CHAPTER – 2

FUZZY IDEALS AND FUZZY FILTERS IN CI-ALGEBRAS

SECTION 2.1

FUNDAMENTAL DEFINITIONS ON FUZZY SETS

Definition : 2.1.1

Let X is a non-empty set. A **fuzzy set** μ is a mapping $\mu : X \to [0, 1]$.

Definition : 2.1.2

A fuzzy set $\mu : X \to [0, 1]$ is said to be **empty** if $\mu(x) = 0$ for all x in X. It is denoted by O_X or Φ_X.

Definition : 2.1.3

A fuzzy set $\mu : X \to [0, 1]$ is said to be **whole fuzzy set** if $\mu(x) = 1$ for all x in X. It is denoted by 1_X or X.

Definition : 2.1.4

For $C \in [0, 1]$, a fuzzy set $K_c : X \to [0, 1]$ is said to be a **constant fuzzy set** if $K_c(x) = c$ for all x in X.

Definition : 2.1.5

The **complement of a fuzzy set** μ in X denoted by μ^c or $\overline{\mu}$ is defined as $\mu^c(x) = 1 - \mu(x)$.

Definition : 2.1.6

If μ is a fuzzy set in X, then **support of μ** denoted by supp μ is defined as, Supp $\mu = \{x \in X / \mu(x) > 0\}$.

Definition : 2.1.7

Two fuzzy sets μ in X and γ in X are **equal**, (written as $\mu = \gamma$), if and only if $\mu(x) = \gamma(x)$ for x in X.

Definition : 2.1.8

Let μ and γ are fuzzy sets in X. μ is contained in γ (or equivalently, μ is a subset of γ, or μ is smaller than or equal to γ) if and only if $\mu(x) \leqq \gamma(x)$ for all x in X. In symbols $\mu \leq \gamma$ iff $\mu(x) \leqq \gamma(x)$.

Definition : 2.1.9

The **union of two fuzzy** sets μ in X and γ in X is a fuzzy set λ (written as $\lambda = \mu \vee \gamma$) defined by $\lambda(x) = (\mu \vee \gamma)(x) = \max\{\mu(x), \gamma(x)\}$, for all $x \in X$.

Definition : 2.1.10

The **intersection of two fuzzy sets** μ in X and γ in X is a fuzzy set λ (written as $\lambda = \mu \wedge \gamma$) defined by $\lambda(x) = (\mu \wedge \gamma)(x) = \min\{\mu(x), \gamma(x)\}$, $\forall\, x \in X$.

Definition : 2.1.11

Let Λ be an indexing set and $\{\mu_\lambda / \lambda \in \Lambda\}$ be a family of fuzzy sets in X. Then their **union** and **intersection** are defined as follows :

$$\bigvee_{\lambda \in \Lambda} \mu_\lambda(x) = \sup\{\mu_\lambda(x) / \lambda \in \Lambda\} \text{ for all } x \in X.$$

$$\bigwedge_{\lambda \in \Lambda} \mu_\lambda(x) = \inf\{\mu_\lambda(x) / \lambda \in \Lambda\} \text{ for all } x \in X.$$

Definition : 2.1.12

A fuzzy set μ in X is said to have the **sup property** if for any subset $T \subset X$ there exists $x_0 \in T$ such that $\mu(x_0) = \sup_{t \in T} \mu(t)$.

Definition : 2.1.13

A fuzzy set μ in X defined by

$$\mu(y) = \begin{cases} t \text{ if } y = x, t \in (0,1] \\ 0 \text{ if } y \neq x \end{cases}$$

is called a **fuzzy point** with support x and value t. It is denoted by (x, t) or x_t.

Definition : 2.1.14

A fuzzy point (x, t) or x_t is said **to belong to** a **fuzzy set** μ in X if $t \leq \mu(x)$ $\forall\, x \in X$. It is denoted by $(x, t) \in \mu$ or $x_t \in \mu$.

Definition : 2.1.15

(i) A fuzzy point (x, t) is said to **be quasi coincident** with a fuzzy set μ in X dented by $(x, t)q\mu$ if $t + \mu(x) > 1$.

(ii) A fuzzy point (x, t) is **not quasi coincident** with a fuzzy set μ in X denoted by $\overline{(x, t)q\mu}$ $(x, t)q\mu$ if $t + \mu(x) \leqq 1$.

(iii) If $(x, t)\in\mu$ or $(x, t)q\mu$ then $(x, t)\in\vee q\mu$.

(iv) If $(x, t)\in\mu$ and $x_t\, q\, \mu$ then $(x, t)\in\wedge q\mu$.

Note

Also $\overline{\in\vee q}$ means $\in\vee q$ does not hold.

Definition : 2.1.16

Let μ be a fuzzy set in a set X. For $t \in [0, 1]$

(i) The set $U(\mu\, ;\, t) = \{x \in X \,/\, \mu(x) \geq t\}$ is called an **upper level subset** of μ. It is also denoted by μ^t.

(ii) The set $L(\mu\, ;\, t) = \{x \in X \,/\, \mu(x) \leq t\}$ is called an **lower level subset** of μ. It is also denoted by μ_t.

Note

$\mu^t \cup \mu_t = X$ for $t \in [0, 1]$. If $t_1 < t_2$ then $\mu_{t_1} \subseteq \mu_{t_2}$.

Definition : 2.1.17

Let $f : X \to Y$ be a function from a set X to a set Y and let μ be a fuzzy set of X. Then the fuzzy set λ of Y is defined by

$$\lambda(y) = \begin{cases} \text{Sup}\,\{\mu(x) / x \in f^{-1}(y)\}, \text{if } f^{-1}(y) = \{x \in X / f(x) = y\} \neq \Phi \\ 0, \quad \text{otherwise} \end{cases}$$

is called the **image of μ under f**, denoted by $f(\mu)$.

If λ is a fuzzy set of Y, then the fuzzy set μ of X is given by $\mu(x) = \lambda(f(x))$, for all $x \in X$, is called the **preimage of λ under f** and is denoted by $f^{-1}(\lambda)$ that is, $f^{-1}(\lambda)(x) = \lambda(f(x))$.

SECTION 2.2
FUZZY IDEALS IN CI-ALGEBRAS

Definition : 2.2.1

A fuzzy set μ in X is called **fuzzy ideal of X** if it satisfies the following :

(FI 1) $\mu(x * y) \geq \mu(y)$, for all x, y $\in$ X.

(FI 2) $\mu((x * (y * z) * z) \geq \min \{\mu(x), \mu(y)\}$, for all x, y, z $\in$ X.

Theorem : 2.2.2

Let μ be a fuzzy set in a CI-algebra X. Then μ is a fuzzy ideal of X iff it satisfies :

$U(\mu ; \alpha) \neq \Phi \Rightarrow U(\mu ; \alpha)$ is an ideal of X, $\forall\ \alpha \in [0, 1]$

where $U(\mu ; \alpha) = \{x \in X / \mu(x) \geq \alpha\}$.

Proof

Assume that μ is a fuzzy ideal of X.

To Prove : $U(\mu ; \alpha)$ is an ideal of X

Let $\alpha \in [0, 1]$ be such that $U(\mu ; \alpha) \neq \Phi$.

Let x, y $\in$ X be such that $y \in U(\mu ; \alpha)$. Then $\mu(y) \geq \alpha$.

Therefore $\mu(x * y) \geq \mu(y) \geq \alpha$ (by FI 1)

Thus $x * y \in U(\mu ; \alpha)$ (1)

Let x $\in$ X and a, b $\in U(\mu ; \alpha)$ then $\mu(a) \geq \alpha$ and $\mu(b) \geq \alpha$.

Then from (FI 2) we have

$\mu((a * (b * x)) * x) \geq \min \{\mu(a), \mu(b)\} \geq \alpha$

$\Rightarrow (a * (b * x)) * x \in U(\mu ; \alpha)$ (2)

Hence by (1) and (2) $U(\mu ; \alpha)$ is an ideal of X.

Conversely, suppose $U(\mu ; \alpha) \neq \Phi \Rightarrow U(\mu ; \alpha)$ is an ideal of X for all $\alpha \in [0, 1]$.

Claim : μ is a fuzzy ideal of X.

If $\mu(a * b) < \mu(b)$ for some $a, b \in X$

then $\mu(a * b) < \alpha_0 < \mu(b)$ by taking $\alpha_0 = \frac{1}{2}[\mu(a * b) + \mu(b)]$

Hence $a * b \notin U(\mu ; \alpha_0)$ and $b \in U(\mu ; \alpha_0)$ which is a contradiction.

Therefore $\mu(a * b) \geq \mu(b)$ for all $a, b \in X$ (1)

Let $a, b, c \in X$ be such that

$\mu((a * (b * c)) * c) < \min \{\mu(a), \mu(b)\}$.

Taking $\beta_0 = \frac{1}{2}\ \mu((a * (b * c)) * c) + \min \{\mu(a), \mu(b)\}$ we have $\beta_0 \in [0, 1]$ and $\mu((a * (b * c)) * c) < \beta_0 < \min \{\mu(a), \mu(b)\}$.

Hence $a, b \in U(\mu ; \beta_0)$ and $(a * (b * c)) * c \notin U(\mu ; \beta_0)$

This is a contradiction.

Therefore, $\mu((a * (b * c)) * c) \geq \min \{\mu(a), \mu(b)\}$ (2)

By (1) and (2), μ is a fuzzy ideal of X.

Lemma : 2.2.3

Every fuzzy ideal μ of X satisfies the following inequality : $\mu(1) \geq \mu(x)$, $\forall x \in X$.

Proof

Let μ be a fuzzy ideal in X.

Then by (CI 1), $\mu(1) = \mu(x * x)\ \forall x \in X$

$\Rightarrow \mu(1) \geq \mu(x)\ \forall x \in X$

Example : 2.2.4

Let X = {1, a, b, c, d, 0} be a set with the following cayley table :

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then (X ; $*$, 1) is a CI-algebra.

(i) Let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} 0.7 & \text{if } x \in \{1, a, b\} \\ 0.2 & \text{if } x \in \{c, d, 0\} \end{cases}$$

$$\text{Then } U(\mu ; \alpha) = \begin{cases} \Phi & \text{if } \alpha \in (0.7, 1], \\ \{1, a, b\} & \text{if } \alpha \in (0.2, 0.7], \\ X & \text{if } \alpha \in [0, 0.2]. \end{cases}$$

Then {1, a, b} and X are ideals of X, and so μ is a fuzzy ideal of X.

(ii) Let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} 0.6 & \text{if } x \in \{1, a\} \\ 0.4 & \text{if } x \in \{b, c, d, 0\} \end{cases}$$

$$\text{Then } U(\mu ; \beta) = \begin{cases} \Phi & \text{if } \beta \in (0.6, 1], \\ \{1, a\} & \text{if } \beta \in (0.4, 0.6], \\ X & \text{if } \beta \in [0, 0.4]. \end{cases}$$

$$\begin{aligned} \text{Then } (a * (a * b)) * b &= (a * a) * b \\ &= 1 * b \\ &= b \notin \{1, a\} \end{aligned}$$

$\Rightarrow$ {1, a} is not an ideal of X.

Hence μ is not a fuzzy ideal in X.

Proposition : 2.2.5

If μ is a fuzzy ideal of X, then $\mu((x * y) * y) \geq \mu(x)$, $\forall$ x, y $\in$ X.

Proof

Let μ be a fuzzy ideal of X.

Taking y = 1 and z = y in $\mu((x * (y * z)) * z) \geq \min \{\mu(x), \mu(y)\}$ $\forall$ x, y, z $\in$ X and

using $1 * x = x$ and $\mu(1) \geq \mu(x)$ we get

$\mu((x * y) * y) = \mu(((x * (1 * y)) * y)$

$\geq \min \{\mu(x), \mu(1)\}$

$= \mu(x)$ $\forall$ x, y $\in$ X

Corollary : 2.2.6

Every fuzzy ideal μ of X is order preserving i.e, μ satisfies :

$x \leq y \Rightarrow \mu(x) \leq \mu(y)$, $\forall$ x, y $\in$ X.

Proof

Let μ be a fuzzy ideal in X. Let x, y $\in$ X be such that $x \leq y$.

Then $x * y = 1$ and so $\mu(y) = \mu(1 * y)$ by (CI 2)

$= \mu((x * y) * y)$

By proposition (2.2.5) $\mu(y) \geq \mu(x)$.

Proposition : 2.2.7

Let μ be a fuzzy set in X which satisfies $\mu(1) \geq \mu(x)$ $\forall$ x $\in$ X and $\mu(x * z) \geq \min \{\mu(x * (y * z)), \mu(y)\}$ $\forall$ x, y, z $\in$ X. Then μ is order preserving.

Proof

Let μ be a fuzzy set in X which satisfies $\mu(1) \geq \mu(x)$ $\forall$ $x \in X$ and $\mu(x * z) \geq \min \{\mu(x * (y * z)), \mu(y)\}$ $\forall$ x, y, z $\in$ X. Let x, y $\in$ X be be such that $x \leq y$. Then $x * y = 1$.

And $\mu(y) = \mu(1 * y) \geq \min \{\mu(1 * (x * y)), \mu(x)\}$.

By putting x = 1, z = y, y =x in (1) we get,

$\mu(y) = \min \{\mu(1 * 1), \mu(x)\} = \min\{\mu(1), \mu(x)\} = \mu(x)$.

$\therefore \mu(x) \leq \mu(y)$

$\therefore x \leq y \Rightarrow \mu(x) \leq \mu(y)$

$\therefore \mu$ is order presenting.

Theorem : 2.2.8

Let X be a transitive CI-algebra. A fuzzy set μ in X is a fuzzy ideal of X. Then it satisfies condition :

(i) $\mu(1) \geq \mu(x)$, $\forall$ $x \in X$ and

(ii) $\mu(x * z) \geq \min \{\mu(x * (y * z)), \mu(y)\}$, for all x, y, z $\in$ X.

Proof

Assume that μ is a fuzzy ideal of X. Then by lemma (2.2.3) μ satisfies, $\mu(1) \geq \mu(x)$. Since X is transitive, we have

$(y * z) * z \leq (x * (y * z)) * (x * z)$

i.e., $((y * z) * z) * ((x * (y * z)) * (x * z)) = 1$ $\forall$ x, y, z $\in$ X.

Consider $\mu(x * z) = \mu(1 * (x * z))$ by (CI 2)

$= \mu[((y * z) * z) * ((x *(y * z)) * (x * z)) * (x * z)]$

$\geq \min \{\mu((y * z) * z), \mu(x * (y * z))\}$

$\geq \min \{\mu(x * (y * z), \mu(y))\}$

Hence μ satisfies (ii)

Corollary : 2.2.9

Let X be a self-distributive CI-algebra. A fuzzy set μ in X is a fuzzy ideal of X, then it satisfies condition

(i) $\mu(1) \geq \mu(x) \quad \forall\, x \in X$

(ii) $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$ for all x, y, z $\in$ X.

Proof : Obvious.

Definition : 2.2.10

For every a, b $\in$ X, let μ_a^b be a fuzzy set in X defined by

$$\mu_a^b(x) = \begin{cases} \alpha & \text{if } a * (b * x) = 1, \\ \beta & \text{otherwise} \end{cases}$$

for all x $\in$ X and $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$.

The following example shows that there exist a, b $\in$ X such that μ_a^b is not a fuzzy ideal of X.

Example : 2.2.11

Let X = {1, a, b, c, d, 0} be a CI-algebra as in example (2.2.4).

Then $\mu_1^a((a * (a * b)) * b) = \mu_1^a((a * a) * b)$

$= \mu_1^a(1 * b)$

$= \mu_1^a(b)$

since $1 * (a * b) * a \neq 1 = \beta < \alpha = \mu_1^a(a)$

$= \min\{\mu_1^a(a), \mu_1^a(a)\}$

i.e. $\mu_1^a((a * (a * b)) * b) < \min\{\mu_1^a(a), \mu_1^a(a)\}$

$\Rightarrow \mu_1^a$ is not a fuzzy ideal of X.

Lemma : 2.2.12

A non empty subset I of a CI-algebra X is an ideal of X if it satisfies.

(i) $1 \in I$ and

(ii) $x * (y * z) \in I \Rightarrow x * z \in I, \forall x, z \in X$ and $\forall y \in I$

Proof

Let I be an ideal of a CI-algebra X.

Since $x * x = 1$ and if $x \in X$, $a \in I$ then $x * a \in I$,

we have $1 = a * a \in I \ \forall a \in I$

$\therefore 1 \in I$

Claim : $x * y \in I \Rightarrow y \in I, \ \forall x \in I, \ \forall y \in X$

Let $x \in I$ and $y \in X$ be such that $x * y \in I$.

Then $y = 1 * y$

$= ((x * y) * (x * y)) * y \in I$ by (1.1.17)

Now let $x, z \in X$ and $y \in I$ be such that $x * (y * z) \in I$

Then $y * (x * z) \in I$ [by (CI 3)]

Since $y \in I$, we have $x * z \in I$

That is, $x * (y * z) \in I \Rightarrow x * z \in I \ \forall x, z \in X$ and $\forall y \in I$

This completes the proof.

Theorem : 2.2.13

Let μ be a fuzzy set in a CI-algebra X. Then μ is a fuzzy ideal of X iff μ satisfies the following assertion :

For all $a, b \in X$ and $\alpha \in [0, 1]$ $a, b \in U(\mu ; \alpha) \Rightarrow A(a, b) \subseteq U(\mu ; \alpha)$.

Proof

Assume that μ is a fuzzy ideal of a CI-algebra X. Let $a, b \in U(\mu ; \alpha)$ then $\mu(a) \geq \alpha$ and $\mu(b) \geq \alpha$. Let $x \in A(a, b)$. Then $a * (b * x) = 1$.

Hence $\mu(x) = \mu(1 * x)$

$= \mu[a * (b * x) * x]$

$\geq \min \{\mu(a), \mu(b)\}$

$\geq \alpha$

$\therefore\ x \in U(\mu ; \alpha)$

Thus $A(a, b) \subseteq U(\mu ; \alpha)$

Conversely, suppose $A(a, b) \subseteq U(\mu ; \alpha)$ for all $a, b \in U(\mu ; \alpha)$.

Claim : μ is a fuzzy ideal of X.

Obviously $1 \in A(a, b) \subseteq U(\mu ; \alpha)\ \ \forall\ a, b \in X$.

Let $x, y, z \in X$ be such that $x * (y * z) \in U(\mu ; \alpha)$ and $y \in U(\mu ; \alpha)$

Since $(x * (y * z)) * (y * (x * z))$

$= (x * (y * z)) * (x * (y * z))$ [by (CI 3)]

$= 1$ [by (CI 1)]

$\Rightarrow x * z \in A(x * (y * z), y) \subseteq U(\mu ; \alpha)$

$\therefore\ U(\mu ; \alpha)$ is an ideal of X

Hence by theorem (2.2.2) μ is a fuzzy ideal of X.

Corollary : 2.2.14

If μ is a fuzzy ideal of a CI-algebra X.

Then $U(\mu ; \alpha) \neq \Phi \Rightarrow U(\mu ; \alpha) = \bigcup_{a, b \in U(\mu ; \alpha)} A(a, b), \quad \forall\ \alpha \in [0, 1]$.

Proof

Let μ be a fuzzy ideal of a CI-algebra X.

Let $\alpha \in [0, 1]$ be such that $U(\mu ; \alpha) \neq \Phi$.

Since $1 \in U(\mu ; \alpha)$ we have

$$U(\mu ; \alpha) \subseteq \bigcup_{a \in U(\mu ; \alpha)} A(a, 1) \subseteq \bigcup_{a, b \in U(\mu ; \alpha)} A(a, b).$$

Let $x \in \bigcup_{a, b \in U(\mu ; \alpha)} A(a, b)$.

Then there exists a, b $\in$ U(μ ; α) such that x $\in$ A(a, b) $\subseteq$ U(μ ; α), by theorem (2.2.13).

Thus $\bigcup_{a,b \in U(\mu;\alpha)} A(a,b) \subseteq U(\mu ; \alpha)$.

Definition : 2.2.15

A fuzzy filter μ of a CI-algebra X is said to be **normal** if there exists x $\in$ X such that μ(x) = 1.

Example : 2.2.16

Let X = {1, a, b, c} be a set with the following table :

*	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	c	c	c	1

Then (X ; $*$, 1) is a CI-algebra. Then the fuzzy filter μ in X defined by μ(1) = μ(a) = 1, μ(b) = μ(c) = 0.3 is a fuzzy normal filter.

Note

If μ is a normal fuzzy filter of a CI-algebra X, then clearly μ(1) = 1, and hence μ is normal if and only if μ(1) = 1.

Theorem : 2.2.17

Let μ is a fuzzy filter of a CI-algebra X. Let μ^+ be a fuzzy set in X defined by $\mu^+(x) = \mu(x) + 1 - \mu(1)$ for all x $\in$ X. Then μ^+ is a normal fuzzy filter of X which contains μ.

Proof

Let μ is a fuzzy filter of a CI-algebra X. Let μ^+ be a fuzzy set in X defined by $\mu^+(x) = \mu(x) + 1 - \mu(x)$ for all $x \in X$.

Then $\mu^+(1) = \mu(1) + 1 - \mu(1) = 1 \geq \mu^+(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\min \{\mu^+(x), \mu^+(x * y)\} = \min \{\mu(x) + 1 - \mu(1), \mu(x * y) + 1 - \mu(1)\}$$
$$= \min \{\mu(x), \mu(x * y)\} + 1 - \mu(1)$$
$$\leq \mu(y) + 1 - \mu(1) = \mu^+(y).$$

Hence μ^+ is a normal fuzzy filter of X. Clearly $\mu \subseteq \mu^+$.

Note

Let μ and μ^+ be as in Theorem (2.2.17). If there exists $x \in X$ such that $\mu^+(x) = 1$, then $\mu(x) = 1$.

Theorem : 2.2.18

A fuzzy filter μ of a CI-algebra X is normal if and only if $\mu^+ = \mu$.

Proof

The sufficiency is obvious.

Assume that μ is a normal fuzzy CI-algebra of X and let $x \in X$.

Then $\mu^+(x) = \mu(x) + 1 - \mu(1) = \mu(x)$, hence $\mu^+ = \mu$.

Proposition : 2.2.19

If μ is a fuzzy filter of X, then $(\mu^+)^+ = \mu^+$.

Proof

For any $x \in X$, we have

$(\mu^+)^+ (x) = \mu^+(x) + 1 - \mu^+(1) = \mu^+(x)$.

SECTION 2.3
ON ($\in$, $\in\vee$qk)-FUZZY FILTERS OF CI-ALGEBRAS

Definition : 2.3.1

A fuzzy set μ in a CI-algebra X is called a **fuzzy filter of X** if it satisfies the following conditions :

(FF 1) $\mu(1) \geq \mu(x)$ for all $x \in X$.

(FF 2) $\mu(y) \geq \min \{\mu(x * y), \mu(x)\}$ for all $x, y \in X$.

Definition : 2.3.2

Let X be a CI-algebra and $k \in [0, 1]$ for $\alpha \in \{\in, \in\vee qk\}$, define the following

(i) $(x, t)\, qk\mu$, we mean $\mu(x) + t + k > 1$,

(ii) $(x, t) \in \vee qk\mu$, means that $(x, t) \in \mu$ or $(x, t)\, qk\mu$.

Note

For $(x, t)\ \overline{\alpha}\ \mu$, we mean $(x, t)\alpha\mu$ does not hold.

Definition : 2.3.3

A fuzzy set μ in X is called an **($\in$, $\in\vee$qk)-fuzzy filter of X** if it satisfies

(C 1) $(x, t) \in \mu \Rightarrow (1, t) \in\vee qk\mu$,

(C 2) $(x, t) \in \mu$ and $(x * y, r) \in \mu \Rightarrow (y, \min \{t, r\}) \in\vee qk\mu$ for all $x, y \in X$ and $t, r \in (0, 1]$.

Note

An ($\in$, $\in\vee$qk)-fuzzy filter of X with k = 0 is called ($\in$, $\in\vee$q)-fuzzy filter of X.

Example : 2.3.4

Let X = {1, a, b, c} be a CI-algebra as in example (2.2.16). Then define a fuzzy set $\mu : X \to [0, 1]$ on X, by $\mu(1) = 0.8$, $\mu(a) = \mu(b) = 0.4$ and $\mu(c) = 0.3$. Then μ is a $(\in, \in\vee qk)$-fuzzy filter of X.

Theorem : 2.3.5

A fuzzy set μ in X is an $(\in, \in\vee qk)$-fuzzy filter of X if and only if it satisfies two conditions :

(C 3) $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\}$, for all $x \in X$.

(C 4) $\mu(y) \geq \min\{\mu(x), \mu(x * y), \frac{1-k}{2}\}$, for all $x, y \in X$.

Proof

Assume that μ is an $(\in, \in\vee qk)$-fuzzy filter of X, if (C3) is not valid, then $\mu(1) < t_a \leq \min\{\mu(a), \frac{1-k}{2}\}$, for some $a \in X$ and $t_a \in (0, \frac{1-k}{2}]$.

Thus $(a, t_a) \in \mu$ but $(1, t_a) \overline{\in} \mu$.

Also, $\mu(1) + t_a < 2\, t_a \leq 1 - k$, i.e., $(1, t_a)\ \overline{qk}\mu$.

Therefore $(1, t_a)\ \overline{\in \vee qk}\mu$, which is a contradiction.

Consequently $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\}$ for all $x \in X$.

Assume that (C4) is not valid.

Then there exist $a, b \in X$ and $t \in (0, \frac{1-k}{2}]$

such that $\mu(b) \leq t \leq \min\{\mu(a), \mu(a * b), \frac{1-k}{2}\}$.

If $\min\{\mu(a), \mu(a * b)\} < \frac{1-k}{2}$, then $\mu(b) < t \leq \min\{\mu(a), \mu(a * b)\}$.

Hence $(a, t) \in \mu$ and $(a * b) \in \mu$ but $(y, t) \overline{\in} \mu$.

Moreover, $\mu(b) + t < 2t \leq 1 - k$, and so $(b, t)\ \overline{qk}\ \mu$.

Therefore $(b, t)\ \overline{\in \vee qk}\mu$, which is a contradiction,

if $\min \{\mu(a), \mu(a + b) \geq \frac{1-k}{2}\}$ then $(a, \frac{1-k}{2}) \in \mu$ and $(a + b, \frac{1-k}{2}) \in \mu$ but $(b, \frac{1-k}{2})\ \overline{\in}\ \mu$.

Also, $\mu(b) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$. i.e. $(b, \frac{1-k}{2})\ \overline{qk}\mu$.

Hence $(b, \frac{1-k}{2})\ \overline{\in \vee qk}\mu$, which is a contradiction.

Consequently, $\mu(b) \geq \min \{\mu(a), \mu(a * b), \frac{1-k}{2}\}$ for all $x, y \in X$.

Conversely, let μ be a fuzzy set in X satisfying (C3) and (C4).

Let $x \in X$ and $t \in (0, 1]$ be such that $(x, t) \in \mu$.

Then $\mu(x) \geq t$, and so $\mu(1) \geq \min \{\mu(x), \frac{1-k}{2}\} \geq \min \{t, \frac{1-k}{2}\}$.

If $t \leq \frac{1-k}{2}$, then $\mu(1) \geq t$, i.e, $(1, t) \in \mu$.

If $t > \frac{1-k}{2}$, then $\mu(1) \geq \frac{1-k}{2}$.

Thus $\mu(1) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$. i.e., $(1, t)\ qk\mu$.

Hence $(1, t) \in \vee qk\mu$, which proves (C1).

Let $x, y \in X$ and $t, r \in (0, 1]$ be such that $(x, t) \in \mu$ and $(x * y, r) \in \mu$.

Then $\mu(x) \geq t$ and $\mu(x * y) \geq r$.

It follows from (C4) that $\mu(y) \geq \min \{\mu(x), \mu(x, y), \frac{1-k}{2}\}$

$$\geq \min \{t, r, \frac{1-k}{2}\} = \begin{cases} \min\{t, r\} & \text{if } t \leq \frac{1-k}{2} \text{ or } r \leq \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } t > \frac{1-k}{2} \text{ or } r > \frac{1-k}{2} \end{cases}$$

The case $\mu(y) \geq \min \{t, r\}$ implies that $(y, \min \{t, r\}) \in \mu$.

From $\mu(y) \geq \frac{1-k}{2}$ we have

$\mu(y) + \min\{t, r\} > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, i.e. $(y, \min\{t, r\})\ qk\mu$.

Hence $(y, \min\{t, r\}) \in \vee qk\mu$.

Therefore the condition (C2) is valid.

Consequently, μ is an $(\in, \in\vee qk)$-fuzzy filter of L.

Theorem : 2.3.6

A fuzzy set μ in a CI-algebra X is an $(\in, \in\vee qk)$-fuzzy filter of X if and only if it satisfies :

$(*)$ $U(\mu\ ; t) \neq \Phi \Rightarrow U(\mu\ ; t)$ is a filter of L, for all $t \in (0, \frac{1-k}{2}]$.

Proof

Let μ be an $(\in, \in\vee qk)$-fuzzy filter of X.

Let $t \in (0, \frac{1-k}{2}]$ be such that $U(\mu\ ; t) \neq \Phi$.

Obviously, $1 \in U(\mu\ ; t)$ for all $t \in (0, \frac{1-k}{2}]$.

Let $x, y \in X$ be such that $x \in U(\mu\ ; t)$ and $x * y \in U(\mu\ ; t)$.

Then $\mu(x) \geq t$ and $\mu(x * y) \geq t$.

It follows from(C4) that

$\mu(x * y) \geq \min\{\mu(x), \mu(x * y), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\} = t$

so that $y \in U(\mu\ ; t)$. Hence $U(\mu\ ; t)$ is a filter of X.

Conversely, let μ be a fuzzy set in X in which $(*)$ is valid.

If there exists $a \in X$ such that $\mu(1) < \min\{\mu(a), \frac{1-k}{2}\}$, then $\mu(1) < t_a \leq \min\{\mu(a), \frac{1-k}{2}\}$ for some $t_a \in (0, \frac{1-k}{2}]$. Thus $(a, t_a) \in \mu$ but $(1, t_a) \overline{\in} \mu$.

Also, $\mu(1) + t_a < 2t_a \le 1 - k$, i.e,$(1, t_a)\ \overline{qk}\mu$. Hence $(1, t_a)\ \overline{\in \vee qk}\mu$, which is a contradiction. Therefore $\mu(1) \ge \min\{\mu(x), \frac{1-k}{2}\}$ for all $x \in X$. Assume that there exists $a, b \in X$ such that $\mu(b) < \min\{\mu(a), \mu(a * b), \frac{1-k}{2}\}$.

Then $\mu(b) < t \le \min\{\mu(a), \mu(a * b), \frac{1-k}{2}\}$ for some $t \in (0, \frac{1-k}{2}]$, and so $a \in U(\mu ; t)$ and $a * b \in U(\mu ; t)$, but $b \notin U(\mu; t)$.

Since $U(\mu ; t)$ is a filter of X, it is a contradiction.

Therefore $\mu(y) \ge \min\{\mu(x), \mu(x * y), \frac{1-k}{2}\}$ for all $x, y \in X$.

Consequently, μ is an $(\in, \in\vee qk)$-fuzzy filter of X.

Corollary : 2.3.7

A fuzzy set μ in X is an $(\in, \in\vee q)$ fuzzy filter of X if and only if it satisfies: $U(\mu ; t) \ne \Phi \Rightarrow U(\mu ; t)$ is a filter of X, for all $t \in (0, 0.5]$.

Proof : Obvious.

Theorem : 2.3.8

If A is a filter of X, then a fuzzy set μ in X defined by $\mu : X \to [0, 1]$, $x = \begin{cases} t_1 & \text{if } x \in A \\ t_2 & \text{otherwise} \end{cases}$ where $t_1 \in [\frac{1-k}{2}, 1]$ and $t_2 \in (0, \frac{1-k}{2})$, is an $(\in, \in\vee qk)$-fuzzy filter of X.

Proof

Obviously, $U(\mu ; r) = \begin{cases} A & \text{if } r \in (t_2, \frac{1-k}{2}], \\ X & r \in (0, t_2] \end{cases}$ is a filter of X.

By Theorem (2.3.6), μ is an $(\in, \in\vee qk)$-fuzzy filter of X.

Corollary : 2.3.9

If A is a filter of X, then a fuzzy set μ in X defined by $\mu : L \rightarrow [0, 1]$, $x = \begin{cases} t_1 & \text{if } x \in A, \\ t_2 & \text{otherwise} \end{cases}$ where $t_1 \in [0.5, 1]$ and $t_2 \in (0, 0.5)$, is an $(\in, \in\vee qk)$-fuzzy filter of X.

Proof : Obvious.

Theorem : 2.3.10

Every fuzzy filter is an $(\in, \in\vee qk)$-fuzzy filter.

Proof : Obvious.

Note

The converse of Theorem (2.3.10) is not true in general which can be shown in following example.

Example : 2.3.11

Let X = {1, a, b, c} be a set with the following table :

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

Then $(X ; *, 0)$ is a CI-algebra. Define a fuzzy set $\mu : X \rightarrow [0, 1]$ on X, by $\mu(1) = \mu(a) = \mu(b) = 0.6$ and $\mu(c) = 0.5$ and $k = 0.2$ then μ is a $(\in, \in\vee qk)$-fuzzy filter of X but is not a fuzzy filter because $\mu(c)$ is not $\geq \min \{\mu(b), \mu(c * b)\}$.

Theorem : 2.3.12

If μ is an $(\in, \in\vee qk)$-fuzzy filter satisfying $\mu(1) < \frac{1-k}{2}$, then μ is a fuzzy filter.

Proof

Let μ be an $(\in, \in\vee qk)$-fuzzy filter of X such that $\mu(1) < \frac{1-k}{2}$, using (C3) we have $\min\{\mu(x), \frac{1-k}{2}\} \leq \mu(1) < \frac{1-k}{2}$ and so $\mu(x) \leq \frac{1-k}{2}$ for all $x \in X$.

It follows from (C3) and (C4) that $\mu(1) \geq \mu(x)$ and $\mu(y) \geq \min\{\mu(x), \mu(y * x)\}$ for all $x, y \in X$.

Hence μ is a fuzzy filter of X.

Corollary : 2.3.13

If μ is an $(\in, \in\vee q)$-fuzzy filter satisfying $\mu(1) < 0.5$, then μ is a fuzzy filter.

Proposition : 2.3.14

For any $k_1, k_2 \in (0, 1]$ with $k_1 < k_2$, every $(\in, \in\vee qk_1)$-fuzzy filter is an $(\in, \in\vee qk_2)$-fuzzy filter.

Example : 2.3.15

In Example (2.3.11), define a fuzzy set $\mu : X \to [0, 1]$ on X, by $\mu(1) = \mu(a) = \mu(b) = 0.6$ and $\mu(c) = 0.5$ and $k_1 = 0.1$. Then μ is a $(\in, \in\vee qk_1)$ fuzzy filter of X. Also by $\mu(1) = \mu(a) = \mu(b) = 0.6$ and $\mu(c) = 0.4$ and $k_2 = 0.4$, then μ is a $(\in, \in\vee qk_2)$-fuzzy filter of X, but $(\in, \in\vee qk_2)$-fuzzy filter is not an $(\in, \in\vee qk_1)$-fuzzy filter.