Chapter VI

CHAPTER – VI

SOME NEW OPERATORS ON TRIANULAR FUZZY NUMBERS AND TRIANGULAR FUZZY NUMBER MATRICES

In this chapter, some new elementary operators on Triangular Fuzzy Numbers (TFNs) and some new operators on Triangular Fuzzy Number Matrices (TFNMs) are defined. Using these operators, some important properties of TFNs and TFNMs are presented.

Some New Operators on Triangular Fuzzy Numbers

Definition : 6.1

Let $\tilde{M} = < m, \rho, \beta >$ and $\tilde{N} = < n, \gamma, \delta >$ be two TFNs where m, ρ, β, n, γ, $\delta \in [0, 1]$. Then the following operators are defined :

(1) $\tilde{M} \oplus \tilde{N} \;=\; < m + n - mn, \rho + \gamma - \rho\gamma, \beta + \delta - \beta\delta >$

(2) $\tilde{M} \odot \tilde{N} \;=\; < mn, \rho\gamma, \beta\delta >$

(3) $\tilde{M} \vee \tilde{N} \;=\; < m \vee n, \rho \vee \gamma, \beta \vee \delta >$ where $x \vee y$ means max {x, y}. That is, $x \vee y = \max\{x, y\}$.

(4) $\tilde{M} \wedge \tilde{N} \;=\; < m \wedge n, \rho \wedge \gamma, \beta \wedge \delta >$ where $x \wedge y$ means min {x, y}. That is, $x \wedge y = \min\{x, y\}$.

(5) $\tilde{M} \ominus \tilde{N} \;=\; < m \ominus n, \rho \ominus \gamma, \beta \ominus \delta >$ where

$$x \ominus y = \begin{cases} x, & \text{if } x > y \\ 0, & \text{if } x \le y \end{cases}$$

(6) $\tilde{M} \ge \tilde{N}$ if $m \ge n$, $\rho \ge \gamma$ and $\beta \ge \delta$.

Some New Operators on Triangular Fuzzy Number Matrices

Definition : 6.2 :

Let $M = (\tilde{M}_{ij})_{mxn}$ and $N = (\tilde{N}_{ij})_{mxn}$ be two TFNMs of the same order. Then the following operators are defined.

(1) $M \oplus N = (\tilde{M}_{ij} \oplus \tilde{N}_{ij})$

(2) $M \odot N = (\tilde{M}_{ij} \odot \tilde{N}_{ij})$

(3) $M \vee N = (\tilde{M}_{ij} \vee \tilde{N}_{ij})$

(4) $M \wedge N = (\tilde{M}_{ij} \wedge \tilde{N}_{ij})$

(5) $M \ominus N = (\tilde{M}_{ij} \ominus \tilde{N}_{ij})$

(6) $M^{[1]} = M$

(7) $M^{[K+1]} = M^{[K]} \odot M$, K = 1, 2, …

(8) $[1]\,M = M$

(9) $[K+1]\,M = [K]\,M \oplus M$, K = 1, 2, …

(10) $M \geq N$ iff $\tilde{M}_{ij} \geq \tilde{N}_{ij}$ $\forall$ i = 1 to m, j = 1 to n.

Example : 6.3

Consider two triangular fuzzy number matrices M and N as follows :

$$M = \begin{bmatrix} <0.5, 0.1, 0.4> & <0.1, 0.7, 0.6> \\ <0.3, 0.5, 0.2> & <0.6, 0.3, 0.65> \end{bmatrix} \text{ and}$$

$$N = \begin{bmatrix} <0.3, 0.3, 0.6> & <0.8, 0.1, 0.3> \\ <0.4, 0.9, 0.5> & <0.2, 0.1, 0.3> \end{bmatrix}$$

Then

(1) $$M \oplus N = \begin{bmatrix} <0.65, 0.37, 0.76> & <0.82, 0.73, 0.72> \\ <0.58, 0.95, 0.60> & <0.68, 0.37, 0.75> \end{bmatrix}$$

(2) $$M \odot N = \begin{bmatrix} <0.15, 0.03, 0.24> & <0.08, 0.07, 0.18> \\ <0.12, 0.45, 0.10> & <0.12, 0.03, 0.195> \end{bmatrix}$$

(3) $$M \vee N = \begin{bmatrix} <0.5, 0.3, 0.6> & <0.8, 0.7, 0.6> \\ <0.4, 0.9, 0.5> & <0.6, 0.3, 0.65> \end{bmatrix}$$

(4) $$M \wedge N = \begin{bmatrix} <0.3, 0.1, 0.4> & <0.1, 0.1, 0.3> \\ <0.3, 0.5, 0.2> & <0.2, 0.1, 0.3> \end{bmatrix}$$

(5) $$M \ominus N = \begin{bmatrix} <0.5, 0.0, 0.0> & <0.0, 0.7, 0.6> \\ <0.0, 0.0, 0.0> & <0.6, 0.3, 0.65> \end{bmatrix}$$

Now we define some special types of Triangular Fuzzy Number Matrices.

Definition : 6.4

Let M = $(\tilde{M}_{ij})$ be an n x n triangular fuzzy number matrix. Then

(i) M is **Reflexive** if and only if $\tilde{M}_{ii} = \tilde{1}$ for all i = 1, 2, …, n and where $\tilde{1}$ = < 1, 0, 0 >.

(ii) M is **Irreflexive** iff $\tilde{M}_{ii} = \tilde{0}$ for i = 1, 2, …, n and where $\tilde{0}$ = < 0, 0, 0 >.

(iii) M is **Nearly irreflexive** iff $\tilde{M}_{ii} \leq \tilde{M}_{ij}$ for all i, j = 1, 2, …, n.

(iv) M is **Weakly reflexive** iff $\tilde{M}_{ii} \geq \tilde{M}_{ij}$ for all i, j.

(v) M is **Diagonal** iff $\tilde{M}_{ii} \geq 0$ and $\tilde{M}_{ij} = 0$ $(i \neq j)$ for all i, j.

(vi) M is **Constant** iff $\tilde{M}_{ij} = \tilde{M}_{kj}$ for all i, j, k = 1, 2, …, n.

Theorem : 6.5

Let M and N be two triangular fuzzy number matrices, then

(i) $M \oplus N \geq M \odot N$

(ii) If M and N are nearly irreflexive then $M \oplus N$ and $M \odot N$ are nearly irreflexive.

(iii) $M \oplus M \geq M$ and

(iv) $M \odot M \leq M$.

Proof

Let M = $(\tilde{M}_{ij})$ where $\tilde{M}_{ij}$ = < m_{ij}, ρ_{ij}, β_{ij} > and N = $(\tilde{N}_{ij})$ where $\tilde{N}_{ij}$ = < n_{ij}, γ_{ij}, δ_{ij} >

(i) The ij[th] element of $M \oplus N$ is $\tilde{M}_{ij} \oplus \tilde{N}_{ij}$ which is equal to

< $m_{ij} + n_{ij} - m_{ij}.n_{ij}$, $\rho_{ij} + \gamma_{ij} - \rho_{ij}.\gamma_{ij}$, $\beta_{ij} + \delta_{ij} - \beta_{ij}.\delta_{ij}$ > and that of $M \odot N$ is

$\tilde{M}_{ij} \odot \tilde{N}_{ij}$ = < $m_{ij}.n_{ij}$, $\rho_{ij}.\gamma_{ij}$, $\beta_{ij}.\delta_{ij}$ >

Consider $m_{ij} + n_{ij} - m_{ij}.n_{ij} - m_{ij}.n_{ij}$

$\Rightarrow m_{ij}.(1 - n_{ij}) + n_{ij}.(1 - m_{ij}) \geq 0$ (since $0 \leq m_{ij} \leq 1$ and $0 \leq n_{ij} \leq 1$)

$\therefore m_{ij} + n_{ij} - m_{ij}.n_{ij} \geq m_{ij}.n_{ij}$.

Similarly, $\rho_{ij} + \gamma_{ij} - \rho_{ij}.\gamma_{ij} \geq \rho_{ij}.\gamma_{ij}$ and

$\beta_{ij} + \delta_{ij} - \beta_{ij}.\delta_{ij} \geq \beta_{ij}.\delta_{ij}$

$\therefore M \oplus N \geq M \odot N$.

(ii) Since M and N are nearly irreflexive, $\tilde{M}_{ii} \leq \tilde{M}_{ij}$ and $\tilde{N}_{ii} \leq \tilde{N}_{ij}$ for all i, j.

$$\left.\begin{aligned}
&\therefore < m_{ii}, \rho_{ii}, \beta_{ii} > \leq < m_{ij}, \rho_{ij}, \beta_{ij} > \\
&\therefore m_{ii} \leq m_{ij}, \rho_{ii} \leq \rho_{ij}, \beta_{ii} \leq \beta_{ij} \\
&\text{Similarly, } < n_{ii}, \gamma_{ii}, \delta_{ii} > \leq < n_{ij}, \gamma_{ij}, \delta_{ij} > \\
&\therefore n_{ii} \leq n_{ij}, \gamma_{ii}, \leq \gamma_{ij}, \delta_{ii} \leq \delta_{ij}.
\end{aligned}\right\} \quad (1)$$

Let $\tilde{C}_{ij}$ and $\tilde{D}_{ij}$ be the ij[th] elements of $M \oplus N$ and $M \odot N$ respectively.

Then

$$\begin{aligned}
\tilde{C}_{ij} - \tilde{C}_{ii} &= < m_{ij} + n_{ij} - m_{ij}.n_{ij}, \rho_{ij} + \gamma_{ij} - \rho_{ij}.\gamma_{ij}, \beta_{ij} + \delta_{ij} - \beta_{ij}.\delta_{ij} > \\
&\quad - < m_{ii} + n_{ii} - m_{ii}.n_{ii}, \rho_{ii} + \gamma_{ii} - \rho_{ii}.\gamma_{ii}, \beta_{ii} + \delta_{ii} - \beta_{ii}.\delta_{ii} > \\
&= < m_{ij} + n_{ij} - m_{ij}.n_{ij} - m_{ii} - n_{ii} + m_{ii}.n_{ii}, \\
&\quad \rho_{ij} + \gamma_{ij} - \rho_{ij}.\gamma_{ij} - \rho_{ii} - \gamma_{ii} + \rho_{ii}.\gamma_{ii}, \\
&\quad \beta_{ij} + \delta_{ij} - \beta_{ij}.\delta_{ij} - \beta_{ii} - \delta_{ii} + \beta_{ii}.\delta_{ii} > \\
&= < (1 - m_{ii}).(1 - n_{ii}) - (1 - m_{ij}).(1 - n_{ij}), \\
&\quad (1 - \rho_{ii}).(1 - \gamma_{ii}) - (1 - \rho_{ij}).(1 - \gamma_{ij}), \\
&\quad (1 - \beta_{ii}).(1 - \delta_{ii}) - (1 - \beta_{ij}).(1 - \delta_{ij}) >
\end{aligned}$$

From (1), we get

$(1 - m_{ii}).(1 - n_{ii}) - (1 - m_{ij}).(1 - n_{ij}) \geq 0$ as $1 - m_{ii} \geq 1 - m_{ij}$

and $1 - n_{ii} \geq 1 - n_{ij}$ (2)

Similarly, $(1 - \rho_{ii}).(1 - \gamma_{ii}) - (1 - \rho_{ij}).(1 - \gamma_{ij}) \geq 0$ (3)

and $(1 - \beta_{ii}).(1 - \delta_{ii}) - (1 - \beta_{ij}).(1 - \delta_{ij}) \geq 0$ (4)

From (2), (3) and (4), we get $\tilde{C}_{ij} \geq \tilde{C}_{ii}$

Hence, $M \oplus N$ is nearly irreflexive.

Now consider,

$\tilde{D}_{ij} - \tilde{D}_{ii} = < m_{ij}.n_{ij}, \rho_{ij}.\gamma_{ij}, \beta_{ij}.\delta_{ij} > - < m_{ii}.n_{ii}, \rho_{ii}.\gamma_{ii}, \beta_{ii}.\delta_{ii} >$

$= < m_{ij}.n_{ij} - m_{ii}.n_{ii}, \rho_{ij}.\gamma_{ij} - \rho_{ii}.\gamma_{ii}, \beta_{ij}.\delta_{ij} - \beta_{ii}.\delta_{ii} >$

But $m_{ij}.n_{ij} - m_{ii}.n_{ii} \geq 0$, $\rho_{ij}.\gamma_{ij} - \rho_{ii}.\gamma_{ii} \geq 0$ and $\beta_{ij}.\delta_{ij} - \beta_{ii}.\delta_{ii} \geq 0$ (By (1))

$\therefore \tilde{D}_{ij} \geq \tilde{D}_{ii}$

Hence, $M \odot N$ is nearly irreflexive.

(iii) The ij[th] element of $M \oplus M$ is $\tilde{M}_{ij} \oplus \tilde{M}_{ij}$ which is equal to

$< m_{ij} + m_{ij} - m_{ij}.m_{ij}, \rho_{ij} + \rho_{ij} - \rho_{ij}.\ \rho_{ij}, \beta_{ij} + \beta_{ij} - \beta_{ij}.\ \beta_{ij} >$

i.e., $< 2m_{ij} - m_{ij}^2, 2\rho_{ij} - \rho_{ij}^2, 2\beta_{ij} - \beta_{ij}^2 >$

consider

$$2m_{ij} - m_{ij}^2 = m_{ij} + m_{ij}(1 - m_{ij}) \geq m_{ij} \quad (1)$$

$$2\rho_{ij} - \rho_{ij}^2 = \rho_{ij} + \rho_{ij}(1 - \rho_{ij}) \geq \rho_{ij} \quad (2)$$

$$2\beta_{ij} - \beta_{ij}^2 = \beta_{ij} + \beta_{ij}(1 - \beta_{ij}) \geq \beta_{ij} \quad (3)$$

From (1), (2) and (3), we get that

$M \oplus M \geq M$.

(iv) The ij[th] element of $M \odot M$ is $\tilde{M}_{ij} \odot \tilde{M}_{ij}$ which is equal to $< m_{ij}.m_{ij}, \rho_{ij}.\ \rho_{ij},$

$\beta_{ij}.\ \beta_{ij} >$

i.e., $< m_{ij}^2, \rho_{ij}^2, \beta_{ij}^2 >$ which is less than (or) equal to $< m_{ij}, \rho_{ij}, \beta_{ij} >$.

i.e., $m_{ij}^2 \leq m_{ij}$, $\rho_{ij}^2 \leq \rho_{ij}$ and $\beta_{ij}^2 \leq \beta_{ij}$.

Therefore, $M \odot M \leq M$.

Theorem : 6.6

Let M, N and Q be three triangular fuzzy number matrices. If $M \leq N$, then

(1) $M \oplus Q \leq N \oplus Q$

(2) $M \odot Q \leq N \odot Q$ and

(3) $M \oplus N \geq M \vee N \geq M \ominus N$.

Proof

(1) Let $\tilde{D}_{ij}$, $\tilde{E}_{ij}$, $\tilde{F}_{ij}$ and $\tilde{G}_{ij}$ be the ij^{th} elements of $M \oplus Q$, $N \oplus Q$, $M \odot Q$ and $N \odot Q$ respectively.

But the ij^{th} element of $M \oplus Q$, $N \oplus Q$, $M \odot Q$ and $N \odot Q$ is $\tilde{M}_{ij} \oplus \tilde{Q}_{ij}$, $\tilde{N}_{ij} \oplus \tilde{Q}_{ij}$, $\tilde{M}_{ij} \odot \tilde{Q}_{ij}$ and $\tilde{N}_{ij} \odot \tilde{Q}_{ij}$.

Then

$$\tilde{M}_{ij} \oplus \tilde{Q}_{ij} = \tilde{D}_{ij} = < m_{ij} + q_{ij} - m_{ij}.q_{ij},\ \rho_{ij} + \gamma_{ij} - \rho_{ij}.\gamma_{ij},\ \beta_{ij} + \delta_{ij} - \beta_{ij}.\delta_{ij} >$$
$$\tilde{N}_{ij} \oplus \tilde{Q}_{ij} = \tilde{E}_{ij} = < n_{ij} + q_{ij} - n_{ij}.q_{ij},\ \rho_{ij} + \gamma_{ij} - \rho_{ij}.\gamma_{ij},\ \beta_{ij} + \delta_{ij} - \beta_{ij}.\delta_{ij} >$$
$$\tilde{M}_{ij} \odot \tilde{Q}_{ij} = \tilde{F}_{ij} = < m_{ij}.q_{ij},\ \rho_{ij}.\gamma_{ij},\ \beta_{ij}.\delta_{ij} >$$
$$\tilde{N}_{ij} \odot \tilde{Q}_{ij} = \tilde{G}_{ij} = < n_{ij}.q_{ij},\ \rho_{ij}.\gamma_{ij},\ \beta_{ij}.\delta_{ij} >$$

Since $M \leq N$, $m_{ij} \leq n_{ij}$.

$\therefore\ m_{ij}(1 - q_{ij}) \leq n_{ij}(1 - q_{ij})$

$\therefore\ m_{ij} - m_{ij}.q_{ij} \leq n_{ij} - n_{ij}.q_{ij}$

i.e., $m_{ij} + q_{ij} - m_{ij}.q_{ij} \leq n_{ij} + q_{ij} - n_{ij}.q_{ij}$

Thus, we get

$< m_{ij} + q_{ij} - m_{ij}.q_{ij},\ \rho_{ij} + \gamma_{ij} - \rho_{ij}.\gamma_{ij},\ \beta_{ij} + \delta_{ij} - \beta_{ij}.\delta_{ij} > \leq$

$< n_{ij} + q_{ij} - n_{ij}.q_{ij},\ \rho_{ij} + \gamma_{ij} - \rho_{ij}.\gamma_{ij},\ \beta_{ij} + \delta_{ij} - \beta_{ij}.\delta_{ij} >$

i.e., $\tilde{D}_{ij} \leq \tilde{E}_{ij}$ for all i, j

$\therefore\ \tilde{M}_{ij} \oplus \tilde{Q}_{ij} \leq \tilde{N}_{ij} \oplus \tilde{Q}_{ij}$ for all i, j.

Hence, $M \oplus Q \leq N \oplus Q$.

(2) Also $< m_{ij}.q_{ij},\ \rho_{ij}.\gamma_{ij},\ \beta_{ij}.\delta_{ij} > \leq < n_{ij}.q_{ij},\ \rho_{ij}.\gamma_{ij},\ \beta_{ij}.\delta_{ij} >$

i.e., $\tilde{F}_{ij} \leq \tilde{G}_{ij}$ for all i, j.

$\therefore\ \tilde{M}_{ij} \odot \tilde{Q}_{ij} \leq \tilde{N}_{ij} \odot \tilde{Q}_{ij}\ \forall$ i, j

Hence, $M \odot Q \leq N \odot Q$.

(3) Let $\tilde{C}_{ij}$, $\tilde{D}_{ij}$ and $\tilde{E}_{ij}$ be the ij^{th} elements of $M \oplus N$, $M \vee N$ and $M \ominus N$ respectively.

But the ijth elements of $M \oplus N$, $M \vee N$ and $M \ominus N$ are $\tilde{M}_{ij} \oplus \tilde{N}_{ij}$, $\tilde{M}_{ij} \vee \tilde{N}_{ij}$ and $\tilde{M}_{ij} \ominus \tilde{N}_{ij}$.

Then

$$\tilde{M}_{ij} \oplus \tilde{N}_{ij} = \tilde{C}_{ij}$$

$$= \ < m_{ij} + n_{ij} - m_{ij}.n_{ij},\ \rho_{ij} + \gamma_{ij} - \rho_{ij}.\gamma_{ij},\ \beta_{ij} + \delta_{ij} - \beta_{ij}.\delta_{ij} >$$

$$= \begin{cases} < m_{ij} + n_{ij}(1-m_{ij}), \rho_{ij} + \gamma_{ij}(1-\rho_{ij}), \beta_{ij} + \delta_{ij}(1-\beta_{ij}) > \\ \text{which is } \geq \ < m_{ij}, \rho_{ij}, \beta_{ij} > \\ \qquad\qquad \geq \tilde{M}_{ij} \\ < n_{ij} + m_{ij}(1-n_{ij}), \gamma_{ij} + \rho_{ij}(1-\gamma_{ij}), \delta_{ij} + \beta_{ij}(1-\delta_{ij}) > \\ \text{which is } \geq \ < n_{ij}, \gamma_{ij}, \delta_{ij} > \\ \qquad\qquad \geq \tilde{N}_{ij} \end{cases} \quad (1)$$

$$\tilde{M}_{ij} \vee \tilde{N}_{ij} = \tilde{D}_{ij} = \max\{\tilde{M}_{ij}, \tilde{N}_{ij}\}$$

$$\leq \tilde{C}_{ij} \text{ (from (1))}$$

Thus, $\tilde{C}_{ij} \geq \tilde{D}_{ij}$ for all i, j. i.e., $\tilde{M}_{ij} \oplus \tilde{N}_{ij} \geq \tilde{M}_{ij} \vee \tilde{N}_{ij}\ \forall$ i, j.

Hence, $M \oplus N \geq M \vee N$ (2)

Now consider

$$\tilde{E}_{ij} = \tilde{M}_{ij} \ominus \tilde{N}_{ij} = \ < m_{ij} \ominus n_{ij},\ \rho_{ij} \ominus \gamma_{ij},\ \beta_{ij} \ominus \delta_{ij} >$$

where $m_{ij} \ominus n_{ij} = \begin{cases} m_{ij}, & m_{ij} > n_{ij} \\ 0, & m_{ij} \leq n_{ij} \end{cases}$

i.e., $m_{ij} \ominus n_{ij} \leq m_{ij}$

$\leq \max\{m_{ij}, n_{ij}\}$

$\rho_{ij} \ominus \gamma_{ij} \leq \rho_{ij}$

$\leq \max\{\rho_{ij}, \gamma_{ij}\}$ and

$\beta_{ij} \ominus \delta_{ij} \leq \beta_{ij}$

$\leq \max\{\beta_{ij}, \delta_{ij}\}$

$$\therefore \tilde{E}_{ij} \leq \max\{< m_{ij}, \rho_{ij}, \beta_{ij} >, < n_{ij}, \gamma_{ij}, \delta_{ij} >\}$$

$$\leq \max\left\{\tilde{M}_{ij}, \tilde{N}_{ij}\right\}$$

$$\leq \tilde{M}_{ij} \vee \tilde{N}_{ij}$$

Thus, $M \ominus N \leq M \vee N$ (3)

From (2) and (3), we get $M \oplus N \geq M \vee N \geq M \ominus N$.