Chapter 4

CHAPTER – IV

ANALYSIS OF (m, N) POLICY OF BATCH ARRIVAL QUEUE WITH MULTI - OPTIONAL SERVICE AND UNRELIABLE SERVER UNDER SINGLE VACATION POLICY

INTRODUCTION

All the existing research works on Bi-level control of batch arrival queues with an early setup have a common assumption that the system contains single server who provides only one kind of service to the customers. However in Chapter III, we have analysed the (m, N) policy for a batch arrival $M^X/G/1$ queueing system with server vacation, in which the server provides First Essential Service (FES) to all the existing customers where as only some of them are provided with Second Optional Service (SOS) when they demand for it.

Server breakdown is a remarkable and unavoidable phenomenon in the service facility of a queueing system. Server breakdowns with multi-optional service are more flexible and applicable in practice than single service counter parts. However, due to their analytic complexity there have been only a few studies carried out on multi stages service queueing systems with server breakdowns. Existing research works, have never investigated the (m, N) policy of batch arrival queues with multi-optional service and vacation taking into account the service failures and repairs.

In this chapter, a repairable, $M^X/G/1$ queue with multi-optional service is considered to analyse the (m, N) policy. It is assumed that the server's lifetime has exponential distribution with mean $1/a_i$ $(1 \le i \le c)$ in the i^{th} type of service. If the server breaks down during the service it is sent for repair immediately and the customer who was being served during server failure remains in the service facility, to complete the remaining service. It is also assumed that the server takes a single vacation whenever the system becomes empty.

4.1 Mathematical Analysis of the Model

I Model Description

Arrival Pattern

Customers arrive in batches, so that the arrival streams form a Poisson process with group arrival rate λ. The actual number of customers in any arriving module is a random variable X which may take any positive integral value k ($< \infty$) with probability $Pr(X = k) = g_k$, $k = 1, 2, 3, \ldots$.

(m, N) Policy and Server's Vacation

As soon as the system becomes empty the server leaves for a vacation of random length V. When the server returns from the vacation and finds that the system size is greater than or equal to a predetermined value **m** then the server starts a setup operation of random length D. This setup time corresponds to the preparatory work of the server before starting the busy period. If after returning from vacation, the server finds less than **m** customers in the system then the server remains idle in the system until the system length reaches at least **m** and then starts the setup operation. The period during which the server remains idle until '**m**' or more customers accumulate in the system is called **buildup period.**

At the end of the setup period if the number of customers in the queue is less than N, then the server remains idle (dormant) in the system waiting for the queue length to reach or exceed N, to start a service. This idle period is called dormant period (or) stand by period. On the other hand, if at the end of the setup period, queue length is greater than or equal to N, then the server begins to serve the customers exhaustively.

Service Process

During busy period, the server provides c-kinds of general heterogeneous service to the customers .The customers are served, one at a time. Just before a service starts each customer has the option to choose the

i^{th} $(1 \le i \le c)$ type of service with probability r_i where $\sum_{i=1}^{c} r_i = 1$. It is assumed that the i^{th} type of service time S_i follows an arbitrary distribution and its distribution function, density function and the first two moments are $S_i(t)$, $s_i(t)$ and $E(S_i^k)(1 \le i \le c)$ respectively. The customers undergo only one type of service according to their choice and leave the system as soon as the service is complete.

The server may breakdown at any time while serving customers. It is assumed that the server's life time has exponential distribution with mean a_i $(1 \le i \le c)$ in the i^{th} type of service. When the server breaks down, it is sent for repair immediately. If the server breaks down during the i^{th} type of service, the customer just being served before server breakdown waits for the server in the service facility to complete the remaining service. The repair time B_i is arbitrarily distributed and has probability distribution function $B_i(t)$ density function $b_i(t)$ and the first and second moments as $E(B_i^k)$ where $(1 \le I \le c)$ and $(k = 1,2)$. Immediately after the server is fixed, it starts to serve the customer who is waiting for his remaining service. It is assumed that the service time for a customer is cumulative and after repair the server is considered as good as new.

Thus in this model, vacation period, build-up period, a setup period and a standby period together give an idle period. The busy period and the breakdown period constitute a completion period. The idle period and a completion period will determine a cycle.

The customers continue to arrive and join the system independent of the system states, following the compound Poisson process. Various stochastic processes involved in the queueing system are assumed to be independent of each other.

This model is denoted by $\mathbf{M^X_{(m,N)}} / \mathbf{G_{i\ (1 \le i \le c)}}$ **/ 1 / SV / BD**

The following notations are used to write the partial differential equations using supplementary variable technique. The remaining service time, setup time, vacation time and repair time are introduced as the supplementary variables.

Let $N_S(t)$ denote the system size at time 't' and (m, N) denote the thresholds. The CDF, pdf and LST (the cumulative distribution function, probability density function and the Laplace Stieltjes transform) of various random variables are listed below.

	RV	**CDF**	**pdf**	**Remaining time of random variable t**	**LST**	**k^{th} moment**
Service time ($1 \le i \le c$)	S_i	$S_i(x)$	$s_i(x)$	$S_i^0(t)$	$S_i^*(\theta)$	$E(S_i^k)$
Vacation time	V	V(x)	v(x)	$V^0(t)$	$V^*(\theta)$	$E(V^k)$
Setup time	D	D(x)	d(x)	$D^0(t)$	$D^*(\theta)$	$E(D^k)$
Repair time ($1 \le i \le c$)	B_i	$B_i(y)$	$b_i(y)$	$B_i^0(t)$	$b_i^{*1}(\theta_1)$	$E(B_i^k)$

II Steady State Equations

At time t, let Y(t) = 0, 1, 2, 3 and 4 respectively denote that the server is on vacation, buildup state, setup, in dormant, busy and in breakdown state respectively. Then the state of the system at time t can be described by the Markov process as $K(t) = \{ N_S(t), \delta(t) \}$ $t \ge 0$ where $\delta(t) = Q^0(t), 0, D^0(t), 0$ $S_i^0(t)$, $B_i^0(t)$ according as Y(t) = 0, 1, 2, 3 and 4 respectively. The transient system states are defined by

$Q_n(x, t)\,dt = \Pr\{N_S(t) = n, x \le V^0(t) \le x + dt, Y(t) = 0\}$

$R_n(t) = \Pr\{N_S(t) = n, Y(t) = 1\}$, $\quad 0 \le n \le m-1$

$D_n(x, t)\,dt = \Pr\{N_S(t) = n, x \le D^0(t) \le x + dt, Y(t) = 2\}$, $\quad n \ge m$

$U_n(t) = \Pr\{N_S(t) = n, Y(t) = 3\}$, $\quad m \le n \le N-1$

$P_{n,i}(x, t)\,dt = \Pr\{N_S(t) = n, x \le S_i^0(t) \le x+dt, Y(t) = 4\}$, $\quad n \ge 1$

$B_{n,i}(x, y, t)\,dt = \Pr\{N_S(t) = n,\ S_i^0(t) = x,\ y \le B_i^0(t) \le y + dt,\ Y(t) = 5\}, \quad n \ge 1$

$i = 1, 2, 3, \ldots.$

Thus, $P_{n,i}(x, t)\,dt$ $(1 \le i \le c)$ denotes the joint probability that at time t, there are n customers in the system, the server is doing i^{th} type of service and the remaining service time of the customer lies in the interval $[x, x + \Delta t]$. $P_{n,i}(0)$ $(1 \le i \le c)$ denotes the corresponding probability at the termination of i^{th} service time.

$B_{n,i}(x, y, t)dt$ denotes the joint probability that at time t, there are n customers in the system, one being provided with i^{th} type of service facility with the remaining service time is equal to x, and the server being repaired has remaining repair time between y and y+dt . The corresponding probability at the termination of the i^{th} repair time is $B_{n,i}(x, 0, t)$ $(1 \le i \le c)$ and the other notations corresponding to buildup, dormant, setup and vacation states are as mentioned in previous chapters. Let the steady state system size probabilities be independent of time t, and given by,

$$\lim_{t \to \infty} R_n(t) = R_n \qquad \lim_{t \to \infty} U_n(t) = U_n$$

$$\lim_{t \to \infty} Q_n(t) = Q_n(x) \qquad \lim_{t \to \infty} D_n(x, t) = D_n(x)$$

$$\lim_{t \to \infty} P_{n,i}(x, t) = P_{n,i}(x) \qquad \lim_{t \to \infty} B_{n,i}(x, y, t) = B_{n,i}(x, y)$$

Following the arguments of Cox (1955) and observing the changes of states during the interval $(t, t + \Delta t)$ for any time t, the steady state equations are :

$$-\frac{d}{dx}(Q_0(x)) = -\lambda\, Q_0(x) + P_1(0)\, v(x) \text{ where } P_1(0) = \sum_{j=1}^{c} P_{1,j}(0) \tag{4.1}$$

$$-\frac{d}{dx}(Q_n(x)) = -\lambda\, Q_n(x) + \lambda \sum_{k=1}^{n} Q_{n-k}(x)\, g_k \qquad n \ge 1 \tag{4.2}$$

$$\lambda R_0 = Q_0(0) \tag{4.3}$$

$$\lambda R_n = Q_n(0) + \lambda \sum_{k=1}^{n} R_{n-k}\, g_k \qquad 1 \le n \le m-1 \qquad (4.4)$$

$$-\frac{d}{dx}(D_m(x)) = -\lambda D_m(x) + Q_m(0)\, d(x) + \lambda \sum_{k=1}^{m} R_{m-k}\, g_k\, d(x) \qquad (4.5)$$

$$-\frac{d}{dx}(D_n(x)) = -\lambda D_n(x) + Q_n(0)\, d(x) + \lambda \sum_{k=n-m+1}^{n} R_{n-k}\, g_k\, d(x) + \lambda \sum_{k=1}^{n-m} D_{n-k}(x)\, g_k, \qquad n \ge m+1 \qquad (4.6)$$

$$\lambda U_m = D_m(0) \qquad (4.7)$$

$$\lambda U_n = D_n(0) + \lambda \sum_{k=1}^{n-m} u_{n-k}\, g_k, \qquad m+1 \le n \le N-1 \qquad (4.8)$$

$$-\frac{d}{dx}(P_{1,i}(x)) = -(\lambda + a_i)\, P_{1,i}(x) + r_i\, S_i(x) \sum_{j=1}^{c} P_{2,j}(0) + B_{1,i}(x, 0) \qquad 1 \le i \le c \qquad (4.9)$$

$$-\frac{d}{dx}(P_{n,i}(x)) = -(\lambda + a_i)\, P_{n,i}(x) + r_i\, s_i(x) \sum_{j=1}^{c} P_{n+1,j}(0) + \lambda \sum_{k=1}^{n-1} P_{n-k,i}(x)\, g_k + B_{n,i}(x, 0) \qquad 2 \le n \le N-1\,;\ 1 \le i \le c \qquad (4.10)$$

$$-\frac{d}{dx}(P_{n,i}(x)) = -(\lambda + a_i)\, P_{n,i}(x) + r_i\, s_i(x) \sum_{j=1}^{c} P_{n+1,j}(0) + \lambda \sum_{k=1}^{n-1} P_{n-k,i}(x)\, g_k + B_{n,i}(x, 0) + \lambda\, r_i\, s_i(x) \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_{\mathbf{k}} + r_i\, s_i(x)\, D_n(0) \qquad n \ge N,\ 1 \le i \le c \qquad (4.11)$$

$$-\frac{d}{dy}(B_{1,i}(x, y)) = -\lambda B_{1,i}(x, y) + a_i\, P_{1,i}(x)\, b_i(y), \qquad i = 1 \text{ to } c \qquad (4.12)$$

$$-\frac{d}{dy}(B_{n,i}(x, y)) = -\lambda B_{n,i}(x, y) + a_i\, P_{n,i}(x)\, b_i(y) + \lambda \sum_{k=1}^{n-1} B_{n-k,i}(x, y)\, g_k \qquad i = 1 \text{ to } c,\ n \ge 2 \qquad (4.13)$$

To obtain the PGF of the system size probabilities LST is applied first with respect to x.

Let $B^{*}_{n,i}(\theta, 0) = \int_{0}^{\infty} e^{-\theta x} B_{n,i}(x, 0)\, dx$

and $B^{*}_{n,i}(\theta, y) = \int_{0}^{\infty} e^{-\theta x} B_{n,i}(x, y)\, dx$

and the other LSTs are as mentioned in previous chapters.

Thus we have

$$\theta\, Q_0^*(\theta) - Q_0(0) = \lambda\, Q_0^*(\theta) - V^*(\theta) \sum_{j=1}^{c} P_{1,j}(0) \qquad (4.14)$$

$$\theta\, Q_n^*(\theta) - Q_n(0) = \lambda\, Q_n^*(\theta) - \lambda \sum_{k=1}^{n} Q_{n-k}^*(\theta) g_k, \qquad n \geq 1 \qquad (4.15)$$

$$\theta\, D_m^*(\theta) - D_m(0) = \lambda\, D_m^*(\theta) - Q_m(0)\, D^*(\theta) - \lambda\, D^*(\theta) \sum_{k=1}^{m} R_{m-k}\, g_k \qquad (4.16)$$

$$\theta\, D_n^*(\theta) - D_n(0) = \lambda\, D_n^*(\theta) - Q_n(0)\, D^*(\theta) - \lambda\, D^*(\theta) \sum_{k=n-m+1}^{n} R_{n-k}\, g_k$$

$$- \lambda \sum_{k=1}^{n-m} D_{n-k}^*(\theta)\, g_k, \qquad n \geq m+1 \qquad (4.17)$$

$$\theta\, P_{1,i}^*(\theta) - P_{1,i}(0) = (\lambda + a_i)\, P_{1,i}^*(\theta) - r_i\, S_i^*(\theta) \sum_{j=1}^{c} P_{2,j}(0) - B_{1,i}^*(\theta, 0) \qquad (4.18)$$

$$\theta\, P_{n,i}^*(\theta) - P_{n,i}(0) = (\lambda + a_i)\, P_{n,i}^*(\theta) - r_i\, S_i^*(\theta) \sum_{j=1}^{c} P_{n+1,j}(0)$$

$$- \lambda \sum_{k=1}^{n-1} P_{n-k,i}^*(\theta)\, g_k - B_{n,i}^*(\theta, 0) \qquad 2 \leq n \leq N-1 \qquad (4.19)$$

$$\theta\, P_{n,i}^*(\theta) - P_{n,i}(0) = (\lambda + a_i)\, P_{n,i}^*(\theta) - r_i\, S_i^*(\theta) \sum_{j=1}^{c} P_{n+1,j}(0) - \lambda \sum_{k=1}^{n-1} P_{n-k,i}^*(\theta)\, g_k$$

$$- \lambda\, r_i\, S_i^*(\theta) \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k - r_i\, S_i^*(\theta)\, D_n(0) - B_{n,i}^*(\theta, 0)$$

$$n \geq N \qquad (4.20)$$

$$-\frac{d}{dy} B_{1,i}^*(\theta, y) = -\lambda\, B_{1,i}^*(\theta, y) + a_i\, P_{1,i}^*(\theta)\, b_i(y) \qquad (4.21)$$

$$-\frac{d}{dy} B^{*}_{n,i}(\theta, y) = -\lambda B^{*}_{n,i}(\theta, y) + a_i P^{*}_{n,i}(\theta) b_i(y) + \lambda \sum_{k=1}^{n-1} B^{*}_{n-k,i}(\theta, y) g_k \qquad n \geq 2 \quad (4.22)$$

Moreover when the server is in repair state, the LSTs with respect to the remaining repair time (y) are needed.

Let
$$B^{**1}_{n,i}(\theta, \theta_1) = \int_0^\infty e^{-\theta_1 y} B^{*}_{n,i}(\theta, y)\, dy \qquad n \geq 1$$

$$B^{*1}_{i}(\theta_1) = \int_0^\infty e^{-\theta_1 y} b_i(y)\, dy$$

and hence taking LST of the equations (4.21) and (4.22) with respect to y , we have,

$$\theta_1 B^{**1}_{1,i}(\theta, \theta_1) - B^{*}_{1,i}(\theta, 0) = \lambda B^{**1}_{1,i}(\theta, \theta_1) - a_i P^{*}_{1,i}(\theta) B^{*1}_{i}(\theta_1) \qquad (4.23)$$

$$\theta_1 B^{**1}_{n,i}(\theta, \theta_1) - B^{*}_{n,i}(\theta, 0) = \lambda B^{**1}_{n,i}(\theta, \theta_1) - a_i P^{*}_{n,i}(\theta) B^{*1}_{i}(\theta_1) - \lambda \sum_{k=1}^{n-1} B^{**1}_{n-k,i}(\theta, \theta_1) g_k \qquad n \geq 2 \qquad (4.24)$$

III The Probability Generating Function

Equations (4.1) to (4.8) are similar to the corresponding equations of section (3.1) of chapter III. Hence the partial generating functions R(z), U(z), D(z, 0), $D^{*}(z,0)$, Q(z, 0) and $Q^{*}(z,0)$ are obtained similarly as

$$R(z) = P_1(0) \sum_{n=0}^{m-1} \frac{\psi_n z^n}{\lambda} = P_1(0)\, \psi(z) \qquad (4.25)$$

$$U(z) = P_1(0) \sum_{n=m}^{N-1} \frac{\phi_n^S z^n}{\lambda} = P_1(0)\, \phi_S(z) \qquad (4.26)$$

$$D(z, 0) = P_1(0)\; D^{*}(w_X(z)) [V^{*}(w_X(z)) - w_X(z)\, \psi(z)] \qquad (4.27)$$

$$D^{*}(z, \theta) = \frac{(D^{*}(w_X(z)) - D^{*}(\theta))}{(\theta - w_X(z))} P_1(0)\; [V^{*}(w_X(z)) - w_X(z)\, \psi(z)] \qquad (4.28)$$

$$Q(z, 0) = P_1(0)\ V^*(w_X(z)) \tag{4.29}$$

and
$$Q^*(z,\theta) = P_1(0)\ \frac{(V^*(w_X(z)) - V^*(\theta))}{(\theta - w_X(z))} \tag{4.30}$$

where $w_X(z) = \lambda(1\text{-}X(z))$, $X(z) = \sum_{k=1}^{\infty} g_k z^k$, $P_1(0) = \sum_{j=1}^{c} P_{1,j}(0)$

and π_n's, ψ_n's and ϕ_n's are given by

$$\pi_0 = 1,\ \pi_n = \sum_{i=1}^{n} g_i\ \pi_{n-i} \qquad 0 \le n \le m-1$$

$$\psi_0 = \alpha_0\ ;\ \psi_n = \sum_{i=0}^{n} \alpha_i\ \pi_{n-i} \qquad 0 \le n \le m-1 \tag{4.31}$$

$$\phi_n^S = \sum_{k=m}^{n} \pi_{n-k} \sum_{i=m}^{k} \xi_i\ h_{k-i} = \sum_{k=m}^{n} \xi_k \sum_{i=0}^{n-k} h_i\ \pi_{n-k-i}$$

with
$$\xi_n = \alpha_n + \sum_{i=0}^{m-1} \psi_i\ g_{n-i} \qquad m \le n \le N\text{-}1 \tag{4.32}$$

α_n and h_n's denote the probabilities that n customers arrive during a vacation time V and setup period D respectively.

Thus the partial generating function of the system size when the server is in idle state (vacation, buildup, setup and dormant) is given by

$$P_I(z) = Q^*(z,0) + R(z) + D^*(z,0) + U(z)$$
$$= P_1(0)\ (I_S(z)) \tag{4.33}$$

where
$$I_S(z) = \left[\left(\frac{1 - D^*(w_X(z)\ V^*(w_X(z))}{w_X(z)}\right) + D^*(w_X(z))\,(\psi(z)) + \phi_S(z)\right] \tag{4.34}$$

To obtain the remaining partial PGFs corresponding to the breakdown state $B_i^{**1}(z,\theta,\theta_1)$ and busy state $P_i^*(z,\theta)$ due to the i^{th} type of service we define, $B_i^{**1}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} B_{n-k,i}^{**1}(\theta,\theta_1)z^n$ and $P_i^*(z,\theta) = \sum_{n=1}^{\infty} P_{n,i}(\theta)z^n$, $1 \le i \le c$

Equations (4.23 and 4.24) imply, for $(1 \le i \le c)$,

$$(\theta_1 - w_X(z))\, B_i^{**1}(z,\theta,\theta_1) = B_i^{*}(z,\theta,0) - a_i\, P_i^{*}(z,\theta)\, B_i^{*1}(\theta_1)$$

when $\theta_1 = w_X(z)$, $B_i^{*}(z,\theta,0) = a_i\, P_i^{*}(z,\theta)\, B_i^{*1}(w_X(z))$ (4.35)

and hence,

$$B_i^{**1}(z,\theta,\theta_1) = \frac{(B_i^{*1}(w_X(z)) - B_i^{*1}(\theta_1))}{(\theta_1 - w_X(z))}\, a_i\, P_i^{*}(z,\theta) \tag{4.36}$$

Similarly equations (4.18) to (4.20) imply for $(1 \le i \le c)$

$$(\theta - w_X(z) - a_i)P_i^{*}(z,\theta) = P_i(z,0) - \frac{r_i S_i^{*}(\theta)}{z}\left[\sum_{j=1}^{c} P_j(z,0) - \sum_{j=1}^{c} P_{1,j}(0)\, z\right] - B_i^{*}(z,\theta,0)$$

$$- r_i S_i^{*}(\theta) \sum_{n=N}^{\infty} D_n z^n - \lambda\, r_i S_i^{*}(\theta) \sum_{n=N}^{\infty} z^n \sum_{k=n-N+1}^{n-m} U_{n-k} g_k \tag{4.37}$$

Equations (4.7) and (4.8) give

$$\lambda\, U(z) = \sum_{n=m}^{N-1} D_n(0)\, z^n + \lambda \sum_{n=m+1}^{N-1} z^n \sum_{k=1}^{n-m} U_{n-k}\, g_k \tag{4.38}$$

Multiplying equation (4.38) by $(- r_i\; S_i^{*}(\theta))$ adding with equation (4.37) and using equation (4.35) the following equation is obtained.

$$\text{(i.e.)}\ (\theta - h_{a_i}(w_X(z)))\, P_i^{*}(z,\theta) = P_i(z,0) - r_i\, S_i^{*}(\theta)\, \Big(\sum_{j=1}^{c} \frac{P_j(z,0)}{z} - P_1(0)$$

$$+ D(z,0) - U(z)\, w_X(z)\Big)$$

where $h_{a_i}(w_X(z)) = w_X(z) + a_i\,(1 - B_i^{*1}(w_X(z)))$ (4.39)

when $\theta = h_{a_i}(w_X(z))$

$$P_i(z,0) = r_i\, H_i^{*}(w_X(z)) \left[\sum_{j=1}^{c} \frac{P_j(z,0)}{z} - P_1(0) + D(z,0) - U(z)\, w_X(z)\right] \tag{4.40}$$

where $H_i^{*}(w_X(z)) = S_i^{*}(h_{a_i}(w_X(z)))$

Thus equation (4.39) can be simplified as

$(\theta - h_{a_i}(w_X(z)))\, P_i^*(z,\theta)$

$$= \quad r_i \;(H_i^*(w_X(z)) - S_i^*(\theta)) \left[\sum_{j=1}^{c} \frac{P_j(z,0)}{z} - P_1(0) + D(z,0) - U(z)\, w_X(z) \right] \quad (4.41)$$

Adding equation (4.40) over i = 1 to c

$$\sum_{i=1}^{c} P_i(z,0) \quad = \quad z \left[\sum_{i=1}^{c} r_i\, H_i^*(w_X(z)) \right] \left[\frac{D(z,0) - P_1(0) - U(z)\, w_X(z)}{z - \sum_{i=1}^{c} r_i\, H_i^*(w_X(z))} \right]$$

$$= \quad z\, H^*(w_X(z)) \left[\frac{D(z,0) - P_1(0) - U(z)\, w_X(z)}{(z - H^*(w_X(z)))} \right] \quad (4.42)$$

where $H^*(w_X(z)) \;=\; \sum_{i=1}^{c} r_i\, H_i^*(w_X(z))$

The equation (4.41) is simplified as

$$(\theta - h_{a_i}(w_X(z)))\, P_i^*(z,\theta) \quad = \quad r_i \;(H_i^*(w_X(z)) - S_i^*(\theta)) \left(\frac{z}{z - H^*(w_X(z))} \right) (D(z,\,0) - P_1(0) - U(z)\; w_X(z)) \quad (4.43)$$

Substituting for D(z, 0) and U(z) from equations (4.26) and (4.27),

$$D(z,\,0) - P_1(0) - U(z)\; w_X(z) \;=\; -\, P_1(0)\; w_X(z)\; I_S(z) \quad (4.44)$$

where $I_S(z)$ is given by (4.34)

Hence the equation (4.43) becomes

$$P_i^*(z,\theta) \quad = \; z\, r_i \,\frac{(H_i^*(w_X(z)) - S_i^*(\theta))}{(\theta - h_{a_i}(w_X(z)))} \;\frac{P_1(0)(-\,w_X(z))\, I_S(z)}{(z - H^*(w_X(z)))} \quad (4.45)$$

Further when $\theta = \theta_1 = 0$,equation (4.36) implies

$$B_i^{**_1}(z,0,0) \quad = \quad \frac{(1 - B_i^{*_1}(w_X(z)))}{(w_X(z))}\, a_i\, P_i^*(z,0) \qquad 1 \le i \le c \quad (4.46)$$

$$\text{Then}\;\; P_i^*(z,0) \;+\; B_i^{**_1}(z,0,0) \;=\; z\, r_i \,\frac{(H_i^*(w_X(z)) - 1)}{(z - H^*(w_X(z)))}\, P_1(0)\; I_S(z) \quad (4.47)$$

Thus the total PGF $\mathbf{P}^{\mathbf{C}}_{\mathrm{br}(m,N)}(z)$ is given by

$$\mathbf{P}^{c}_{br(m,N)}(z) = P_{I}(z) + \sum_{i=1}^{c} (B_i^{**_1}(z,0,0) + P_i^{*}(z,0))$$

$$= P_1(0)\, I_S(z) \left(1 + \frac{z\,(H^{*}(w_X(z)) - 1)}{(z - H^{*}(w_X(z)))}\right)$$

$$\mathbf{P}^{c}_{br(m,N)}(z) = \frac{P_1(0)\,(z-1)\,H^{*}(w_X(z))\,I_S(z)}{(z - H^{*}(w_X(z)))} \qquad (4.48)$$

IV Performance Measures

The steady state system size probabilities at various states are derived as follows. Let P_{build}, P_{set}, P_{dor}, P_{busy}, P_v and P_{br} denote the probability that the server is in buildup, setup, dormant, busy, vacation and in breakdown state respectively. Then

(i) $P_{build} = \underset{z\to 1}{lt}\ R(z) = P_1(0)\ \psi(1)$

(ii) $P_{set} = \underset{z\to 1}{lt}\ D^{*}(z,0) = P_1(0)\ E(D)$

(iii) $P_{dor} = \underset{z\to 1}{lt}\ U(z) = P_1(0)\ \phi_S(1)$

(iv) $P_{v} = \underset{z\to 1}{lt}\ Q^{*}(z,0) = P_1(0)\ E(V)$

Thus the probability that the server is in idle state is given by

(v) $P_{I} = P_1(0)\ I_S(1) = P_1(0)\ D_S(m, N)$

$$\text{where}\quad D_S(m, N) = \left(E(V) + E(D) + \sum_{n=0}^{m-1} \psi_n(1/\lambda) + \sum_{n=m}^{N-1} \varphi_n^{S}(1/\lambda)\right) \qquad (4.49)$$

(vi) Further the probability that the server is in busy state is given by

$$P_{busy} = \underset{z\to 1}{lt} \sum_{i=1}^{c} P_i^{*}(z,0) = \sum_{i=1}^{c} P_{w_i}$$

$$\text{where } P_{w_i} = \underset{z\to 1}{lt}\ P_i^{*}(z,0) = \frac{P_1(0)\,\lambda\,E(X)}{(1-\rho^{c}_{br})}\,(r_i\,E(S_i))\,D_S(m,N)$$

(i.e), $P_{busy} = \dfrac{\rho_c}{(1-\rho_{br}^c)} P_1(0)\, D_S(m, N)$

where $\rho_c = \lambda\, E(X) \sum_{i=1}^{c} r_i\, E(S_i)$ and

$\rho_{br}^c = \lambda\, E(X) \sum_{i=1}^{c} r_i\, E(S_i)\, (1 + a_i\, E(B_i))$

(vii) The probability that the server is in breakdown state is given by

$$P_{br} = \sum_{i=1}^{c} \lim_{z\to 1} B_i^{**1}(z, 0, 0)$$

$$= \lambda\, E(X) \sum_{i=1}^{c} r_i\, E(S_i)\; a_i\, E(B_i)\, \frac{D_S(m, N)}{(1-\rho_{br}^c)}\, P_1(0)$$

Hence $P_{busy} + P_{br} = P_1(0) \left(\dfrac{\rho_{br}^c}{1-\rho_{br}^c} \right) D_S(m, N)$

The normalizing condition implies,

$$P_1(0) = (1-\rho_{br}^c) \,/\, D_S(m, N) \tag{4.50}$$

Thus the probability that the server is in busy state is given by

$$P_{busy} = (\lambda\, E(X))\, E(S) = \rho_c$$

where $E(S) = \sum_{i=1}^{c} r_i\, E(S_i)$

substituting for $P_1(0) = (1-\rho_{br}^c) \,/\, D_S(m, N)$ in equation (4.48),

$$\mathbf{P^c_{br(m,N)}(z)} = \frac{(1-\rho_{br}^c)(z-1)H^*(w_X(z))}{(z-H^*(w_X(z)))} \frac{I_S(z)}{I_S(1)}$$

Thus the ergodic condition for the model is $\rho_{br}^c < 1$.

V Decomposition Property

The PGF of the system size of the model under consideration is decomposed into product of two Probability Generating Functions one of which is $P_{br}^c(z) = \dfrac{(1-\rho_{br}^c)(z-1)H^*(w_X(z))}{(z-H^*(w_X(z)))}$ and this gives the PGF of the corresponding classical breakdown model provided with c-types of heterogeneous service facility without N-policy and $\frac{I_S(z)}{I_S(1)}$ gives the PGF of the conditional system size distribution during the server idle period.

VI Mean Queue Length

Let L_v, L_{build}, L_{set}, L_{dor}, L_{busy} and L_{br} denote the expected system size when the server is in vacation, build up, set up, dormant, busy and in breakdown state respectively. Then

(i) $L_v = \lambda E(X) (E(V^2) / 2) P_1(0)$

(ii) $L_{build} = P_1(0) \sum_{n=0}^{m-1} \frac{n \psi_n}{\lambda}$

(iii) $L_{set} = P_1(0) \lambda E(X) ((E(D^2) / 2) + E(V) E(D) + E(D) \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda})$

(iv) $L_{dor} = P_1(0) \sum_{n=m}^{N-1} \frac{n \phi_n}{\lambda}$, follow by differentiating the equations (4.25) to (4.30) at z = 1.

Thus the expected system size (L_{idle}) when the server is in idle state is obtained by adding (i) to (iv) and is given by

(v) $L_{idle} = L_S (m, N) P_1(0)$

where $L_S(m, N) = L_0 + E(D) \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda} + \sum_{n=0}^{m-1} \frac{n \psi_n}{\lambda} + \sum_{n=m}^{N-1} \frac{n \phi_n^S}{\lambda} = \frac{d}{dz}[I_S(z)]_{z=1}$

with $L_0 = \lambda E(X) (E(V^2) / 2 + E(D^2) / 2 + E(V) E(D))$

(vi) $L_{busy} + L_{br} = \frac{d}{dz}(\sum_{i=1}^{c}[P_i^*(z,0) + B_i^{**}(z,\theta,0)])_{z=1}$

$$= P_1(0) \frac{d}{dz}\left(\frac{(z-1)H^*(w_X(z)-1)I_S(z)}{(z-H^*(w_X(z)))}\right)_{z=1} \quad \text{from equation (4.47)}$$

$$L_{busy} + L_{br} = \rho_{br}^c + P_1(0)\left(\frac{L_S(m,N)}{(1-\rho_{br}^c)} + \frac{D_S(m,N)}{2(1-\rho_{br}^c)^2}(\lambda E(X(X-1))E(H) + (\lambda E(X))^2 E(H^2))\right)$$

where $E(H) = \sum_{i=1}^{c} r_i E(H_i)$ with $E(H_i) = (1 + a_i E(B_i) E(S_i)$

$E(H^2) = \sum_{i=1}^{c} r_i E(H_i^2)$ where , $E(H_i^2) = a_i E(B_i^2) E(S_i) + (1 + a_i E(B_i))^2 E(S_i^2)$

Then the expected system size is given by

$$L^{c}_{br(m,N)} = L_{M^x/G_i(1 \le i \le c)/1/SV/br} + \frac{L_S(m,N)}{D_S(m,N)}$$

where $L_{M^X/G_i(i=1\text{ to }c)/1/SV/br} = \left[\rho^{c}_{br} + \frac{\lambda E(X(X-1))E(H) + (\lambda E(X))^2 E(H^2)}{2(1-\rho^{c}_{br})^2}\right]$

gives the mean system size of the corresponding breakdown model provided with c- types of heterogenous service facility.

VI Other system characteristics

Let E(Cycle), E(Busy),E(I) and $E(W_s)$ denote the expected length of cycle, busy period, idle period and expected waiting time in the system. Then from IV- (ii), (vi) and (v) we get,

(i) $E(\text{Cycle}) = \frac{1}{P_1(0)} = \frac{(1-\rho^{c}_{br})}{D_S(m,N)}$

(ii) $E(\text{Busy}) = P_{busy} E(\text{cycle}) = \frac{\rho_c}{(1-\rho^{c}_{br})} D_S(m, N)$

(iii) $E(\text{idle}) = P_I E(\text{cycle}) = D_S(m, N)$

(iv) $E(W_s) = \frac{L^{c}_{br(m,N)}}{\lambda E(x)}$ (Using little's formula)

VII Queue Size Distribution at Departure Epoch

Theorem If $\pi^+(z)$ denotes the PGF of the system size distribution at departure epoch of the system under consideration, with stability condition $\rho^{c}_{br} < 1$ then $\pi^+(z) = \frac{1}{E(X)}\left(\frac{(X(z)-1)}{z-1}\right) P^{c}_{br(m,N)}(z)$

Proof Let π_j^+ denote the probability that there are j customers in the system at departure epoch then, $\pi_j^+ = D\left(\sum_{i=1}^{c} P_{j+1,i}(0)\right)$ where D is constant then

$$\pi^+(z) = \sum_{j=0}^{\infty} \pi_j^+ z^j = \left(\frac{D}{Z}\right)\left(\sum_{i=1}^{c} P_i(z,0)\right)$$

Substituting for $\sum_{i=1}^{c} P_i(z,0)$ from equation (4.42) and evaluating D from the normalizing condition it is found that

$$\pi^+(z) = \frac{1}{E(X)}\left(\frac{(X(z)-1)}{z-1}\right)\mathbf{P}^{c}_{br(m,N)}(z)$$

4.2 Optimal Management Policy

By following the procedure of section (2.1.2), the optimal threshold values (m^*, N^*) can be obtained.

Recalling the cost structure as ,C_y(startup cost per cycle), C_{set}(setup cost), C_{dor}(standby cost), C_{busy} (operating cost), C_h (holding cost per customer), C_{br}(breakdown cost)and C_v (reward cost) per unit time, the average cost per unit time of the system is given by

$$T_c^{br}(m,N) = \left[\frac{C_y}{E(Cycle)} + C_{set} P_{set} + C_{dor} P_{dor} + C_{busy} P_{busy} + C_{br} P_{br} + C_h \mathbf{L}^{c}_{br(m,N)} - C_v P_v\right]$$

By substituting the corresponding measures,

$$T_c^{br}(m,N) = \frac{(1-\rho_{br}^c)}{D_S(m,N)}\left[A_{br}^c + z_{br}^c(m) + C_{dor}\sum_{n=m}^{N-1}\frac{\phi_n^S}{\lambda} + C_h\sum_{n=m}^{N-1}\frac{n\,\phi_n^S}{\lambda}\right] + A_{br}^{'}$$

where $A_{br}^c = (1-\rho_{br}^c)\,(C_y + C_{set}\,E(D) - C_v\,E(V)) + C_h\,(\lambda\,E(X))\,(E(D^2)/2 + E(D)\,E(V) + E(V^2)/2)$

$$z^{c}_{br}(m) = [(1-\rho^{c}_{br})C_{build} + C_h \lambda E(D) E(X)] \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda} + C_h \sum_{n=0}^{m-1} \frac{n\psi_n}{\lambda}$$

$$A^{'}_{br} = C_{busy}\, \rho^{c} + C_{br} \lambda E(X) \sum_{i=1}^{c} r_i E(S_i)\, a_i E(B_i) + C_h L_{M^X/G_i (i=1 \text{ to } c)/1/SV}$$

By proceeding as in previous chapters it is found that , the conditional optimal threshold policy is given by

$$\mathbf{N^*(m) = Min\ \{k \geq 1 \,/\, H^{c}_{br}(m,k) > 0\}}$$

where $H^{c}_{br}(m,k) = C_h (k\,\ell^{br}_{m} + \sum_{n=m}^{k} (k-n)\frac{\phi^{S}_{n}}{\lambda}) + C_{dor} (1-\rho^{c}_{br})\,\ell^{br}_{m} - (A^{c}_{br} + z^{c}_{br}(m))$

with $\ell^{br}_{m} = E(D) + E(V) + \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda}$

Thus the optional value $(m^*, N^*(m))$ and the corresponding $T_c(m^*, N^*(m))$ can be obtained similarly, as in Chapter III.

4.3 Particular cases

(1) When c = 1 the results of single service breakdown (m, N) policy queueing model with single vacation can be obtained. If the service time, vacation time and setup time follow exponential distribution then the corresponding results of section (2.1) and section (4.1) coincide under the condition that r_i = 1 for every i.

(2) When $\alpha_i = 0$, $(1 \leq i \leq c)$ the results of $M^{X}_{(m,N)}/G_{i\ (1 \leq i \leq c)}/1/SV$ model without server breakdown can be obtained.

4.4 Numerical Analysis

In this section we provide numerical results of (i) Optimal expected system size $(\mathbf{L}^{c}_{br(m^*,N^*)})$ (ii) The optimal threshold values (m*,N*) of (m ,N) (iii) The total expected cost per unit time Tc(m ,N) and (iv) the optimum value Tc(m*,N*) for the $M^X/G_{i\,(i=1\text{ to }3)}/1/SV/BD$ qeueing system.

For computation purpose the following distributions are assumed for setup time D ,vacation time V, service time (S_i) repair time R_i where i=1,2,3 and for the batch size x.

Random Variable (y)	Distribution F(y)	Mean E(y)	Second order moment
S_1	Two stage hyper exponential	$E(S_1)=\frac{a_1}{\mu_{11}}+\frac{1-a_1}{\mu_{12}}$ $0 \le a_1 \le 1$	$E(S_1^2)=2\left(\frac{a_1}{\mu_{11}^2}+\frac{1-a_1}{\mu_{12}^2}\right)$
S_2	Uniform distribution	$E(S_2)=\frac{1}{2}$	$E(S_2^2)=\frac{1}{3}$
S_3	Exponential distribution	$E(S_3)=\frac{1}{\mu_3}$	$E(S_3^2)=\frac{2}{\mu_3^2}$
D	Gamma Distribution $G(2,\mu)$	$E(D)=\frac{2}{v}$	$E(D^2)=\frac{6}{v^2}$
V	Erlang 3 type distribution	$E(V)=\frac{1}{\eta}$	$E(V^2)=\frac{4}{3\eta^2}$
$R_{i\,(i=1,2,3)}$	Exponential distribution	$E(R_i)=\frac{1}{\beta_i}$	$E(R_i^2)=\frac{2}{\beta_i^2}$
X	Geometric distribution	$E(X)=\frac{1}{1-p}$	$E(X(X-1))=\frac{2p}{(1-p)^2}$

r_i $(1 \le i \le 3)$ denotes that the probability that the customers choose the i^{th} type of service. α_i and β_i $(1 \le i \le 3)$ denote the breakdown rate and repair rate respectively.

Tables (4.1) and (4.2) show that as the mean vacation time E (V) $=1/\eta$ or the mean setup time E(D)=$2/v$ decreases, the mean system size also decreases. The table values also show that the mean system size increases with the arrival rate λ. The parametric values chosen to construct the Tables (4.1) and (4.2) are the same. Their corresponding graphical representations are given in figures (4.1) & (4.2) .

Table (4.1) : Mean system size $\mathbf{L}^{c}_{br(m^*,N^*)}$ with respect to λ and η

$(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v, C_{br})$ = (100, 10, 40, 100,1000 ,10000,10, 10) :

$(p, \mu_3, v, E(S_1))$ = (.75,.5,.2,.63) ; (r_1, r_2, r_3) = (.2, .5, .3) ;

$(\alpha_1, \alpha_2, \alpha_3)$ = (.2,.1,.2) ; $(\beta_1, \beta_2, \beta_3)$ = (.2, .3, .4) ;

λ \ η	.03	.04	.05	.25	.35	.45	.5
0.05	5.618	4.637	4.103	2.8871	2.854	2.841	2.837
0.075	9.134	7.563	6.578	4.567	4.504	4.478	4.071
0.1	13.067	10.916	9.379	6.604	6.506	6.464	5.352
0.125	17.588	14.866	13.084	9.16	9.072	9.007	8.088
0.15	23.05	19.771	17.451	12.65	12.556	12.472	11.347
0.175	30.255	26.427	24.075	18.041	17.797	17.689	16.557

Table (4.2) : Mean system size $\mathbf{L}^{c}_{br(m^*,N^*)}$ with respect to λ and v

All the parameters are as in Table (4.1) with η =.35

λ	v = .05	v =.075	v = .1	v = .15	v = .2	v = .25	v = .3
0.05	7.126	5.268	4.332	3.378	2.854	2.542	2.374
0.075	10.841	8.033	6.629	5.223	4.504	4.085	3.773
0.1	14.982	11.179	9.313	7.446	6.506	5.938	5.551
0.125	19.747	14.916	12.554	10.225	9.072	8.361	7.895
0.15	25.498	19.624	16.759	13.946	12.556	11.727	11.161
0.175	33.034	26.109	22.728	19.424	17.796	16.827	16.188

Fig. (4.1) $L^{c}_{br(m^*,N^*)}$ with respect to λ for different η

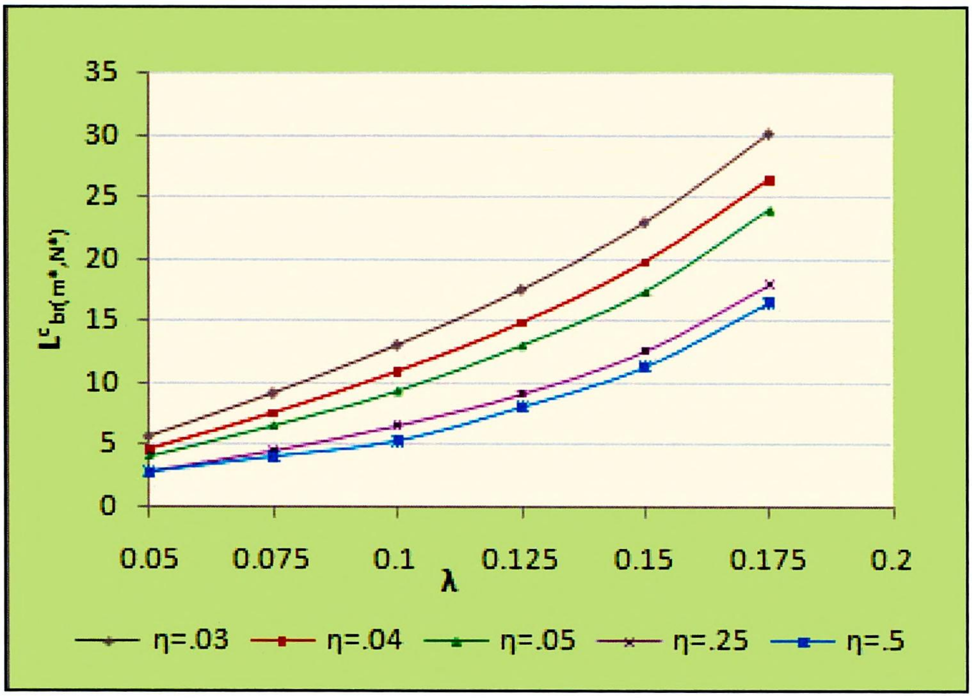
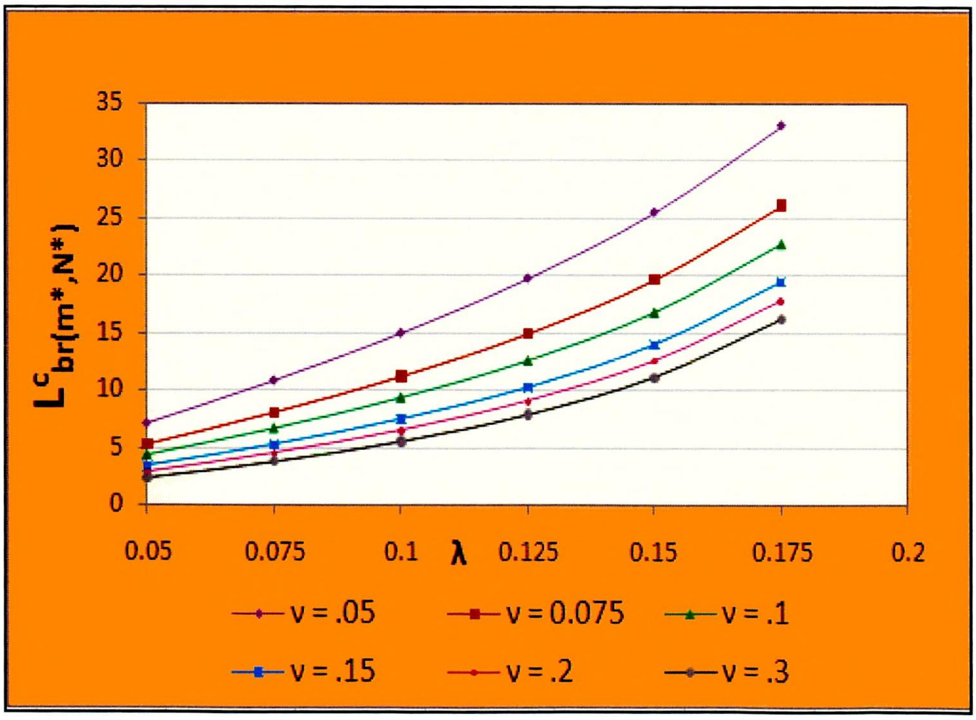


Fig. (4.2) $L^{c}_{br(m^*,N^*)}$ with respect to λ for different v



The values in tables (4.3) and (4.4) give the optimal mean system size as breakdown rate (α_1) and repair rate (β_1) vary. The numerical results are calculated for two different cases namely r_1 = .3 and r_1 = .6 where r_1 denotes the probability of selecting first type of service. Similarly the analysis can be carried out by changing (α_i , β_i and r_i) i= 1,2,3. The table values show that, the system size (i) increases with breakdown rate α_1 and (ii) decreases as the repair rate β_1 increases. The pictorial representation of the above discussion is given in figures (4.3) and (4.4)

Table (4.3): $\mathbf{L}^{c}_{br(m^*,N^*)}$ with respect to α_1 for various β_1

Parameters are as in Table (4.1) with η =.35 and λ=.175, (r_1,r_2,r_3)=(.3 .3, .4)

α_1	β_1 = .1	β_1 = .125	β_1 =.15	β = .175	β_1 = .2	β_1 = .25
0.1	19.25	18.67	18.33	18.103	17.95	17.74
0.15	20.54	19.58	19.03	18.668	18.42	18.09
0.2	21.99	20.59	19.79	19.272	18.92	18.46
0.25	23.63	21.7	20.61	19.92	19.45	18.85
0.3	25.5	22.92	21.5	20.61	20.02	19.25
0.35	27.63	24.23	22.5	21.36	20.62	19.69

Table (4.4): $\mathbf{L}^{c}_{br(m^*,N^*)}$ with respect to α_1 for various β_1 for r_1 = .6

Parameters are as in Table (4.1) with η =.35 and λ=.175, (r_1,r_2,r_3)=(.6 .2, .2)

α_1	β_1= .1	β_1 =.125	β_1 =.15	β_1 =.175	β_1 = .2	β_1 = .25
0.1	24.12	22.521	21.61	21.02	20.61	20.08
0.15	28.17	25.21	23.57	22.56	21.88	21
0.2	33.53	28.51	25.91	24.34	23.31	22.02
0.25	40.89	32.68	28.71	26.41	24.93	23.14
0.3	51.69	38.05	32.11	28.83	26.8	24.38
0.35	68.44	45.23	36.32	31.7	28.93	25.76

Fig. (4.3) $L^{c}_{br}(m^*,N^*)$ with respect to α_1 for various β_1 (r_1 =.3)

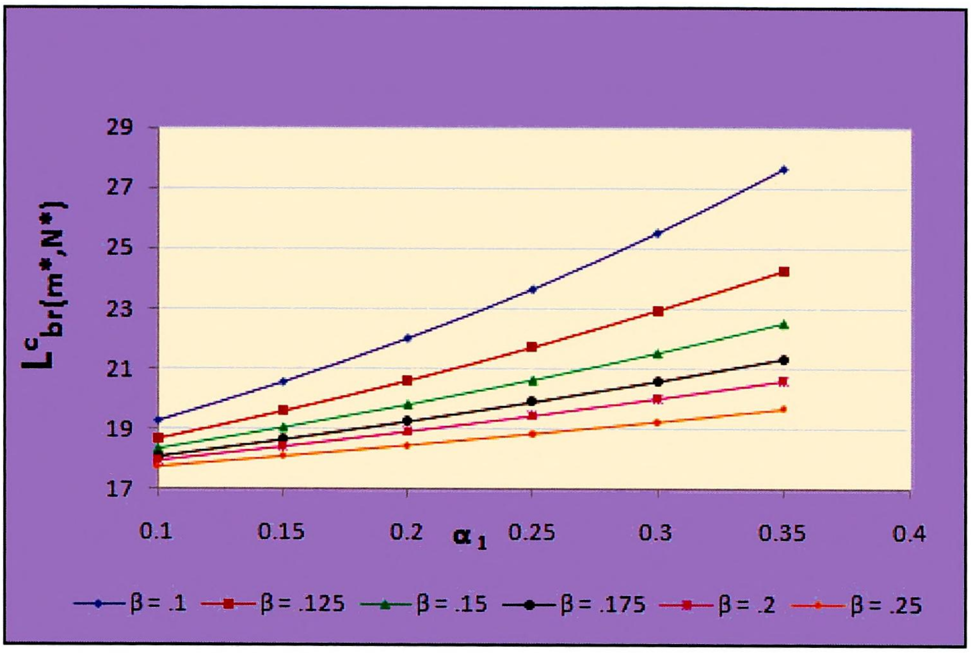
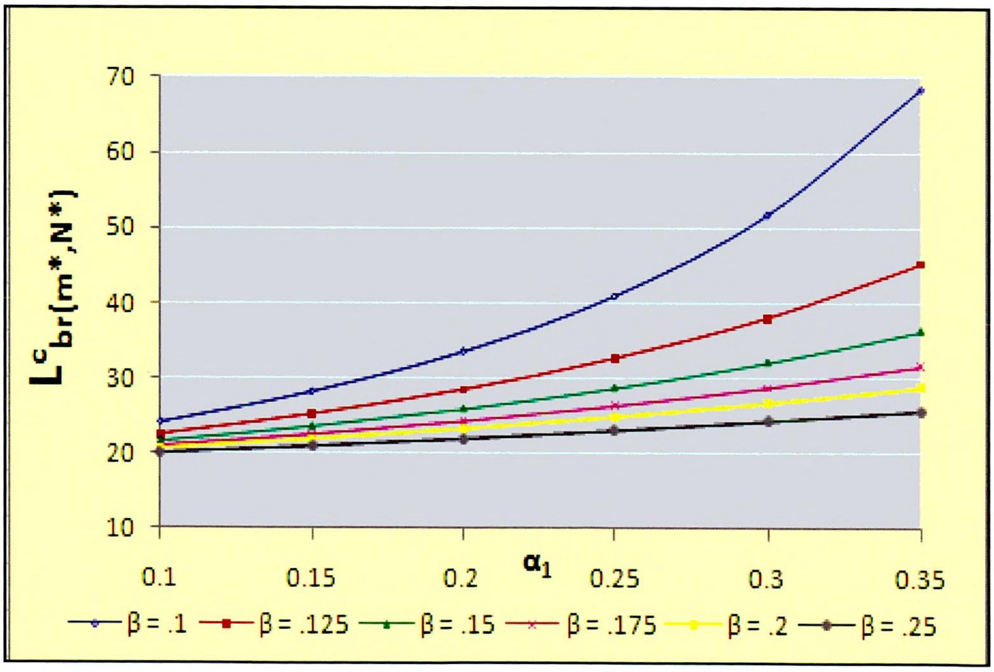


Fig. (4.4) $L^{c}_{br}(m^*,N^*)$ with respect to α_1 for various β_1 (r_1 =.6)



The system size probabilities corresponding to different states of the server are presented in Table (4.5) and figure (4.5). ρ_c and ρ_{br}^{c} are given by $\rho_c = \lambda E(X) \sum_{i=1}^{c} r_i E(S_i)$ and $\rho_{br}^{c} = \lambda E(X) \sum_{i=1}^{c} r_i E(S_i)(1 + a_i E(B_i))$.

Table 4.5 : System size probabilities with respect to λ

All the parameters are as in Table (4.1) with η = .35.

λ	P_{busy}	P_v	P_{set}	P_{build}	P_{dor}	P_{br}	ρ_c	ρ_{br}^{c}
0.05	0.126	0.068	0.237	0.412	0.092	0.066	0.126	0.066
0.075	0.188	0.072	0.251	0.272	0.12	0.098	0.188	0.098
0.1	0.251	0.069	0.242	0.184	0.122	0.131	0.251	0.131
0.15	0.377	0.058	0.201	0.09	0.078	0.197	0.377	0.197
0.155	0.389	0.056	0.196	0.084	0.072	0.203	0.389	0.203
0.175	0.44	0.046	0.163	0.058	0.063	0.23	0.44	0.23
0.2	0.502	0.034	0.12	0.035	0.046	0.262	0.502	0.263
0.225	0.565	0.021	0.073	0.018	0.027	0.295	0.565	0.295
0.25	0.628	0.007	0.024	0.005	0.007	0.328	0.628	0.328

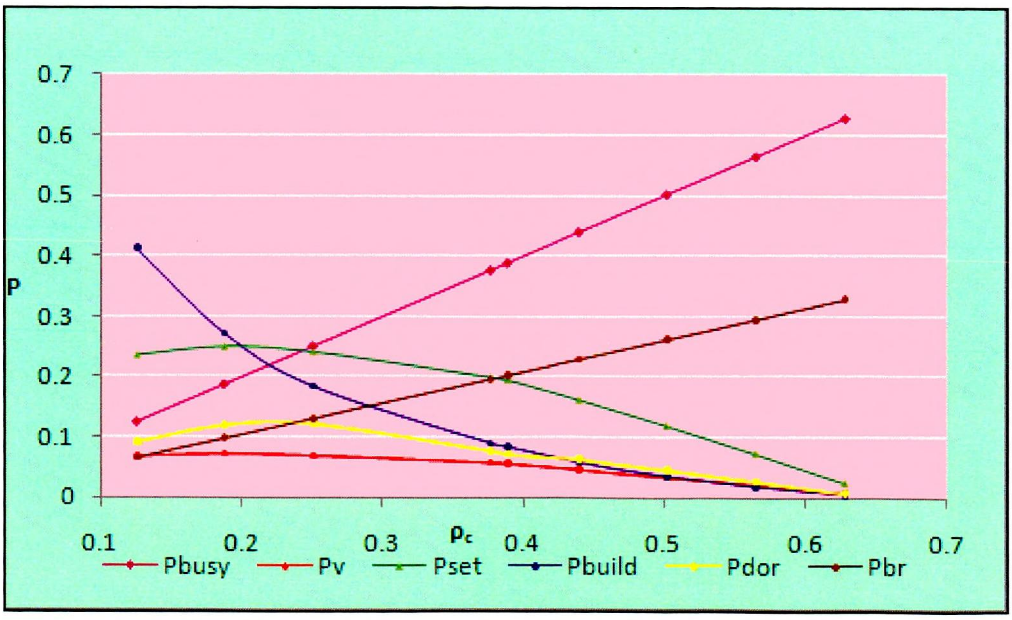


Fig. (4.5) system size probabilities with respect to ρ

The total cost function Tc (m,N) and the optimal cost Tc(m*,N*) are tabulated in table 4.6. Figure 4.6 gives the graphical representation of the cost function. The optimal cost Tc (m*,N*) = 650.821 is observed at m*=16,N* = 22

Table 4.6 : The expected cost Tc(m,N) Vs m and N.

$(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v, C_{br})$ =(8, 8, 100, 100, 1000 , 10000, 8, 100)

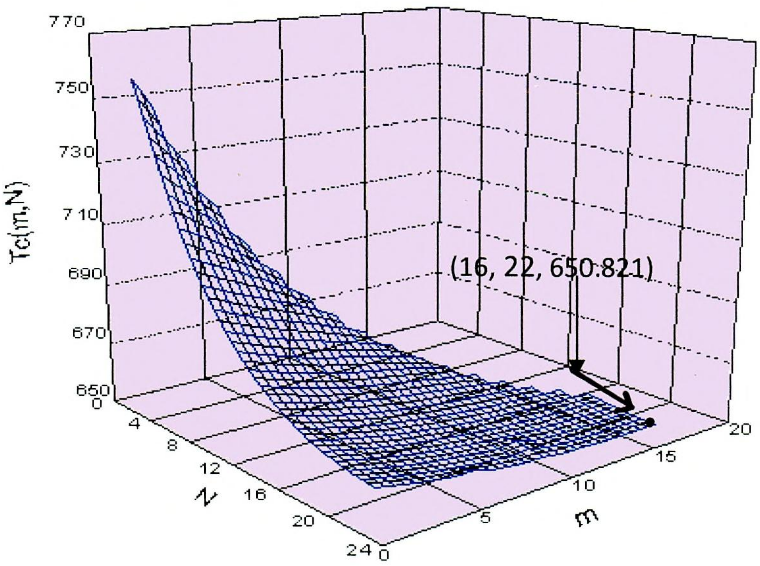
$(p, \lambda, v, \eta, \mu, E(S_1))$ = (.75, .18, .2, .5, .5, .484) ;

(r_1, r_2, r_3) = (.6, .2, .2) ; $(\alpha_1, \alpha_2, \alpha_3)$ = (.25,.15,.25) ;

$(\beta_1, \beta_2, \beta_3)$ = (.2, .3, .4) ; C = 3 .

m \ N	17	18	19	20	21	22	23	24
8	656.567	655.428	654.557	653.936	653.545	653.361	653.387	653.592
9	655.846	654.737	653.894	653.295	652.924	652.762	652.797	653.013
10	655.225	654.144	653.324	654.144	653.324	652.746	652.391	652.516
11	654.701	653.645	652.846	652.286	651.945	651.810	651.866	652.099
12	654.272	653.238	652.457	651.911	651.583	651.458	651.521	651.760
13	653.934	652.919	652.154	651.620	651.302	651.185	651.253	651.497
14	653.614	652.686	651.934	651.411	651.101	650.989	651.061	651.307
15	653.316	652.484	651.795	651.281	650.976	650.868	650.943	651.188
16	653.043	652.319	651.700	651.227	650.927	**650.821**	650.896	651.140
17	652.800	652.194	651.653	651.227	650.950	650.844	650.918	651.160

Fig. 4.6: The expected cost Tc(m,N) for Vs m and N when C = 3 servers



Tables (4.7) and (4.8) exhibit the optimal policies for double and single threshold models, for different values of service selecting probabilities (r_1, r_2, r_3) and breakdown rates (α_1, α_2,α_3). From the Tables (4.7) and (4.8), it is found that as r_i or α_i increases, (m*,N*) decreases and Tc(m*,N*) increases. The expected cyclic length for (m,N) policy is found to be longer than that of N policy in either cases as expected. For Tables (4.7) and (4.8) cost elements and parameters are as in Table (4.6).

Table (4.7) optimum threshold with respect to (r_1, r_2, r_3)

(r_1,r_2,r_3)	(m*,N*)	Tc (m*,N*)	Ecy(m*,N*)	N*	Tc(N*)	Ecy(N*)
(.2,.5,.3)	(7,18)	770.139	123.270	14	773.935	120.535
(.3,.4,.3)	(5.17)	788.778	127.638	13	793.363	126.081
(.4,.3,.3)	(3,17)	809.993	136.64	12	815.52	132.694
(.5,.2,.3)	(3,16)	834.704	144.934	11	841.327	140.714
(.3,.3,.4)	(3,9)	2648.031	938.79	1	2655.695	675.13

Table (4.8) optimum threshold with respect to (α_1, α_2, α_3)

(α_1, α_2, α_3)	(m*,N*)	Tc (m*,N*)	Ecy(m*,N*)	N*	Tc(N*)	Ecy(N*)
(.35,.15,25)	(14,21)	656.438	101.539	18	657.726	99.163
(.35,.45,.35)	(10,19)	676.849	113.882	15	679.662	109.218
(.55,.25,.35)	(5,18)	708.1011	129.622	13	712.615	124.520
(.45,.45,.55)	(4,17)	729.317	136.6866	12	734.763	131.093
(.5,.5,.5)	(2,14)	814.902	169.539	9	823.742	162.214
(.75,.55,.45)	(2,12)	921.323	207.035	8	931.323	201.844