Intuitionistic L-Fuzzy Structures in Z-Algebras

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In 1984, intuitionistic L-fuzzy set was introduced by Atanassov and Stoeva [11] as a generalization of L-fuzzy set where the membership and non-membership function takes values in a bounded lattice with maximal element 1 and minimal element 0, respectively to allow incomparability among elements. Motivated by this, in this chapter, we define the intuitionistic L-fuzzy Z-Subalgebras of Z-algebras and the intuitionistic L-fuzzy Z-Ideals of Z-algebras and establish some of their properties. All the results proved in this chapter are analogous to the results proved in the previous chapter, except the fact that we are considering the intuitionistic L- fuzzy sets of a Z-algebra X.

6.1 Intuitionistic L-fuzzy Z-Subalgebras in Z-algebras

In this section we introduce the notion of Intuitionistic L-fuzzy Z-Subalgebra of a Z-algebra. Also we have obtained some interesting results.

Definition 6.1.1: An **Intuitionistic L-fuzzy Set** $A = (\mu_A, \nu_A)$ in a Z-algebra $(X, *, 0)$ is called an **Intuitionistic L-fuzzy Z-Subalgebra** of X if it satisfies the following conditions:

(i) $\mu_A(x * y) \geq \mu_A(x) \wedge \mu_A(y)$

(ii) $\nu_A(x * y) \leq \nu_A(x) \vee \nu_A(y)$, for all $x, y \in X$.

One can easily prove the following theorems.

Theorem 6.1.2: Let A_1 and A_2 be two intuitionistic L-fuzzy Z- Subalgebras of a Z-algebra X. Then $A_1 \cap A_2$ is an intuitionistic L-fuzzy Z-Subalgebra of X.

We generalize the above theorem as follows.

Corollary 6.1.3: Let $\{A_i \mid i \in \Omega\}$ be a family of intuitionistic L-fuzzy Z-Subalgebras of a Z-algebra X. Then $\bigcap_{i\in\Omega} A_i$ is an intuitionistic L-fuzzy Z-Subalgebra of X.

In the same way and by the definition of A^c, we can prove the following result.

Theorem 6.1.4: An intuitionistic L-fuzzy set $A = (\mu_A, \nu_A)$ is an intuitionistic L-fuzzy Z-Subalgebra of a Z-algebra X if and only if the L-fuzzy sets μ_A and $(\nu_A)^c$ are L-fuzzy Z-Subalgebras of X.

Theorem 6.1.5: $A = (\mu_A, \nu_A)$ is an intuitionistic L-fuzzy Z-Subalgebra of a Z-algebra X if and only if $\oplus A = (\mu_A, (\mu_A)^c)$ and $\otimes A = ((\nu_A)^c, \nu_A)$, both are intuitionistic L-fuzzy Z-Subalgebras of X.

Theorem 6.1.6: If $A = (\mu_A, \nu_A)$ is an intuitionistic L-fuzzy Z-Subalgebra of a Z-algebra X, then $U(\mu_A; s)$ and $L(\nu_A; t)$ are Z-Subalgebras of X for all $s, t \in L$.

Theorem 6.1.7: Let $A = (\mu_A, \nu_A)$ be an intuitionistic L-fuzzy set in a Z-algebra X such that the sets $U(\mu_A; s)$ and $L(\nu_A; t)$ are Z-Subalgebras of X for every $s, t \in L$. Then $A = (\mu_A, \nu_A)$ is an intuitionistic L-fuzzy Z-Subalgebra of X.

Theorem 6.1.8: Any Z-Subalgebra of a Z-algebra X can be realized as both the upper s-level and lower t-level Z-Subalgebras of some intuitionistic L-fuzzy Z-Subalgebras of X.

Theorem 6.1.9: Let Q be a subset of a Z-algebra X and A be an intuitionistic L-fuzzy set on X which is given in the proof of Theorem 6.1.8. If Q be realized as upper s- level Z- Subalgebra , lower t-level Z- Subalgebra of intuitionistic L-fuzzy Z-Subalgebra A of X, then Q is a Z-Subalgebra of X.

As a generalization of Theorem 6.1.8, we prove the following theorem:

Theorem 6.1.10: Let $(X, *, 0)$ be a Z-algebra. Then any given chain of Z-Subalgebras $Q_0 \subset Q_1 \subset \ldots \subset Q_r = X$, there exists an intuitionistic L-fuzzy Z-Subalgebra A of X whose level(upper level and lower level) Z-Subalgebras are exactly the Z-Subalgebras of this chain.

Note: If X is a finite Z-algebra, then the number of Z-Subalgebra of X is finite, but on the other hand the number of level Z-Subalgebras of an intuitionistic L-fuzzy Z-Subalgebra A appears to be infinite. But since every level Z-Subalgebras is indeed a Z-Subalgebra of X, not all these Z-Subalgebras are distinct. The next theorem characterizes this aspect.

Theorem 6.1.11: Let A be an intuitionistic L-fuzzy Z-Subalgebra of a Z-algebra X. Then

(i) two upper s-level Z- Subalgebras $U(\mu_A; s_1)$ and $U(\mu_A; s_2)$ (with $s_1 < s_2$) of A are equal if and only if there is no $x \in X$ such that $s_1 \le \mu_A(x) < s_2$.

(ii) two lower t-level Z- Subalgebras $L(\nu_A; t_1)$ and $L(\nu_A; t_2)$ (with $t_1 > t_2$) of A are equal if and only if there is no $x \in X$ such that $t_1 \ge \nu_A(x) > t_2$.

Theorem 6.1.12: Let X be a finite Z-algebra and A be an intuitionistic L-fuzzy Z-Subalgebra of X.

(i) If $Im(\mu_A) = \{s_1, ..., s_n\}$, then the family of Z-Subalgebras $U(\mu_A; s_i), i = 1,2,\dots,n$ constitutes all the upper s-level Z-Subalgebras of A.

(ii) If $Im(\nu_A) = \{t_1, ..., t_r\}$, then the family of Z-Subalgebras $L(\nu_A; t_i)$, $i = 1,2,\dots,n$ constitutes all the lower t-level Z-Subalgebras of A.

Theorem 6.1.13: Let A be an intuitionistic L-fuzzy Z-Subalgebra of a Z-algebra X. Then

(i) If $Im(\mu_A)$ is finite , say $\{s_1, ..., s_n\}$, then for any $s_i, s_j \in Im(\mu_A)$, $U(\mu_A; s_i) = U(\mu_A; s_j)$ implies $s_i = s_j$.

(ii) If $Im(\nu_A)$ is finite , say $\{t_1, ..., t_n\}$, then for any $t_i, t_j \in Im(\nu_A)$, $L(\nu_A; t_i) = L(\nu_A; t_j)$ implies $t_i = t_j$.

Theorem 6.1.14: Let h be a Z-homomorphism from a Z-algebra $(X,*,0)$ onto a Z-algebra $(Y,*',0')$ and $A = (\mu_A, \nu_A)$ be an intuitionistic L-fuzzy Z-Subalgebra of X with sup-inf property. Then the image $h(A) = \{\langle y, \mu_{h(A)}(y), \nu_{h(A)}(y)\rangle \mid y \in Y\}$ of A under h is an intuitionistic L-fuzzy Z-Subalgebra of Y.

Theorem 6.1.15 : Let $h:(X,*,0) \to (Y,*',0')$ be a Z-homomorphism of Z-algebras and B be an intuitionistic L-fuzzy Z-Subalgebra of Y. Then the inverse image of B, $h^{-1}(B) = \{\langle x, \mu_{h^{-1}(B)}(x), \nu_{h^{-1}(B)}(x)\rangle \mid x \in X\}$ is an intuitionistic L-fuzzy Z-Subalgebra of X. Converse is true if h is an Z-epimorphism.

Definition 6.1.16: Let $h:(X,*,0) \to (X,*,0)$ be an Z-endomorphism of Z-algebras and A be an intuitionistic L-fuzzy set in X. We define a **new intuitionistic L-fuzzy set** $A^h = (\mu_{A^h}, \nu_{A^h})$ in X as $\mu_{A^h}(x) = \mu_A(h(x))$ and $\nu_{A^h}(x) = \nu_A(h(x))$ for all $x \in X$.

Theorem 6.1.17: Let h be an Z-endomorphism of a Z-algebra $(X,*,0)$. If A be an intuitionistic L-fuzzy Z-Subalgebra of X. Then intuitionistic L-fuzzy set $A^h = (\mu_{A^h}, \nu_{A^h})$ is also an intuitionistic L-fuzzy Z-Subalgebra of X.

Proof: Let x, y $\in$ X. Then

(i) $\mu_{A^h}(x*y) = \mu_A(h(x*y)) = \mu_A(h(x)*h(y)) \geq \mu_A(h(x)) \wedge \mu_A(h(y)) = \mu_{A^h}(x) \wedge \mu_{A^h}(y)$

$\Rightarrow \quad \mu_{A^h}(x*y) \geq \mu_{A^h}(x) \wedge \mu_{A^h}(y)$

(ii) $\nu_{A^h}(x*y) = \nu_A(h(x*y)) = \nu_A(h(x)*h(y)) \leq \nu_A(h(x)) \vee \nu_A(h(y)) = \nu_{A^h}(x) \vee \nu_{A^h}(y)$

$\Rightarrow \quad \nu_{A^h}(x*y) \leq \nu_{A^h}(x) \vee \nu_{A^h}(y)$

From (i) and (ii) we get, A^h is an intuitionistic L-fuzzy Z-Subalgebra of a Z-algebra X.

Analogously, we can prove the following theorem.

Theorems 6.1.18: Let A and B be any two intuitionistic L-fuzzy Z-Subalgebras of a Z-algebra X. Then $A \times B$ is an intuitionistic L-fuzzy Z-Subalgebra of $X \times X$.

Proof: Take (x_1, y_1) and $(x_2, y_2) \in X \times X$. Then

$$\begin{aligned}
\mu_{A\times B}[(x_1,y_1)*(x_2,y_2)] = \mu_{A\times B}[(x_1*x_2),(y_1*y_2)] &= \mu_A(x_1*x_2) \wedge \mu_B(y_1*y_2) \\
&\geq (\mu_A(x_1) \wedge \mu_A(x_2)) \wedge (\mu_B(y_1) \wedge \mu_B(y_2)) \\
&= (\mu_A(x_1) \wedge \mu_B(y_1)) \wedge (\mu_A(x_2) \wedge \mu_B(y_2)) \\
&= \mu_{A\times B}(x_1,y_1) \wedge \mu_{A\times B}(x_2,y_2)
\end{aligned}$$

$$\begin{aligned}
\nu_{A\times B}[(x_1,y_1)*(x_2,y_2)] = \nu_{A\times B}[(x_1*x_2),(y_1*y_2)] &= \nu_A(x_1*x_2) \vee \nu_B(y_1*y_2) \\
&\leq (\nu_A(x_1) \vee \nu_A(x_2)) \vee (\nu_B(y_1) \vee \nu_B(y_2)) \\
&= (\nu_A(x_1) \vee \nu_B(y_1)) \vee (\nu_A(x_2) \vee \nu_B(y_2)) \\
&= \nu_{A\times B}(x_1,y_1) \vee \nu_{A\times B}(x_2,y_2)
\end{aligned}$$

This proves that the Cartesian product of two intuitionistic L-fuzzy Z-Subalgebras is again an intuitionistic L-fuzzy Z-Subalgebra of a Z-algebra X.

6.2 Intuitionistic L-fuzzy Z-Ideals in Z-algebras

In this section we introduce the notion of Intuitionistic L-fuzzy Z-ideal of a Z-algebra and some interesting results are obtained.

Definition 6.2.1: An intuitionistic L-fuzzy set $A=(\mu_A,\nu_A)$ in a Z-algebra $(X,*,0)$ is called an **intuitionistic L-fuzzy Z-ideal** of X if it satisfies the following conditions:

(i) $\mu_A(0)\ge\mu_A(x)$ and $\nu_A(0)\le\nu_A(x)$

(ii) $\mu_A(x)\ \ge\ \mu_A(x*y)\ \wedge\ \mu_A(y)$

(iii) $\nu_A(x)\ \le\ \nu_A(x*y)\ \vee\ \nu_A(y)$, for all $x,y\in X$.

One can easily prove the following theorems.

Theorem 6.2.2: Intersection of any two intuitionistic L-fuzzy Z-ideals of a Z-algebra X is again an intuitionistic L-fuzzy Z-ideal of X.

We generalize the above theorem as follows.

Theorem 6.2.3: Let $\{A_i\mid i\in\Omega\}$ be a family of intuitionistic L-fuzzy Z-ideals of a Z-algebra X. Then $\bigcap_{i\in\Omega}A_i$ is an intuitionistic L-fuzzy Z-ideal of X.

Lemma 6.2.4: An intuitionistic L-fuzzy set $A=(\mu_A,\nu_A)$ is an intuitionistic L-fuzzy Z-ideal of a Z-algebra X if and only if the L-fuzzy sets μ_A and $(\nu_A)^c$ are L-fuzzy Z-ideals of X.

Theorem 6.2.5: Let $A=(\mu_A,\nu_A)$ be an intuitionistic L-fuzzy set in a Z-algebra X. Then $A=(\mu_A,\nu_A)$ is an intuitionistic L-fuzzy Z-ideal of X if and only if $\oplus A=(\mu_A,(\mu_A)^c)$ and $\otimes A=((\nu_A)^c,\nu_A)$ are intuitionistic L-fuzzy Z-ideals of X.

Theorem 6.2.6: An intuitionistic L-fuzzy set $A=(\mu_A,\nu_A)$ is an intuitionistic L-fuzzy Z-ideal of a Z-algebra X if and only if for all $s,t\in L$, the sets $U(\mu_A;s)$ and $L(\nu_A;t)$ are either empty or Z-ideals of X.

Theorem 6.2.7: Let h be a homomorphism from a Z-algebra $(X,*,0)$ onto a Z-algebra $(Y,*',0')$ and A be an intuitionistic L-fuzzy Z-ideal of X with sup-inf property. Then image of A, $h(A)=\{\langle y,\mu_{h(A)}(y),\nu_{h(A)}(y)\rangle\mid y\in Y\}$ is an intuitionistic L-fuzzy Z-ideal of Y.

Theorem 6.2.8: Let $h:(X,*,0)\to(Y,*',0')$ be a Z-homomorphism of Z-algebras and B be an intuitionistic L-fuzzy Z-ideal of Y. Then the inverse image of B, $h^{-1}(B)=\{\langle x,\mu_{h^{-1}(B)}(x),\nu_{h^{-1}(B)}(x)\rangle | x\in X\}$ is an intuitionistic L-fuzzy Z-ideal of X.

Theorem 6.2.9: Let $h:(X,*,0)\to(Y,*',0')$ be an Z-epimorphism of Z-algebras. Let B be an intuitionistic L-fuzzy set of Y. If $h^{-1}(B)$ is an intuitionistic L-fuzzy Z-ideal of X then B is an intuitionistic L-fuzzy Z-ideal of Y.

Theorem 6.2.10: Let h be an Z-endomorphism of a Z-algebra $(X,*,0)$. If A be an intuitionistic L-fuzzy Z-ideal of X. Then, $A^h=(\mu_{A^h},\nu_{A^h})$ is also an intuitionistic L-fuzzy Z-ideal of X.

Theorem 6.2.11: Let A and B be two intuitionistic L-fuzzy Z-ideals in a Z-algebra X. Then $A\times B$ is an intuitionistic L-fuzzy Z-ideal of $X\times X$.

Proof: Take $(x_1,x_2)\in X\times X$.

Then $\mu_{A\times B}(0,0)=\mu_A(0)\wedge\mu_B(0)\ge\mu_A(x_1)\wedge\mu_B(x_2)=\mu_{A\times B}(x_1,x_2)$

and $\nu_{A\times B}(0,0)=\nu_A(0)\vee\nu_B(0)\le\nu_A(x_1)\vee\nu_B(x_2)=\nu_{A\times B}(x_1,x_2)$

Now take $(x_1,x_2),(y_1,y_2)\in X\times X$. Then

$$\begin{aligned}\mu_{A\times B}(x_1,x_2)=\mu_A(x_1)\wedge\mu_B(x_2)&\ge(\mu_A(x_1*y_1)\wedge\mu_A(y_1))\wedge(\mu_B(x_2*y_2)\wedge\mu_B(y_2))\\&=(\mu_A(x_1*y_1)\wedge\mu_B(x_2*y_2))\wedge(\mu_A(y_1)\wedge\mu_B(y_2))\\&=\mu_{A\times B}((x_1*y_1),(x_2*y_2))\wedge\mu_{A\times B}(y_1,y_2)\\&=\mu_{A\times B}((x_1,x_2)*(y_1,y_2))\wedge\mu_{A\times B}(y_1,y_2)\end{aligned}$$

$$\begin{aligned}\nu_{A\times B}(x_1,x_2)=\nu_A(x_1)\vee\nu_B(x_2)&\le(\nu_A(x_1*y_1)\vee\nu_A(y_1))\vee(\nu_B(x_2*y_2)\vee\nu_B(y_2))\\&=(\nu_A(x_1*y_1)\vee\nu_B(x_2*y_2))\vee(\nu_A(y_1)\vee\nu_B(y_2))\\&=\nu_{A\times B}((x_1*y_1),(x_2*y_2))\vee\nu_{A\times B}(y_1,y_2)\\&=\nu_{A\times B}((x_1,x_2)*(y_1,y_2))\vee\nu_{A\times B}(y_1,y_2)\end{aligned}$$

Hence $A\times B$ is an intuitionistic L-fuzzy Z-ideal of $X\times X$.

Theorem 6.2.12: Let A and B be two intuitionistic L-fuzzy sets in a Z-algebra X. If $A\times B$ is an intuitionistic L-fuzzy Z-ideal of $X\times X$, the following are true.

(i) $\mu_A(0)\ge\mu_B(y)$ and $\mu_B(0)\ge\mu_A(x)$ for all $x,y\in X$.

(ii) $\nu_A(0) \leq \nu_B(y)$ and $\nu_B(0) \leq \nu_A(x)$ for all $x, y \in X$.

Proof : Assume that $\mu_B(y) > \mu_A(0)$ and $\mu_A(x) > \mu_B(0)$ for some $x, y \in X$.

Then $\mu_{A\times B}(x, y) = \mu_A(x) \wedge \mu_B(y) > \mu_B(0) \wedge \mu_A(0)$

$= \mu_{A\times B}(0,0)$ which is a contradiction.

Similarly, assume that $\nu_A(x) < \nu_B(0)$ and $\nu_B(y) < \nu_A(0)$ for some $x, y \in X$.

Then $\nu_{A\times B}(x, y) = \nu_A(x) \vee \nu_B(y) < \nu_B(0) \vee \nu_A(0)$

$= \nu_{A\times B}(0,0)$ which is also a contradiction thus proving the result.

Theorem 6.2.13: Let A and B be two intuitionistic L-fuzzy sets in a Z-algebra X such that $A \times B$ is an intuitionistic L-fuzzy Z-ideal of $X \times X$. Then either A or B is an intuitionistic L-fuzzy Z-Ideal of X.

Proof : Now by above Theorem 6.2.12 if we take $\mu_A(0) \geq \mu_B(y)$ and $\nu_A(0) \leq \nu_B(y)$ for all $y \in X$,

$\mu_{A\times B}(0, y) = \mu_A(0) \wedge \mu_B(y) = \mu_B(y)$ and $\nu_{A\times B}(0, y) = \nu_A(0) \vee \nu_B(y) = \nu_B(y)$ (1)

Take $(x_1, y_1), (x_2, y_2) \in X \times X$.

Since $A \times B$ is an intuitionistic L-fuzzy Z-ideal of $X \times X$.

$\mu_{A\times B}(x_1, y_1) \geq \mu_{A\times B}((x_1, y_1) * (x_2, y_2)) \wedge \mu_{A\times B}(x_2, y_2)$

$= \mu_{A\times B}(x_1 * x_2, y_1 * y_2) \wedge \mu_{A\times B}(x_2, y_2)$ (2)

Putting $x_1 = x_2 = 0$ in (2) we get,

$\mu_{A\times B}(0, y_1) \geq \mu_{A\times B}(0, y_1 * y_2) \wedge \mu_{A\times B}(0, y_2)$ and by (1),

$\mu_B(y_1) \geq \mu_B(y_1 * y_2) \wedge \mu_B(y_2)$

Analogously, we can prove $\nu_B(y_1) \leq \nu_B(y_1 * y_2) \vee \nu_B(y_2)$.

Hence B is an intuitionistic L-fuzzy Z-ideal of a Z-algebra X.

By Theorem 6.2.12, assume that $\mu_B(0) \geq \mu_A(x)$ and $\nu_B(0) \leq \nu_A(x)$ then A is an intuitionistic L-fuzzy Z-ideal of a Z-algebra X.

Therefore, either A or B is an intuitionistic L-fuzzy Z-ideal of a Z-algebra X.

This completes the proof.