$

$$= \lambda[\lambda^4 - 8\lambda^3 + 24\lambda^2 - 31\lambda + 14 + \lambda - \lambda + 2 - 2]$$

$$= \lambda[(\lambda-2)^4 + \lambda - 2]$$

$$P_5(\lambda) = \lambda[(\lambda-2)^4 + (-1)^4(\lambda-2)]$$



R-red, B-blue, Y-yellow, G-green

W_5- Wheel graph of 5 vertices

c_i is the number of ways of coloring the given graph using i colors.

Here $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 120, c_5 = 600, c_6 = 6!, n = 6.$

$$P_6(\lambda) = \sum_{1}^{6} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$
$$+ c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$
$$+ c_6 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$= 120 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 600 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$
$$+ 6! \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)[5 + 5(\lambda-4) + (\lambda-4)(\lambda-5)]$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)[\lambda^2 - 4\lambda + 5]$$

$$= \lambda[\lambda^5 - 10\lambda^4 + 40\lambda^3 - 80\lambda^2 + 79\lambda - 30 + \lambda - \lambda + 2 - 2]$$

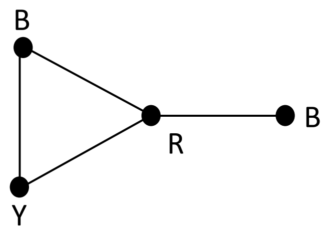
$$= \lambda[(\lambda-2)^5 - \lambda + 2]$$

$$P_6(\lambda) = \lambda[(\lambda-2)^5 + (-1)^5(\lambda-2)]$$

Similarly we can find the chromatic polynomial for W_7, W_8 and so on.

Therefore the general form of wheel graph W_n is given by $\lambda[(\lambda-2)^{n-1} + (-1)^n(\lambda-2)]$.

Chromatic Polynomial of Pan Graph: 2.2.7



R-red, B-blue, Y-yellow

P_3- Pan graph of 4 vertices

c_i is the number of ways of coloring the given graph using i colors.

Here $c_1 = 0, c_2 = 0, c_3 = 12, c_4 = 4!, n = 3.$

$$P_3(\lambda) = \sum_1^4 c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= 12 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 4! \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

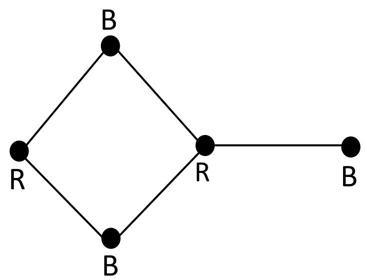
$$= \lambda(\lambda-1)(\lambda-2)[2 + \lambda - 3]$$

$$= (\lambda-1)^2(\lambda^2 - 2\lambda + 1 - 1)$$

$$= (\lambda-1)^2((\lambda-1)^2 - 1)$$

$$= (\lambda-1)^4 - (\lambda-1)^2$$

$$P_3(\lambda) = (\lambda-1)^4 + (-1)^3(\lambda-1)^2$$



R-red, B-blue

P_4- Pan graph of 5 vertices

c_i is the number of ways of coloring the given graph using i colors.

Here $c_1 = 0, c_2 = 2, c_3 = 30, c_4 = 120, c_5 = 5!, n = 4.$

$$P_5(\lambda) = \sum_1^5 c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$= 2\frac{\lambda(\lambda-1)}{2!} + 30\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 120\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 5!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$= \lambda(\lambda-1)\begin{bmatrix} 1 + 5(\lambda-2) + 5(\lambda-2)(\lambda-3) \\ +(\lambda-2)(\lambda-3)(\lambda-4) \end{bmatrix}$$

$$= \lambda(\lambda-1)[\lambda^3 - 4\lambda^2 + 6\lambda - 3]$$

$$= (\lambda-1)[\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 - 1 + \lambda - \lambda]$$

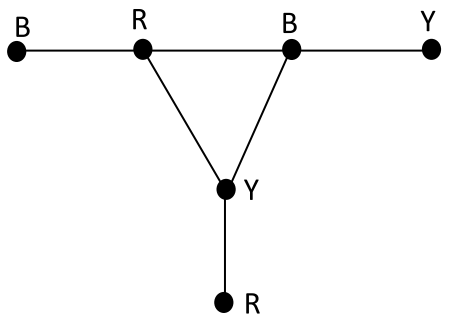
$$= (\lambda-1)[(\lambda-1)^4 + (\lambda-1)]$$

$$P_4(\lambda) = (\lambda-1)^5 + (-1)^4(\lambda-1)^2$$

Similarly we can find the chromatic polynomial for P_5, P_6 and soon.

Therefore the general form of pan graph P_n is given by $[(\lambda-1)^{n+1} + (-1)^n(\lambda-1)^2]$.

Chromatic Polynomial of Sunlet graph: 2.2.8



R-red, B-blue, Y-yellow

3- Sunlet graph of 6 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 0, c_3 = 60, c_4 = 480, c_5 = 1080, c_6 = 61, n = 3$

$$P_6(\lambda) = \sum_{1}^{6} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$
$$+ c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$
$$+ c_6 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$= 60 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 480 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$
$$+ 1080 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$
$$+ 6! \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$= \lambda(\lambda-1)(\lambda-2)[10 + 20(\lambda-3) + 9(\lambda-3)(\lambda-4) + (\lambda-3)(\lambda-4)(\lambda - 5)]$$

$$= (\lambda^2-\lambda)(\lambda-2)[10 + 20(\lambda-3) + 9(\lambda^2-7\lambda+12) + (\lambda^3-12\lambda^2+47\lambda - 60]$$

$$= (\lambda^3-3\lambda^2+2\lambda)\ [10 + 20\lambda - 60 + 9\lambda^2 - 63\lambda + 108 + \lambda^3 - 12\lambda^2 + 47\lambda - 60]$$

$$= (\lambda^3-3\lambda^2+2\lambda)[\lambda^3-3\lambda^2+4\lambda-2]$$

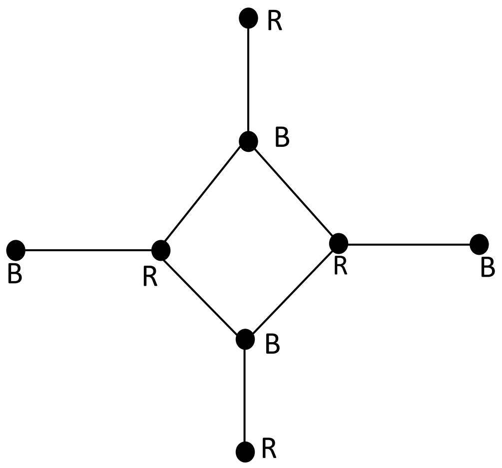
$$= \lambda^6 - 3\lambda^5 + 4\lambda^4 - 2\lambda^3 - 3\lambda^5 + 9\lambda^4 - 12\lambda^3 + 6\lambda^2 + 2\lambda^4 - 6\lambda^3 + 8\lambda^2 - 4\lambda$$

$$= \lambda^6 - 6\lambda^5 + 15\lambda^4 - 20\lambda^3 + 14\lambda^2 - 4\lambda$$

$$= \lambda^6 - 6\lambda^5 + 15\lambda^4 - 20\lambda^3 + 14\lambda^2 - 4\lambda + 1 - 1 + \lambda^2 - \lambda^2 + 2\lambda - 2\lambda$$

$$= \lambda^6 - 6\lambda^5 + 15\lambda^4 - 20\lambda^3 + 15\lambda^2 - 6\lambda + 1 - (1 - 2\lambda + \lambda^2)$$

$$P_6(\lambda) = (\lambda-1)^6 - (1-\lambda)^2$$



R-red, B-blue

4- Sunlet graph of 8 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 2, c_3 = 258, c_4 = 5544, c_5 = 35280, c_6 = 90720,$

$c_7 = 100800, c_8 = 8!, n = 4.$

$$P_8(\lambda) = \sum_{1}^{8} c_i \binom{\lambda}{i}$$

$$P_8(\lambda) = c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$
$$+ c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$
$$+ c_6 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$
$$+ c_7 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)}{7!}$$
$$+ c_8 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{8!}$$

$$= (0)\frac{\lambda}{1!} + (2)\frac{\lambda(\lambda-1)}{2!} + 258\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$+5544\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$+35280\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$+90720\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$+100800\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)}{7!}$$

$$+8!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{8!}$$

$$\begin{aligned}= \lambda(\lambda-1)[1 &+ 43(\lambda-2) + 231(\lambda^2-5\lambda+6) + 294(\lambda^3-9\lambda^2+26\lambda-24)\\ &+ 126(\lambda^4-14\lambda^3+17\lambda^2-154\lambda+120)\\ &+ 20(\lambda^5-20\lambda^4+155\lambda^3-58\lambda^2+1044\lambda-720) + (\lambda^6-27\lambda^5\\ &+ 295\lambda^4-1665\lambda^3+5104\lambda^2-8028\lambda+5040)]\end{aligned}$$

$$\begin{aligned}= (\lambda^2-\lambda)[1 &+ 43\lambda - 86 + 231\lambda^2 - 1155\lambda + 1386 + 294\lambda^3 - 2646\lambda^2\\ &+ 7644\lambda - 7056 + 126\lambda^4 - 1764\lambda^3 + 8946\lambda^2 - 19404\lambda + 15120\\ &+ 20\lambda^5 - 400\lambda^4 + 3100\lambda^3 - 11600\lambda^2 + 20880\lambda\lambda - 14400 + \lambda^6\\ &- 27\lambda^5 + 295\lambda^4 - 1665\lambda^3 + 5104\lambda^2 - 8028\lambda + 5040]\end{aligned}$$

$$= (\lambda^2-\lambda)[\lambda^6 - 7\lambda^5 + 21\lambda^4 - 35\lambda^3 + 35\lambda^2 - 20\lambda + 5]$$

$$\begin{aligned}= \lambda^8 &- \lambda^7 - 7\lambda^7 + 7\lambda^6 + 21\lambda^6 - 21\lambda^5 - 35\lambda^5 + 35\lambda^4 + 35\lambda^4 - 35\lambda^3\\ &- 20\lambda^3 + 20\lambda^2 + 5\lambda^2 - 5\lambda\end{aligned}$$

$$\begin{aligned}= \lambda^8 &- 8\lambda^7 + 28\lambda^6 - 56\lambda^5 + 70\lambda^4 - 55\lambda^3 + 25\lambda^2 - 5\lambda + 1 - 1 + 3\lambda - 3\lambda\\ &+ 3\lambda^2 - 3\lambda^2 + \lambda^3 - \lambda^3\end{aligned}$$

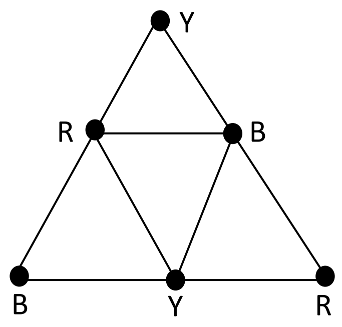
$$\begin{aligned}= (\lambda^8 &- 8\lambda^7 + 28\lambda^6 - 56\lambda^5 + 70\lambda^4 - 56\lambda^3 + 28\lambda^2 - 8\lambda + 1) + -(1 - 3\lambda\\ &+ 3\lambda^2 + \lambda^3)\end{aligned}$$

$$= (\lambda-1)^8 - (1-\lambda)^3$$

Similarly we can find the chromatic polynomial for 5-sunlet graph, 6-sunlet graph and so on.

Therefore the general form of chromatic polynomial of *n*- Sunlet graph is given by $(\lambda-1)^{2n} - (1-\lambda)^{n-1}$.

Chromatic Polynomial of Sun graph: 2.2.9



R-red color , B-blue color,

Y-yellow color

3-sun graph of 6 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 0, c_3 = 6, c_4 = 168, c_5 = 720, c_6 = 6!, n = 3.$

$$P_6(\lambda) = \sum_{1}^{6} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!} + c_6 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

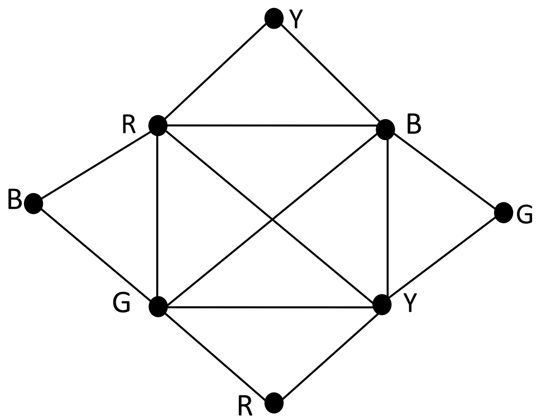
$$= 6 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 168 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 720 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!} + 6! \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$= \lambda(\lambda-1)(\lambda-2)\begin{bmatrix} 1 + 7(\lambda-3) + 6(\lambda-3)(\lambda-4) \\ +(\lambda-3)(\lambda-4)(\lambda-5) \end{bmatrix}$$

$$= \lambda(\lambda-1)(\lambda-2)[1 + 7(\lambda-3) + 6(\lambda^2 - 7\lambda + 12) + (\lambda^3 - 12\lambda^2 + 47\lambda - 60)]$$

$$= \lambda(\lambda-1)(\lambda-2)[1+7\lambda-21+6\lambda^2-42\lambda+72+\lambda^3-12\lambda^2+47\lambda-60]$$

$$P_6(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-2)^3$$



R- red color,

B –blue color ,

Y- yellow color,

G – green color.

4- Sun graph of 8 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 23424, c_5 = 72600, c_6 = 82800,$

$c_7 = 70560, c_8 = 8!, n = 4.$

$$P_8(\lambda) = \sum_{1}^{8} c_i \binom{\lambda}{i}$$

$$P_8(\lambda) = c_1\frac{\lambda}{1!} + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$
$$+ c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$
$$+ c_6\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$
$$+ c_7\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)}{7!}$$
$$+ c_8\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{8!}$$

$$= 23424\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 72600\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$
$$+ 82800\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$
$$+ 70560\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)}{7!}$$
$$+ 8!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{8!}$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)[976 + 605(\lambda-4) + 115(\lambda-4)(\lambda-5)$$
$$+ 14(\lambda-4)(\lambda-5)(\lambda-6) + (\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)]$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)[976 + 605(\lambda-4) + 115(\lambda^2-9\lambda+20)$$
$$+ 14(\lambda^3-15\lambda^2+74\lambda-120) + (\lambda^4-22\lambda^3+179\lambda^2-638\lambda$$
$$+ 840)]$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)[976 + 605\lambda - 2420 + 115\lambda^2 - 1035\lambda + 2300$$
$$+ 14\lambda^3 - 210\lambda^2 + 1036\lambda - 1680 + \lambda^4 - 22\lambda^3 + 179\lambda^2 - 638\lambda$$
$$+ 840]$$

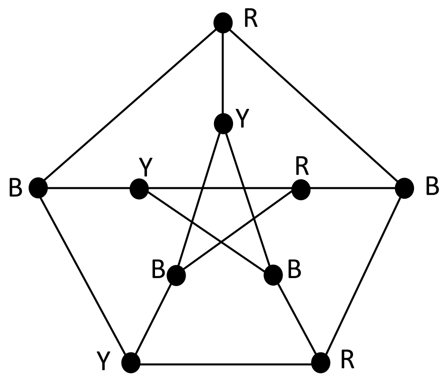
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)[\lambda^4 - 8\lambda^3 + 84\lambda^2 - 32\lambda + 16]$$

$$P_8(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-2)^4$$

Similarly we can find the chromatic polynomial for 5-sun graph,6-sun graph and so on.

Therefore the general form of chromatic polynomial of n-sun graph is given by $\lambda(\lambda-1)(\lambda-2)\ldots\ldots\ldots\ldots\big(\lambda-(n-1)\big)(\lambda-2)^n$ or $(\lambda)_n(\lambda-2)^n$ where $(\lambda)_n$ is a falling factorial.

Chromatic Polynomial of Petersen Graph: 2.2.10



R- red color,

B –blue color ,

Y- yellow color.

Petersen graph with 10 vertices and 15 edges

Here $c_1 = 0, c_2 = 0, c_3 = 120, c_4 = 12480, c_5 = 269280, c_6 = 2062800,$

$c_7 = 7232400, c_8 = 12700800, c_9 = 10886400, c_{10} = 10!\, n = 10.$

$$P_{10}(\lambda) = \sum_{i=1}^{10} c_i \binom{\lambda}{i}$$

$$P_{10}(\lambda) = c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$+c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{4!}$$

$$+c_6 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$+c_7 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)}{7!}$$

$$+c_8 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{8!}$$

$$+C_9 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)}{9!}$$

$$+C_{10}\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)(\lambda-9)}{10!}$$

$$= (0)\frac{\lambda}{1!} + (0)\frac{\lambda(\lambda-1)}{2!} + (120)\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$+(12480)\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$+(269280)\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$+(2062800)\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$+(7232400)\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)}{7!}$$

$$+(12700800)\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{8!}$$

$$+(10886400)\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)}{9!}$$

$$+(10!)\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)(\lambda-9)}{10!}$$

$$= \lambda(\lambda-1)(\lambda-2)[20 + 520(\lambda-3) + 2244(\lambda^2 - 7\lambda + 12)$$
$$+ 2865(\lambda^3 - 12\lambda^2 + 47\lambda)$$
$$+ 1435(\lambda^4 - 18\lambda^3 + 119\lambda^2 - 342\lambda + 360)$$
$$+ 315(\lambda^5 - 25\lambda^4 + 245\lambda^3 - 1175\lambda^2 + 2754\lambda - 2520)$$
$$+ 30(\lambda^6 - 33\lambda^5 + 445\lambda^4 - 3135\lambda^3 + 12154\lambda^2 - 24552\lambda$$
$$+ 20160) + (\lambda^7 - 42\lambda^6 + 742\lambda^5 - 7140\lambda^4 + 40369\lambda^3 - 133938\lambda^2$$
$$+ 241128\lambda - 181440]$$

$$= \lambda(\lambda-1)(\lambda-2)\,[20 + 520\lambda - 1560 + 2244\lambda^2 - 15708\lambda + 25928$$
$$+ 2865\lambda^3 - 34380\lambda^2 + 134655\lambda - 171900 + 1435\lambda^4 - 25830\lambda^3$$
$$+ 170765\lambda^2 - 490770\lambda + 516600 + 315\lambda^5 - 7875\lambda^4 + 77175\lambda^3$$
$$- 370125\lambda^2 + 867510\lambda - 793800 + 36\lambda^6 - 990\lambda^5 + 13350\lambda^4$$
$$- 94050\lambda^3 + 364620\lambda^2 - 73560\lambda + 604800 + \lambda^7 - 42\lambda^6 + 742\lambda^5$$
$$- 7140\lambda^4 + 40369\lambda^3 - 113938\lambda^2 + 241128\lambda - 181440]$$

$$P_{10}(\lambda) = \lambda(\lambda-1)(\lambda-2)[\lambda^7 - 12\lambda^6 + 67\lambda^5 - 230\lambda^4 + 529\lambda^3 - 814\lambda^2 + 775\lambda - 352]$$

Therefore the general form of chromatic polynomial of Peterson graph is given by

$$\lambda(\lambda-1)(\lambda-2)[\lambda^7 - 12\lambda^6 + 67\lambda^5 - 230\lambda^4 + 529\lambda^3 - 814\lambda^2 + 775\lambda - 352]$$

Chapter - 3

CHAPTER – III

Chromatic Polynomial for Lexicographic Product of Graphs

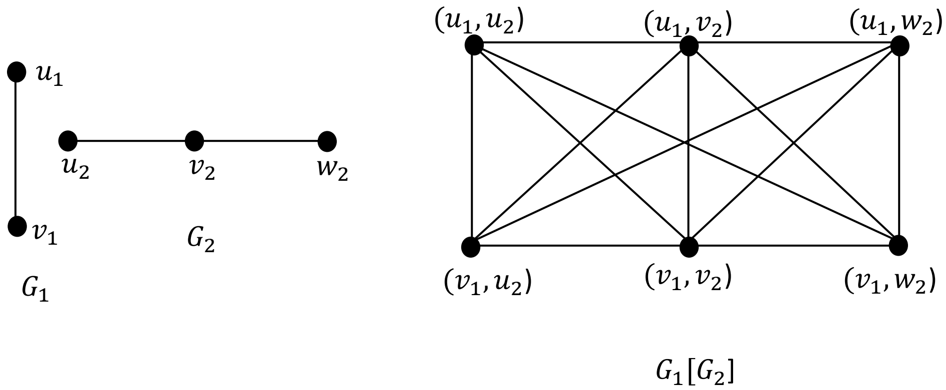
Section 3.1

Preliminaries

Definition: 3.1.1 [1]

The composition $G = G_1[G_2]$ of graphs $G_1$and $G_2$with disjoint point set $V_1$and $V_2$and edge sets $X_1$and X_2 is the graph with point vertex $V_1 \times V_2$and $U = (u_1, u_2)$ adjacent with $V = (v, v_2)$when ever $[u_1 adj v_1]$ or $[u_{1=} v_1 and\ u_2 adj\ v_2]$. It is also called the *graph composition* or *lexicographic product.*

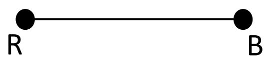
Example: 3.1.2



Section 3.2

Chromatic Polynomial for Lexicographic Product of Some graphs

Chromatic Polynomial of Ladder graph: 3.2.1



R-red, B-blue

$n = 1$ Ladder graph of 2 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

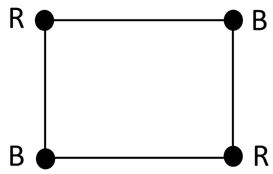
Here $c_1 = 0, c_2 = 2,$

$$P_2(\lambda) = \sum_{1}^{2} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!}$$

$$= 2! \frac{\lambda(\lambda-1)}{2!}$$

$$P_2(\lambda) = \lambda(\lambda-1)$$



R-red, B-blue

$n = 2$ ladder graph of 4 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 2, c_3 = 12, c_4 = 4!, n = 4.$

$$P_4(\lambda) = \sum_{1}^{4} c_i \binom{\lambda}{i}$$

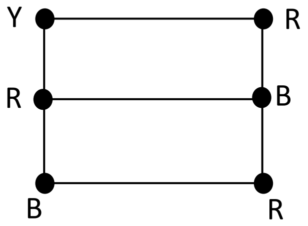
$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= 2 \frac{\lambda(\lambda-1)}{2!} + 12 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 4! \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$= \lambda(\lambda-1)[1 + 2(\lambda-2) + (\lambda-2)(\lambda-3)]$$

$$= \lambda(\lambda-1)[1 + 2\lambda - 4 + \lambda^2 - 5\lambda + 6]$$

$$P_4(\lambda) = \lambda(\lambda-1)[\lambda^2 - 3\lambda + 3]$$



R-red, B-blue,
Y-yellow

$n = 3$ ladder graph of 6 vertices

c_i is the number of ways of coloring the given graph using *i* colors.

Here $c_1 = 0, c_2 = 2, c_3 = 48, c_4 = 384, c_5 = 960, c_6 = 61$

$$P_6(\lambda) = \sum_{1}^{6} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!} + c_6 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$= 2! \frac{\lambda(\lambda-1)}{2!} + 48 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 384 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 960 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5! \quad c_2} + 6! \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$= \lambda(\lambda-1)[1 + 8(\lambda-2) + 16(\lambda-2)(\lambda-3) + 8(\lambda-2)(\lambda-3)(\lambda-4) + (\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)]$$

$$= \lambda(\lambda-1)[1 + 8(\lambda-2) + 16(\lambda^2 - 5\lambda + 6) + 8(\lambda^3 - 9\lambda^2 + 26\lambda - 24)(\lambda-3) + (\lambda^4 - 14\lambda^3 + 72\lambda^2 - 154\lambda + 120)]$$

$$= \lambda(\lambda - 1)[1 + 8\lambda - 16 + 16\lambda^2 - 80\lambda + 96 - 8\lambda^3 - 72\lambda^2 + 208\lambda - 192 + \lambda^4 - 14\lambda^3 + 71\lambda^2 - 154\lambda + 120]$$

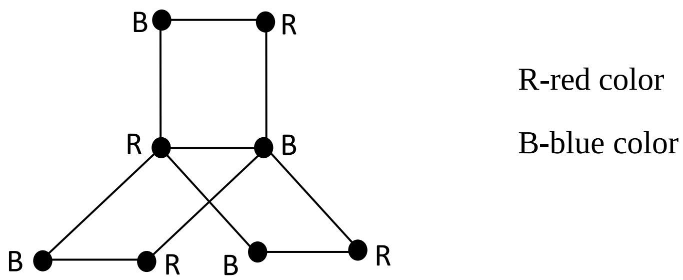
$$= \lambda(\lambda - 1)[\lambda^4 - 6\lambda^3 + 15\lambda^2 - 18\lambda + 9]$$

$$P_6(\lambda) = \lambda(\lambda - 1)[\lambda^2 - 3\lambda + 2]^2$$

Similarly we can find the chromatic polynomial for 4-Ladder graph,5-Ladder graph and so on.

Therefore the General form of chromatic polynomial of n-Ladder graph is given by $\lambda(\lambda - 1)[\lambda^2 - 3\lambda + 2]^{n-1}$.

Chromatic Polynomial of Book graph: 3.2.2



$\boldsymbol{B_3}$**-** book graph of 8 vertices

c_i is the number of ways of coloring the given graph using i colors.

Here $c_1 = 0, c_2 = 2, c_3 = 156, c_4 = 3480, c_5 = 24960, c_6 = 72720,$

$c_7 = 90720, c_8 = 8!, n = 3.$

$$P_8(\lambda) = \sum_{1}^{8} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda - 1)}{2!} + c_3 \frac{\lambda(\lambda - 1)(\lambda - 2)}{3!} + c_4 \frac{\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)}{4!} + c_5 \frac{\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)}{5!}$$

$$+c_6\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$+c_7\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)}{7!}$$

$$+c_8\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{8!}$$

$$=(0)\frac{\lambda}{1!}+(2)\frac{\lambda(\lambda-1)}{2!}+156\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$
$$+3480\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$+24960\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$+72720\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

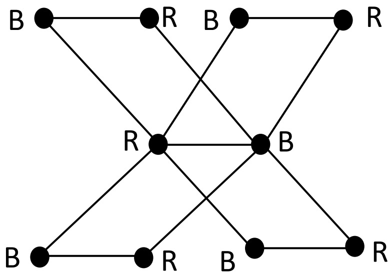
$$+90720\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)}{7!}$$

$$+8!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{8!}$$
$$=\lambda(\lambda-1)[1+26(\lambda-2)+145(\lambda^2-5\lambda+6)+208(\lambda^3-9\lambda^2+26\lambda-24)$$
$$+101(\lambda^4-14\lambda^3+17\lambda^2-154\lambda+120)$$
$$+18(\lambda^5-20\lambda^4+155\lambda^3-58\lambda^2+1044\lambda-720)+(\lambda^6-27\lambda^5$$
$$+295\lambda^4-1665\lambda^3+5104\lambda^2-8028\lambda+5040)]$$

$$=\lambda(\lambda-1)[1+26\lambda-52+145\lambda^2-725\lambda+870+208\lambda^3-1872\lambda^2$$
$$+5408\lambda-4992+101\lambda^4-1414\lambda^3+7171\lambda^2-1555\lambda+12120$$
$$+18\lambda^5-360\lambda^4+2790\lambda^3-10440\lambda^2+18792\lambda-12960+\lambda^6$$
$$-27\lambda^5+295\lambda^4-1665\lambda^3+5104\lambda^2-8028\lambda+5040]$$

$$=\lambda(\lambda-1)[\lambda^6-9\lambda^5+36\lambda^4-81\lambda^3+108\lambda^2-81\lambda+27]$$

$$P_8(\lambda)=\lambda(\lambda-1)(\lambda^2-3\lambda+3)^3$$



R-red color

B-blue color

$\boldsymbol{B_4}$- book graph of 10 vertices

c_i is the number of ways of coloring the given graph using i colors.

Here$c_1 = 0, c_2 = 2, c_3 = 480, c_4 = 26880, c_5 = 432000,$

$c_6 = 2829600, c_7 = 8951040, c_8 = 14515200, c_9 = 1162160, c_9 = 9!$

$$P_{10}(\lambda) = \sum_{1}^{10} c_i \binom{\lambda}{i}$$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$+c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{4!}$$

$$+c_6 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$+c_7 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)}{7!}$$

$$+c_8 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{8!}$$

$$+C_9 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)}{9!}$$

$$+C_{10} \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)(\lambda-9)}{10!}$$

$$= (0)\frac{\lambda}{1!} + 2\frac{\lambda(\lambda-1)}{2!} + 480\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

$$+26880\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$+432000\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{4!}$$

$$+2829600\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{6!}$$

$$+8951040\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)}{7!}$$

$$+14515200\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)}{8!}$$

$$+11612160\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)}{9!}$$

$$+10!\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)(\lambda-9)}{10!}$$

$$\begin{aligned}
= \lambda(\lambda-1)[&1 + 80(\lambda-2) + 1120(\lambda^2-5\lambda+6) \\
&+ 3600(\lambda^3-9\lambda^2+26\lambda-24) \\
&+ 3930(\lambda^4-14\lambda^3+17\lambda^2-154\lambda+120) \\
&+ 1776(\lambda^5-20\lambda^4+155\lambda^3-58\lambda^2+1044\lambda-720) \\
&+ 360(\lambda^6-27\lambda^5+295\lambda^4-1665\lambda^3+5104\lambda^2-8028\lambda+5040) \\
&+ 32(\lambda^7-35\lambda^6+511\lambda^5-4025\lambda^4+18424\lambda^3-48860\lambda^2 \\
&+ 69264\lambda-40320) + (\lambda^8-44\lambda^7+826\lambda^6-8624\lambda^5+54649\lambda^4 \\
&- 214676\lambda^3+509004\lambda^2-663696\lambda+362880)]
\end{aligned}$$

$= \lambda(\lambda-1)[1 + 80\lambda - 160 + 1120\lambda^2 - 5600\lambda + 6720 + 3600\lambda^3$
$- 32400\lambda^2 + 93600\lambda - 86400 + 3930\lambda^4 - 55020\lambda^3 + 279030\lambda^2$
$- 605220\lambda + 471600 + 1776\lambda^5 - 35560\lambda^4 + 275280\lambda^3$
$- 1030080\lambda^2 + 1854144\lambda - 1278720 + 360\lambda^6 - 9720\lambda^5$
$+ 106200\lambda^4 - 599400\lambda^3 + 1837440\lambda^2 - 2890080\lambda + 194400$
$+ 32\lambda^7 - 1120\lambda^6 + 16352\lambda^5 - 128800\lambda^4 + 589568\lambda^3$
$- 1563520\lambda^2 + 2216448\lambda - 1290240 + \lambda^8 - 44\lambda^7 + 826\lambda^6$
$- 8624\lambda^5 + 54649\lambda^4 - 214676\lambda^3 + 509004\lambda^2 - 663696\lambda$
$+ 362880]$

$= \lambda(\lambda-1)[\lambda^8 - 12\lambda^7 + 66\lambda^6 - 216\lambda^5 + 459\lambda^4 - 648\lambda^3 + 594\lambda^2 - 324\lambda$
$+ 81]$

$P_{10}(\lambda) = \lambda(\lambda-1)(\lambda^2 - 3\lambda + 3)^4$

Similarly we can find the chromatic polynomial for B_5, B_6 graphs and so on.

Therefore the general form of chromatic polynomial of B_n-Book graph with 2n+2 vertices is given by $\lambda(\lambda-1)(\lambda^2-3\lambda+3)^n$

Section 3.3

Lexicographic Product of Two Circulant graph

Theorem: 3.2.1

Let $G = C_{n,S_1}$, and $H = C_{m,S_2}$, be circulant graphs define

$$S = \left(\bigcup_{t=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor} tn + S_1\right) \cup \left(\bigcup_{t=1}^{\left\lfloor\frac{m}{2}\right\rfloor} tn - S_1\right) \cup (nS_2)$$

where $tn \pm S_1 = \{tn + r : r \in S_1\}$ and $nS_2 = \{nq : q \in S_2\}$.Then $G[H]$ is isomorphic to the circulant graph $C_{nm,S}$.

Proof:

Relabel the vertices of $G[H]$so that (g,h) is assigned the new vertex $(g+nh)$.

Lemma: 3.2.2 [3]

Let $= C_{n,S_1}$, and $H = C_{m,S_2}$, be circulant graphs , construct the lexicographic product graph $G[H]$and relabel each vertex (g, h) with the integer $(g + nh)$,Then

a) This relabeling of vertices ensures that each of the nm vertices in $G[H]$is assigned a unique integer label between 0 and $nm - 1$ inclusive.

b) Let $x = g_1 + nh_1$ and $y = g_2 + nh_2$be the new labels assigned to the vertices $(g, h)$and$(g + nh)$in $G[H]$.

Suppose that $x > y$ so that $1 \leq x - y \leq nm - 1$ among these $nm - 1$ possible values for $x - y$, $(g_1, h_1) \backsim (g_2, h_2)$ in $G[H]$ if and only if $x - y \equiv \pm r(mod\ n)$ for some $r \in S_1$ or n divides$x - y$ and $\frac{x-y}{n} = \pm q(mod\ m)$ for some $q \in S_2$.

By part a) of lemma the nm vertices of $G[H]$ are the integers from 0 to $nm - 1$ inclusive.

By part of lemma

If $x = g_1 + nh_1$ and $y = g_2 + nh_2$for some $0 \leq g_1, g_2 \leq n - 1$ and $0 \leq h_1 h_2 \leq m - 1$,then $x > y$ in $G[H]$ if and only if $x - y \equiv \pm r(mod\ n)$ for some $r \in S_1$ or $\frac{x-y}{n} \equiv \pm q(mod\ m)$ for some $q \in S_2$.

Let S' denote the set of possible values $x - y$ (where $1 \leq x - y \leq nm - 1$) satisfying the congruence equation above.

We note that each integer of the form nq or $n(m - q)$ is in S', where $q \in S_2$. It is simple to check that each such integer lies in the required interval [1, nm-1] , since $1 \leq q \leq \left\lfloor \frac{m}{2} \right\rfloor$.

In addition to these values already in S', we must also include each value of C_{n+r},over all choices of $0 \leq C \leq m - 1$ and $r \in S_1$ and each value of d_{n-r}, over all choices of $1 \leq d \leq m$. We note that all of these values lie in the required interval [1, nm-1] for any choice of c,d and r. We remark that any c and d not satisfying the given inequality (i.e., $c < 0, c > m-1,\ \ d < 1, d > m$) will make $cn + r$ and |or $dn - r$ full outside of the required interval of for any choices of r , since $1 \le r \le \left\lfloor \frac{n}{2} \right\rfloor$.

As a final note, remark that $v \in S'$ if and only if $nm - v \in S'$. This will make it very lays to compute the set of circular distances $|x - y|_{nm}$we have shown that $x \sim y$ in $G[H]$ iff $x - y \in S'$ by letting $S = \{|x - y|_{nm} : x - y \in S\}$ be the generating set produced by computing the circular distance (mod nm) or each value of S', we have $x \sim y$ in $G[H]$ iff $|x - y|_{nm} \in S$.

This proves that $G[H]$ is a circulant $C_{nm,s}$ for this generating set S. Given that we know which elements are in S', we can describe exactly the set of element in S.since $v \in S'$ iff $nm - v \in S'$,the element of S are precisely those elements in S' that are in the interval $\left[1, \left\lfloor \frac{nm}{2} \right\rfloor\right]$.

Let us given an explicit characterization of this generating set. First look at all the multiples of n,each integer of the form nq and $nm - nq$ belong to S' where $q \in S_2$, clearly each $nq \in S$, since $nq \le n \left\lfloor \frac{m}{2} \right\rfloor \le \left\lfloor \frac{nm}{2} \right\rfloor$. Thus each of $n, 2n, 3n, \ldots \ldots \ldots . n \left\lfloor \frac{m}{2} \right\rfloor$ are counted in S.

Conversely, all of the elements in the latter set are greater than $\left\lfloor \frac{nm}{2} \right\rfloor$ and hence do not belong to the set S .

The only exception to this occurs when m is even and $q = \frac{m}{2}$, however this value has already been added to our set S,since $n - \frac{m}{2} = nm - n\frac{m}{2}$. Among the multiples of n the only values in S are the elements nq, over all $q \in S_2$.

Define $tn \pm S_1 = \{tn + r : r \in S_1\}$ and $nS_2 = \{nq : q \in S_2\}$ From our analysis above, we have proven that $G[H] \simeq C_{nm,s}$ where $S = (\bigcup_{t=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} tn + S_1) \cup \left(\bigcup_{t=1}^{\left\lfloor \frac{m}{2} \right\rfloor} tn - S_2\right) \cup n S_2$ for some indices c and d. Recall for the full set S' we had $0 \le c \le m - 1$and $1 \le d \le m$, we will show that for this reduced set S,we $0 \le c \le \left\lfloor \frac{m-1}{2} \right\rfloor$ and $1 \le d \le \left\lfloor \frac{m}{2} \right\rfloor$.

Given that we are selecting the elements of S' in the bottom half ,this result is completely unsurprising, we do not need to consider the lower bound, $s \geq 0$ and $d \geq 1$, as they have already been dealt with previously, we now prove the upper bounds for c and d by establishing that all possible values of the form C_{n+r} and d_{n-r}belong to S (for those two bounds) and proving that we have not missed any other values .

If $C \leq \left\lfloor\frac{m-1}{2}\right\rfloor$, then $C_{n+r} \leq n\left\lfloor\frac{m-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor \leq \left\lfloor\frac{nm}{2}\right\rfloor$ for all choices of r and if $C > \left\lfloor\frac{m-1}{2}\right\rfloor$,this is equivalent to the inequaling $C \geq \frac{m}{2}$. Since C and are both integer then$C_{n+r} \geq \frac{mn}{2} + r > \left\lfloor\frac{nm}{2}\right\rfloor$.

If $d \leq \left\lfloor\frac{nm}{2}\right\rfloor$ then $d_{n-r} \leq n\left\lfloor\frac{m}{2}\right\rfloor + r < \left\lfloor\frac{nm}{2}\right\rfloor$ and if $d > \left\lfloor\frac{m}{2}\right\rfloor$ this is equivalent to the inequality $d \geq \frac{m+1}{2}$, since d and m are integer we have $d_{n-r} \geq \frac{mn}{2} + \frac{n-r}{2} > \left\lfloor\frac{nm}{2}\right\rfloor$ except in the one special case when the following form conditions hold n is even $r = \frac{n}{2}$, m is odd and $d = \frac{m+1}{2}$.

However this case is easily dealt with, we have $d_{n-r} = \frac{mn+n}{2} - \frac{n}{2} = \frac{nm}{2}$ ∈S. But this value was already taken into account for the case $C = \left\lfloor\frac{m-1}{2}\right\rfloor$ and $r = \frac{n}{2}$ since$C_{n+r} = \frac{m-n}{2}n + \frac{n}{2} = \frac{nm}{2}$.

Hence the desired upper bound is connect no elements of S have been included. Therefore we have proven that $G[H]$ is isomorphic to $C_{nm,s}$where

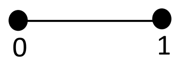
$$S = \left(\bigcup_{t=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor} \mathrm{tn} + S_1\right) \cup \left(\bigcup_{t=1}^{\left\lfloor\frac{m}{2}\right\rfloor} \mathrm{tn} - S_1\right) \cup (nS_2)$$

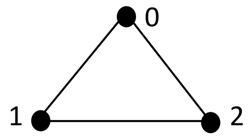
Where $tn \pm S_1 = \{tn + \mathrm{r} : \mathrm{r} \in S_1\}$ and $nS_2 = \{\mathrm{nq} : \mathrm{q} \in S_2\}$ our proof is complete.

Example: 3.2.3

To find the lexicographic product graph of $C_{2,\{1\}}$ and $C_{3,\{1\}}$

Consider,





$C_{2,\{1\}}$ Circulant graph of 2 vertices

$C_{3,\{1\}}$ Circulant graph of 3 vertices

Here $n = 2, m = 3, S_1 = \{1\}, S_2 = \{1\}$

Number of vertices $= mn = 2 \times 3 = 6$

The generating set is given by the theorem

$$S = \left(\bigcup_{t=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} tn + S_1 \right) \cup \left(\bigcup_{t=1}^{\left\lfloor \frac{m}{2} \right\rfloor} tn - S_1 \right) \cup (nS_2)$$

$$= \left(\bigcup_{t=0}^{1} t2 + 1 \right) \cup \left(\bigcup_{t=1}^{1} t2 - 1 \right) \cup (2\{1\})$$

$$= (\{2 \times 0\} + 1, \{2 \times 1\} + 1) \cup (\{2 \times 0\} - 1, \{2 \times 1\} - 1) \cup (2)$$

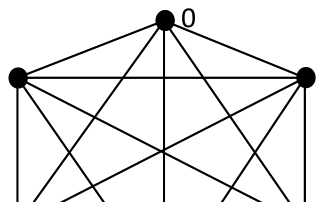
$$= (1,3) \cup (3) \cup (2)$$

$$S = (1,2,3)$$

Where S denote the circulant graph (1, 2, 3) vertices

Thus {1, 2, 3} is the generating set of the product graph of $C_{2,\{1\}}$ and $C_{2,\{1\}}$.

(0, 0)

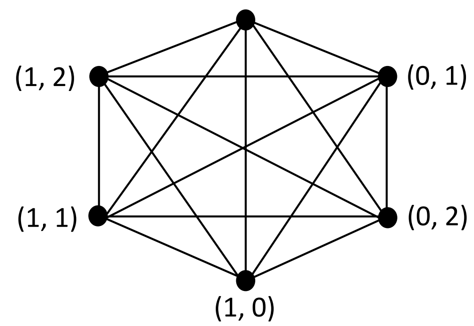


5 1

4 2

3

$C_{6,\{1,2,3\}}$ Circulant graph



Lexicographic product $C_{2,\{1\}}$ and $C_{3,\{1\}}$

Hence the lexicographic product of two circulant graphs $C_{2,\{1\}}$ and $C_{2,\{1\}}$ is again a circulant graph of $C_{6,\{1,2,3\}}$.

Summary and Conclusion

SUMMARY AND CONCLUSION

This thesis is an attempt to study the chromatic polynomials of some commonly known the graphs and lexicographic product of graphs.

In chapter 1, we have provided the preliminary definitions of graph theory and each has been illustrated with an example.

In chapter 2, we have studied the chromatic polynomial of some graphs such as Complete graph, Path graphs, Tree, Cycle graph, Star graph, Wheel graph, Pan graph, Sunlet graph, Sun graph, and Petersen graph.

In chapter 3, Comprises the main results of this thesis, the chromatic polynomial of lexicographic product of graphs such as Ladder graph, Book graph and chromatic polynomial of lexicographic product of circulant graphs.

Bibliography

BIBLIOGRAPHY

1. **Biggs, N.,** "*Algebraic Graph Theory*", Cambridge university press, Cambridge, 1974.
2. **Biggs, N., C, klin. M. H, Reinfeid. P,** "*Algebraic Methods for Chromatic Polynomials, Eurup, J. combinatoris*", 25 (2004) 147-160.
3. **Chia, G. L.,** "*Some Problem on Chromatic Polynomials, Discrete Mathematics*", 172(1997), 39-44.
4. **Douglas B. West.,** "*Introduction to Graph Theory*", Second Edition, 2006.
5. **Godsil, C. D.,** "*Real Graph Polynomial, Progress in Graph Theory*" (Academic press Canada), (1984) 281-292.
6. **Godsil, C. D., Gutman, I.,** "*On the Theory of the Matching Polynomial, J. Graph Theory*", 5 (1981) 137-144.
7. https:\\en.wikipedia.org\wiki\chromatic polynomial
8. mathworld.wolfram.com\chromatic polynomial.html