

# Chapter 1

## Preliminaries

### 1.1 Closed and Open sets

**Definition 1.1.1.** For a subset  $A$  of a topological space  $(X, \tau)$ ,

1. the intersection of all closed sets containing  $A$  is called the *closure of  $A$*  and denoted by  $cl(A)$ .
2. the union of all open sets contained in  $A$  is called the *interior of  $A$*  and denoted by  $int(A)$ .

**Definition 1.1.2.** A subset  $A$  of a topological space  $(X, \tau)$  is called

- (i) *regular closed* (Stone, 1937) if  $A = cl(int(A))$ .
- (ii) *semi-closed* (Levine, 1963) if  $int(cl(A)) \subseteq A$ .
- (iii)  *$\alpha$ -closed* (Njastad, 1965) if  $cl(int(cl(A))) \subseteq A$ .
- (iv) *pre-closed* (Mashhour, 1982) if  $cl(int(A)) \subseteq A$ .

The complements of the above mentioned sets are called regular open, semi-open,  $\alpha$ -open and pre-open respectively.

The intersection of all regular closed (resp. semi-closed,  $\alpha$ -closed and pre-closed) subsets of  $(X, \tau)$  containing  $A$  is called the *regular closure* (resp. *semi-closure*,  *$\alpha$ -closure*

and *pre-closure*) of  $A$  and is denoted by  $rcl(A)$  (resp.  $scl(A)$ ,  $\alpha cl(A)$  and  $pcl(A)$ ).

A subset  $A$  of  $(X, \tau)$  is called *clopen* if it is both open and closed in  $(X, \tau)$ .

**Definition 1.1.3.** A subset  $A$  of a topological space  $(X, \tau)$  is called  **$\delta$ -open** (Velicko, 1968) if  $A$  is the union of regular open sets. The complement of  $\delta$ -open is called  **$\delta$ -closed**.

**Definition 1.1.4.** For a subset  $A$  of a topological space  $(X, \tau)$ ,

1.  **$cl_\delta(A)$**  (Velicko, 1968) is the intersection of all  $\delta$ -closed sets containing  $A$  and  $int_\delta(A)$  is the union of all  $\delta$ -open sets contained in  $A$ .

2.  **$pInt(A)$** (Andrijevic, 1986) is the union of all pre-open sets contained in  $A$ . Equivalently,

$$pint(A) = A \cup int(cl(A)) = A$$

3.  **$sCl(A)$** (Andrijevic, 1986) is the intersection of all semi-closed sets containing  $A$ . Equivalently,

$$scl(A) = A \cap int(cl(A)) = A$$

**Definition 1.1.5.** A subset  $A$  of  $(X, \tau)$  is called

1.  **$\Lambda$ -set** (Maki, 1986) if it is equal to its kernel.

2.  **$\lambda$ -closed** (Arenas, 1997) if  $A = T \cap C$ , where  $T$  is a  $\Lambda$ -set and  $C$  is a closed set.

3.  **$\Lambda_\delta$ -set** (Georgiou, 2004) if  $\Lambda_\delta(A) = A$ , where  $\Lambda_\delta(A) = \cap\{O \in \delta O(X, \tau) \mid A \subseteq O\}$ .

4.  **$(\Lambda, \delta)$ -closed** (Georgiou, 2004) if  $A = T \cap C$ , where  $T$  is a  $\Lambda_\delta$ -set and  $C$  is a  $\delta$ -closed set.

The complements of the above mentioned sets are called their respective open sets.

**Definition 1.1.6.** The intersection of all  $\lambda$ -closed sets containing  $A$  is called  **$\lambda$ -closure of  $A$**  (Caldas, 2007) and is denoted by  **$cl_\lambda(A)$**

**Definition 1.1.7.** A subset  $A$  of  $(X, \tau)$  is called

1.  **$g$ -closed** (Levine, 1970) if  $cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $(X, \tau)$ .
2.  **$\delta g$ -closed** (Dontchev, 1996) if  $cl_\delta(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $(X, \tau)$ .
3.  **$g\delta$ -closed** (Dontchev, 2004) if  $cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $\delta$ -open in  $(X, \tau)$ .
4.  **$\delta g_s$ -closed** (Park, 2007) if  $\delta-scl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $\delta$ -open in  $(X, \tau)$ .
5.  **$g\delta s$ -closed** (Benchalli, 2012) if  $scl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $\delta$ -open in  $(X, \tau)$ .
6.  **$\alpha g$ -closed** (Maki, 1994) if  $\alpha cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $(X, \tau)$ .
7.  **$\Lambda_g$ -closed** (Caldas, 2008) if  $cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $\lambda$ -open in  $(X, \tau)$ .
8.  **$\Lambda-g$ -closed** (Caldas, 2008) if  $cl_\lambda(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $\lambda$ -open in  $(X, \tau)$ .

The complements of the above mentioned sets are called their respective open sets.

## 1.2 Separation Axioms

**Definition 1.2.1.** A topological space  $(X, \tau)$  is called

1. ***almost weakly Hausdorff*** (Dontchev, 2004) if its semi-regularization is  $T_{1/2}$ . In other words,  $g$ -closed sets of  $(X, \tau_s)$  are  $\delta$ -closed in  $(X, \tau)$ .
2. ***weakly Hausdorff*** (Dontchev, 2004) iff each singleton is  $\delta$ -closed.
3.  **$T_{3/4}$ -space** (Dontchev, 1996) if every  $\delta g$ -closed set is  $\delta$ -closed in  $(X, \tau)$ .
4.  **$\delta T_{3/4}$ -space** (Benchalli, 2012) if every  $g\delta s$ -closed set is  $\delta$ -closed in  $(X, \tau)$ .
5.  **$T_\delta$ -space** (Dontchev, 2004) if every  $g\delta$ -closed set is  $\delta$ -closed in  $(X, \tau)$ .
6.  **$\delta-R_0$**  (Georgiou, 2004) if for each  $\delta$ -open set  $U$  and  $x \in U$ ,  $cl_\delta\{x\} \subseteq U$ .

7.  $\delta\text{-T}_0$  (Georgiou, 2004) if for any distinct pair of points in  $X$ , there is a  $\delta$ -open set  $U$  containing one of the points but not the other.
8.  $\delta\text{-T}_1$  (Georgiou, 2004) if for any distinct pair of points  $x$  and  $y$  in  $X$ , there is a  $\delta$ -open  $U$  in  $X$  containing  $x$  but not  $y$  and a  $\delta$ -open set  $V$  in  $X$  containing  $y$  but not  $x$ .
9.  $\delta\text{-T}_2$  (Georgiou, 2004) if for any distinct pair of points  $x$  and  $y$  in  $X$ , there are  $\delta$ -open sets  $U_1$  and  $U_2$  such that  $x \in U_1$ ,  $y \in U_2$  and  $U_1 \cap U_2 = \phi$ .
10.  $\delta\text{-door}$  (Sudha, 2014) if every subset of  $X$  is either  $\delta$ -open or  $\delta$ -closed in  $(X, \tau)$ .
11. *partition space* (Nieminen, 1977) if every open set is closed.
12. *semi-regular* (Dontchev, 2000) if its regular open sets form a base. In a semi-regular space, open sets
13.  $\text{T}_{1/2}\text{-space}$  (Levine, 1970) if every  $g$ -closed set is closed.
14. *pointwise semi-regular* (Dontchev, 2000) if every closed singleton of  $X$  is  $\delta$ -closed.

**Definition 1.2.2.** The *Khalimsky line* (Khalimsky, 1990) or the so-called digital line is the set of integers  $Z$ , equipped with the topology  $K$  having  $\{\{2n - 1, 2n, 2n + 1\} : n \in Z\}$  as a subspace.

### 1.3 Functions

**Result 1.3.1.** Let  $A$  be a subset of  $X$ ,  $B$  be a subset of  $Y$  and  $f : X \rightarrow Y$  be a function, then

1.  $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$
2.  $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$
3.  $A \subseteq f^{-1}(f(A))$ , [If  $f$  is injective then  $A = f^{-1}(f(A))$ ]

4.  $f(f^{-1}(B)) \subseteq B$ , [If  $f$  is surjective then  $B = f^{-1}(f(B))$ ]
5. If  $f$  is surjective then  $(f(A))^c \subseteq f(A^c)$
6. If  $f$  is bijective then  $(f(A))^c = f(A^c)$

**Definition 1.3.2.** A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is called

1. **continuous** (Levine, 1970) if  $f^{-1}(V)$  is a closed set in  $(X, \tau)$  for every closed set  $V$  in  $(Y, \sigma)$ .
2. **semi-continuous** (Levine, 1963) if  $f^{-1}(V)$  is a semi-closed set in  $(X, \tau)$  for every closed set  $V$  in  $(Y, \sigma)$ .
3. **super continuous** (Munshi, 1982) if  $f^{-1}(V)$  is a  $\delta$ -closed set in  $(X, \tau)$  for every closed set  $V$  in  $(Y, \sigma)$ .
4. **pre-continuous** (Mashhour, 1982) if  $f^{-1}(V)$  is a pre-closed set in  $(X, \tau)$  for every closed set  $V$  in  $(Y, \sigma)$ .
5.  **$g\delta$ -continuous** (Dontchev, 2000) if  $f^{-1}(V)$  is a  $g\delta$ -closed set in  $(X, \tau)$  for every closed set  $V$  in  $(Y, \sigma)$ .
6.  **$\delta gs$ -continuous** (Park, 2007) if  $f^{-1}(V)$  is a  $\delta gs$ -closed set in  $(X, \tau)$  for every closed set  $V$  in  $(Y, \sigma)$ .
7.  **$g\delta s$ -continuous** (Benchalli, 2011) if  $f^{-1}(V)$  is a  $g\delta s$ -closed set in  $(X, \tau)$  for every closed set  $V$  in  $(Y, \sigma)$ .
8.  **$\alpha$ -continuous** (Mashhour, 1983) if  $f^{-1}(V)$  is a  $\alpha$ -closed set in  $(X, \tau)$  for every closed set  $V$  in  $(Y, \sigma)$ .

**Definition 1.3.3.** A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is called

1. **strongly continuous** (Levine, 1960) if  $f^{-1}(V)$  is clopen in  $(X, \tau)$  for every subset  $V$  in  $(Y, \sigma)$ .
2. **perfectly continuous** (Noiri, 1978) if  $f^{-1}(V)$  is clopen in  $(X, \tau)$  for every closed subset  $V$  in  $(Y, \sigma)$ .

3. **totally continuous** (Jain, 1980) if  $f^{-1}(V)$  is clopen in  $(X, \tau)$  for every open subset  $V$  in  $(Y, \sigma)$ .
4. **contra continuous** (Levine, 1960) if  $f^{-1}(V)$  is closed in  $(X, \tau)$  for every open subset  $V$  in  $(Y, \sigma)$ .

**Definition 1.3.4.** A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is called

1. **irresolute** (Crossley, 1971) if  $f^{-1}(V)$  is semi-closed in  $(X, \tau)$  for every semi-closed set  $V$  in  $(Y, \sigma)$ .
2.  **$\delta$ -continuous** (Noiri, 1980) if  $f^{-1}(V)$  is  $\delta$ -closed in  $(X, \tau)$  for every  $\delta$ -closed set  $V$  in  $(Y, \sigma)$ .
3.  **$g\delta$ -irresolute** (Dontchev, 2004) if  $f^{-1}(V)$  is  $g\delta$ -closed in  $(X, \tau)$  for every  $g\delta$ -closed set  $V$  in  $(Y, \sigma)$ .
4.  **$\delta g s$ -irresolute** (Park, 2007) if  $f^{-1}(V)$  is  $\delta g s$ -closed in  $(X, \tau)$  for every closed set  $V$  in  $(Y, \sigma)$ .

**Definition 1.3.5.** A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is called

1.  **$\delta$ -closed** (Noiri, 1980) if  $f(V)$  is  $\delta$ -closed in  $Y$  for every closed set  $V$  in  $X$ .

**Definition 1.3.6.** (Raja, 2014) Let  $A$  be a subset of a topological space  $(X, \tau)$ . A point  $x \in A$  is said to be a  **$\delta$ -limit point** of  $A$  if for each  $\delta$ -open set  $U$  containing  $x$ ,  $U \cap (A \setminus \{x\}) \neq \phi$ . The set of all  $\delta$ -limit points of  $A$  is called the  **$\delta$ -derived set** of  $A$  and is denoted by  $D_\delta(A)$ .

**Definition 1.3.7.** (Raja, 2014) Let  $A$  be a subset of a topological space  $(X, \tau)$ .  **$\delta$ -frontier** of  $A$  is defined by  $Fr_\delta(A) = cl_\delta(A) \setminus int_\delta(A)$ .

**Definition 1.3.8.** (Raja, 2014) Let  $A$  be a subset of a topological space  $(X, \tau)$ .  **$\delta$ -exterior** of  $A$  is defined by  $Ext_\delta(A) = X \setminus cl_\delta(A)$ .

**Definition 1.3.9.** (Choquet, 1947) A **grill**  $\mathcal{G}$  on a topological space  $X$  is a non-empty collection of non-empty subsets of  $X$  satisfying two conditions:

- (i)  $A \in \mathcal{G}$  and  $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$ .
- (ii)  $A, B \subseteq X$  and  $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$  or  $B \in \mathcal{G}$ .

**Definition 1.3.10.** (Roy, 2009) Let  $X \neq \phi$ , the *principal grill* generated by a non-empty subset  $A$  of  $X$  is defined as  $[A] = \{B \subseteq X \mid A \cap B \neq \phi\}$ .

If  $(X, \tau)$  is a topological space with a grill  $\mathcal{G}$  on  $X$ . Then, we call it a grill topological space and denote it by  $(X, \tau, \mathcal{G})$ . (Roy, 2007)

Throughout this thesis,  $(X, \tau)$ ,  $(Y, \sigma)$  and  $(Z, \eta)$  (or simply,  $X$ ,  $Y$  and  $Z$ ) denote topological spaces on which no separation axioms are imposed, unless it is stated explicitly.

## 1.4 Fuzzy Topological Spaces

**Definition 1.4.1.** Let  $X$  be a non-empty set. A *fuzzy set*  $A$  in  $X$  is a function from  $X$  into the closed unit interval  $[0, 1]$ . We write  $A \leq B$ , if  $A(x) \leq B(x)$  for all  $x \in X$ . By  $A = B$ , we mean  $A \leq B$  and  $B \leq A$ . If  $\{A_i \mid i \in I\}$  is a collection of fuzzy sets in  $X$ , then  $\vee A_i$  and  $\wedge A_i$  are given by  $(\vee A_i)(x) = \sup\{A_i(x) \mid i \in I\}$  for all  $x \in X$  and  $(\wedge A_i)(x) = \inf\{A_i(x) \mid i \in I\}$  for all  $x \in X$ .

- The complement  $A^c$  of a fuzzy set  $A$  in  $X$  is given by  $A^c(x) = 1 - A(x)$ , for all  $x \in X$ .
- The fuzzy sets  $0$  and  $1$  are given by  $0(x) = 0$  and  $1(x) = 1$ , for all  $x \in X$ .

**Definition 1.4.2.** A *fuzzy topology* is a family  $\mathcal{F}$  of fuzzy sets in  $X$  which satisfies the following conditions:

- (i)  $0, 1 \in \mathcal{F}$
- (ii) If  $A, B \in \mathcal{F}$  then  $A \wedge B \in \mathcal{F}$ .
- (iii) If  $A_i \in \mathcal{F}$  for each  $i \in I$ , then  $\vee A_i \in \mathcal{F}$ , where  $I = [0, 1]$ .

- Members of  $\mathcal{F}$  are called *fuzzy open sets* and the pair  $(X, \mathcal{F})$  is called a *fuzzy topological space* (Chang, 1968).

- Complements of fuzzy open sets are called **fuzzy closed sets**.
- The **closure** and **interior** of a fuzzy set  $A$  in a fuzzy topological space are denoted by  $fcl(A)$  and  $fint(A)$  respectively and they are defined as follows:

$$fcl(A) = \wedge \{B \mid B \text{ is a fuzzy closed set such that } A \leq B\} \text{ and}$$

$$fint(A) = \vee \{B \mid B \text{ is a fuzzy open set such that } B \leq A\}.$$

**Definition 1.4.3.** A fuzzy point  $x_p \in A$  is said to be **quasi-coincident** (Pu, 1980) with a fuzzy set  $A$  denoted by  $x_p q A$  iff  $p + A(x) > 1$ . A fuzzy set  $A$  is quasi-coincident with a fuzzy set  $B$  denoted by  $A q B$  iff there exists  $x \in X$  such that  $A(x) + B(x) > 1$ . If  $A$  and  $B$  are not quasi-coincident then we write  $A \bar{q} B$ . Note that  $A \leq B \Leftrightarrow A \bar{q} (1 - B)$ .

**Definition 1.4.4.** A fuzzy subset  $A$  of a fuzzy topological space  $(X, \mathcal{F})$  is called

- Regular open** (Azad, 1981) if  $int(cl(A)) = A$ .
- $\delta$ -open** (Petricevic, 1991) if  $A = \bigvee_{i \in I} A_i$ , where  $A_i$  is a fuzzy regular open set for each  $i$  in  $(X, \mathcal{F})$ .

A fuzzy subset  $A$  of a fuzzy topological space  $(X, \mathcal{F})$  is called **fuzzy regular closed** (Azad, 1981) (resp. **fuzzy  $\delta$ -closed**) if  $1 - A$  is fuzzy regular open (resp. fuzzy  $\delta$ -open). Let the family of all fuzzy regular open, fuzzy regular closed, fuzzy  $\delta$ -open and fuzzy  $\delta$ -closed sets be represented by  $FRO(X, \mathcal{F})$ ,  $FRC(X, \mathcal{F})$ ,  $F\delta O(X, \mathcal{F})$  and  $F\delta C(X, \mathcal{F})$  respectively.

**Definition 1.4.5.** The  **$\delta$ -closure of a fuzzy set  $A$**  (Petricevic, 1991) is the intersection of all fuzzy regular closed sets containing  $A$  (shortly,  $cl_\delta(A)$ ). A fuzzy point  $y_t \in cl_\delta(A)$  iff every fuzzy regular open set which is  $q$ -coincident with  $x_t$  is also  $q$ -coincident with  $A$ .

**Definition 1.4.6.** A fuzzy function  $f : X \rightarrow Y$  is called

- fuzzy super-continuous** (Petricevic, 1991) if  $f^{-1}(V)$  is a fuzzy  $\delta$ -open set on  $X$  for any fuzzy open set  $V$  on  $Y$ .

2. ***somewhat fuzzy  $\delta$  continuous*** (Thangaraj, 2015) if  $B \in \mathcal{G}$  and  $f^{-1}(B) \neq 0$ , there exists a  $\delta$ -open set in  $(X, \mathcal{F})$  such that  $A \neq 0$  and  $A \leq f^{-1}(B)$ . That is,  $int_{\delta}(f^{-1}(B)) \neq 0$ .