CHAPTER I

CHAPTER I

CLOSURE AND BICLOSURE SPACES

In this chapter preliminary results of closure spaces, biclosure spaces and $T_{1/2}$ – biclosure spaces are studied.

SECTION 1.1

CLOSURE SPACES

This section deals with some basic definitions and results on closure spaces.

Definition: 1.1.1

A map u: P(X)→P(X) defined on the power set P(X) of a set X is called a **closure operator** on X if the following axioms are satisfied:

(i) $u\phi = \phi$

(ii) $A \subseteq uA$ for every $A \subseteq X$

(iii) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

The pair (X, u) is called a **closure space.**

Definition: 1.1.2

A closure operator u on a set X is called **additive** if for subsets A, B of **X**, $u(A \cup B) = uA \cup uB$.

Definition: 1.1.3

A closure operator u on a set X is called **idempotent** if $A \subseteq X \Rightarrow uuA = uA$.

Definition: 1.1.4

A subset $A \subseteq X$ is **closed** in the closure space (X, u) if $uA = A$.

Definition: 1.1.5

A subset $A \subseteq X$ is **open** if its complement in X is closed.

Remark: 1.1.6

The empty set and the whole space are both open and closed.

Definition: 1.1.7

A closure space (Y, v) is said to be a **subspace** of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$.

Remark: 1.1.8

If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed.

Definition: 1.1.9

Let (X, u) and (Y, v) be closure spaces. A map $f: (X, u) \rightarrow (Y, v)$ is said to be **continuous** if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

Proposition: 1.1.10

A map $f: (X, u) \rightarrow (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$.

Proof:

Let $f : (X, u) \rightarrow (Y, v)$ be continuous.

i.e., $f(uA) \subseteq vf(A)$ for each subset $A \subseteq X$.

To prove: $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$.

Let $B \subseteq Y$ and $f^{-1}(B) = A$. Then $ff^{-1}(B) = f(A)$.

Since $f(uA) \subseteq vf(A)$, $f(uf^{-1}(B)) \subseteq vf(f^{-1}(B)) \subseteq vB$. $(\because ff^{-1}(B) \subset B)$

Therefore, $f(uf^{-1}(B)) \subseteq vB$. $uf^{-1}(B) \subseteq f^{-1}(vB)$.

Conversely, let $uf^{-1}(B) \subseteq f^{-1}(vB)$, $\forall B \subseteq Y$.

To prove: $f: (X, u) \rightarrow (Y, v)$ is continuous.

Let A be a subset of X and $B = f(A)$.

Since $uf^{-1}(B) \subseteq f^{-1}(vB)$

$uf^{-1}(f(A)) \subseteq f^{-1}(v\, f(A))$

$uA \subseteq uf^{-1}(f(A)) \subseteq f^{-1}(v\, f(A))$ $(\because A \subseteq f^{-1}(f(A))$

$uA \subseteq f^{-1}(v\, f(A))$

$f(uA) \subseteq f(f^{-1}(v\, f(A))) \subseteq v\, f(A)$

Therefore, $f(uA) \subseteq v\, f(A)$

Therefore, f is continuous.

Proposition: 1.1.11

If $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Proof:

Let $f : (X, u) \to (Y, v)$ be continuous.

To prove: $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Let F be a closed subset of (Y, v).

Therefore, $vF = F$

By continuity, $uf^{-1}(B) \subseteq f^{-1}(vB)$

$uf^{-1}(F) \subseteq f^{-1}(vF) = f^{-1}(F)$

Therefore, $uf^{-1}(F) \subseteq f^{-1}(F)$ (1)

Since, $f^{-1}(F) \subseteq uf^{-1}(F)$ (2)

From (1) and (2), $f^{-1}(F) = uf^{-1}(F)$.

Therefore, $f^{-1}(F)$ is a closed subset of (X, u).

Definition: 1.1.12

Let (X, u) and (Y, v) be closure spaces. A map f: (X, u) → (Y, v) is said to be a **closed map** if f(F) is a closed subset of (Y, v) whenever F is a closed subset of (X, u).

Definition: 1.1.13

Let (X, u) and (Y, v) be closure spaces. A map f: (X, u) → (Y, v) is said to be an **open map** if f(F) is an open subset of (Y, v) whenever F is an open subset of (X, u).

Definition: 1.1.14

The product of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ is the closure space $(\prod_{\alpha\in I} X_\alpha, u)$ where $\prod_{\alpha\in I} X_\alpha$ denotes the Cartesian product of sets X_α, $\alpha \in I$ and u is the closure operator generated by the **projection** $\pi_\alpha : \prod_{\alpha\in I} X_\alpha \to X_\alpha$, $\alpha \in I$. i.e., is defined by $uA = \prod_{\alpha\in I} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha\in I} X_\alpha$

Proposition: 1.1.15

If $\{(X_\alpha, u_\alpha): \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta: \prod_{\alpha\in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Proof:

Let the projection map be $\pi_\beta: \prod_{\alpha\in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$.

Claim: 1 π_β is closed

Let A be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Then A = uA and by definition uA = $\prod_{\alpha\in I} u_\alpha \pi_\alpha(A)$.

To prove: $\pi_\beta(A)$ is closed in (X_β, u_β).

Since A is closed, $\pi_\beta(A) = \pi_\beta(uA)$

$\pi_\beta(A) = \pi_\beta(\prod_{\alpha\in I} u_\alpha \pi_\alpha(A))$

Therefore, $\pi_\beta(A) = u_\beta\, \pi_\beta(A)$

Therefore, $\pi_\beta(A)$ is closed in (X_β, u_β).

Therefore, π_β is closed.

Claim:2 π_β is continuous

i.e., to prove: $\pi_\beta(uA) \subseteq u_\beta\, \pi_\beta(A)$

Consider $\pi_\beta(uA)$

$\pi_\beta(uA) = \pi_\beta(\prod_{\alpha\in I} u_\alpha \pi_\alpha(A))$

$\pi_\beta(uA) = u_\beta(\pi_\beta(A))$

$\pi_\beta(uA) \subseteq u_\beta\, \pi_\beta(A)$

Therefore, π_β is continuous.

Proposition: 1.1.16

Let $\{(X_\alpha, u_\alpha): \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a closed subset of (X_β, u_β) if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

Proof:

Let F be a closed subset of (X_β, u_β).

Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

But $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$. Hence $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ be a closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha) = F$, a closed subset of (X_β, u_β).

Therefore, F is a closed subset of (X_β, u_β).

Proposition: 1.1.17

Let $\{(X_\alpha, u_\alpha): \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is an open subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

Proof:

Let G be an open subset of (X_β, u_β).

Then G^c is a closed subset of (X_β, u_β).

Since π_β is continuous, $\pi_\beta^{-1}(G^c)$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

$\pi_\beta^{-1}(G^c) = G^c \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$

Therefore, $G^c \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Therefore, $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ be an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Then $G^c \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Since π_β is closed, $\pi_\beta(G^c \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha) = G^c$ is a closed subset of (X_β, u_β).

Therefore, G is an open subset of (X_β, u_β).

Definition: 1.1.18

Let (X, u) be a closure space. A subset $A \subseteq X$ is called a **generalized closed set (g-closed set)**, if $uA \subseteq G$ whenever G is an open subset of (X, u) with $A \subseteq G$.

Definition: 1.1.19

A subset $A \subseteq X$ is called a **generalized open set (g-open set)**, if its complement is g-closed.

Proposition: 1.1.20

Let (X, u) be a closure space and let (Y, v) be a closed subspace of (X, u). If F is a g-closed subset of (Y, v), then F is a g-closed subset of (X, u).

Proof:

Let (X, u) be a closure space and let (Y, v) be a closed subspace of (X, u).

Let F be a g-closed subset of (Y,v).

To prove: F is a g-closed subset of (X, u).

Let G be an open subset of (X, u) and $F \subseteq G$.

Since Y is closed in (X, u) and $F \subset Y$, $uF \subset uY = Y$.

$uF = uF \cap Y = vF$

Since F is g-closed in (Y, v), $vF \subset G \cap Y \subset G$

Therefore, $uF \subset G$.

Hence F is a g-closed subset of (X, u).

Proposition: 1.1.21

Let (X, u) be a closure space. If F is a closed subset of (X, u) and F' is a closed subset of (X, u) then $F \cap F'$ is a g-closed subset of (X, u).

Proof:

Let (X, u) be a closure space.

Let F and F' be closed subsets of (X, u).

Then F ∩ F' is a closed subset of (X, u).

Let G be an open subset of (X, u) such that $F \cap F' \subseteq G$.

Since F ∩ F' is a closed subset of (X, u), $u(F \cap F') = F \cap F' \subseteq G$.

Therefore, F ∩ F' is a g-closed subset of (X, u).

Proposition: 1.1.22

Let $\{(X_\alpha, u_\alpha): \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a g- closed subset of (X_β, u_β) if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g-closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

Proof:

Let F be a g-closed subset of (X_β, u_β).

To prove: $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g-closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

Let $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \subseteq G$,

where G is an open subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

$G = \prod_{\alpha \in I} G_\alpha$, where $F \subset G_\beta$ and $G_\alpha = X_\alpha$.

To prove: $u(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha) \subseteq G$

Consider $u(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha)$

$u(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha) = u_\beta F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} u_\alpha X_\alpha$

$u(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha) \subset \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} G_\beta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \subseteq G$

Therefore, $u(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha) \subseteq G$

Therefore, $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g-closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ be a g-closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

i.e., $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \subset G$

To prove: F is a g-closed subset of (X_β, u_β).

Let G_β be an open subset of X_β such that $F \subset G_\beta$.

Then $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \subset G$ where $G_\alpha = X_\alpha$ for $\alpha \neq \beta$.

Since $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g-closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

$u(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha) \subset G$

$$u_\beta F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha u_\alpha \subset G = \prod_{\alpha\in I} G_\alpha$$

Therefore, $u_\beta F \subset G_\beta$.

Therefore, F is a g-closed subset of (X_β, u_β).

Proposition: 1.1.23

Let $\{(X_\alpha, u_\alpha): \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is a g-open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Proof:

Let G be a g-open subset of (X_β, u_β).

Then G^c is a g-closed subset of (X_β, u_β).

By Proposition 1.1.22, $G^c \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Therefore, $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ be a g-open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Then $G^c \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

By Proposition 1.1.22, G^c is a g-closed subset of (X_β, u_β).

Therefore, G is a g-open subset of (X_β, u_β).

Proposition: 1.1.24

Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces for each $\beta \in I$, let $\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta$ be the projection map. Then

(i) If F is a g-closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$, then $\pi_\beta(F)$ is a g- closed subset of (X_β, u_β).

(ii) If F is a g-closed subset of (X_β, u_β) then $\pi_\beta^{-1}(F)$ is a g-closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

Proof:

Let $\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta$ be the projection map, for each $\beta \in I$.

(i) Assume F is a g-closed subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.

Therefore, $uF \subseteq G$ where G is an open subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$ with $F \subseteq G$.

Claim: 1 $\pi_\beta(F)$ is a g-closed subset of (X_β, u_β).

Let $\pi_\beta(uF) \subseteq G$

$\pi_\beta(\prod_{\alpha \in I} u_\alpha \pi_\beta (F) \subseteq G$

Therefore, $u_\beta \pi_\beta (F) \subseteq G$

Therefore, $\pi_\beta(F)$ is a g-closed subset of (X_β, u_β).

(ii) Assume F is a g-closed subset of (X_β, u_β).

Therefore, $u_\beta F \subseteq G$ where G is an open subset of X_β.

Claim: 2 $\pi_\beta^{-1}(F)$ is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Let $\pi_\beta^{-1}(uF) \subseteq G$ where G is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

$\pi_\beta^{-1}(F) \subset G$ where $G = \Pi G_\alpha$.

To prove: $u(\pi_\beta^{-1}(F)) \subseteq G$

$$\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha \subset G = G_\beta \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha, \quad F \subseteq G_\beta$$

Therefore, $u_\beta F \subseteq G_\beta$

$$\text{Let } u(\pi_\beta^{-1}(F)) = u(F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha) = u_\beta F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} u_\alpha X_\alpha$$

$$u(\pi_\beta^{-1}(F)) \subset G_\beta \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha = G$$

Therefore, $\pi_\beta^{-1}(F)$ is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Definition: 1.1.25

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \rightarrow (Y, v)$ is called **g- irresolute**, if $f^{-1}(F)$ is a g-closed subset of (X, u) for every g- closed subset F of (Y, v).

Remark: 1.1.26

A map $f : (X, u) \rightarrow (Y, v)$ is g-irresolute if and only if $f^{-1}(G)$ is a g- open subset of (X, u) for every g- open subset G of (Y, v).

SECTION 1.2
BICLOSURE SPACES

In this section some basic definitions and results on biclosure spaces are analyzed.

Definition: 1.2.1

A **biclosure space** is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two closure operators on X.

Definition: 1.2.2

A subset A of a biclosure space (X, u_1, u_2) is called **closed** if $u_1u_2A = A$.

Note: 1.2.3

The complement of a closed set is **open**.

Remark: 1.2.4

A is a closed subset of a biclosure space (X, u_1, u_2) if and only if A is both a closed subset of (X, u_1) and (X, u_2).

Remark: 1.2.5

Let A be a closed subset of a biclosure space (X, u_1, u_2). Then the following conditions are equivalent:

(i) $u_2u_1A = A$

(ii) $u_1A = A$, $u_2A = A$.

Definition: 1.2.6

Let (X, u_1, u_2) be a biclosure space. A biclosure space (Y, v_1, v_2) is called a **subspace** of (X, u_1, u_2) if Y $\subseteq$ X and v_iA = u_iA ∩ Y for each i ∈ {1,2} and each subset A $\subseteq$ Y.

Proposition: 1.2.7

Let (X, u_1, u_2) be a biclosure space and (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If F is a closed subset of (Y, v_1, v_2), then F is a closed subset of (X, u_1, u_2).

Proof:

Let F be a closed subset of (Y, v_1, v_2).

Then v_1F = F and v_2F = F.

Since (Y, v_1, v_2) is a closed subset of (X, u_1, u_2), F = v_1F = u_1F ∩ Y = u_1F ∩ u_1Y = u_1(F ∩ Y) = u_1F and F = v_2F = u_2F ∩ Y = u_2F ∩ u_2Y = u_2(F ∩ Y) = u_2F.

Therefore, F is both a closed subset of (X, u_1) and (X, u_2).

Therefore, F is a closed subset of (X, u_1, u_2).

Definition: 1.2.8

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let i ∈ {1,2}. A map f : (X, u_1, u_2) → (Y, v_1, v_2) is called **i-continuous** if the map f : (X, u_i) → (Y, v_i) is continuous.

Definition: 1.2.9

A map f : (X, u_1, u_2) → (Y, v_1, v_2) is called **continuous** if f is i- continuous for each i ∈ {1,2}.

Definition: 1.2.10

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let i ∈ {1,2}. A map f : (X, u_1, u_2) → (Y, v_1, v_2) is called **i-g-irresolute** if the map f : (X, u_i) → (Y, v_i) is g-irresolute.

Definition: 1.2.11

A map f : (X, u_1, u_2) → (Y, v_1, v_2) is called **g-irresolute** if f is i- g- irresolute for each i ∈ {1,2}.

Definition: 1.2.12

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let i ∈ {1,2}. A map f : (X, u_1, u_2) → (Y, v_1, v_2) is called **i-closed** if the map f : (X, u_i) → (Y, v_i) is closed.

Definition: 1.2.13

A map f : (X, u_i) → (Y, v_i) is called **closed** if f is i-closed for each i ∈ {1,2}.

Definition: 1.2.14

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1,2\}$. A map $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ is called **i-open** if the map $f : (X, u_i) \rightarrow (Y, v_i)$ is open.

Definition: 1.2.15

A map $f : (X, u_i) \rightarrow (Y, v_i)$ is called **open** if f is i- open for each $i \in \{1,2\}$.

Proposition: 1.2.16

Let (X, u_1, u_2) be a biclosure space and let $A \subseteq X$. Then

(i) A is open if and only if $A = X - u_1u_2(X - A)$.

(ii) If G is open and $G \subseteq A$, then $G \subseteq X - u_1u_2(X - A)$.

Proof:

(i) Let A be open.

To prove: $A = X - u_1u_2(X - A)$.

Given A is an open subset of a biclosure space (X, u_1, u_2), then A^c is closed.

Therefore, $u_1u_2A^c = A^c$.

Implies $u_1u_2(X - A) = X - A$

Therefore, $A = X - u_1u_2(X - A)$.

Conversely, let $A = X - u_1u_2(X - A)$.

To prove: A is open

$X - A = u_1u_2(X - A)$

$A^c = u_1u_2(A^c)$

Implies A^c is closed

Therefore, A is open.

(ii) Let G be open and $G \subseteq A$.

To prove: $G \subseteq X - u_1u_2(X - A)$.

Since G is open and $G \subseteq A$, $X - G \supset X - A$ and $G = X - u_1u_2(X - G)$.

Then $u_1u_2(X - G) \supset u_1u_2(X - A)$

$X - u_1u_2(X - G) \subseteq X - u_1u_2(X - A)$

Therefore, $G \subseteq X - u_1u_2(X - A)$.

Proposition: 1.2.17

Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. Then F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ if and only if F is both a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1)$ and $\prod_{\alpha\in I}(X_\alpha, u_\alpha^2)$.

Proof:

Let F be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$.

Then $F = \prod_{\alpha\in I} u_\alpha^1\pi_\alpha(\prod_{\alpha\in I} u_\alpha^2\pi_\alpha(F))$.

Since $F \subseteq \prod_{\alpha\in I} u_\alpha^2\pi_\alpha(F)$, $\prod_{\alpha\in I} u_\alpha^1\pi_\alpha(F) \subseteq \prod_{\alpha\in I} u_\alpha^1\pi_\alpha(\prod_{\alpha\in I} u_\alpha^2\pi_\alpha(F)) = F$.

Therefore, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1)$.

Since $\prod_{\alpha\in I} u_\alpha^2\pi_\alpha(F) \subseteq \prod_{\alpha\in I} u_\alpha^2\pi_\alpha(F)$, $\prod_{\alpha\in I} u_\alpha^2\pi_\alpha(F) \subseteq \prod_{\alpha\in I} u_\alpha^1\pi_\alpha(\prod_{\alpha\in I} u_\alpha^2\pi_\alpha(F)) = F$.

Therefore, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^2)$.

Conversely, let F be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1)$ and $\prod_{\alpha\in I}(X_\alpha, u_\alpha^2)$.

Then $F = \prod_{\alpha\in I} u_\alpha^1\pi_\alpha(F)$ and $F = \prod_{\alpha\in I} u_\alpha^2\pi_\alpha(F)$.

Therefore, $F = \prod_{\alpha\in I} u_\alpha^1\pi_\alpha(\prod_{\alpha\in I} u_\alpha^2\pi_\alpha(F))$.

Therefore, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$.

Definition: 1.2.18

A subset A of a biclosure space (X, u_1, u_2) is called **generalized closed (g- closed)**, if $u_1 A \subseteq G$ whenever G is an open subset of (X, u_2) with $A \subseteq G$.

Definition: 1.2.19

The complement of a g- closed set is called **g- open**.

Remark: 1.2.20

If A is a closed subset of a biclosure space (X, u_1, u_2), then A is g- closed. The converse need not be true.

Example: 1.2.21

Let X = {a, b} and define a closure operator u_1 on X by $u_1\phi = \phi$ and $u_1\{a\} = u_1\{b\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\phi = \phi$ and $u_2\{a\} = \{a\}$ and $u_2\{b\} = u_2X = X$. Then {a} is g- closed but it is not closed.

Proposition: 1.2.22

Let (X, u_1, u_2) be a biclosure space. Then A is a g- open subset of (X, u_1, u_2) if and only if $F \subseteq X - u_1(X - A)$ for every F is a closed subset of (X, u_2) with $F \subseteq A$.

Proof:

Let A be g- open and F be a closed subset of (X, u_2) such that

$F \subseteq A$. Then $X - A \subseteq X - F$.

Since $X - A$ is g- closed and $X - F$ is an open subset of (X, u_2),

$u_1(X - A) \subseteq X - F$.

Therefore, $F \subseteq X - u_1(X - A)$.

Conversely, let U be an open subset of (X, u_2) such that $X - A \subseteq U$.

Then $X - U \subseteq A$.

Since $X - U$ is a closed subset of (X, u_2), $X - U \subseteq X - u_1(X - A)$.

Therefore, $u_1(X - A) \subseteq U$.

Therefore, X – A is g- closed.

Therefore, A is g- open.

Proposition: 1.2.23

Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then G is a g- open subset of $(X_\beta, u_\beta^1, u_\beta^2)$ if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g- open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2)$.

Proof:

Let $\beta \in I$ and let G be a g- open subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

Let F be a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha^2)$ such that $F \subseteq G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$.

Then $\pi_\beta(F) \subseteq G$.

Since $\pi_\beta(F)$ is a closed subset of (X_β, u_β^2) and $\pi_\beta(F) \subseteq X_\beta - u_\beta^1(X_\beta - G)$.

Therefore, $F \subseteq \pi_\beta^{-1}(X_\beta - u_\beta^1(X_\beta - G))$

Therefore, $F = \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} u_\alpha^1 \pi_\alpha(\prod_{\alpha \in I} X_\alpha - G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha)$

By Proposition 1.2.22, $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g- open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2)$.

Conversely, let $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ be a g- open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ and let F be a closed subset of (X_β, u^2_β) such that $F \subseteq G$.

Then $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$.

Since $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is closed and $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is g- open,

$F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha \subseteq \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u^1_\alpha \pi_\alpha \left(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha \right)$

by Proposition 1.2.22.

Therefore, $\prod_{\alpha\in I} u^1_\alpha \pi_\alpha((X_\beta - G) \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha) \subseteq \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$

Therefore, $\prod_{\alpha\in I} u^1_\alpha \pi_\alpha((X_\beta - G) \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha) = (X_\beta - F) \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$.

Therefore, $u^1_\beta(X_\beta - G) \subseteq X_\beta - F$.

Implies $F \subseteq X_\beta - u^1_\beta(X_\beta - G)$.

Hence G is a g- open subset of $(X_\beta, u^1_\beta, u^2_\beta)$.

Proposition: 1.2.24

Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a g- closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a g- closed subset of $\prod_{\alpha\in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

Proof:

Let $\beta \in I$ and let F be a g- closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then X_β - F is a g- open subset of $(X_\beta, u^1_\beta, u^2_\beta)$.

By Proposition 1.2.23, $(X_\beta - F) \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha = \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a g- open subset of $\prod_{\alpha\in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

Hence $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a g- closed subset of $\prod_{\alpha\in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ be a g- closed subset of $\prod_{\alpha\in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$ and let G be an open subset of (X_β, u^2_β) such that $F \subseteq G$.

Then $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$.

Since F $\times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is g- closed and G $\times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is open, $\prod_{\alpha\in I} u_\alpha^1 \pi_\alpha(F\times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\beta) \subseteq G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$.

Therefore, $u_\beta^1 F \subseteq G$. Hence F is a g- closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

Proposition: 1.2.25

Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. For each $\beta \in I$, let $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2) \rightarrow (X_\beta, u_\beta^1, u_\beta^2)$ be the projection map. Then

(i) If F is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$, then $\pi_\beta(F)$ is a g-closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

(ii) If F is a g-closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$, then $\pi_\beta^{-1}(F)$ is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$.

Proof:

(i) Let F be a g- closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ and let G be an open subset of (X_β, u_β^2) such that $\pi_\beta(F) \subseteq G$.

Then $F \subseteq \pi_\beta^{-1}(G) = G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$.

Since G $\times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^2)$,

$\prod_{\alpha\in I} u_\alpha^1 \pi_\alpha(F) \subseteq G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$. Therefore, $u_\beta^1 \pi_\beta(F) \subseteq G$.

Hence $\pi_\beta(F)$ is a g- closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

(ii) Let F be a g- closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$ then

$\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$.

By Proposition 1.2.24, $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a g- closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$

Hence $\pi_\beta^{-1}(F)$ is a g- closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$.

Lemma: 1.2.26

Let (X, u_1, u_2) be a biclosure space and (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If G is an open subset of (X, u_1) and an open subset of (X, u_2), then $G \cap Y$ is an open subset of (Y, v_1) and an open subset of (Y, v_2).

Proof:

Let G be an open subset of (X, u_1).

Claim: $Y - (G \cap Y) = Y \cap (X - G)$

Let $x \in Y - (G \cap Y)$

Therefore, $x \in Y$ and $x \notin (G \cap Y)$

i.e., $x \notin G$ and $x \notin Y$ but $x \in Y$.

$x \notin G \Rightarrow x \in X - G$ and $x \in Y$.

Therefore, $x \in Y \cap (X - G)$.

Therefore, $Y - (G \cap Y) \subseteq Y \cap (X - G)$ (3)

Let $z \in Y \cap (X - G)$

Therefore, $z \in Y$ and $z \in X - G$

i.e., $z \in X$ and $x \notin G$ but $x \in Y$

Implies $z \notin G \cap Y$

Therefore, $z \in Y - (G \cap Y)$

Therefore, $Y \cap (X - G) \subseteq Y - (G \cap Y)$ (4)

From (3) and (4), $Y - (G \cap Y) = Y \cap (X - G)$.

As Y and (X – G) are closed subsets of (X, u_1),

$Y \cap (X - G) = u_1 Y \cap u_1(X - G) = u_1(Y \cap (X - G))$

$Y \cap (X - G) = u_1(Y \cap (X - G)) \cap Y$

$Y \cap (X - G) = v_1(Y \cap (X - G))$

$Y \cap (X - G) = v_1(Y - (G \cap Y))$. ($\because Y - (G \cap Y) = Y \cap (X - G)$)

Hence $Y - (G \cap Y)$ is a closed subset of (Y, v_1) and therefore $G \cap Y$ is an open subset of (Y, v_1). Similarly, if G is an open subset of (X, u_2), then $G \cap Y$ is an open subset of (Y, v_2).

Lemma: 1.2.27

Let (X, u_1, u_2) be a biclosure space and (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If G is a g- open subset of (X, u_1) and a g- open subset of (X, u_2), then $G \cap Y$ is a g- open subset of (Y, v_1) and a g- open subset of (Y, v_2).

Proof:

Let G be a g- open subset of (X, u_1). Then $X - G$ is a g- closed subset of (X, u_1).Since Y is a closed subset of (X, u_1), $(X - G) \cap Y$ is a g- closed subset of (X, u_1).

Claim: $(X - G) \cap Y = Y - (G \cap Y)$.

Consider $x \in (X - G) \cap Y$, implies $x \in X - G$ and $x \in Y$

i.e., $x \notin G$ and $x \in Y \Rightarrow x \notin G \cap Y$.

$x \in Y \Rightarrow x \notin G \cap Y \Rightarrow x \in Y - (G \cap Y)$

Therefore, $(X - G) \cap Y = Y - (G \cap Y)$.

Therefore, $G \cap Y$ is a g- open subset of (Y, v_1).

Similarly, if G is a g- open subset of (X, u_2), then $G \cap Y$ is a g- open subset of (Y, v_2).

Proposition: 1.2.28

Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a ∂-closed subset of (X_β, u_β) if and only if $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a ∂- closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Proposition: 1.2.29

Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is a ∂-open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a ∂- open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Proposition: 1.2.30

Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces. For each $\beta \in I$, let $\pi_\beta : \prod_{\alpha\in I} X_\alpha \rightarrow X_\beta$ be the projection map. Then

(*i*) If F is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$, then π_β (F) is a ∂- closed subset of (X_β, u_β).

(*ii*) If F is a ∂-closed subset of (X_β, u_β), Then $\pi_\beta^{-1}(F)$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

SECTION 1.3

$T_{1/2}$ - BICLOSURE SPACES

In this section discusses the concept of $T_{1/2}$ - biclosure spaces.

Definition: 1.3.1

A biclosure space (X, u_1, u_2) is called a **$T_{1/2}$ - biclosure space** if every g-closed subset of (X, u_1, u_2) is a closed subset of (X, u_1).

Proposition: 1.3.2

Let (X, u_1, u_2) be a biclosure space. Then (X, u_1, u_2) is a $T_{1/2}$ - biclosure space if and only if every singleton subset of X is either a closed subset of (X, u_2) or an open subset of (X, u_1).

Proof:

Let (X, u_1, u_2) be a $T_{1/2}$ – biclosure space.

Let $x \in X$ and suppose that $\{x\}$ is not a closed subset of (X, u_2). Then $X - \{x\}$ is not an open subset of (X, u_2). The only open subset of (X, u_2) containing $X - \{x\}$ is X, hence $X - \{x\}$ is a g-closed subset of (X, u_1, u_2).

Since (X, u_1, u_2) is a $T_{1/2}$ - biclosure space, $X - \{x\}$ is a closed subset of (X, u_1).

Hence $\{x\}$ is an open subset of (X, u_1).

Conversely, let A be a g- closed subset of (X, u_1, u_2). Suppose that $x \notin A$. Then $\{x\} \subseteq X - A$ and we have $A \subseteq x - \{x\}$.

If $\{x\}$ is an open subset of (X, u_1), then $X - \{x\}$ is a closed subset of (X, u_1).

Therefore, $u_1A \subseteq u_1(X - \{x\}) = X - \{x\}$, thus $x \notin u_1A$.

If $\{x\}$ is a closed subset of (X, u_2), then $X - \{x\}$ is an open subset of (X, u_2). Since A is g- closed, $u_1A \subseteq X - \{x\}$.

Therefore, $x \notin u_1A \Rightarrow A \subseteq u_1A$.

Since, $u_1A \subseteq A$.

Therefore, $u_1A = A$.

i.e., A is a closed subset of (X, u_1).

Therefore, (X, u_1, u_2) is a $T_{1/2}$ - biclosure space.

Proposition: 1.3.3

Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If (X, u_1, u_2) is a $T_{1/2}$ - biclosure space, then (Y, v_1, v_2) is a $T_{1/2}$ - biclosure space too.

Proof:

Let (X, u_1, u_2) is a $T_{1/2}$ - biclosure space.

Let F be a g- closed subset of (Y, v_1, v_2). Then F is a g- closed subset of (X, u_1, u_2).

Since (X, u_1, u_2) is a $T_{1/2}$ - biclosure space, F is a closed subset of (X, u_1).

Implies F is a closed subset of (Y, v_1).

Therefore, (Y, v_1, v_2) is a $T_{1/2}$ - biclosure space.

Proposition: 1.3.4

Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. Then $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a $T_{1/2}$ - biclosure space if and only if $(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a $T_{1/2}$ - biclosure space for each $\alpha \in I$.

Proof:

Let $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ be a $T_{1/2}$ - biclosure space.

Claim 1: $(X_\beta, u_\beta^1, u_\beta^2)$ is a $T_{1/2}$ - biclosure space for each $\beta \in I$.

Let $\beta \in I$ and let F be a g- closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

Then $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a g- closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$.

Since $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a $T_{1/2}$ - biclosure space, $F \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ and hence F is a closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

Therefore, $(X_\beta, u_\beta^1, u_\beta^2)$ is a $T_{1/2}$ - biclosure space.

Conversely, let $(X_\alpha, u_\alpha^1, u_\alpha^2)$ be a $T_{1/2}$ - biclosure space for each $\alpha \in I$.

Claim 2: $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a $T_{1/2}$ - normal biclosure space.

Let F be a g- closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ and let $(x_\alpha)_{\alpha\in I} \notin F$.

Then there exists a $\beta \in I$ such that $x_\beta \notin \pi_\beta(F)$.

Since $\pi_\beta(F)$ is g- closed and $(X_\beta, u_\beta^1, u_\beta^2)$ is a $T_{1/2}$ - biclosure space, $\pi_\beta(F)$ is a closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

Thus $x_\beta \notin u_\beta^1\pi_\beta(F) \Rightarrow (x_\alpha)_{\alpha\in I} \notin \prod_{\alpha\in I} u_\alpha^1\pi_\alpha(F)$.

Therefore, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1)$.

Hence $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a $T_{1/2}$ - biclosure space.