# Chapter - 2

# CHAPTER 2

# INTUITIONISTIC GRADATION OF OPENNESS

In this chapter, the concepts of intuitionistic fuzzy sets, intuitionistic fuzzy topological spaces and intuitionistic gradation of openness introduced by Atanassov [5], Coker [12] and Mondal, Samanta [28] respectively are discussed.

In section one, some important operations and their relations on intuitionistic fuzzy sets are collected.

In section two, some interesting properties and results on intuitionistic fuzzy topological spaces are discussed.

In section three, some basic definitions and results on intuitionistic gradation of openness that are needed for our study are collected.

## SECTION: 2.1

## INTUITIONISTIC FUZZY SETS

### Definition: 2.1.1

Let X be a nonempty fixed set. An **intuitionistic fuzzy set** (**IFS**, for short) A is an object having the form

$$A = \{< x, \mu_A(x), \gamma_A(x) > | x \in X\}$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

**Definition: 2.1.2**

Let X be a nonempty set, and the IFSs A and B be in the form $A = \{< x, \mu_A(x), \gamma_A(x) > | x \in X\}$, $B = \{< x, \mu_B(x), \gamma_B(x) > | x \in X\}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;

(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(c) $A^C = \{< x, \gamma_A(x), \mu_A(x) > | x \in X\}$ where $A^C$ is the complement of A;

(d) $A \cap B = \{< x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x) > | x \in X\}$;

(e) $A \cup B = \{< x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x) > | x \in X\}$;

(f) $[\,]A = \{< x, \mu_A(x)\ 1 - \mu_A(x) > | x \in X\}$;

(g) $\langle\rangle A = \{< x 1 - \gamma_A(x), \gamma_A(x) > | x \in X\}$.

**Definition: 2.1.3**

Let X be a non empty set and let $\{A_i \,/\, i \in J\}$ be an arbitrary family of IFSs in X. Then

(a) $\cap A_i = \{< x, \wedge \mu_{A_i}(x), \vee \gamma_{A_i}(x) > | x \in X\}$;

(b) $\cup A_i = \{< x, \vee \mu_{A_i}(x), \wedge \gamma_{A_i}(x) > | x \in X\}$.

**Definition: 2.1.4**

Let X be a non empty set and IFSs $\tilde{0}$ and $\tilde{1}$ in X be defined as

$\tilde{0} = \{\langle x,0,1\rangle / x \in X\}$ and $\tilde{0} = \{\langle x,1,0\rangle / x \in X\}$.

**Proposition: 2.1.5**

Let A, B, C be IFSs in X. Then

(a) $A \subseteq B$ and $C \subseteq D \Rightarrow A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$

(b) $A \subseteq B$ and $A \subseteq C \Rightarrow A \subseteq B \cap C$

(c) $A \subseteq C$ and $B \subseteq C \Rightarrow A \cup B \subseteq C$

(d) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

(e) $(A \cup B)^C = A^C \cap B^C$

(f) $(A \cap B)^C = A^C \cup B^C$

(g) $(A^C)^C = A$

(h) $(\tilde{1})^C = \tilde{0}$

(i) $(\tilde{0})^C = \tilde{1}$.

**Definition: 2.1.6**

Let X and Y be two non empty sets and $f : X \to Y$.

If $B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle / y \in Y\}$ is an IFS in Y, then **the preimage of B under f**, denoted by $\mathbf{f^{-1}(B)}$, is the IFS in X defined by

$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)\rangle / x \in X\}$.

**Definition: 2.1.7**

Let X and Y be two non empty sets and $f : X \rightarrow Y$.

If $A = \{\langle x, \lambda_A(x), \gamma_A(x)\rangle / x \in X\}$ is an IFS in X, **the image of A under f**, denoted by **f(A)**, is the IFS in Y defined by

$f(A) = \{\langle y, f(\lambda_A)(y), (1\text{-}f(1\text{-}\gamma_A))(y)\rangle / y \in Y\}$

where $f(\lambda_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise,} \end{cases}$

and $(1 - f(1 - \gamma_A))(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \gamma_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 1 & \text{otherwise,} \end{cases}$

We denote $1 - f(1 - \gamma_A)$ as $f - (\gamma_A)$.

**Proposition: 2.1.8**

Let A, $A_i(i \in J)$ be IFSs in X, B, $B_j(j \in K)$ IFSs in Y and $f : X \rightarrow Y$, a function. Then

(a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$

(b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(c) $A \subseteq f^{-1}(f(A)$ [ If f is injective, then $A = f^{-1}(f(A))$ ],

(d) $f(f^{-1}(B)) \subseteq B$ [ If f is surjective, then $f(f^{-1}(B)) = B$ ],

(e) $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$,

(f) $f^{-1}(\cap Bj) = \cap f^{-1}(B_j)$

(g) $f(\cup A_i) = \cup f(A_i)$

(h) $f(\cap A_i) \subseteq \cap f(A_i)$ [ If f is injective, then $f(\cap A_i) = \cap f(A_i)$ ]

(i) $f^{-1}(\tilde{1}) = \tilde{1}$, (j) $f^{-1}(\tilde{0}) = \tilde{0}$,

(k) $f(\tilde{1}) = \tilde{1}$ if f is surjective,

(l) $f(\tilde{0}) = \tilde{0}$

(m) $(f(A))^C \subseteq f(A^C)$, If f is surjective

(n) $f^{-1}(B^C) = (f^{-1}(B))^C$.

**SECTION: 2.2**

**INTUITIONISTIC FUZZY TOPOLOGICAL SPACES**

**Definition: 2.2.1**

An **intuitionistic fuzzy topology** (**IFT**, for short) on a nonempty set X is a family $\tau$ of IFSs in X satisfying the following axioms:

(i) $\tilde{0}, \tilde{1} \in \tau$.

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$

(iii) $\cup G_i \in \tau$ for any arbitrary family $\{G_i / i \in J\} \subseteq \tau$.

The pair (X, $\tau$) is called an **intuitionistic fuzzy topological space** (**IFTS**, for short) and any IFS in $\tau$ is known as an **intuitionistic fuzzy open set** (**IFOS** for short) in X.

**Proposition: 2.2.2**

Let (X, $\tau$) be an IFTS.

(a) $\tau_1 = \{\mu_G / G \in \tau\}$ is a fuzzy topological space on X in Chang's sense.

(b) $\tau_2^* = \{\gamma_G \ / \ G \in \tau\}$ is the family of all fuzzy closed sets of the fuzzy topological space $\tau_2 = \{1-\gamma_G \ / \ G \in \tau\}$ on X in Chang's sense.

(c) Since $0 \le \mu_G(x) + \gamma_G(x) \le 1$ for each $x \in X$ and each $G \in \tau$, we obtain $\mu_G \le 1 - \gamma_G$.

(d) Using (a) and (b), we conclude that (X, $\tau_1$, $\tau_2$) is a bifuzzy topological space.

**Definition: 2.2.3**

Let (X, $\tau_1$), (X, $\tau_2$) be two IFTSs on X. Then **$\tau_1$ is contained in $\tau_2$** (in symbols, **$\tau_1 \subseteq \tau_2$**) if G $\in \tau_2$ for each G $\in \tau_1$. We also say that $\tau_1$ is coarser than $\tau_2$.

**Proposition: 2.2.4**

Let $\{\tau_i \ / \ i \in J\}$ be a family of IFTs on X. Then $\cap \tau_i$ is an IFT on X. Furthermore, $\cap \tau_i$ is the coarsest IFT on X containing $\tau_i$'s.

**Definition: 2.2.5**

The complement $A^C$ of an IFOS A in an IFTS (X, $\tau$) is called an **intuitionistic fuzzy closed set** (**IFCS**, for short) in X.

**Definition: 2.2.6**

Let (X, $\tau$) be an IFTS and A = $\{\langle x, \mu_A(x), \gamma_A(x)\rangle$ be an IFS in X. Then the **fuzzy closure** and **fuzzy interior of A** are defined respectively by

$cl(A) = \cap\{K / K$ is an IFCS in X and $A \subseteq K\}$

$int(A) = \cup\{G / G$ is an IFOS in X and $G \subseteq A\}$.

**Remark: 2.2.7**

Let $(X, \tau)$ be an IFTS and $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle$ be an IFS in X. Then cl(A) is an IFCS and int(A) is an IFOS in X, and

(a) A is an IFCS in X iff $cl(A) = A$;

(b) A is an IFOS in X iff $int(A) = A$.

**Proposition: 2.2.8**

For any IFS A in $(X, \tau)$, we have

(a) $cl(A^C) = (int(A))^C$

(b) $int(A^C) = (cl(A))^C$.

**Proposition: 2.2.9**

Let $(X, \tau)$ be an IFTS and A, B be IFSs in X. Then the following properties hold:

(a) $int(A) \subseteq A$

(b) $A \subseteq cl(A)$

(c) $A \subseteq B \Rightarrow int(A) \subseteq int(B)$

(d) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$

(e) $int(int(A)) = int(A)$

(f) $cl(cl(A)) = cl(A)$

(g) $int(A \cap B) = int(A) \cap int(B)$

(h) $cl(A \cup B) = cl(A) \cup cl(B)$

(i) $int(\tilde{1}) = \tilde{1}$

(j) $cl(\tilde{0}) = \tilde{0}$.

**Definition: 2.2.10**

Let $(X, \tau)$ and $(Y, \phi)$ be two IFTSs and let $f : X \rightarrow Y$ be a function. Then f is said to be **fuzzy continuous** iff the preimage of each IFS in $\phi$ is an IFS in $\tau$.

**Definition: 2.2.11**

Let $(X, \tau)$ and $(Y, \phi)$ be two IFTSs and let $f : X \rightarrow Y$ be a function. Then f is said to be **fuzzy open** iff the image of each IFS in $\tau$ is an IFS in $\phi$.

**Proposition: 2.2.12**

$f : (X, \tau) \rightarrow (Y, \phi)$ is fuzzy continuous iff the preimage of each IFCS in $\phi$ is an IFCS in $\tau$.

**Proposition: 2.2.13**

The following are equivalent to each other:

(a) $f : (X, \tau) \rightarrow (Y, \phi)$ is fuzzy continuous.

(b) $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$ for each IFS B in Y.

(c) $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$for each IFS B in Y.

## SECTION: 2.3

## INTUITIONISTIC GRADATION OF OPENNESS

**Definition: 2.3.1**

Let X be a nonempty set. An **intuitionistic gradation of openness (IGO,** for short) of fuzzy subsets of X (i.e. IGO on X) is an ordered pair $(\tau, \tau^*)$ of functions from $I^X$ to I such that

(i) $\tau(\lambda) + \tau^*(\lambda) \leq 1, \forall \lambda \in I^X$

(ii) $\tau(\tilde{0}) = \tau(\tilde{1}) = 1, \tau^*(\tilde{0}) = \tau^*(\tilde{1}) = 0$

(iii) $\tau(\lambda_1 \cap \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and

$\tau^*(\lambda_1 \cap \lambda_2) \leq \tau^*(\lambda_1) \wedge \tau^*(\lambda_2)$, $\lambda_i \in I^X$, i = 1, 2

(iv) $\tau\left(\underset{i\in\Delta}{\cup} \lambda_i\right) \geq \underset{i\in\Delta}{\wedge} \tau(\lambda_i)$ and

$\tau^*\left(\underset{i\in\Delta}{\cup} \lambda_i\right) \leq \underset{i\in\Delta}{\vee} \tau^*(\lambda_i)$, $\lambda_i \in I^X$, $i\in\Delta$.

The triplet $(X, \tau, \tau^*)$ is called an IFTS. $\tau$ and $\tau^*$ may be interpreted as gradation of openness and gradation of nonopeness, respectively.

**Definition: 2.3.2**

Let X be a nonempty set and $\mathcal{F}$, $\mathcal{F}^* : I^X \to I$ be two mappings satisfying

(i) $\mathcal{F}(\lambda) + \mathcal{F}^*(\lambda) \leq 1, \forall \lambda_i \in I^X$

(ii) $\mathcal{F}(\tilde{0}) = \mathcal{F}^*(\tilde{1}) = 1, \mathcal{F}^*(\tilde{0}) = \mathcal{F}^*(\tilde{1}) = 0$

(iii) $\mathscr{F}(\lambda_1 \cup \lambda_2) \geq \mathscr{F}(\lambda_1) \wedge \mathscr{F}(\lambda_2)$ and

$\mathscr{F}^*(\lambda_1 \cup \lambda_2) \leq \mathscr{F}^*(\lambda_1) \vee \mathscr{F}^*(\lambda_2)$, $\lambda_i \in I^X$ i = 1, 2

(iv) $\mathscr{F}\left(\bigcap_{i\in\Delta} \lambda_i\right) \geq \bigwedge_{i\in\Delta} \mathscr{F}(\lambda_i)$ and

$\mathscr{F}^*\left(\bigcap_{i\in\Delta} \lambda_i\right) \leq \bigvee_{i\in\Delta} \mathscr{F}^*(\lambda_i)$, $\lambda_i \in I^X$, $i \in \Delta$.

Then the pair $(\mathscr{F}, \mathscr{F}^*)$ is an **intuitionistic gradation of closedness** on X (briefly **IGC** on X).

**Definition: 2.3.3**

For two pairs of mappings $(\tau, \tau^*)$ and $(\mathscr{F}, \mathscr{F}^*)$ from $I^X \to I$, define

$\tau_{\mathscr{F}}(\lambda) = \mathscr{F}(\lambda^C)$, $\tau^*_{\mathscr{F}^*}(\lambda) = \mathscr{F}^*(\lambda^C)$

$\mathscr{F}_\tau(\lambda) = \tau(\lambda^C)$, $\mathscr{F}^*_{\tau^*}(\lambda) = \tau^*(\lambda^C)$.

**Theorem: 2.3.4**

(a) $(\tau, \tau^*)$ is an IGO on X iff $(\mathscr{F}_\tau, \mathscr{F}^*_{\tau^*})$ is an IGC on X,

(b) $(\mathscr{F}, \mathscr{F}^*)$ is an IGC on X iff $(\tau_{\mathscr{F}}, \tau^*_{\mathscr{F}^*})$ is an IGO on X.

(c) $\tau_{\mathscr{F}_\tau} = \tau$, $\tau^*_{\mathscr{F}^*_{\tau^*}} = \tau^*$, $\mathscr{F}_{\tau_{\mathscr{F}}} = \mathscr{F}$, $\mathscr{F}^*_{\tau^*_{\mathscr{F}^*}} = \mathscr{F}^*$.

**Definition: 2.3.5**

Let $\{(\tau_i, \tau_i^*)\}_{i\in\Delta}$ be a family of IGOs on X. Then their intersection is defined by

$$\bigcap_{i\in\Delta}(\tau_i, \tau_i^*) = (\bigwedge_{i\in\Delta}\tau_i, \bigvee_{i\in\Delta}\tau_i^*),$$

where $(\bigwedge_{i\in\Delta}\tau_i)(\mu) = \bigwedge_{i\in\Delta}(\tau_i(\mu))$, $(\bigvee_{i\in\Delta}\tau_i^*)(\mu) = \bigvee_{i\in\Delta}(\tau_i^*(\mu))$, $\mu \in I^X$.

**Theorem: 2.3.6**

An arbitrary intersection of IGOs is an IGO.

**Definition: 2.3.7**

Let (X, $\tau$, $\tau^*$) be an IFTS and $Y \subset X$. Define two mapping $\tau_Y$, $\tau_Y^*$ : $I^Y \to I$ by the rule

$$\tau_Y(\mu) = \vee\{\tau(\lambda) : \lambda \in I^X, \lambda /Y = \mu\},$$

$$\tau_Y^*(\mu) = \wedge\{\tau^*(\lambda) : \lambda \in I^X, \lambda /Y = \mu\}, \forall \mu \in I^Y.$$

Then ($\tau_Y$, $\tau_Y^*$) is an IGO on Y and $\tau_Y(\mu) \geq \tau(\mu_X)$, $\tau_Y^*(\mu) \leq \tau^*(\mu_X)$.

**Definition: 2.3.8**

Let (X, $\tau$, $\tau^*$) and (Y, $\mathcal{U}$, $\mathcal{U}^*$) be two IFTSs and f : X $\to$ Y be a mapping. Then f is called a **gradation preserving map (gp-map**, for short) if for each $\mu \in I^Y$,

$$\mathcal{U}(\mu) \leq \tau(f^{-1}(\mu)) \text{ and } \mathcal{U}^*(\mu) \geq \tau^*(f^{-1}(\mu)).$$

**Definition: 2.3.9**

Let f : $(X, \tau, \tau^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ be a mapping, where $(X, \tau, \tau^*)$ and $(Y, \mathcal{U}, \mathcal{U}^*)$ are two intuitionistic fuzzy topological spaces of fuzzy subsets. Then f is said to **intuitionistic continuous** if f: $(X, \tau) \to (Y, \mathcal{U})$ and f : $(X, \tau^*) \to (Y, \mathcal{U}^*)$ are continuous.