

CHAPTER – VI

Bipolar Spherical Neutrosophic Graph Coloring

6.1 Introduction

A vertex coloring of a graph G is assigning colors to each vertices such that no two adjacent vertices share the same color. It is applicable in many real life problems such as computer networks, bioinformatics, telecommunications, scheduling etc. Susana Munoz, Teresa Ortuno, Javier Ramirez and Javier Yanez (2005) proposed the chromatic number of fuzzy graph. Later Eslahchi and Onagh (2006) extended fuzzy graph coloring. Prasanna, Rifayathali and Isamil Mohideen (2017) described the strong and complete intuitionistic fuzzy graphs. Now in this chapter, we propose the concept of coloring the spherical fuzzy graph and describe some of their properties such as union, strong spherical fuzzy graph, complement of strong spherical fuzzy graph and complete spherical fuzzy graph with illustrative example.

6.2 Vertex coloring of Spherical Fuzzy Graph

In this section, we have defined vertex coloring of spherical neutrosophic graph, complement of spherical neutrosophic graph and the union of spherical neutrosophic graph.

Definition 6.2.1. A spherical neutrosophic graph on an underlying set V is a pair $G = (A, B)$ where A is a spherical neutrosophic set in V and B is a spherical neutrosophic relation on $V \times V$ such that

$$T_B(x, y) \leq (T_A(x) \wedge T_A(y)),$$

$$I_B(x, y) \leq (I_A(x) \wedge I_A(y)),$$

$$F_B(x, y) \leq (F_A(x) \vee F_A(y))$$

where T_B denote the truth membership function, I_B denote the indeterminacy membership function, F_B denote the falsity membership function and fulfils the following condition: $0 \leq \left((T_B)^2 + (I_B)^2 + (F_B)^2 \right) \leq \sqrt{3}$ and where A is a spherical neutrosophic vertex set and B is a spherical neutrosophic edge set of G .

Definition 6.2.2. The Spherical neutrosophic graph $G = (A, B)$ consisting of a family $P = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of spherical neutrosophic set A is called a k -vertex coloring if

$$(i) \max\{\alpha_n(a)\} = A, \quad \forall a \in A$$

$$(ii) \min\{\alpha_n, \alpha_m\} = 0$$

$$(iii) \forall \text{ strong edge } ab \text{ of } G,$$

$$\alpha_n(T_1(a)) \wedge \alpha_n(T_1(b)) = 0,$$

$$\alpha_n(I_1(a)) \wedge \alpha_n(I_1(b)) = 0, \text{ and}$$

$$\alpha_n(F_1(a)) \vee \alpha_n(F_1(b)) = 1, (1 \leq n \leq k)$$

The minimum colors for G with k -vertex coloring is denoted by $\chi_\alpha(G)$, is called the chromatic number of the spherical neutrosophic graph G .

Example 6.2.3. Consider the spherical neutrosophic graph $G = (A, B)$ with vertex set $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and the edge set

$B = \{a_{12}, a_{14}, a_{16}, a_{23}, a_{25}, a_{34}, a_{36}, a_{45}, a_{56}\}$, the membership functions defined as follows,

$$(T_1(a_n), I_1(a_n), F_1(a_n)) = \begin{cases} (0.7, 0.4, 0.8) & \text{for } n = 1 \\ (0.6, 0.8, 0.5) & \text{for } n = 2 \\ (0.8, 0.5, 0.9) & \text{for } n = 3 \\ (0.6, 0.7, 0.9) & \text{for } n = 4 \\ (0.6, 0.6, 0.7) & \text{for } n = 5 \\ (0.8, 0.9, 0.4) & \text{for } n = 6 \end{cases}$$

$$(T_2(a_{nm}), I_2(a_{nm}), F_2(a_{nm})) = \begin{cases} (0.6, 0.4, 0.8) & \text{for } nm = 12 \\ (0.6, 0.4, 0.9) & \text{for } nm = 14 \\ (0.7, 0.4, 0.8) & \text{for } nm = 16 \\ (0.6, 0.5, 0.9) & \text{for } nm = 23 \\ (0.6, 0.6, 0.7) & \text{for } nm = 25 \\ (0.6, 0.5, 0.9) & \text{for } nm = 34 \\ (0.8, 0.5, 0.9) & \text{for } nm = 36 \\ (0.6, 0.6, 0.9) & \text{for } nm = 45 \\ (0.6, 0.6, 0.7) & \text{for } nm = 56 \end{cases}$$

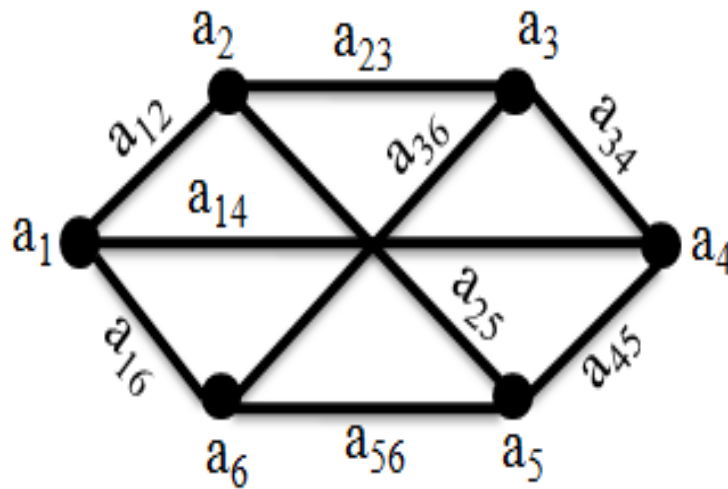


Fig. 6.1: SNG

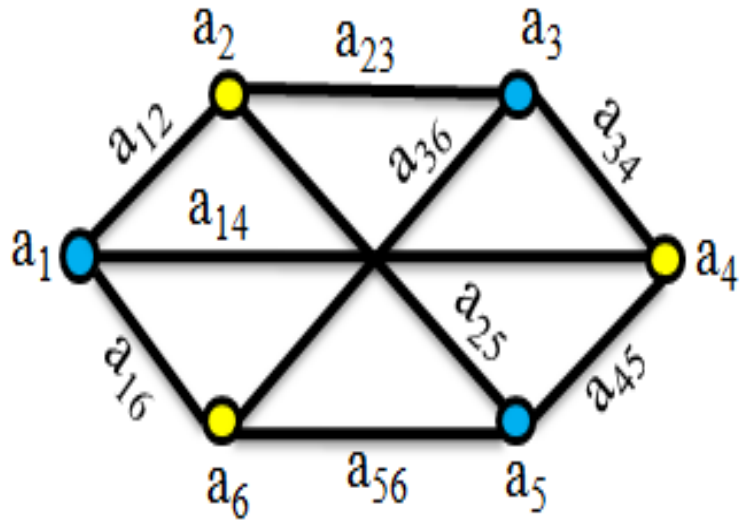


Fig. 6.2: Vertex coloring of SNG

Vertex	Strongly Adjacent Vertices
a_1	a_2, a_4, a_6
a_2	a_1, a_3, a_5
a_3	a_2, a_4, a_6
a_4	a_1, a_3, a_5
a_5	a_2, a_4, a_6
a_6	a_1, a_3, a_5

Here $\deg_s(A)$ = Number of edges incident on a vertices that are strongly adjacent to A. Now $\deg_s(a_1) = 3$, $\deg_s(a_2) = 3$, $\deg_s(a_3) = 3$, $\deg_s(a_4) = 3$, $\deg_s(a_5) = 3$ and $\deg_s(a_6) = 3$. Therefore the above results satisfies the following condition, $\deg(A) \geq \deg_s(A)$.

Since a_2, a_4, a_6 are strongly adjacent to a_1 so they do not have same color. Thus $\phi = \{\phi_1, \phi_2\}$ be the family of spherical neutrosophic sets.

$$\phi_1(a_n) = \begin{cases} (0.7, 0.4, 0.8) & \text{for } n = 1 \\ (0.8, 0.5, 0.9) & \text{for } n = 3 \\ (0.6, 0.6, 0.7) & \text{for } n = 5 \\ (0, 0, 1) & \text{otherwise} \end{cases},$$

$$\phi_2(a_n) = \begin{cases} (0.6, 0.8, 0.5) & \text{for } n = 2 \\ (0.6, 0.7, 0.9) & \text{for } n = 4 \\ (0.8, 0.9, 0.4) & \text{for } n = 6 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

Hence the spherical neutrosophic chromatic number of G is $\chi_\alpha(G) = 2$.

Definition 6.2.4. The complement of a spherical neutrosophic graph $G = (A, B)$ is a $\bar{G} = (\bar{A}, \bar{B})$, where

$$(i) \bar{A} = A$$

$$(ii) \forall a \in A, \bar{T}_A(a) = T_A(a), \bar{I}_A(a) = I_A(a) \text{ and } \bar{F}_A(a) = F_A(a)$$

$$(iii) \forall a, b \in A, \bar{T}_A(ab) = \begin{cases} T_A(a) \wedge T_A(b) & \text{if } T_B(ab) = 0 \\ (T_A(a) \wedge T_A(b)) - T_B(ab) & \text{if } T_B(ab) > 0 \end{cases}$$

$$\bar{I}_A(ab) = \begin{cases} I_A(a) \wedge I_A(b) & \text{if } I_B(ab) = 0 \\ (I_A(a) \wedge I_A(b)) - I_B(ab) & \text{if } I_B(ab) > 0 \end{cases},$$

$$\bar{F}_A(ab) = \begin{cases} F_A(a) \vee F_A(b) & \text{if } F_B(ab) = 0 \\ (F_A(a) \vee F_A(b)) - F_B(ab) & \text{if } F_B(ab) > 0 \end{cases}$$

Definition 6.2.5. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be spherical neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. The union of G_1 and G_2 is defined as follows:

$$T_A(a) = \begin{cases} T_{A_1}(a) & \text{if } a \in V_1 \text{ and } a \notin V_2 \\ T_{A_2}(a) & \text{if } a \notin V_1 \text{ and } a \in V_2 \\ T_{A_1}(a) \vee T_{A_2}(a) & \text{if } a \in V_1 \cap V_2 \end{cases}$$

$$\begin{aligned}
I_A(a) &= \begin{cases} I_{A_1}(a) & \text{if } a \in V_1 \text{ and } a \notin V_2 \\ I_{A_2}(a) & \text{if } a \notin V_1 \text{ and } a \in V_2 \\ I_{A_1}(a) \vee I_{A_2}(a) & \text{if } a \in V_1 \cap V_2 \end{cases} \\
F_A(a) &= \begin{cases} F_{A_1}(a) & \text{if } a \in V_1 \text{ and } a \notin V_2 \\ F_{A_2}(a) & \text{if } a \notin V_1 \text{ and } a \in V_2 \\ F_{A_1}(a) \wedge F_{A_2}(a) & \text{if } a \in V_1 \cap V_2 \end{cases} \\
T_B(ab) &= \begin{cases} T_{B_1}(ab) & \text{if } ab \in E_1 \text{ and } a \notin E_2 \\ T_{B_2}(ab) & \text{if } ab \in E_2 \text{ and } a \notin E_1 \\ T_{B_1}(a) \vee T_{B_2}(a) & \text{if } a \in E_1 \cap E_2 \end{cases} \\
I_B(ab) &= \begin{cases} I_{B_1}(ab) & \text{if } ab \in E_1 \text{ and } a \notin E_2 \\ I_{B_2}(ab) & \text{if } ab \in E_2 \text{ and } a \notin E_1 \\ I_{B_1}(a) \vee I_{B_2}(a) & \text{if } a \in E_1 \cap E_2 \end{cases} \\
F_B(ab) &= \begin{cases} F_{B_1}(ab) & \text{if } ab \in E_1 \text{ and } a \notin E_2 \\ F_{B_2}(ab) & \text{if } ab \in E_2 \text{ and } a \notin E_1 \\ F_{B_1}(a) \wedge F_{B_2}(a) & \text{if } a \in E_1 \cap E_2 \end{cases}
\end{aligned}$$

For any spherical neutrosophic graphs G_1 and G_2 ,
 $\chi_\alpha(G_1 \cup G_2) = \chi_\alpha(G_1) \vee \chi_\alpha(G_2)$.

Definition 6.2.6. A spherical neutrosophic graph $G = (A, B)$ is called Complete Spherical neutrosophic Graph (CSNG) if the following conditions are satisfied:

$$\begin{aligned}
T_B(ab) &= T_A(a) \wedge T_A(b), \\
I_B(ab) &= I_A(a) \wedge I_A(b), \\
F_B(ab) &= F_A(a) \vee F_A(b) \quad \forall a, b \in A.
\end{aligned}$$

Example 6.2.7. Consider the complete spherical neutrosophic graph $G = (A, B)$ with vertex set $A = \{a_1, a_2, a_3, a_4, a_5\}$ and the edge set $B = \{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}, a_{34}, a_{35}, a_{45}\}$, the membership functions defined as follows,

$$(T_1(a_n), I_1(a_n), F_1(a_n)) = \begin{cases} (0.8, 0.2, 0.9) & \text{for } n = 1 \\ (0.9, 0.4, 0.8) & \text{for } n = 2 \\ (0.7, 0.6, 0.2) & \text{for } n = 3 \\ (0.8, 0.4, 0.9) & \text{for } n = 4 \\ (0.6, 0.7, 0.3) & \text{for } n = 5 \end{cases}$$

$$(T_2(a_{nm}), I_2(a_{nm}), F_2(a_{nm})) = \begin{cases} (0.8, 0.2, 0.9) & \text{for } nm = 12, 14 \\ (0.7, 0.2, 0.9) & \text{for } nm = 13 \\ (0.6, 0.2, 0.9) & \text{for } nm = 15 \\ (0.7, 0.4, 0.8) & \text{for } nm = 23 \\ (0.8, 0.4, 0.9) & \text{for } nm = 24 \\ (0.6, 0.4, 0.8) & \text{for } nm = 25 \\ (0.7, 0.4, 0.9) & \text{for } nm = 34 \\ (0.6, 0.6, 0.3) & \text{for } nm = 35 \\ (0.6, 0.4, 0.9) & \text{for } nm = 45 \end{cases}$$

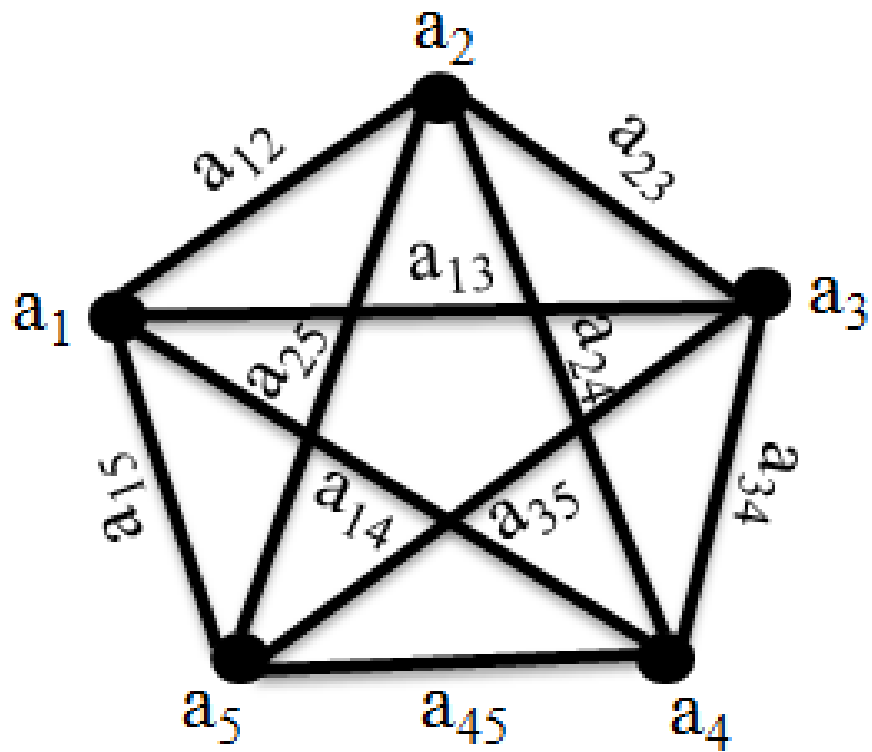


Fig. 6.3: CSNG

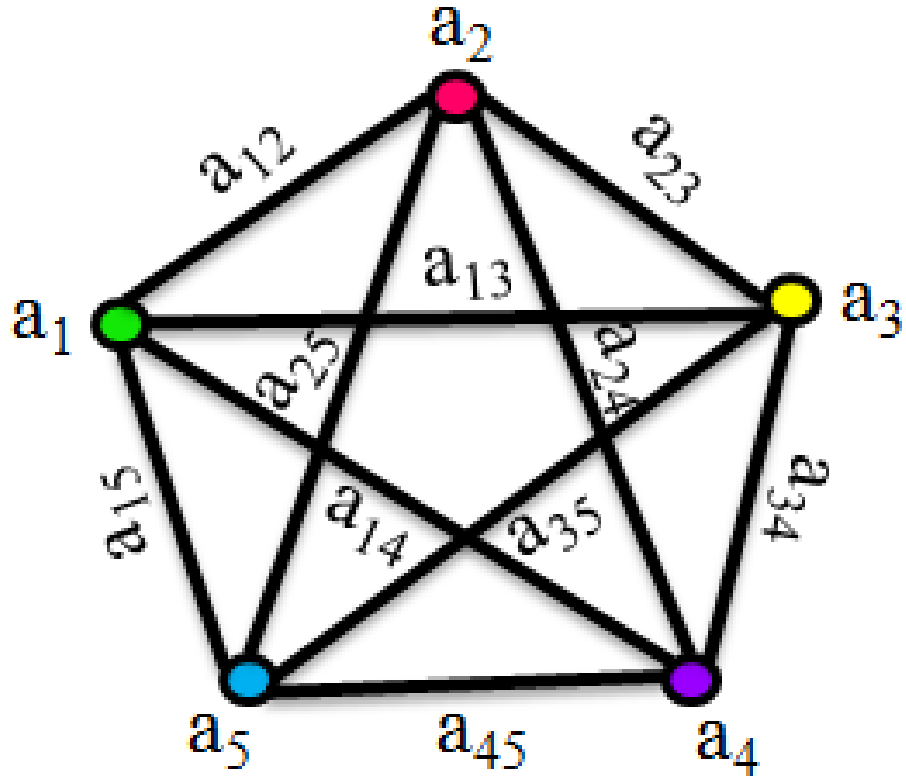


Fig. 6.4: Vertex coloring of CSNG

Vertex	Strongly Adjacent Vertices
a_1	a_2, a_3, a_4, a_5
a_2	a_1, a_3, a_4, a_5
a_3	a_1, a_2, a_4, a_5
a_4	a_1, a_2, a_3, a_5
a_5	a_1, a_2, a_3, a_4

Here $\deg_s(A)$ = Number of edges incident on a vertices that are strongly adjacent to A. Now $\deg_s(a_1) = 4$, $\deg_s(a_2) = 4$, $\deg_s(a_3) = 4$, $\deg_s(a_4) = 4$, $\deg_s(a_5) = 4$ and $\deg_s(a_6) = 4$. Therefore the above results satisfies the following condition, $\deg(A) \geq \deg_s(A)$.

Since a_2, a_3, a_4, a_5 are strongly adjacent to a_1 so they do not have same color.

Thus $\phi = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\}$ be the family of spherical neutrosophic sets.

$$\begin{aligned}\phi_1(a_n) &= \begin{cases} (0.8, 0.2, 0.9) & \text{for } n = 1 \\ (0, 0, 1) & \text{otherwise} \end{cases} \\ \phi_2(a_n) &= \begin{cases} (0.9, 0.4, 0.8) & \text{for } n = 2 \\ (0, 0, 1) & \text{otherwise} \end{cases} & \phi_3(a_n) &= \begin{cases} (0.7, 0.6, 0.2) & \text{for } n = 3 \\ (0, 0, 1) & \text{otherwise} \end{cases} \\ \phi_4(a_n) &= \begin{cases} (0.8, 0.4, 0.9) & \text{for } n = 4 \\ (0, 0, 1) & \text{otherwise} \end{cases} & \phi_5(a_n) &= \begin{cases} (0.6, 0.7, 0.3) & \text{for } n = 5 \\ (0, 0, 1) & \text{otherwise} \end{cases}\end{aligned}$$

Hence the family ϕ fulfilled the conditions of complete spherical neutrosophic vertex coloring of G . Hence the complete spherical neutrosophic chromatic number of G is $\chi_\alpha(G) = 5$.

Definition 6.2.8. A spherical neutrosophic graph $G = (A, B)$ is called Strong Spherical neutrosophic Graph (SSNG) if the following conditions are satisfied:

$$\begin{aligned}T_B(ab) &= T_A(a) \wedge T_A(b), \\ I_B(ab) &= I_A(a) \wedge I_A(b), \\ F_B(ab) &= F_A(a) \vee F_A(b) \quad \forall a, b \in B.\end{aligned}$$

Definition 6.2.9. The complement of a spherical neutrosophic graph $G = (A, B)$ is defined by $\bar{G} = (\bar{A}, \bar{B})$, where

$$\begin{aligned}(i) \quad \bar{A} &= A \\ (ii) \quad \forall a \in A, \quad \bar{T}_A(a) &= T_A(a), \quad \bar{I}_A(a) = I_A(a) \quad \text{and} \quad \bar{F}_A(a) = F_A(a) \\ (iii) \quad \forall a, b \in A, \quad \bar{T}_A(ab) &= \begin{cases} T_A(a) \wedge T_A(b) & \text{if } T_B(ab) = 0 \\ 0 & \text{if } T_B(ab) > 0 \end{cases} \\ \bar{I}_A(ab) &= \begin{cases} I_A(a) \wedge I_A(b) & \text{if } I_B(ab) = 0 \\ 0 & \text{if } I_B(ab) > 0 \end{cases} \\ \bar{F}_A(ab) &= \begin{cases} F_A(a) \vee F_A(b) & \text{if } F_B(ab) = 0 \\ 0 & \text{if } F_B(ab) > 0 \end{cases}\end{aligned}$$

Example 6.2.10. Consider the strong spherical neutrosophic graph $G=(A,B)$ with vertex set $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and the edge set $B = \{a_{12}, a_{16}, a_{25}, a_{34}, a_{36}, a_{45}\}$, the membership functions defined as follows,

$$(T_1(a_n), I_1(a_n), F_1(a_n)) = \begin{cases} (0.7, 0.5, 0.1) & \text{for } n = 1 \\ (0.8, 0.4, 0.5) & \text{for } n = 2 \\ (0.3, 0.6, 0.7) & \text{for } n = 3 \\ (0.5, 0.5, 0.9) & \text{for } n = 4 \\ (0.2, 0.7, 0.3) & \text{for } n = 5 \\ (0.4, 0.8, 0.2) & \text{for } n = 6 \end{cases}$$

$$(T_2(a_{nm}), I_2(a_{nm}), F_2(a_{nm})) = \begin{cases} (0.7, 0.4, 0.5) & \text{for } nm = 12 \\ (0.4, 0.5, 0.2) & \text{for } nm = 16 \\ (0.2, 0.4, 0.5) & \text{for } nm = 25 \\ (0.3, 0.5, 0.9) & \text{for } nm = 34 \\ (0.3, 0.6, 0.7) & \text{for } nm = 36 \\ (0.2, 0.5, 0.9) & \text{for } nm = 45 \end{cases}$$

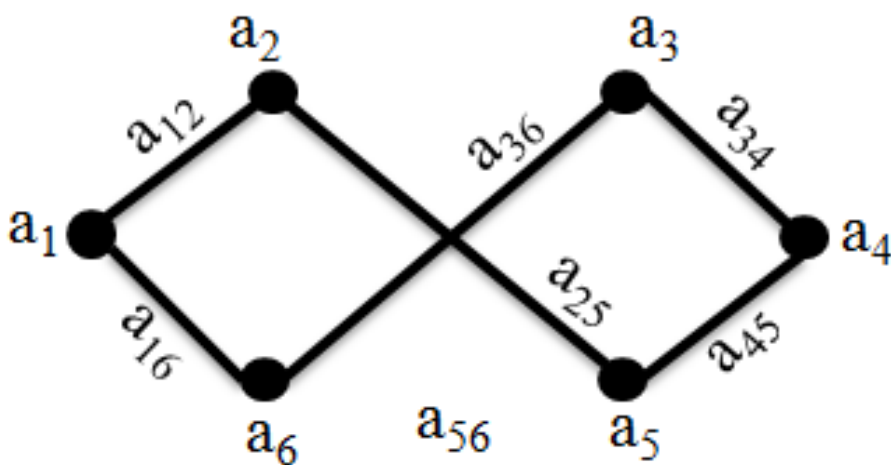


Fig. 6.5: SSNG

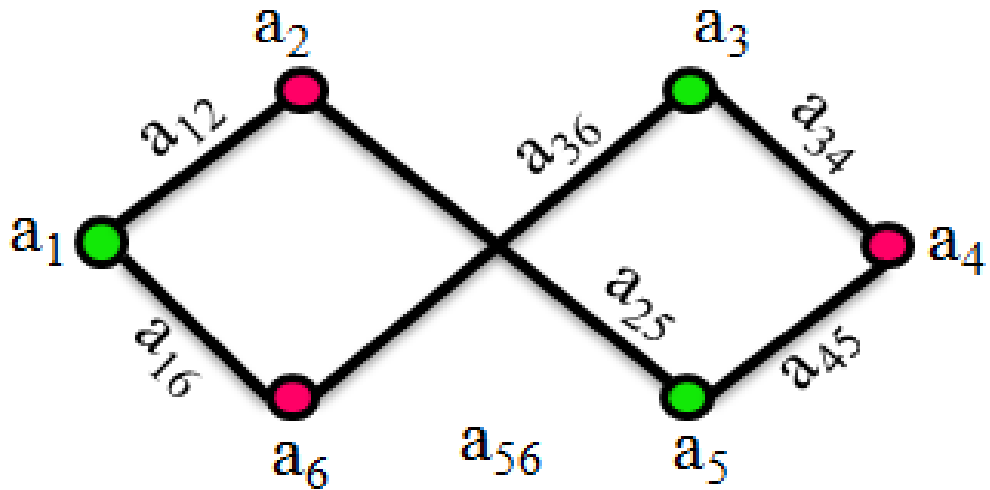


Fig. 6.6: Vertex coloring of SSNG

Vertex	Strongly Adjacent Vertices
a_1	a_2, a_6
a_2	a_1, a_5
a_3	a_4, a_6
a_4	a_3, a_5
a_5	a_2, a_4
a_6	a_1, a_3

Here $\deg_s(A)$ = Number of edges incident on a vertices that are strongly adjacent to A. Now $\deg_s(a_1) = 2$, $\deg_s(a_2) = 2$, $\deg_s(a_3) = 2$, $\deg_s(a_4) = 2$, $\deg_s(a_5) = 2$ and $\deg_s(a_6) = 2$. Therefore the above results satisfies the following condition, $\deg(A) \geq \deg_s(A)$.

Since a_2, a_6 are strongly adjacent to a_1 so they do not have same color. Thus $\phi = \{\phi_1, \phi_2\}$ be the family of spherical neutrosophic sets.

$$\phi_1(a_n) = \begin{cases} (0.7, 0.5, 0.1) & \text{for } n = 1 \\ (0.3, 0.6, 0.7) & \text{for } n = 3 \\ (0.2, 0.7, 0.3) & \text{for } n = 5 \\ (0, 0, 1) & \text{otherwise} \end{cases},$$

$$\phi_2(a_n) = \begin{cases} (0.8, 0.4, 0.5) & \text{for } n = 2 \\ (0.5, 0.5, 0.9) & \text{for } n = 4 \\ (0.4, 0.8, 0.2) & \text{for } n = 6 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

Hence the family ϕ fulfilled the conditions of strong spherical neutrosophic vertex coloring of G . Hence the strong spherical neutrosophic chromatic number of G is $\chi_\alpha(G) = 2$.

Theorem 6.2.11. For any complete spherical neutrosophic graph with n vertices, $\chi_\alpha(G) \leq n$.

Proof. By the definition of complete spherical neutrosophic graph, all the vertices are strongly adjacent to each other. The complete spherical neutrosophic graph with n vertices has spherical neutrosophic vertex coloring is n . By the observations, all the other graphs with n vertices are sub graphs of complete spherical neutrosophic graph. Hence $\chi_\alpha(G) \leq n$.

Theorem 6.2.12. For a path graph P_m , $\chi_\alpha(P_m) = 2$ where $m \geq 2$.

Proof. Consider a family $P = \{\alpha_1, \alpha_2\}$ of spherical neutrosophic sets defined on V as follows:

$$\alpha_1(a_n) = \begin{cases} (T(a_n), I(a_n), F(a_n)) & \text{for } n = \text{odd} \\ (0, 0, 1) & \text{for } n = \text{even} \end{cases}$$

$$\alpha_2(a_n) = \begin{cases} (T(a_n), I(a_n), F(a_n)) & \text{for } n = \text{even} \\ (0, 0, 1) & \text{for } n = \text{odd} \end{cases}$$

Hence the family P fulfilled the conditions of Spherical neutrosophic vertex coloring of G . Hence the spherical neutrosophic chromatic number of P_m is $\chi_\alpha(P_m) = 2$.

Theorem 6.2.13. For a cycle graph C_m , $\chi_\alpha(C_m) = \begin{cases} 2 & \text{if } m = \text{even} \\ 3 & \text{if } m = \text{odd} \end{cases}$ where $m \geq 3$

.

Proof.

For m is even: Consider a family $P = \{\alpha_1, \alpha_2\}$ of spherical neutrosophic sets defined on V as follows:

$$\alpha_1(a_n) = \begin{cases} (T(a_n), I(a_n), F(a_n)) & \text{for } n = \text{odd} \\ (0, 0, 1) & \text{for } n = \text{even} \end{cases}$$

$$\alpha_2(a_n) = \begin{cases} (T(a_n), I(a_n), F(a_n)) & \text{for } n = \text{even} \\ (0, 0, 1) & \text{for } n = \text{odd} \end{cases}$$

Hence the family P fulfilled the conditions of spherical neutrosophic vertex coloring of G . Hence the spherical neutrosophic chromatic number of C_m is $\chi_\alpha(C_m) = 2$.

For m is odd: Consider a family $P = \{\alpha_1, \alpha_2, \alpha_3\}$ of spherical neutrosophic sets defined on V as follows:

$$\alpha_1(a_n) = \begin{cases} (T(a_n), I(a_n), F(a_n)) & \text{for } n = 1, 3, \dots, m-2 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\alpha_2(a_n) = \begin{cases} (T(a_n), I(a_n), F(a_n)) & \text{for } n = 2, 4, \dots, m-1 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\alpha_3(a_n) = \begin{cases} (T(a_n), I(a_n), F(a_n)) & \text{for } n = m \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

Hence the family P fulfilled the conditions of spherical neutrosophic vertex coloring of G . Hence the spherical neutrosophic chromatic number of C_m is $\chi_\alpha(C_m) = 3$.

6.3 Vertex coloring of Bipolar Spherical Neutrosophic Graph

Definition 6.3.1. The bipolar spherical neutrosophic graph $G = (A, B)$ consisting of a family $P = (P^+, P^-) = (\{\alpha_1^+, \alpha_2^+, \dots, \alpha_{k_1}^+\}, \{\alpha_1^-, \alpha_2^-, \dots, \alpha_{k_2}^-\})$ of bipolar spherical neutrosophic set A is called a (k_1, k_2) - bipolar spherical neutrosophic coloring if

$$(i) \max\{\alpha_n^+(a)\} = A, \min\{\alpha_n^-(a)\} = A, \quad \forall a \in A$$

$$(ii) \min\{\alpha_n^+, \alpha_m^+\} = 0, \max\{\alpha_n^-, \alpha_m^-\} = 0$$

$$(iii) \forall \text{ strong edge } ab \text{ of } G,$$

$$\alpha_n^+(T_1(a)) \wedge \alpha_n^+(T_1(b)) = 0, \alpha_n^-(T_1(a)) \vee \alpha_n^-(T_1(b)) = 0$$

$$\alpha_n^+(I_1(a)) \wedge \alpha_n^+(I_1(b)) = 0, \alpha_n^-(I_1(a)) \vee \alpha_n^-(I_1(b)) = 0 \text{ and}$$

$$\alpha_n^+(F_1(a)) \vee \alpha_n^+(F_1(b)) = 1, \alpha_n^-(F_1(a)) \wedge \alpha_n^-(F_1(b)) = 1 \quad (1 \leq n \leq k_1, k_2)$$

The minimum colors for G with (k_1, k_2) -bipolar spherical neutrosophic coloring is denoted by $\chi_\alpha(G) = (\chi_{\alpha^+}(G), \chi_{\alpha^-}(G))$, is called the chromatic number of the bipolar spherical neutrosophic graph G .

Example 6.3.2. Consider the spherical neutrosophic graph $G = (A, B)$ with vertex set $A = \{a_1, a_2, a_3, a_4\}$ and the edge set $B = \{a_{12}, a_{14}, a_{23}, a_{24}, a_{34}\}$, the membership functions defined as follows,

$$(T_1(a_n), I_1(a_n), F_1(a_n)) = \begin{cases} (0.5, 0.7, 0.9, -0.8, -0.6, -0.4) & \text{for } n = 1 \\ (0.6, 0.2, 0.5, -0.7, -0.1, -0.9) & \text{for } n = 2 \\ (0.8, 0.6, 0.4, -0.3, -0.4, -0.6) & \text{for } n = 3 \\ (0.2, 0.9, 0.5, -0.6, -0.7, -0.4) & \text{for } n = 4 \end{cases}$$

$$(T_2(a_{nm}), I_2(a_{nm}), F_2(a_{nm})) = \begin{cases} (0.5, 0.2, 0.9, -0.7, -0.1, -0.9) & \text{for } nm = 12 \\ (0.2, 0.7, 0.9, -0.6, -0.6, -0.4) & \text{for } nm = 14 \\ (0.6, 0.2, 0.5, -0.3, -0.1, -0.9) & \text{for } nm = 23 \\ (0.2, 0.2, 0.5, -0.6, -0.1, -0.9) & \text{for } nm = 24 \\ (0.2, 0.6, 0.5, -0.3, -0.4, -0.6) & \text{for } nm = 34 \end{cases}$$

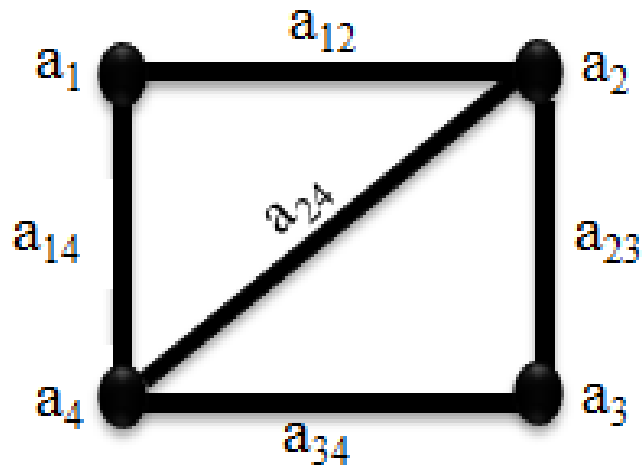


Fig. 6.7: BSNG

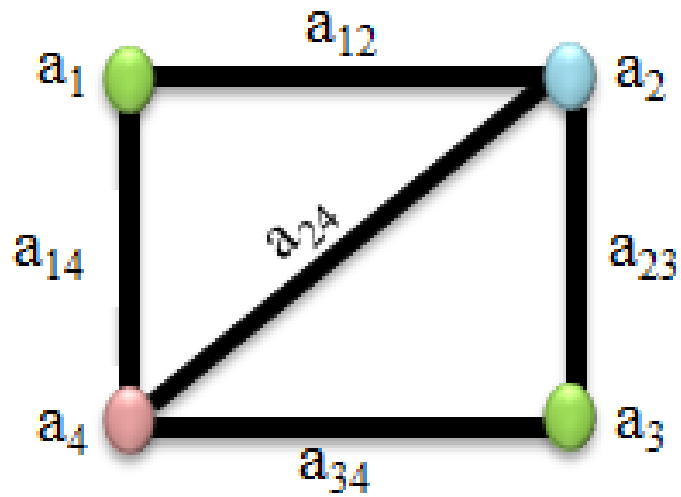


Fig. 6.8: Vertex coloring of BSNG

Vertex	Strongly Adjacent Vertices
a_1	a_2, a_4
a_2	a_1, a_3, a_4
a_3	a_2, a_4
a_4	a_1, a_2, a_3

Here $\deg_s(A)$ = Number of edges incident on a vertices that are strongly adjacent to A. Now $\deg_s(a_1) = 2$, $\deg_s(a_2) = 3$, $\deg_s(a_3) = 2$ $\deg_s(a_4) = 3$.

Therefore the above results satisfies the following condition, $\deg(A) \geq \deg_s(A)$

Since a_2, a_4 are strongly adjacent to a_1 so they do not have same color. Thus

$\phi = (\{\phi_1^+, \phi_2^+, \phi_3^+\}, \{\phi_1^-, \phi_2^-, \phi_3^-\})$ be the family of spherical neutrosophic sets.

$$\phi_1^+(a_n) = \begin{cases} (0.5, 0.7, 0.9) & \text{for } n = 1 \\ (0.8, 0.6, 0.4) & \text{for } n = 3 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\phi_1^-(a_n) = \begin{cases} (-0.8, -0.6, -0.4) & \text{for } n = 1 \\ (-0.3, -0.4, -0.6) & \text{for } n = 3 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\phi_2^+(a_n) = \begin{cases} (0.6, 0.2, 0.5) & \text{for } n = 2 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\phi_2^-(a_n) = \begin{cases} (-0.7, -0.1, -0.9) & \text{for } n = 2 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\phi_3^+(a_n) = \begin{cases} (0.2, 0.9, 0.5) & \text{for } n = 4 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\phi_3^-(a_n) = \begin{cases} (-0.6, -0.7, -0.4) & \text{for } n = 4 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

Hence the spherical neutrosophic chromatic number of G is $\chi_\alpha(G) = (\chi_{\alpha^+}(G), \chi_{\alpha^-}(G)) = (3, 3)$.

Definition 6.3.3. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar spherical neutrosophic graphs of G_1^* and G_2^* respectively. Then $G_1 \cup G_2$ is defined as follows:

$$T_A^+(a) = \begin{cases} T_{A_1}^+(a) & \text{if } a \in V_1 \text{ and } a \notin V_2 \\ T_{A_2}^+(a) & \text{if } a \notin V_1 \text{ and } a \in V_2 \\ T_{A_1}^+(a) \vee T_{A_2}^+(a) & \text{if } a \in V_1 \cap V_2 \end{cases}$$

$$T_A^-(a) = \begin{cases} T_{A_1}^-(a) & \text{if } a \in V_1 \text{ and } a \notin V_2 \\ T_{A_2}^-(a) & \text{if } a \notin V_1 \text{ and } a \in V_2 \\ T_{A_1}^-(a) \wedge T_{A_2}^-(a) & \text{if } a \in V_1 \cap V_2 \end{cases}$$

$$I_A^+(a) = \begin{cases} I_{A_1}^+(a) & \text{if } a \in V_1 \text{ and } a \notin V_2 \\ I_{A_2}^+(a) & \text{if } a \notin V_1 \text{ and } a \in V_2 \\ I_{A_1}^+(a) \vee I_{A_2}^+(a) & \text{if } a \in V_1 \cap V_2 \end{cases}$$

$$I_A^-(a) = \begin{cases} I_{A_1}^-(a) & \text{if } a \in V_1 \text{ and } a \notin V_2 \\ I_{A_2}^-(a) & \text{if } a \notin V_1 \text{ and } a \in V_2 \\ I_{A_1}^-(a) \wedge I_{A_2}^-(a) & \text{if } a \in V_1 \cap V_2 \end{cases}$$

$$F_A^+(a) = \begin{cases} F_{A_1}^+(a) & \text{if } a \in V_1 \text{ and } a \notin V_2 \\ F_{A_2}^+(a) & \text{if } a \notin V_1 \text{ and } a \in V_2 \\ F_{A_1}^+(a) \wedge F_{A_2}^+(a) & \text{if } a \in V_1 \cap V_2 \end{cases}$$

$$F_A^-(a) = \begin{cases} F_{A_1}^-(a) & \text{if } a \in V_1 \text{ and } a \notin V_2 \\ F_{A_2}^-(a) & \text{if } a \notin V_1 \text{ and } a \in V_2 \\ F_{A_1}^-(a) \vee F_{A_2}^-(a) & \text{if } a \in V_1 \cap V_2 \end{cases}$$

$$T_B^+(ab) = \begin{cases} T_{B_1}^+(ab) & \text{if } ab \in E_1 \text{ and } a \notin E_2 \\ T_{B_2}^+(ab) & \text{if } ab \in E_2 \text{ and } a \notin E_1 \\ T_{B_1}^+(a) \vee T_{B_2}^+(a) & \text{if } a \in E_1 \cap E_2 \end{cases}$$

$$T_B^-(ab) = \begin{cases} T_{B_1}^-(ab) & \text{if } ab \in E_1 \text{ and } a \notin E_2 \\ T_{B_2}^-(ab) & \text{if } ab \in E_2 \text{ and } a \notin E_1 \\ T_{B_1}^-(a) \wedge T_{B_2}^-(a) & \text{if } a \in E_1 \cap E_2 \end{cases}$$

$$I_B^+(ab) = \begin{cases} I_{B_1}^+(ab) & \text{if } ab \in E_1 \text{ and } a \notin E_2 \\ I_{B_2}^+(ab) & \text{if } ab \in E_2 \text{ and } a \notin E_1 \\ I_{B_1}^+(a) \vee I_{B_2}^+(a) & \text{if } a \in E_1 \cap E_2 \end{cases}$$

$$I_B^-(ab) = \begin{cases} I_{B_1}^-(ab) & \text{if } ab \in E_1 \text{ and } a \notin E_2 \\ I_{B_2}^-(ab) & \text{if } ab \in E_2 \text{ and } a \notin E_1 \\ I_{B_1}^-(a) \wedge I_{B_2}^-(a) & \text{if } a \in E_1 \cap E_2 \end{cases}$$

$$F_B^+(ab) = \begin{cases} F_{B_1}^+(ab) & \text{if } ab \in E_1 \text{ and } a \notin E_2 \\ F_{B_2}^+(ab) & \text{if } ab \in E_2 \text{ and } a \notin E_1 \\ F_{B_1}^+(a) \wedge F_{B_2}^+(a) & \text{if } a \in E_1 \cap E_2 \end{cases}$$

$$F_B^-(ab) = \begin{cases} F_{B_1}^-(ab) & \text{if } ab \in E_1 \text{ and } a \notin E_2 \\ F_{B_2}^-(ab) & \text{if } ab \in E_2 \text{ and } a \notin E_1 \\ F_{B_1}^-(a) \vee F_{B_2}^-(a) & \text{if } a \in E_1 \cap E_2 \end{cases}$$

For any bipolar spherical neutrosophic graphs G_1 and G_2 ,
 $\chi_\alpha(G_1 \cup G_2) = \chi_\alpha(G_1) \vee \chi_\alpha(G_2)$.

Definition 6.3.4. A bipolar spherical neutrosophic graph $G = (A, B)$ is called Complete Bipolar Spherical Neutrosophic Graph (CBSNG) if the following conditions are satisfied:

$$T_B^+(ab) = T_A^+(a) \wedge T_A^+(b), \quad T_B^-(ab) = T_A^-(a) \vee T_A^-(b)$$

$$I_B^+(ab) = I_A^+(a) \wedge I_A^+(b), \quad I_B^-(ab) = I_A^-(a) \vee I_A^-(b)$$

$$F_B^+(ab) = F_A^+(a) \vee F_A^+(b), \quad F_B^-(ab) = F_A^-(a) \wedge F_A^-(b) \quad \forall a, b \in A.$$

Example 6.3.5. Consider the complete bipolar spherical neutrosophic graph $G = (A, B)$ with vertex set $A = \{a_1, a_2, a_3, a_4\}$ and the edge set $B = \{a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}\}$, the membership functions defined as follows,

$$(T_1(a_n), I_1(a_n), F_1(a_n)) = \begin{cases} (0.4, 0.6, 0.5, -0.3, -0.5, -0.1) & \text{for } n = 1 \\ (0.2, 0.5, 0.7, -0.1, -0.4, -0.8) & \text{for } n = 2 \\ (0.8, 0.7, 0.1, -0.9, -0.2, -0.1) & \text{for } n = 3 \\ (0.1, 0.4, 0.3, -0.7, -0.5, -0.2) & \text{for } n = 4 \end{cases}$$

$$(T_2(a_{nm}), I_2(a_{nm}), F_2(a_{nm})) = \begin{cases} (0.2, 0.5, 0.7, -0.1, -0.4, -0.8) & \text{for } nm = 12 \\ (0.4, 0.6, 0.5, -0.3, -0.2, -0.1) & \text{for } nm = 13 \\ (0.1, 0.4, 0.5, -0.3, -0.5, -0.2) & \text{for } nm = 14 \\ (0.2, 0.5, 0.7, -0.1, -0.2, -0.8) & \text{for } nm = 23 \\ (0.1, 0.4, 0.7, -0.1, -0.4, -0.8) & \text{for } nm = 24 \\ (0.1, 0.4, 0.3, -0.7, -0.2, -0.2) & \text{for } nm = 34 \end{cases}$$

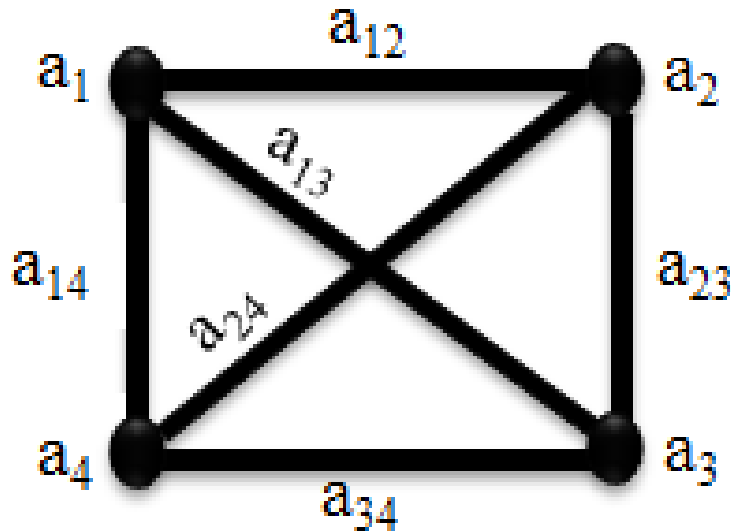


Fig. 6.9: BCSNG

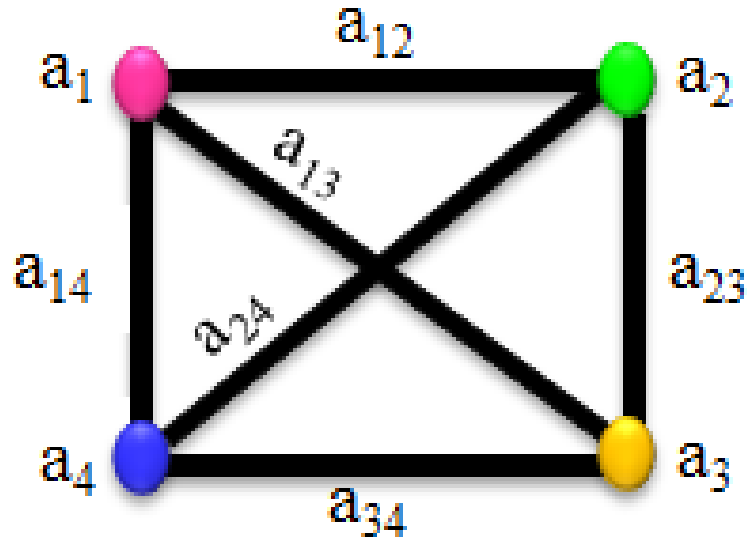


Fig. 6.10: Vertex coloring of BCSNG

Vertex	Strongly Adjacent Vertices
a_1	a_2, a_3, a_4
a_2	a_1, a_3, a_4
a_3	a_1, a_2, a_4
a_4	a_1, a_2, a_3

Here $\deg_s(A)$ = Number of edges incident on a vertices that are strongly adjacent to A . Now $\deg_s(a_1) = 3$, $\deg_s(a_2) = 3$, $\deg_s(a_3) = 3$, $\deg_s(a_4) = 3$. Therefore the above results satisfies the following condition, $\deg(A) \geq \deg_s(A)$. Since a_2, a_3, a_4 are strongly adjacent to a_1 so they do not have same color. Thus $\phi = (\{\phi_1^+, \phi_2^+, \phi_3^+, \phi_4^+\}, \{\phi_1^-, \phi_2^-, \phi_3^-, \phi_4^-\})$ be the family of spherical neutrosophic sets.

$$\phi_1^+(a_n) = \begin{cases} (0.4, 0.6, 0.5) & \text{for } n = 1 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\phi_1^-(a_n) = \begin{cases} (-0.3, -0.5, -0.1) & \text{for } n = 1 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\begin{aligned}\phi_2^+(a_n) &= \begin{cases} (0.2, 0.5, 0.7) & \text{for } n = 2 \\ (0, 0, 1) & \text{otherwise} \end{cases} \\ \phi_2^-(a_n) &= \begin{cases} (-0.1, -0.4, -0.8) & \text{for } n = 2 \\ (0, 0, 1) & \text{otherwise} \end{cases} \\ \phi_3^+(a_n) &= \begin{cases} (0.8, 0.7, 0.1) & \text{for } n = 3 \\ (0, 0, 1) & \text{otherwise} \end{cases} \\ \phi_3^-(a_n) &= \begin{cases} (-0.9, -0.2, -0.1) & \text{for } n = 3 \\ (0, 0, 1) & \text{otherwise} \end{cases} \\ \phi_4^+(a_n) &= \begin{cases} (0.1, 0.4, 0.3) & \text{for } n = 4 \\ (0, 0, 1) & \text{otherwise} \end{cases} \\ \phi_4^-(a_n) &= \begin{cases} (-0.7, -0.5, -0.2) & \text{for } n = 4 \\ (0, 0, 1) & \text{otherwise} \end{cases}\end{aligned}$$

Hence the family ϕ fulfilled the conditions of complete spherical neutrosophic vertex coloring of G . Hence the complete spherical neutrosophic chromatic number of G is $\chi_\alpha(G) = (\chi_{\alpha^+}^C(G), \chi_{\alpha^-}^C(G)) = (4, 4)$.

Theorem 6.3.6. The bipolar spherical neutrosophic vertex chromatic number of complete bipolar spherical neutrosophic graph G is (n, n) , where n is the number of vertices of G . *ie* $\chi_\alpha(G) = (\chi_{\alpha^+}(G), \chi_{\alpha^-}(G)) \leq (n, n)$.

Proof. By the definition of complete bipolar spherical neutrosophic graph, all the vertices are strongly adjacent to each other. The complete bipolar spherical neutrosophic graph with (n, n) vertices has bipolar spherical neutrosophic vertex coloring is (n, n) . By the observations, all the other graphs with n vertices are sub graphs of complete spherical neutrosophic graph. Hence $\chi_\alpha(G) = (\chi_{\alpha^+}(G), \chi_{\alpha^-}(G)) \leq (n, n)$.

Definition 6.3.7. A bipolar spherical neutrosophic graph $G = (A, B)$ is called Strong Bipolar Spherical Neutrosophic Graph (SBSNG) if the following conditions are satisfied:

$$T_B^+(ab) = T_A^+(a) \wedge T_A^+(b), \quad T_B^-(ab) = T_A^-(a) \vee T_A^-(b)$$

$$I_B^+(ab) = I_A^+(a) \wedge I_A^+(b), \quad I_B^-(ab) = I_A^-(a) \vee I_A^-(b)$$

$$F_B^+(ab) = F_A^+(a) \vee F_A^+(b), \quad F_B^-(ab) = F_A^-(a) \wedge F_A^-(b) \quad \forall a, b \in B.$$

Example 6.3.8. Consider the strong bipolar spherical neutrosophic graph $G = (A, B)$ with vertex set $A = \{a_1, a_2, a_3, a_4\}$ and the edge set $B = \{a_{13}, a_{14}, a_{23}, a_{24}\}$, the membership functions defined as follows,

$$(T_1(a_n), I_1(a_n), F_1(a_n)) = \begin{cases} (0.6, 0.5, 0.2, -0.5, -0.4, -0.1) & \text{for } n = 1 \\ (0.8, 0.7, 0.1, -0.2, -0.7, -0.3) & \text{for } n = 2 \\ (0.4, 0.7, 0.7, -0.9, -0.4, -0.5) & \text{for } n = 3 \\ (0.7, 0.6, 0.8, -0.6, -0.3, -0.9) & \text{for } n = 4 \end{cases}$$

$$(T_2(a_{nm}), I_2(a_{nm}), F_2(a_{nm})) = \begin{cases} (0.4, 0.5, 0.7, -0.5, -0.4, -0.5) & \text{for } nm = 13 \\ (0.6, 0.5, 0.8, -0.5, -0.3, -0.9) & \text{for } nm = 14 \\ (0.4, 0.7, 0.7, -0.2, -0.4, -0.5) & \text{for } nm = 23 \\ (0.7, 0.6, 0.8, -0.2, -0.3, -0.9) & \text{for } nm = 24 \end{cases}$$

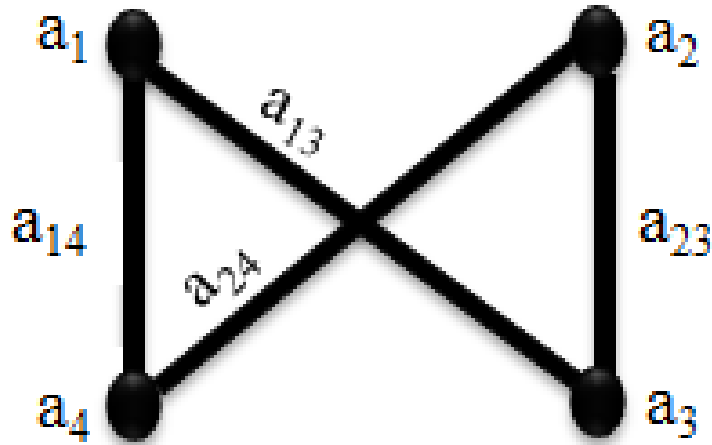


Fig. 6.11: BSSNG

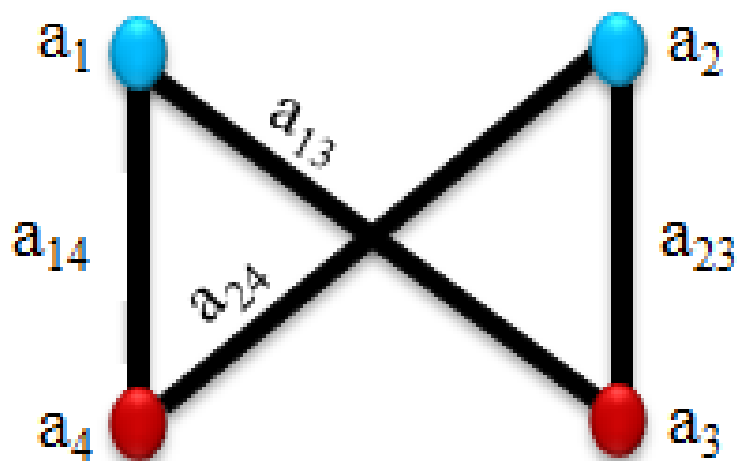


Fig. 6.12: Vertex coloring of BSSNG

Vertex	Strongly Adjacent Vertices
a_1	a_3, a_4
a_2	a_3, a_4
a_3	a_1, a_2
a_4	a_1, a_2

Here $\deg_s(A)$ = Number of edges incident on a vertices that are strongly adjacent to A. Now $\deg_s(a_1) = 2$, $\deg_s(a_2) = 2$, $\deg_s(a_3) = 2$, $\deg_s(a_4) = 2$.

Therefore the above results satisfies the following condition, $\deg(A) \geq \deg_s(A)$.

Since a_3, a_4 are strongly adjacent to a_1 so they do not have same color. Thus

$\phi = (\{\phi_1^+, \phi_2^+\}, \{\phi_1^-, \phi_2^-\})$ be the family of spherical neutrosophic sets.

$$\phi_1^+(a_n) = \begin{cases} (0.6, 0.5, 0.2) & \text{for } n = 1 \\ (0.8, 0.7, 0.1) & \text{for } n = 2 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\phi_1^-(a_n) = \begin{cases} (-0.5, -0.4, -0.1) & \text{for } n = 1 \\ (-0.2, -0.7, -0.3) & \text{for } n = 2 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\phi_2^+(a_n) = \begin{cases} (0.4, 0.7, 0.7) & \text{for } n = 3 \\ (0.7, 0.6, 0.8) & \text{for } n = 4 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

$$\phi_2^-(a_n) = \begin{cases} (-0.9, -0.4, -0.2) & \text{for } n = 3 \\ (-0.6, -0.3, -0.9) & \text{for } n = 4 \\ (0, 0, 1) & \text{otherwise} \end{cases}$$

Hence the family ϕ fulfilled the conditions of complete spherical neutrosophic vertex coloring of G . Hence the complete spherical neutrosophic chromatic number of G is $\chi_\alpha(G) = (\chi_{\alpha^+}^S(G), \chi_{\alpha^-}^S(G)) = (2, 2)$.