Chapter 2

CHAPTER – II

THE (m,N) POLICY OF SINGLE SERVER BATCH ARRIVAL MARKOVIAN QUEUEING SYSTEMS WITH SERVER VACATIONS AND BREAKDOWNS UNDER RESTRICTED ADMISSIBILITY

INTRODUCTION

Queueing systems with bi-level control (m,N) policy have been studied extensively in the literature by number of authors (Lee and Park 1997, Lee *et al* .,1998, 2003 and J.C. Ke 2004a). Most of the batch arrival queueing systems under bi-level control policy are analyzed by using decomposition property of vacation queues and they do not involve cost model (or) numerical analysis. Hence in Chapter II to V , the main objectives are

(i) to analyze some general bulk arrival queueing systems with server vacations and early setup and to derive the PGF of the system size by solving the difference differential equations satisfied by the system size probabilities.

(ii) to study the effect of various parameters on system size through numerical results.

(iii) to develop a cost model and a procedure to find the optimal values of the decision variables m and N that minimize the linear average cost and to illustrate the solution procedure through some examples.

Past work regarding queues may be divided into two categories (i) the case of controlling the service and (ii) the case of controlling the arrivals. Regarding the control policy of service, Yadin and Naor (1963) introduced an N-policy for M/M/1 queueing system, which turns the server on whenever N (predetermined value) or more customers present in the system and turns off the server when the system becomes empty. Lee *et al.,* (1994a) successively combined the batch arrival queues with N-policy and later Lee and Srinivasan (1989), Lee *et al.,* (1994b and 1995) studied the behavioral characteristics of batch arrival queues with N-policy and server vacations. But these research works do not involve setup operations.

In many real world production systems, setup operations are needed in several occasions. For example, when the machine changes its production type, the operator of the machine changes the tools and adjusts the machine speed. Sometimes, a setup operation takes several days and is very costly. One way to reduce the setup cost per unit time is to delay the production until some number of raw materials accumulates. But when the setup cost is very high, the operator may not need to wait until the accumulated items reach the usual single threshold N (i.e.) the server can start the setup operation when **m** ($m \leq N$) customers accumulate in the queue. And after the setup, if there are less than N customers in the queue, then the server remains dormant in the system until the number of customers reaches N. If N (or) more customers are in the system, after the setup, the server begins to serve the customers immediately. This policy is called bi-level threshold policy (or) (m, N) policy. This policy is more general than the usual (N,N) policy in which the server starts a setup when N customers have piled up in the queue and then starts his service as soon as the setup is complete.

Lee and Park (1997) introduced the double threshold policy for M/G/1 queues and showed that the policy was more beneficial than the conventional single threshold N policy when the startup cost was excessively high compared with the inventory holding cost. Later Lee, Park and Jeon (2003) have applied the factorization property of vacation queues to derive the queue length for the batch arrival of bi-level control queueing system with or without server vacations. The research works mentioned above do not investigate cases involving both server breakdowns and vacations. In the literature of queueing, most of the papers deal with the queueing system wherein the service stations are reliable (i.e.) do not fail. However in practice, we often meet the case where the service station may fail and can be repaired. The performance of the system may be heavily affected by the server break downs and limited repair capacity. On such situations queueing system with unreliable service stations are worth to investigate from performance prediction view point.

J.C.Ke (2004a) studied a like queue production system under bi-level control policy where an unreliable server operates an (m, N) policy. This paper also applies the stochastic decomposition property to derive the system size distribution and some system characteristics.

Regarding the control policy of arrivals many authors (Rue and Roshen shine 1981, Neuts 1984 and Stidham 1985) deal with the policy, where not all arriving batches are allowed to join the system at all times. Such restrictions may be necessary in many real life situations, particularly in the over saturated queue with arrivals occurring faster than the departures.

J.C.Ke and Pearn(2004) discussed the optimal management policy for heterogeneous arrival queueing systems with server breakdowns and multiple vacations for an M/M/1 queue, and derived the system size distribution and employed the PGF to obtain the system characteristics. But the concept of heterogeneous arrivals is not combined with the double threshold policy, batch arrival queues under server breakdowns and vacations in literature.

In the present Chapter, batch arrival Markovian queueing systems along with server breakdowns and vacations are analysed under bi-level threshold policy for service and restricted admissibility policy for arrivals. The PGF of the system size is obtained through the Chapman-Kolmogorov balanced equations satisfied by the steady state system size probabilities. The PGF is presented in closed form so that various performance measures can be calculated easily. Two types of vacation policies (single and multiple) are considered in two different sections (2.1) and (2.2) of this chapter.

SECTION : 2.1

SINGLE VACATION QUEUING SYSTEM

2.1.1 Mathematical Analysis of the System

I Model Description

The System has the following specifications:

a) Customers arrive in batches in accordance with the time homogeneous Poisson process with group arrival rate λ. The batch size X is a random variable with probability distribution $Pr(X = k) = g_k$, k=1,2,3... (i.e.) the probability that a batch of k units arrive in an infinitesimal interval (t,t+h) is $\lambda\, g_k\, h + O(h)$.

b) Not all arriving batches are allowed to join the system at all times. The probability that an arriving batch is allowed to join the system varies according to the system state which falls into one of the 3-categories namely idle (build up, setup, dormant and vacation) or busy or breakdown period. r_1 $(0 \leq r_1 \leq 1)$ denotes the probability that an arriving batch is allowed to join the system while the server is idle and r_i $(i = 2, 3)$ $(0 \leq r_i \leq 1)$ denotes the probability with which an arriving batch joins the system during the server's busy and breakdown periods respectively.

(c) The customers who arrive and join the system form a single waiting line based on the order of the batches. It is further assumed that the customers within a batch are pre-ordered for service. The customers are served one by one according to the order in the queue.

(d) A cycle starts whenever the system empties, the server is deactivated and leaves the system for a vacation of random length V, following an exponential distribution of parameter η.

(e) After returning from the vacation, if the server finds **m** (or) more customers in the system, then the server immediately starts a setup operation of random length D. Otherwise the server joins the system and remains idle in the system until the system size reaches atleast **m** and then starts the setup work (i.e.,) single vacation policy is adopted. The period during which the server remains idle in the system before starting the setup work is called **buildup period**. The setup time is assumed to be an exponentially distributed random variable with mean $(1/\gamma)$.

(f) At the end of the setup period, if the queue length is greater than or equal to N, then the server begins to serve the customers, one at a time. Otherwise the server remains idle (**dormant**) in the system waiting for the queue length to reach at least N, to start the service. The service time of each customer is an independent and identically distributed random variable, following exponential distribution $(1 - e^{-\mu t})$.

(g) The server is subject to breakdowns at any time while working, with Poisson rate α. Whenever the system fails, the server is sent immediately for repair at a repair facility where the repair time is an independent and identically distributed random variable B_r following an exponential distribution $(1 - e^{-\beta t})$. The customer, who is just being served when the server breaks down, joins the head of the waiting line and resumes the service as soon as the server returns from the repair facility. This type of service continues until the system becomes empty again. Thus vacation period, buildup period, setup period and dormant period together represent an idle period and the sum of busy period and the breakdown period gives the completion period. Thus an idle period and completion period constitute a cycle.

This model is denoted by $\mathbf{M^X_{i(m,N)} / M / 1 / BD / SV}$ in which BD denotes breakdown and SV denotes single vacation.

In this model, the first threshold **m**, is used to control the starting condition of a setup operation, and the second threshold **N** is used to control the starting condition of service. If m = N, the model becomes the usual setup time queueing model with N-policy and vacations. It is also assumed that all the stochastic processes involved in the model are independent of each other.

In this section, the PGF of the system size distribution is derived. The following notations are used to write the steady state system size equations.

m,N	:	Bilevel thresholds
λ	:	Group arrival rate
X	:	Group size random variable
Pr (X = k)	:	g_k ($k \geq 1$)
X(z)	:	$\sum_{k=1}^{\infty} g_k z^k$ the PGF of X
$g_n^{(i)}$	:	i-fold convolution of g_n's itself , where $g_n^{(0)}$ is 1, if n = 0, and 0 if n >0
r_i (i = 1 to 3)	:	The probability that the arriving batch is allowed to join the system while the server is idle (buildup, setup, dormant and vacation) busy and in breakdown state respectively.
μ, η, γ and β	:	The respective parameters of the exponential distributions of the random variables namely service time (S), vacation time (V), setup time (D) and repair time (B_r).
$V^*(\theta)$, $D^*(\theta)$ and $B_r^*(\theta)$	:	The Laplace-Stieltjes transforms of the corresponding exponential distributions, (i.e.) $V^*(\theta) = \eta / (\eta+\theta)$; $D^*(\theta) = \nu / (\nu+\theta)$ and $B_r^*(\theta) = \beta / (\beta+\theta)$.
N(t)	:	The number of customers in the system at time t, including the one in service.

Ω (0, 1, 2, 3, 4 and 5) : according as the system is in vacation, buildup, setup, dormant, busy and in breakdown state respectively.

$\lambda_i = \lambda r_i, \; i = 1,2,3$

$Q_n(t) = Pr(N(t) = n, \Omega = 0) \; n \geq 0$

$R_n(t) = Pr(N(t) = n, \Omega = 1) \; 0 \leq n \leq m-1$

$D_n(t) = Pr(N(t) = n, \Omega = 2) \; m \geq n$

$U_n(t) = Pr(N(t) = n, \Omega = 3) \; m \leq n \leq N-1$

$P_n(t) = Pr(N(t) = n, \Omega = 4) \; n \geq 1$

$B_n(t) = Pr(N(t) = n, \Omega = 5) \; n \geq 1.$

$(N(t), \Omega)$ follows a Markov process. Further let Q_n, R_n, D_n, U_n, P_n and B_n denote the steady state probabilities (independent of time t) that there are n customers in the system and the system is in vacation, buildup, setup, dormant, busy and in breakdown state respectively.

II The System Size Distribution

The forward set of Kolmogorov equations satisfied by the steady state probabilities are given by

$$(\lambda_1 + \eta) Q_0 = \mu P_1 \tag{2.1.1}$$

$$(\lambda_1 + \eta) Q_n = \lambda_1 \sum_{k=1}^{n} Q_{n-k}\, g_k, \qquad n \geq 1 \tag{2.1.2}$$

$$\lambda_1 R_0 = \eta Q_0 \tag{2.1.3}$$

$$\lambda_1 R_n = \eta Q_n + \lambda_1 \sum_{k=1}^{n} R_{n-k}\, g_k, \qquad 1 \leq n \leq m-1 \tag{2.1.4}$$

$$(\lambda_1 + \gamma) D_m = \lambda_1 \sum_{k=1}^{m} R_{m-k}\, g_k + \eta Q_m \tag{2.1.5}$$

$$(\lambda_1 + \gamma) D_n = \lambda_1 \sum_{k=n-m+1}^{n} R_{n-k}\, g_k + \lambda_1 \sum_{k=1}^{n-m} D_{n-k}\, g_k + \eta Q_n, \quad n \geq m+1 \tag{2.1.6}$$

$$\lambda_1 U_m = \gamma D_m \tag{2.1.7}$$

$$\lambda_1 U_n = \gamma D_n + \lambda_1 \sum_{k=1}^{n-m} U_{n-k}\, g_k \qquad m+1 \le n \le N-1 \tag{2.1.8}$$

$$(\lambda_2 + \mu + \alpha) P_1 = \beta B_1 + \mu P_2 \tag{2.1.9}$$

$$(\lambda_2 + \mu + \alpha) P_n = \beta B_n + \mu P_{n+1} + \lambda_2 \sum_{k=1}^{n-1} P_{n-k}\, g_k, \qquad 2 \le n \le N-1 \tag{2.1.10}$$

$$(\lambda_2 + \mu + \alpha) P_n = \beta B_n + \mu P_{n+1} + \lambda_2 \sum_{k=1}^{n-1} P_{n-k}\, g_k + \gamma D_n + \lambda_1 \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k, \qquad n \ge N \tag{2.1.11}$$

$$(\lambda_3 + \beta) B_1 = \alpha P_1 \tag{2.1.12}$$

$$(\lambda_3 + \beta) B_n = \alpha P_n + \lambda_3 \sum_{k=1}^{n-1} B_{n-k}\, g_k, \qquad n \ge 2 \tag{2.1.13}$$

III Probability Generating Function

To obtain the system size distribution of the model the following partial PGFs of R_n, D_n, U_n, P_n and B_n are defined.

$$\left.\begin{aligned} R(z) &= \sum_{n=0}^{m-1} R_n z^n ; & D(z) &= \sum_{n=m}^{\infty} D_n z^n \\ U(z) &= \sum_{n=m}^{N-1} U_n z^n ; & P_w(z) &= \sum_{n=1}^{\infty} P_n z^n \\ Q(z) &= \sum_{n=0}^{\infty} Q_n z^n \text{ and} & B(z) &= \sum_{n=1}^{\infty} B_n z^n \end{aligned}\right\} \tag{2.1.14}$$

Multiplying the equations by suitable powers of z and adding the corresponding equations, the partial PGFs are obtained through some algebraic manipulations.

Equations (2.1.1) and (2.1.2) imply

$$Q(z) = \frac{\mu P_1}{\eta} \left(\frac{\eta}{\eta + w_X^1(z)} \right) \tag{2.1.15}$$

where $w_X^i(z) = \lambda_i (1 - X(z))$ i = 1, 2, 3

If α_n denotes the probability that n customers arrive during a single vacation then,

$$\alpha_n = \int_0^\infty \sum_{i=0}^{n} e^{-\lambda_1 t} \frac{(\lambda_1 t)^i}{i!} g_n^{(i)} dV(t)$$

and this gives $V^*\left(w_X^1(z)\right) = \sum_{n=0}^{\infty} \alpha_n z^n = \frac{\eta}{(\eta + w_X^1(z))}$ (Gross and Harris 1985)

Hence $\eta\, Q(z) = \mu P_1 \left[V^*(w_X^1(z))\right]$ (2.1.16)

and $\eta\, Q_n = \mu P_1 \alpha_n \qquad n \geq 0$ (2.1.17)

Q(z) can also be written as $Q(z) = \mu P_1 \left[\frac{1 - V^*(w_X^1(z))}{w_X^1(z)}\right]$ (2.1.18)

Equations (2.1.3) and (2.1.4) imply,

$$\lambda_1 R(z) = \eta \sum_{n=0}^{m-1} Q_n z^n + \lambda_1 \sum_{n=1}^{m-1} z^n \sum_{k=1}^{n} R_{n-k}\, g_k$$

Equations (2.1.5) and (2.1.6) imply,

$$(\lambda_1 + v) \quad D(z) = \lambda 1 \sum_{n=m}^{\infty} z^n \sum_{k=n-m+1}^{n} R_{n-k} g_k + \sum_{n=m+1}^{\infty} z^n \sum_{k=1}^{n-m} D_{n-k}\, g_k + \eta \sum_{n=m}^{\infty} Q_n z^n$$

By adding the above two equations, we have,

$$(\gamma + w_X^1(z))\, D(z) = -\, w_X^1(z)\, R(z) + \eta\, Q(z)$$

Substituting for Q(z) from equation (2.1.16), the above equation becomes

$$D(z) = \frac{\mu P_1}{\gamma} D^*(w_X^1(z)) \left[V^*(w_X^1(z)) - \frac{w_X^1(z) R(z)}{\mu P_1}\right] \tag{2.1.19}$$

$$\text{(or) } D(z) = \mu P_1 \left[\frac{1 - D^*(w_X^1(z))}{w_X^1(z)}\right]\left[V^*(w_X^1(z)) - \frac{w_X^1(z) R(z)}{\mu P_1}\right] \tag{2.1.20}$$

Since $h_k = \int_0^\infty \sum_{i=1}^{k} e^{-\lambda_1 t} \frac{(\lambda_1 t)^i}{i!} g_k^{(i)}\, dD(t)$ gives the probability that k customers arrive during a setup period,

$$D^*(w_X^1(z)) \quad = \quad \frac{\gamma}{\gamma + w_X^1(z)} \quad = \quad \sum_{k=0}^{\infty} h_k \, z^k \qquad (2.1.21)$$

For further simplification of D(z) the following result of Lee *et al.* ,(1995) is used.

Theorem: 2.1.1

$$\text{Let } \pi_0 = 1, \ \pi_n = \sum_{k=1}^{n} \pi_{n-k} \, g_k \qquad \text{and} \qquad (2.1.22)$$

$$\psi_0 = \alpha_0, \ \psi_n = \sum_{i=0}^{n} \alpha_i \, \pi_{n-i} \qquad (1 \le n \le m-1) \qquad (2.1.23)$$

$$\text{then} \quad \lambda_1 R(z) \ = \ \mu \, P_1 \sum_{n=0}^{m-1} \psi_n \, z^n \qquad (2.1.24)$$

Proof

Using, the equations (2.1.3), (2.1.4), (2.1.17) and (2.1.23), it can be shown by recursion that $\lambda_1 R_n = \mu P_1 \psi_n \quad 0 \le n \le m-1$.

$$\text{Thus} \qquad R(z) \ = \ \mu P_1 (1/\lambda_1) \sum_{n=0}^{m-1} \psi_n \, z^n \ = \ \mu P_1 \, \psi(z) \qquad (2.1.25)$$

$$\text{where} \qquad \psi(z) \ = \ (1/\lambda_1) \sum_{n=0}^{m-1} \psi_n \, z^n$$

Corollary: For $(0 \le n \le m-1)$

$$\text{(i)} \qquad \sum_{i=0}^{n-1} \psi_i \, g_{n-i} \ = \ \sum_{i=0}^{n-1} \alpha_i \, \pi_{n-i}$$

$$\text{(ii)} \qquad \alpha_n + \sum_{i=0}^{n-1} \psi_i \, g_{n-i} \ = \ \psi_n$$

Proof

Substituting for ψ_{n-k} from (2.1.23), and using the definition of π_n

$$\sum_{k=1}^{n} g_k \, \psi_{n-k} \ = \ \sum_{k=1}^{n} g_k \left[\sum_{i=0}^{n-k} \alpha_i \, \pi_{n-k-i} \right]$$

$$= \sum_{k=0}^{n-1} \alpha_k \sum_{i=1}^{n-k} g_i \, \pi_{n-k-i} = \sum_{k=0}^{n-1} \alpha_k \, \pi_{n-k} \qquad \text{(from 2.1.22)}$$

Hence

$$\alpha_n + \sum_{i=0}^{n-1} \psi_i \, g_{n-i} \quad = \quad \sum_{k=0}^{n} \alpha_k \, \pi_{n-k} \quad = \quad \psi_n$$

Thus equation (2.1.20) can be written as

$$D(z) \quad = \quad \mu \, P_1 \left[\frac{1 - D^*(w_X^1(z))}{w_X^1(z)} \right] \left[V^*(w_X^1(z)) - w_X^1(z) \, \psi(z) \right] \qquad (2.1.26)$$

$$\text{(or)} \;\; D(z) \; = \; \left(\frac{\mu P_1}{\nu} \right) \left(D^*(w_X^1(z)) \right) \left(V^*(w_X^1(z)) - w_X^1(z) \, \psi(z) \right) \qquad (2.1.27)$$

Next equations (2.1.12) and (2.1.13) imply

$$B(z) \quad = \quad \frac{\alpha}{(\beta + w_X^3(z))} \, P_W(z) \quad = \quad \frac{\alpha}{\beta} \, B_r^*(w_X^3(z)) \, P_w(z) \qquad (2.1.28)$$

where $B_r^* \, (w_X^3(z)) \; = \; \dfrac{\beta}{(\beta + w_X^3(z))}$

By using equations (2.1.7) to (2.1.11), $P_w(z)$ is calculated as

$$(\lambda_2 + \mu + \alpha) \, P_w(z) + \lambda_1 \, U(z) \; = \; \beta \, B(z) + (\mu / z) \, (P_w(z) - zP_1) + \lambda_2 \, X(z) \, P_w(z)$$

$$+ \, \gamma \, D(z) + \lambda_1 \, X(z) \, U(z) \qquad (2.1.29)$$

Substituting for D(z) and B(z) from equations (2.1.27) and (2.1.28) and simplifying, the above equation leads to

$$\left[\mu(z-1) + z \left(w_X^2(z) + \alpha \left(1 - B_r^* \, (w_X^3(z)) \right) \right) \right] Pw(z) \; = \; z \left[\gamma \, D(z) - w_X^1(z) \, U(z) - \mu \, P_1 \right]$$

$$= z \, \mu P_1 \left[\left(D^* \left(w_X^1(z) \right) V^* \left(w_X^1(z) \right) - 1 \right) - w_X^1(z) \left(D^* \left(w_X^1(z) \right) \psi(z) + U(z)(1/\mu P_1) \right) \right]$$

$$(2.1.30)$$

For further simplification the following Theorem (2.1.2) is used.

Theorem : 2.1.2

(i) $$\left[V^*(w_X^1(z)) - w_X^1(z)\psi(z)\right] = \sum_{n=m}^{\infty} \xi_n z^n \qquad (2.1.31)$$

where $\xi_n = \alpha_n + \sum_{i=0}^{m-1} \psi_i \, g_{n-i}$

(ii) $$D^*\left(w_X^1(z)\right)\left(V^*\left(w_X^1(z)\right) - w_X^1(z)\,\psi(z)\right) = \sum_{k=m}^{\infty} z^k \sum_{i=m}^{k} \xi_i \, h_{k-i} \qquad (2.1.32)$$

where α_n denotes the probability that n customers arrive during a vacation and $\psi_0 = \alpha_0, \quad \psi_n = \sum_{i=0}^{n} \alpha_i \, \pi_{n-i} \qquad (0 \le n \le m-1)$

Proof

$V^*\left(w_X^1(z)\right) - w_X^1(z)\,\psi(z)$

$$= \left(\sum_{n=0}^{\infty} \alpha_n z^n - \sum_{n=0}^{m-1} \psi_n z^n\right) + \left(\sum_{k=1}^{\infty} g_k z^k\right)\left(\sum_{n=0}^{m-1} \psi_n z^n\right)$$

$$= \left(\sum_{n=0}^{\infty} \alpha_n z^n - \sum_{n=0}^{m-1} \psi_n z^n\right) + \left(\sum_{n=1}^{m-1} z^n \sum_{i=0}^{n-1} \psi_i \, g_{n-i} + \sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \psi_i \, g_{n-i}\right)$$

$$= \sum_{n=0}^{m-1}\left(\alpha_n + \sum_{i=0}^{n-1} \psi_i \, g_{n-i}\right) z^n - \sum_{n=0}^{m-1} \psi_n z^n + \sum_{n=m}^{\infty} z^n \left(\alpha_n + \sum_{i=0}^{m-1} \psi_i \, g_{n-i}\right) \qquad (2.1.33)$$

$= \sum_{n=m}^{\infty} \xi_n z^n$ using the corollary (ii) of Theorem (2.1.1)

where $\xi_n = \alpha_n + \sum_{i=0}^{m-1} \psi_i \, g_{n-i}$

Thus $$D^*\left(w_X^1(z)\right)\left(V^*\left(w_X^1(z)\right) - w_X^1(z)\,\psi(z)\right) = \left(\sum_{k=0}^{\infty} h_k z^k\right)\left(\sum_{n=m}^{\infty} \xi_n z^n\right)$$

$$= \sum_{k=m}^{\infty} z^k \sum_{i=m}^{k} \xi_i \, h_{k-i}$$

To obtain the PGF U(z) of U_n's, equations (2.1.7) and (2.1.8) are used recursively which result in

$$\lambda_1 U_n = \gamma \sum_{k=m}^{n} D_k \pi_{n-k} \qquad (m \le n \le N-1)$$

where D_k = co-efficient of z^k in D(z)

$$= (\mu P_1 / \gamma) \text{ coefficient of } z^k \text{ in } D^*\left(w_X^1(z)\right)\left(V^*\left(w_X^1(z)\right) - w_X^1(z)\psi(z)\right)$$

$$= (\mu P_1 / \gamma) \sum_{i=m}^{k} \xi_i h_{k-i} \qquad \text{from Theorem (2.1.2 (ii))}$$

Thus, $\lambda_1 U_n = \mu P_1 \phi_n^S$

$$\text{where } \phi_n^S = \sum_{k=m}^{n} \pi_{n-k} \sum_{i=m}^{k} \xi_i h_{k-i} \text{(2.1.34)}$$

$$= \sum_{k=m}^{n} \xi_k \sum_{i=0}^{n-k} h_i \pi_{n-k-i}$$

$$\text{(i.e.)} \quad U(z) = \mu P_1 \sum_{n=m}^{N-1} \frac{\phi_n^S z^n}{\lambda_1} = \mu P_1 \phi^S(z) \qquad (2.1.34)$$

Substituting for U(z), the equation (2.1.30), yields

$$\left[\mu(z-1) + z\left(w_X^2(z) + \alpha(1 - B_r^*(w_X^3(z)))\right)\right] P_w(z)$$

$$= -z\mu P_1 w_X^1(z)\left[\left(\frac{1 - D^*(w_X^1(z))\, V^*(w_X^1(z))}{w_X^1(z)}\right) + D^*(w_X^1(z))\psi(z) + \phi^S(z)\right] \qquad (2.1.35)$$

Thus the total PGF $\mathbf{P}^S_{(m,N)}(z)$ is given by

$$\mathbf{P}^S_{(m,N)}(z) = P_I(z) + P_w(z) + B(z)$$

where $P_I(z)$ = PGF of the system size when the server is idle

$$= Q(z) + R(z) + D(z) + U(z)$$

By adding equations (2.1.18), (2.1.25), (2.1.26) and (2.1.34)

$$P_I(z) = \mu P_1 I_S(z)$$

$$\text{where } I_S(z) = \left[\frac{1 - V^*(w_X^1(z))\, D^*(w_X^1(z))}{w_X^1(z)} + D^*(w_X^1(z))\psi(z) + \phi^S(z)\right] \qquad (2.1.35a)$$

and using (2.1.28), $P_w(z) + B(z) = \left(1+(\alpha/\beta)B_r^* \, w_X^3(z)\right)P_w(z)$

Thus by substituting for $P_w(z)$ from equation (2.1.35), the total PGF,

$$\mathbf{P}^S_{(m,N)}(z) = \mu P_1 \, I_S(z)\left[1 - \frac{z\, w_X^1(z)(1+(\alpha/\beta)B_r^*(w_X^3(z)))}{\mu(z-1)+z\left(w_X^2(z)+(\alpha/\beta)\, w_X^3(z)\, B_r^*(w_X^3(z))\right)}\right] \tag{2.1.36}$$

The value of μP_1 can be calculated by using the normalizing condition $P^S_{(m,N)}(1) = 1$.

IV Performance Measures

Let the steady state system size probabilities P_v, P_{build}, P_{set}, P_{dor}, P_I, P_{busy} and P_{Br} denote the probability that the server is in vacation, buildup, setup, dormant, busy and in breakdown state respectively.

Then (i) $P_v = \lim_{z\to 1} Q(z) = Q(1) = (\mu P_1/\eta)$

(ii) $P_{build} = \lim_{z\to 1} \mu P_1 \psi(z) = \mu P_1 \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda_1}$

(iii) $P_{set} = (\mu P_1/\gamma)$

(iv) $P_{dor} = \lim_{z\to 1} \mu P_1 Q_S(z) = \mu P_1 \sum_{n=m}^{N-1} \frac{\phi_n^S}{\lambda_1}$

(v) $P_I = \lim_{z\to 1} P_I(z) = \mu P_1 \, D_S(m, N)$

(vi) $P_{busy} = \lim_{z\to 1} P_w(z) = \frac{\mu P_1 \rho_1}{(1-\rho_{23}^{br})} D_S(m, N)$

and (vii) $P_{Br} = \lim_{z\to 1} B(z) = (\alpha/\beta) P_{busy}$

Then the normalizing condition $P^S_{(m,N)}(1) = 1$ implies

$$\mu P_1 = \frac{1-\rho_{23}^{br}}{[1-(\rho_{23}^{br}-\rho_1^{br})]} \frac{1}{D_S(m,N)} = \frac{R}{D_S(m,N)} \tag{2.1.37}$$

where $D_S(m, N) = I_S(1) = E(D) + E(V) + \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda_1} + \sum_{n=m}^{N-1} \frac{\phi_n^S}{\lambda_1}$ (2.1.38)

$$R = \frac{1-\rho_{23}^{br}}{((1-\rho_{23}^{br})+\rho_1^{br})}, \quad \rho_i = (\lambda_i / \mu)\, E(X);\ i = 1, 2, 3, \ldots$$

$$\rho_1^{br} = \rho_1 (1 + (\alpha / \beta)) \text{ and } \rho_{23}^{br} = \rho_2 + \rho_3 (\alpha / \beta)$$

Thus, $P_{busy} = \dfrac{\rho_1}{1 - \rho_{23}^{br} + \rho_1^{br}}$

By substituting for μP_1 from equation (2.1.37) the equation (2.1.36) can be written as

$$\mathbf{P^S_{(m,N)}(z)} = R \left[1 - \frac{z\, w_X^1(z)\left(1+(\alpha/\beta)\left(B_r^*\left(w_X^3(z)\right)\right)\right)}{\mu(z-1)+z\left(w_X^2(z)+(\alpha/\beta)\; w_X^3(z) B_r^*\left(w_X^3(z)\right)\right)} \right] \frac{I_S(z)}{I_S(1)} \quad (2.1.39)$$

where $I_S(z)$ is as in (2.1.35a)

V Decomposition Property

Equation (2.1.39) implies that under the condition $\rho_{23}^{br} < 1$, the total PGF of the system size probabilities of the system is the product of the PGF of two random variables one of which is

$$P_{M_i^X/M/1/BD}(z) = \frac{(1-\rho_{23}^{br})}{(1-\rho_{23}^{br}+\rho_1^{br})} \left[1 - \frac{z\, w_X^1(z)\left(1+\frac{\alpha}{\beta}\left(B_r^*\left(w_X^3(z)\right)\right)\right)}{\mu(z-1)+z\left(w_X^2(z)+\frac{\alpha}{\beta} w_X^3(z) B_r^*\left(w_X^3(z)\right)\right)} \right]$$

This gives the PGF of the system size for the batch arrival $M_i^X/M/1/BD$ queueing model with heterogeneous arrivals and unreliable server and $\delta(z) = \dfrac{I_S(z)}{I_S(1)}$ gives the PGF of the conditional system size distribution during the idle period (vacation + buildup + setup + dormant).

VI Expected System Size

Let L_v, L_{build}, L_{set}, L_{dor}, L_{busy} and L_{Br} denote the expected system size when the server is in vacation, buildup, setup, dormant, busy and in breakdown state respectively. Then

(i) $L_v = \frac{d}{dz}(Q(z))_{z=1} = \mu P_1 \lambda_1 E(X)(E(V^2)/2)$

(ii) $L_{build} = \frac{d}{dz}(R(z))_{z=1} = \mu P_1 \sum_{n=0}^{m-1} n\psi_n (1/\lambda_1)$

(iii) $L_{set} = \frac{d}{dz}(D(z))_{z=1} = \mu P_1 \lambda_1 E(X)\left((E(D^2)/2) + E(D)\left(E(V) + \psi(1)\right)\right)$

(iv) $L_{dor} = \frac{d}{dz}(U(z))_{z=1} = \mu P_1 \sum_{n=m}^{N-1} n\,\phi_n^S (1/\lambda_1)$

(v) $L_{Br} = \frac{d}{dz}(B(z))_{z=1} = \frac{\alpha}{\beta}\left[L_{busy} + \lambda_3 (E(X)/\beta) P_{busy}\right]$ and

(vi) $L_{busy} = \frac{d}{dz}(P_w(z))_{z=1}$

$$= P_{busy} + \frac{\mu P_1 \rho_1}{(1-\rho_{23}^{br})} L_S(m,N)$$

$$+ \frac{P_1 D_S(m,N)}{2\mu(1-\rho_{23}^{br})^2}\left[\lambda_1 \mu E(X(X-1)) + 2\lambda_1 E(X)(\alpha(\lambda_3 E(X))^2 (E(B_r^2)/2) + \mu\rho_{23}^{br})\right]$$

where $L_S(m,N) = L_0 + \lambda_1 E(X)E(D)\psi(1) + \sum_{n=0}^{m-1} \frac{n\psi_n}{\lambda_1} + \sum_{n=m}^{N-1} \frac{n\phi_n^S}{\lambda_1}$ (2.1.40)

with $L_0 = \lambda_1 E(X)\left((E(D^2)/2) + E(D)E(V) + (E(V^2)/2)\right)$ (2.1.41)

Let $\mathbf{L^S_{(m,N)}}$ denote the expected system size of $M^X_{i(m,N)}/M/1/SV/BD$ queueing system under consideration .Then

$\mathbf{L^S_{(m,N)}} = L_v + L_{build} + L_{set} + L_{dor} + L_{busy} + L_{Br}$ implies,

$$\mathbf{L^S_{(m,N)}} = \frac{L_S(m,N)}{D_S(m,N)} + L_1 \quad (2.1.42)$$

where $L_1 = \frac{1}{(1 - \rho_{23}^{br} + \rho_1^{br})}\left[\rho_1^{br} + \frac{\lambda_1 E(X(X-1))E(H) + \lambda_1\lambda_3 (E(X))^2 E(H^2)}{2(1-\rho_{23}^{br})}\right]$ (2.1.43) with $E(H) = (1 / \mu)(1 + (\alpha / \beta)) = E(S)(1 + (\alpha / \beta))$

and $E(H^2) = \alpha E(B_r^2)E(S)(1 + \rho_3 - \rho_2) + E(S^2)(1 + (\alpha / \beta))[(\lambda_2 / \lambda_3) + (\alpha / \beta)]$

L_1 gives the mean system size of classical $M_i^X / M / 1 / BD$ queueing model with unreliable server under restricted admissibility.

VIII Other System Characteristics

Definition

The **busy period** begins, when the system size becomes at least N soon after the setup work (or) at the end of the dormant period (the server starts serving the customers) and it ends, when the system next becomes empty and the server leaves for a vacation.

Let E(Cycle),E(Busy) and E(I) denote the expected length of cycle, busy period and expected idle period . Then the long run fraction of time when the server is in setup state implies, $P_{set} = \dfrac{E(D)}{E(Cycle)} = \mu P_1 E(D)$.

Then by using IV - (iii), (vi) and (v) we get

(i) $E(Cycle) = \dfrac{1}{\mu P_1}$

(ii) $E(Busy) = P_{busy} E(cycle) = D_S(m, N) \dfrac{\rho_1}{(1 - \rho_{23}^{br})}$

(iii) $E(idle) = P_I E(cycle) = D_S(m, N)$.

Let $E(W_s)$ denote the expected waiting time in the system. Then the Little's formula implies

(iv) $E(W_s) = \dfrac{L^s_{(m,N)}}{\lambda a \; E(x)}$

where λ_a denotes the actual arrival rate into the system which is given by,

$$\lambda_a = (\lambda r_1 P_I + \lambda r_2 P_{busy} + \lambda r_3 P_{Br})$$

$$= \lambda R(r_1 + (r_2 + r_3 (\alpha/\beta))) \frac{\rho_1}{(1 - \rho_{23}^{br})}$$

2.1.2 Optimal Management Policy

In this section, the optimal values of m and N that minimize a linear cost function are obtained. To do this, the cost structure that has been widely used in the literature is employed (refer Yadin and Naor 1963, Ke 2003b, and Arumuganathan and Jeyakumar 2005).

C_y = start up cost per cycle

C_{build} = Cost per unit time for keeping the server idle (buildup)

C_h = Holding cost per customer per unit time.

C_{set} = Server set up cost per unit time for the preparatory work of the server before starting the service

C_{dor} = Server standby cost per unit time.

C_{busy} = Cost per unit time for keeping the server on and in operation

C_v = Reward per unit time due to vacation

C_{Br} = Breakdown cost per unit time for a failed server

Let $T_C^S(m, N)$ denote the total average cost per unit time for the system. Then

$$T_C^S(m, N) = \left[\frac{C_y}{E(Cycle)} + C_h L^S_{(m,N)} + C_{build} P_{build} + C_{set} P_{set} + C_{dor} P_{dor} + C_{Br} P_{Br} + C_{busy} P_{busy} - C_v P_v\right] \quad (2.1.44)$$

Substituting for the system performances, we have,

$$T_C^S(m, N) = A_1^S + \frac{1}{D_S(m,N)}\left[A^S + z_S(m) + C_{dor} R \sum_{n=m}^{N-1} \frac{\phi_n^S}{\lambda_1} + C_h \sum_{n=m}^{N-1} \frac{n\phi_n^S}{\lambda_1}\right]$$

Where $A_1^S = (C_{busy} + \frac{\alpha}{\beta} C_{br}) P_{busy} + C_h L_1$

$$A_S = R\,(C_y + C_{set} E(D) - C_v E(v)) + C_h L_0$$

$$Z_s(m) = (R\, C_{build} + C_h \lambda_1 E(X) E(D)) \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda_1} + C_h \sum_{n=0}^{m-1} \frac{n\,\psi_n}{\lambda_1}$$

and L_0, L_1 as in equations (2.1.41) and (2.1.43).

In order to find the optimal control values (m^*, N^*) that minimize $T_C^S(m, N)$, a two-dimensional search over the non-negative integer space must be made. Due to the mathematical complexity it is difficult to prove the convexity or unimodality of the cost function $T_C^S(m, N)$. Thus following the concept of the dynamic optimization (due to Bellman1957, J.C.Ke 2001 2003b, and Lee and Park 1997), we consider the procedure that makes it possible to calculate the optimal thresholds (m^*, N^*).

Let $J(m, k) = \sum_{i=m}^{k} \phi_k$; $M(m, k) = \sum_{i=m}^{k} i\,\phi_i^S$

then $$T_C^S(m, k+1) - T_C^S(m, k) = \frac{\phi_k^S}{\lambda_1 D_S(m, k+1) D_S(m, k)} H_S(m, k) \qquad (2.1.45)$$

where $$H_S(m, k) = C_h \left(k\, \ell_m^S + \sum_{n=m}^{k-1} (k-n) \frac{\phi_n^S}{\lambda_1}\right) + R\, C_{dor}\, \ell_m^S - (A + z_s(m)) \qquad (2.1.46)$$

$$\ell_m^S = E(D) + E(V) + \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda_1}$$

The equation (2.1.45) implies, for a given m, the sign of $H_S(m, k)$ determines whether $T_C^S(m, k)$ increases or decreases with respect to k.

Let n be the first k for which $H_S(m, k) > 0$.

Then $H_S(m, n+1) = H_S(m, n) + C_h \left(\ell_m^S + \sum_{i=m}^{k} \phi_i^S\right) > 0$ implies

$$T_C^S(m, k) > T_C^S(m, n) \text{ for } k > n$$

This means that for a given m, the optimal value $N^*(m)$ of N is given by the first k for which $H_S(m, k) > 0$ and that , once $T_C^S(m, N(m))$ increases with respect to N, it keeps on increasing thereafter. Therefore for a given m, $T_C^S(m, N(m))$ is conditionally unimodal and thereby $N^*(m)$ is conditionally optimal.

Thus , $\mathbf{N^*(m) = \min\{\ k \geq 1\ /\ H_S(m, k) > 0\ \}}$, where $H_S(m.k)$ is given as in equation (2.1.46)

Therefore for each m, $T_C^S(m, N)$ has a relative minimum value at $(m, N^*(m))$. Thus the pair $(m, N^*(m))$ gives the relative optimal policy for a given m. Though it is difficult to prove mathematically that $T_C^S(m, N)$ is convex (or) unimodular, the computer experiments show that, the expected cost function is convex. Thus the optimal (m^*, N^*) can be obtained by using the following algorithm.

Algorithm

Step 1 : Set m = 1, Determine $N^*(m)$ from equation (2.1.46). Calculate $T_C^S(m, N^*(m))$. GO TO Step 2.

step 2 : Calculate $N^*(m+1)$ and $T_C^S(m+1, N^*(m+1))$. GO TO Step 3.

Step 3 : If $T_C^S(m+1, N^*(m+1)) > T_C^S(m, N^*(m))$, STOP.

The optimal threshold is $(m^*, N^*) = (m, N^*(m))$ and $T_C^S(m, N^*(m))$ is the required optimal value. Otherwise set $m = m+1$ and GO TO Step 2.

SECTION : 2.2

MULTIPLE VACATION QUEUING SYSTEM

2.2.1 Mathematical Analysis of System

I Model Description

Model of section (2.2) differs from the model of section (2.1) only in vacation policy. The server leaves for a vacation of random length V, whenever the system empties (as in model I). After returning from the vacation, if the server finds **m** or more customers in the system then he immediately starts the setup operation of random length D. Otherwise the server takes repeated number of vacations, until he finds **m** (or) more customers accumulate in the system. It is assumed that the sequence of vacations $\{V_1, V_2, \ldots\}$ are independent and identically distributed and are denoted by V. In multiple vacation model, the buildup period is 0.

II System Size distributions :

Let $\Omega' = (\Omega - \{1\})$. Then Ω' takes values 0 and 2 to 5 and the state space $(N(t), \Omega')$ follows Markov process. The other notations followed in this section are same as in section (2.1). The steady state system size equations are then given by

$$\lambda_1 Q_0 = \mu P_1 \tag{2.2.1}$$

$$\lambda_1 Q_n = \lambda_1 \sum_{k=1}^{n} Q_{n-k}\, g_k \qquad 1 \le n \le m-1 \tag{2.2.2}$$

$$(\lambda_1 + \eta) Q_n = \lambda_1 \sum_{k=1}^{n} Q_{n-k}\, g_k \qquad n \ge m \tag{2.2.3}$$

$$(\lambda_1 + \gamma) D_m = \eta Q_m \tag{2.2.4}$$

$$(\lambda_1 + \gamma) D_n = \eta Q_n + \lambda_1 \sum_{k=1}^{n-m} D_{n-k}\, g_k \qquad n \ge m+1 \tag{2.2.5}$$

$$\lambda_1 U_m = \gamma D_m \tag{2.2.6}$$

$$\lambda_1 U_n = \gamma D_n + \lambda_1 \sum_{k=1}^{n-m} U_{n-k}\, g_k \qquad m+1 \le n \le N-1 \tag{2.2.7}$$

$$(\lambda_2 + \mu + \alpha) P_1 = \beta B_1 + \mu P_2 \tag{2.2.8}$$

$$(\lambda_2 + \mu + \alpha) P_n = \beta B_n + \mu P_{n+1} + \lambda_2 \sum_{k=1}^{n-1} P_{n-k}\, g_k \quad 2 \le n \le N-1 \tag{2.2.9}$$

$$(\lambda_2 + \mu + \alpha) P_n = \beta B_n + \mu P_{n+1} + \lambda_2 \sum_{k=1}^{n-1} P_{n-k}\, g_k + \gamma D_n + \lambda_1 \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k, \qquad n \ge N \tag{2.2.10}$$

$$(\lambda_3 + \beta) B_1 = \alpha P_1 \tag{2.2.11}$$

$$(\lambda_3 + \beta) B_n = \alpha P_n + \lambda_3 \sum_{k=1}^{n-1} B_{n-k}\, g_k, \qquad n \ge 2 \tag{2.2.12}$$

III Probability Generating Functions

Equations (2.2.6) to (2.2.12) are similar to the equations (2.1.7) to (2.1.13) of section (2.1). Thus following the arguments of section (2.1), these equations lead to

$$\lambda_1 U_n = \gamma \sum_{k=m}^{n} D_k \pi_{n-k} \tag{2.2.13}$$

$$\left[\mu(z-1) + z\left(w_X^2(z) + \frac{\alpha}{\beta} w_X^3(z) B_r^*\left(w_X^3(z)\right)\right)\right] P_w(z)$$

$$= z\,(\gamma D(z) - w_X^1(z)\,U(z) - \mu P_1) \tag{2.2.14}$$

and $$B(z) = \frac{\alpha P_w(z)}{(\beta + w_X^3(z))} = (\alpha / \beta)\, B_r^*\,(w_X^3(z))\, P_w(z) \tag{2.2.15}$$

For further simplification the following procedure is adopted.

Equations (2.2.4) and (2.2.5) give

$$D(z) = \frac{D^*(w_X^1(z))}{\gamma}\left(\eta\, Q(z) - \eta \sum_{n=0}^{m-1} Q_n z^n\right) \tag{2.2.16}$$

and equations (2.2.1) to (2.2.3) imply

$$\lambda_1 Q(z) + \eta\,(Q(z)) - \sum_{n=0}^{m-1} Q_n z^n\,) = \lambda_1\, x(z) Q(z) + \mu P_1$$

which implies

$$\eta\, Q(z) = q(z)\;\; V^*(w_X^1(z)) \tag{2.2.17}$$

where $$q(z) = \mu P_1 + \eta \sum_{n=0}^{m-1} Q_n z^n$$ and

$$V^*(w_X^1(z)) = \frac{\eta}{\eta + w_X^1(z)}$$

By selecting $\delta_0 = \mu P_1 + \eta Q_0$ and $\delta_n = \eta Q_n$ $(1 \le n \le m-1)$

$$q(z) = \sum_{n=0}^{m-1} \delta_n z^n$$

Thus $\eta\, Q(z) = \left(\sum_{n=0}^{\infty} \alpha_n z^n\right)\left(\sum_{n=0}^{m-1} \delta_n z^n\right)$

where α_n denotes the probability that n customers arrive during vacation V.

Thus, $\eta\, Q(z) = \sum_{n=0}^{m-1} z^n \sum_{i=0}^{n} \alpha_i \delta_{n-i} + \sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \delta_i \alpha_{n-i}$

The coefficient of z^n in the above equation implies

$$\eta\, Q_n = \sum_{i=0}^{n} \alpha_i \delta_{n-i} \qquad 0 \le n \le m-1$$

$$\eta\, Q_n = \sum_{i=0}^{m-1} \delta_i \alpha_{n-i} \qquad n \ge m \qquad (2.2.18)$$

For further simplification the following theorem is used.

Theorem : 2.2.1

$$\text{Let } \beta_0 = 1,\ \beta_n = \sum_{j=1}^{n} \frac{\alpha_j \beta_{n-j}}{(1-\alpha_0)} \qquad 1 \le n \le m-1 \qquad (2.2.19)$$

then $\delta_n = \mu P_1 \dfrac{\beta_n}{(1-\alpha_0)}$ where $0 \le n \le m-1$ (2.2.20)

Proof

By definition of δ_0 , $\delta_0 = \mu P_1 + \eta\, Q_0$

and substituting for Q_0, from equation (2.2.18),

$$\delta_0 = \frac{\mu P_1}{(1-\alpha_0)} = \frac{\mu P_1}{(1-\alpha_0)} \beta_0$$

This shows that the result of δ_n is true for n = 0 and

for $1 \le n \le m-1$ $\delta_n = \eta\, Q_n = \sum_{i=0}^{n} \alpha_i \delta_{n-i}$ implies

$$(1-\alpha_0)\, \delta_n = \sum_{k=1}^{n} \alpha_k \delta_{n-k}$$

Thus by induction, substituting for δ_{n-k}

$$(1-\alpha_0)\, \delta_n = \mu P_1 \sum_{k=1}^{n} \alpha_k \frac{\beta_{n-k}}{(1-\alpha_0)} = \mu P_1 \beta_n$$

(i.e.) $\delta_n = \mu P_1 \dfrac{\beta_n}{(1-\alpha_0)} \qquad 0 \le n \le m-1$ and

$$q(z) = \sum_{n=0}^{m-1} \delta_n z^n = \mu P_1 \beta(z), \text{ where } \beta(z) = \sum_{n=0}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)}$$

substituting for q(z) in (2.2.17) and (2.2.16)

$$\eta Q(z) = \mu P_1 V^*(w_X^1(z))\, \beta(z) \tag{2.2.21}$$

and $$D(z) = \left(D^*(w_X^1(z))/\gamma\right)\left((V^*(w_X^1(z))-1)\beta(z)+1\right)\mu P_1 \tag{2.2.22}$$

For further simplification of U_n the following theorem is used.

Theorem : 2.2.2

$$(V^*(w_X^1(z))-1)\beta(z) + 1 = \sum_{n=m}^{\infty} S_n z^n \text{ where } S_n = \sum_{i=0}^{m-1} \frac{\alpha_{n-i}\beta_i}{(1-\alpha_0)} \tag{2.2.23}$$

$$\left(D^*(w_X^1(z))\right)\left((V^*(w_X^1(z))-1)\beta(z)+1\right) = \sum_{n=m}^{\infty} z^n \sum_{i=m}^{n} S_i h_{n-i} \tag{2.2.24}$$

where h_k denotes the probability that k customers arrive during setup period.

Proof :

$$\left((V^*(w_X^1(z))-1)\beta(z) + 1\right) = (1-\beta(z)) + \left(\sum_{n=0}^{\infty} \alpha_n z^n\right)\left(\sum_{n=0}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)}\right)$$

$$= -\frac{\alpha_0}{(1-\alpha_0)} - \left(\sum_{n=1}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)}\right) + \sum_{n=0}^{m-1} z^n \sum_{i=0}^{n} \frac{\alpha_i \beta_{n-i}}{(1-\alpha_0)} + \sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \frac{\alpha_{n-i}\beta_i}{(1-\alpha_0)}$$

$$= -\left(\sum_{n=1}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)}\right) + \sum_{n=1}^{m-1} z^n \left(\sum_{i=1}^{n} \frac{\alpha_i \beta_{n-i}}{(1-\alpha_0)} + \frac{\alpha_0 \beta_n}{(1-\alpha_0)}\right) + \sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \frac{\alpha_{n-i}\beta_i}{(1-\alpha_0)}$$

$$= \sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \frac{\alpha_{n-i}\beta_i}{(1-\alpha_0)} = \sum_{n=m}^{\infty} S_n z^n \quad \text{where } S_n = \sum_{i=0}^{m-1} \frac{\alpha_{n-i}\beta_i}{(1-\alpha_0)}$$

$$\left(D^*(w_X^1(z))\right)\left((V^*(w_X^1(z))-1)\beta(z)+1\right) = \left(\sum_{k=0}^{\infty} h_k z^k\right)\left(\sum_{n=m}^{\infty} S_n z^n\right)$$

$$= \sum_{n=m}^{\infty} z^n \sum_{i=m}^{n} S_i h_{n-i}$$

Now U(z) can be calculated by using equation (2.2.13), namely,

$$\lambda_1 U_n = \gamma \sum_{k=m}^{n} D_k \pi_{n-k} \qquad (m \le n \le N-1)$$

where D_k = co-eff. of z^k in D(z)

(i.e) $= (\mu P_1 / \gamma)$ co-eff. of z^k in $\left(D^*(w_X^1(z))\right)\left((V^*(w_X^1(z))-1)\beta(z)+1\right)$

$$= (\mu P_1/\gamma) \sum_{i=m}^{k} S_i h_{k-i} \text{ from equation (2.2.24)}$$

$$\text{Then } \lambda_1 U_n = \mu P_1 \sum_{k=m}^{n} \pi_{n-k} \left(\sum_{i=m}^{k} S_i h_{k-i} \right)$$

$$= \mu P_1 \sum_{r=m}^{n} S_r \sum_{k=0}^{n-r} h_k \pi_{n-r-k} = \mu P_1 \phi_n^R$$

$$\text{Thus } U(z) = \mu P_1 \sum_{n=m}^{N-1} (\phi_n^R z^n)/\lambda_1 = \mu P_1 \phi^R(z) \qquad (2.2.25)$$

$$\text{where } \phi_n^R = \sum_{r=m}^{n} S_r \sum_{k=0}^{n-r} h_k \pi_{n-r-k}$$

By substituting for D(z) and U(z) from equations (2.2.21) and (2.2.25),

$$\gamma D(z) - w_X^1(z) U(z) - \mu P_1 = -\mu P_1 z\, w_X^1(z)\, I_R(z)$$

$$\text{where } I_R(z) = \left[D^*(w_X^1(z)) \left(\frac{1-V^*(w_X^1(z))}{w_X^1(z)} \right) \beta(z) + \left(\frac{1-D^*(w_X^1(z))}{w_X^1(z)} \right) + \phi^R(z) \right] \qquad (2.2.26)$$

Thus the partial PGF's are obtained as

$$Q(z) = \mu P_1 \left[\frac{1-V^*(w_X^1(z))}{w_X^1(z)} \right] \beta(z)$$

$$D(z) = \left(\frac{1-D^*(w_X^1(z))}{w_X^1(z)} \right) \left((V^*(w_X^1(z))-1)\beta(z)+1 \right) \mu P_1$$

$$U(z) = \mu P_1 \sum_{n=m}^{N-1} \frac{\phi_n^R z^n}{\lambda_1} \qquad (2.2.27)$$

$$B(z) = (\alpha / \beta)\, B_r^*\left(w_X^3(z)\right) P_w(z) \quad \text{and}$$

$$\left[\mu(z-1) + z\,(w_X^2(z) + (\alpha/\beta)\, w_X^3(z))\, B_r^*(w_X^3(z))\right] P_w(z) = -\,\mu P_1\, z\, w_X^1(z)\, I_R(z)$$

If $P_I^R(z)$ denotes the PGF of the system size when the server is idle (vacation + dormant + setup) then

$$P_I^R(z) = Q(z) + D(z) + U(z) = \mu P_1\, I_R(z)$$

Hence the total PGF $\mathbf{P}^R_{(m,N)}(z) = P_I^R(z) + P_w(z) + B(z)$ is given by

$$\mathbf{P}^R_{(m,N)}(z) = \mu P_1\, I_R(z)\left(1 - \frac{z\,(1 + (\alpha/\beta) B_r^*(w_X^3(z)))(w_X^1(z))}{\mu(z-1) + z\,(w_X^2(z) + (\alpha/\beta)\, w_X^3(z) B_r^*(w_X^3(z)))}\right) \quad (2.2.28)$$

Further μP_1 can be calculated by using the normalizing condition.

IV Performance Measures

Let P_v, P_{set}, P_{dor}, P_{busy}, P_{Br} and P_I denote the probability that the server is on vacation, setup, dormant, busy ,breakdown and in idle state respectively then, proceeding as in section (2.1)

(i) $P_v = \mu P_1 \left(E(V) \sum_{n=0}^{m-1} \frac{\beta_n}{(1-\alpha_0)} \right)$

(ii) $P_{set} = \mu P_1\, E(D)$

(iii) $P_{dor} = \mu P_1\, \phi_R(1)$

(iv) $P_{busy} = \frac{\mu P_1 \rho_1}{(1-\rho_{23}^{br})} D_R(m, N)$ and

(v) $P_{Br} = (\alpha/\beta)\, P_{busy}$

where ρ_i and ρ_{23}^{br} as in section (2.1)

and $$D_R(m, N) = I_R(1) = E(D) + E(V) \sum_{n=0}^{m-1} \frac{\beta_n}{1-\alpha_0} + \sum_{n=m}^{N-1} \frac{\phi_n^R}{\lambda_1} \quad (2.2.29)$$

(vi) $P_I = \mu P_1\, D_R(m, N)$

Then the normalizing condition implies

$$\mu P_1 = \frac{(1-\rho_{23}^{br})}{(1-(\rho_{23}^{br}-\rho_{1}^{br}))}\frac{1}{D_R(m,N)} = \frac{R}{D_R(m,N)} \quad (2.2.30)$$

Then the equation (2.2.28) gives,

$$\mathbf{P^R_{(m,N)}(z)} = R\left(1-\frac{z(1+(\alpha/\beta)B_r^*(w_X^3(z)))(w_X^1(z))}{\mu(z-1)+z(w_X^2(z)+(\alpha/\beta)w_X^3(z)B_r^*(w_X^3(z)))}\right)\frac{I_R(z)}{I_R(1)} \quad (2.2.31)$$

where $I_R(z)$ is as in (2.2.26).

V Decomposition Property

Equation (2.2.31) implies the total PGF of the system probabilities of the model is the product of the PGF of two random variables one of which is given by

$$P_{M_i^X/M/1/BD}(z) = R\left[1-\frac{z\left(w_X^1(z)\left(1+(\alpha/\beta)(B_r^*(w_X^3(z))\right)\right)}{\mu(z-1)+z\left(w_X^2(z)+(\alpha/\beta)w_X^3(z)B_r^*(w_X^3(z))\right)}\right]$$

as in section (2.1), and $\delta(z) = I_R(z) / I_R(1)$ gives the PGF of the conditional system size distribution during the idle period (vacation + setup + dormant) under the condition $\rho_{23}^{br} < 1$.

VI Expected System Size

Let L_V, L_{set}, L_{dor}, L_{busy} and L_{Br} denote the expected system size when the server is in vacation, setup, dormant, busy and in breakdown state respectively. Then

$$\text{(i)} \quad L_V = \frac{\mu P_1}{(1-\alpha_0)}\left[\lambda_1 E(X)\frac{E(V^2)}{2}\sum_{n=0}^{m-1}\beta_n + E(V)\sum_{n=0}^{m-1} n\beta_n\right]$$

$$\text{(ii)} \quad L_{set} = \mu P_1 \lambda_1 E(X)\left[\frac{E(D^2)}{2} + E(D)\,E(V)\sum_{n=0}^{m-1}\frac{\beta_n}{1-\alpha_0}\right]$$

(iii) $L_{dor} = \mu P_1 \sum_{n=m}^{N-1} \frac{n\phi_n^R}{\lambda_1}$

(iv) $L_{busy} = P_{busy} + \mu P_1 \frac{\rho_1}{(1-\rho_{23}^{br})} L_R(m, N)$

$$+ \frac{P_1 D_R(m,N)}{2\mu(1-\rho_{23}^{br})^2}\left[\lambda_1\mu E(X(X-1)) + 2\lambda_1 E(X)(\alpha(\lambda_3 E(X))^2 \frac{E(Br^2)}{2} + \mu\rho_{23}^{br})\right]$$

and (v) $L_{Br} = \frac{\alpha}{\beta}\left(L_{busy} + \frac{\lambda_3 E(x) P_{busy}}{\beta}\right)$

where $D_R(m, N)$ is as in (2.2.29) and

$$L_R(m, N) = I_R'(1)$$

$$= \lambda_1 E(X)\left[\frac{E(D^2)}{2} + E(D)\,E(V)\sum_{n=0}^{m-1}\frac{\beta_n}{1-\alpha_0} + \frac{E(V^2)}{2}\sum_{n=0}^{m-1}\frac{\beta_n}{1-\alpha_0}\right.$$

$$\left. + E(V)\sum_{n=0}^{m-1}\frac{n\beta_n}{1-\alpha_0} + \sum_{n=0}^{m-1}\frac{n\phi_n^R}{\lambda_1}\right] \qquad (2.2.32)$$

Let $\mathbf{L^R_{(m,N)}}$ denote the expected system size of $M^X_{i(m,N)}/M/1/MV/BD$ queueing system under consideration .

Then $\mathbf{L^R_{(m,N)}} = L_V + L_{set} + L_{dor} + L_{busy} + L_{Br}$ implies

$$\mathbf{L^R_{(m,N)}} = \frac{L_R(m,N)}{D_R(m,N)} + L_1 \qquad (2.2.33)$$

where L_1 gives the mean system size of the classical $M_i^X/M/1/BD$ model as in (2.1.43) with $E(H) = E(S)\,[1 + (\alpha/\beta)]$;

$$E(H^2) = \alpha E(Br^2)E(S)(1+\rho_3-\rho_2) + E(S^2)[1+(\alpha/\beta)][(\lambda_2/\lambda_3)+(\alpha/\beta)]$$

where $D_R(m, N)$ and $L_R(m, N)$ are as in (2.2.29) and (2.2.32).

VII Other System Characteristics

Let E(Cycle), E(Busy),E(I) and $E(W_s)$ denote the expected length of cycle, busy period, idle period and expected waiting time in the system. Then by using IV- (ii), (iv) and (vi) we get,

(i) $E(\text{Cycle}) = \dfrac{1}{\mu P_1}$

(ii) $E(\text{Busy}) = P_{busy}\, E(\text{cycle}) = D_R(m, N) \dfrac{\rho_1}{(1-\rho_{23}^{br})}$

(iii) $E(\text{idle}) = P_I\, E(\text{cycle}) = D_R(m, N)$

(iv) $E(W_s) = \dfrac{L^R_{(m,N)}}{\lambda a\ E(x)}$ (Using little's formula)

where $\lambda_a = \lambda R(r_1 +(r_2 +r_3\, (\alpha/\beta))) \dfrac{\rho_1}{(1-\rho_{23}^{br})})$

2.2.2 Optimal Management Policy

By following the procedure of section 2.1.2, the optimal threshold values (m^*, N^*) can be obtained for the multiple vacation model also.

Recalling the cost structure as ,C_y (startup cost per cycle), C_{set} (setup cost), C_{dor} (standby cost), C_{busy} (operating cost), C_{br} (breakdown cost), C_v (reward cost) and C_h (holding cost per customer) per unit time.

The total average cost of the multiple vacation model is given by

$$T_C^R(m, N) = [\frac{C_y}{E(\text{Cycle})} + C_{set} P_{set} + C_{dor} P_{dor} + C_{busy} P_{busy} + C_{br} P_{Br} + C_h L^R_{(m,N)} - C_v P_v]$$

Substituting for various measures, the above equation can be written as

$$T_C^R(m, N) = A_1^R + \frac{1}{D_R(m,N)}[A^R + z_R(m) + R\, C_{dor} \sum_{n=m}^{N-1} \frac{\phi_n^R}{\lambda_1} + C_h \sum_{n=m}^{N-1} \frac{n\,\phi_n^R}{\lambda_1}]$$

where $A_1^R = C_h L_1 + (C_h\,(\alpha / \beta) + C_{busy})\, P_{busy}$

$$A = R\,(C_y + E(D)\,C_{set}) + C_h\,L_0\ ,\quad L_0 = \lambda_1\,E(X)\,\frac{E(D^2)}{2}$$

$$z_R(m) = [C_h\,(\lambda_1\,E(X)\,(\frac{E(V^2)}{2} + E(D)\,E(V)) - R\,E(V)\,C_v]\sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)}$$

$$+ C_h\,E(V)\sum_{n=0}^{m-1}\frac{n\,\beta_n}{(1-\alpha_0)}$$

By calculation $T_C^R(m, k+1) - T_C^R(m, k) = \dfrac{\phi_k^R}{\lambda_1 D_R(m,k) D_R(m,k+1)}\,H_R(m, k)$

where $$H_R(m, k) = C_h\,(k\,\ell_m^R + \sum_{n=m}^{k-1}(k-n)\,\frac{\phi_n^R}{\lambda_1}) + R\,C_{dor}\,\ell_m^R$$

$$\ell_m^R = E(D) + E(V) + \sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)}$$

Then the value of first k, for which $H_R(m, k) > 0$ decides the optional policy of the model.

(i.e.) $\mathbf{N_R^*(m) = Min\,\{k \geq 1 \,/\, H_R(m, k) > 0\}}$.

The other procedure follows similarly.

2.3 PARTICULAR CASES

(1) The homogeneous batch arrival (m,N) policy single vacation model

$M^X_{(m,N)}/M/1/SV/BD$.

By letting, $\lambda_i = \lambda$ and i=1,2,3 in equation (2.1.39),

$$P_S(z) = \left[\frac{(1-\rho^{br})(z-1)\,H^*\,w_X(z)}{(z-H^*\,w_X(z))}\right]\frac{I_S(z)}{I_S(1)}$$

where $\rho^{br} = \dfrac{\lambda\,E(X)}{\mu}(1+\dfrac{\alpha}{\beta})$; $H^*\,w_X(z) = \dfrac{\mu}{(\mu + w_X(z)(1+\dfrac{\alpha}{\beta + w_X(z)}))}$

and the mean system size is $L_S = L_1 + \dfrac{L_S(m,N)}{D_S(m,N)}$,

$$\text{where } L_1 = \left[\rho_1^{br} + \frac{\lambda E(X(X-1))E(H) + (\lambda E(X))^2 E(H^2)}{2(1-\rho_1^{br})}\right]$$

(2) When $\alpha = 0$ the PGF of the single vacation reliable queueing system $M^X_{(m,N)}/M/1/SV$ is given by

$$P(z) = \left[\frac{(1-\rho)(z-1)\,S^*\,w_X(z)}{(z-S^*\,w_X(z))}\right]\frac{I_S(z)}{I_S(1)}$$

$$\text{where } S^*\,w_X(z) = \frac{\mu}{(\mu - w_X(z))} \quad ; \quad \rho = \frac{\lambda E(X)}{\mu}$$

(3) Similarly, the PGF and the mean system size of the homogeneous batch arrival (m,N) policy multiple vacation non-reliable server queueing system $M^X_{(m,N)}/M/1/MV/BD$ are obtained from (2.2.28) and (2.2.23)

$$\text{as,} \quad P_R(z) = \left[\frac{(1-\rho^{br})(z-1)\,H^*\,w_X(z)}{(z-H^*\,w_X(z))}\right]\frac{I_R(z)}{I_R(1)}$$

$$\text{where} \quad L_R = L_1 + \frac{L_R(m,N)}{D_R(m,N)}$$

(4) The corresponding reliable multiple vacation model $M^X_{(m,N)}/M/1/MV$ has the PGF as,

$$P_R(z) = \left[\frac{(1-\rho)(z-1)\,S^*\,w_X(z)}{(z-S^*\,w_X(z))}\right]\frac{I_R(z)}{I_R(1)}$$

2.4 NUMERICAL ANALYSIS

In this section some numerical results are obtained to study the effects of various parameters namely (1) Batch arrival rate λ (2) mean setup time $E(D)=1/\nu$ (3) mean vacation time $E(V) = 1/\eta$ (4) breakdown rate α (5) mean repair time $E(B_r) = 1/\beta$ and (6) the probabilities with which the arriving batch joins the system (r_1,r_2,r_3) on the mean system size for single vacation $\left(L^S_{(m,N)}\right)$ and multiple vacation $\left(L^R_{(m,N)}\right)$ queueing models. The batch size X is assumed to follow geometric distribution with mean $E(X) = 1/(1-p)$ and $E(X(X-1) = 2p/(1-p)^2$

The optimum thresholds (m^*, N^*) for (m,N) policy and the minimum average cost per unit time $Tc(m^*,N^*)$ are also obtained for both the models using the procedures given in sections (2.1.2) and (2.2.2). The bi-level policy is compared with the corresponding N policy also. The parameters are so chosen that the stability condition $\rho_{23}^{br} < 1$ obtained in sections 2.1 and 2.2 is satisfied.

The total expected cost values $Tc(m,N)$ per unit time with respect to m and N are presented in table (2.1) and figures (2,1a and 2.1b) for the single vacation model. The optimum values are given by

1) $m^* = 9, N^*=13$ with $Tc^s(m^*,N^*) = 1004.88$ for $r_1= r_2 = r_3=.2$ (fig 2.1a).

2) $m^*=11, N^*=18$ with $Tc^s(m^*,N^*) = 1055.14$ for $(r_1,r_2,r_3)=((.1,.2,.3)$ (in fig.2.1b).

Similarly the table (2.2) and figures(2.2a and 2.2b) show the value for multiple vacation model. Their optimum values are given by

3) $m^* = 11, N^* = 14$ for $Tc^R(m^*,N^*) = 943.03$ for the case $r_1,= r_2 = r_3= .2$.

4) $m^* = 9, N^*=16$ with $Tc^R(m^*,N^*) = 1124.69$ for the case $(r_1,r_2,r_3)=((.3,.2,.1)$.

Fig. (2.1) The expected cost $Tc^S(m,N)$ Vs m and N for single vacation model

Fig. (2.1a) when $r_1= r_2=r_3=.2$

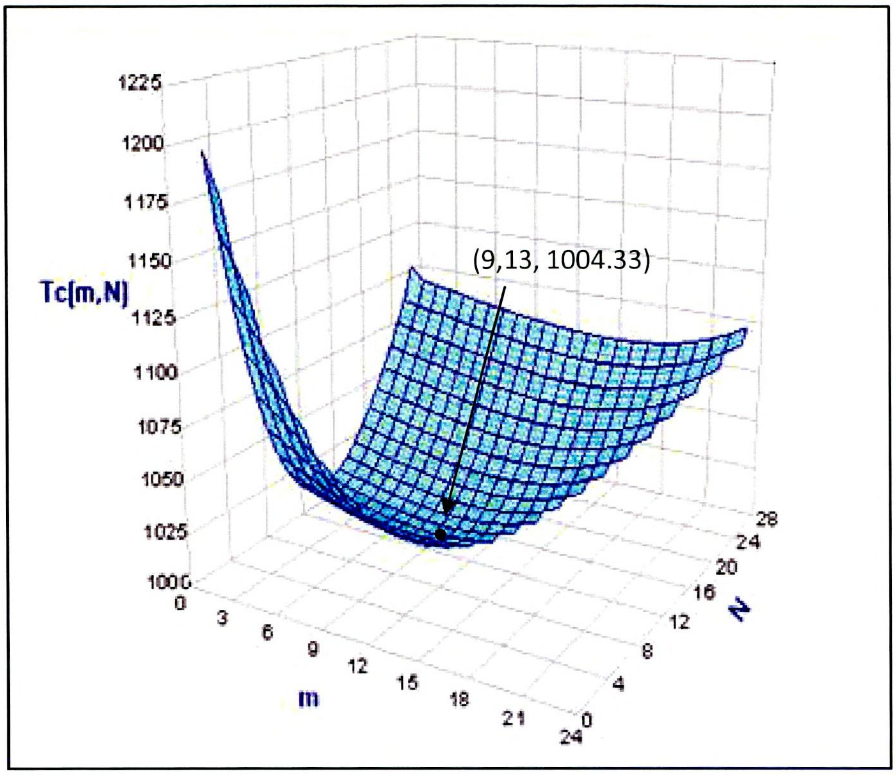

Fig. (2.1b) when $(r_1, r_2,r_3)=(.1,.2,.3)$

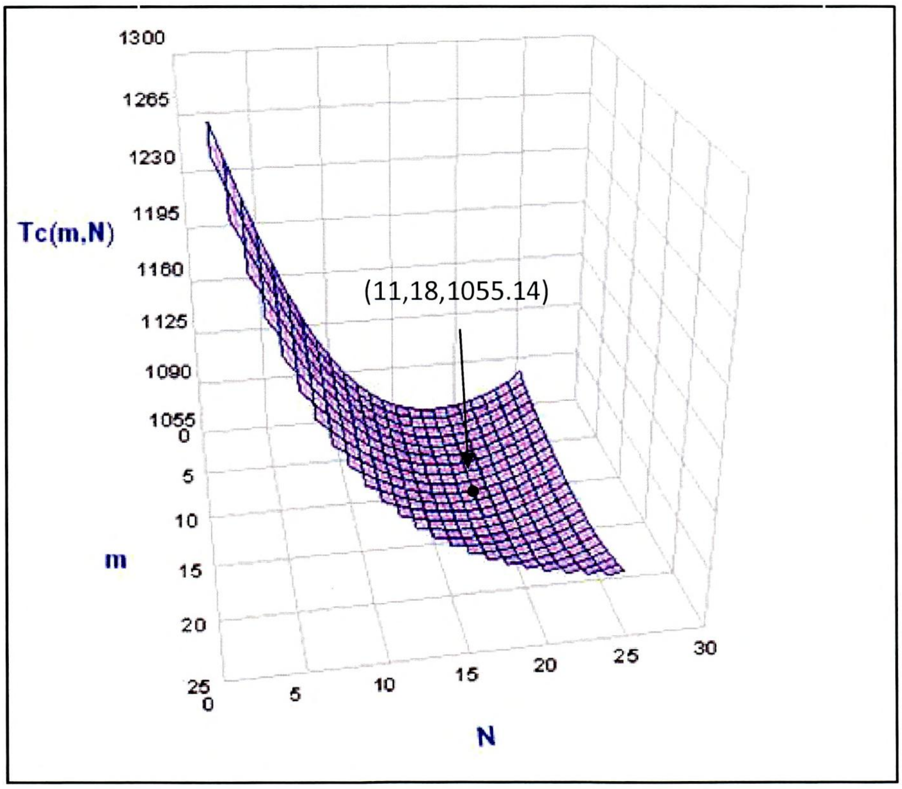

Table (2.1): The total expected cost $Tc^S(m,N)$ Vs m and N for single vacation model

$(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v\ C_{br})$=(35, 8, 100, 100,1000,10000, 8,10); $(p, \lambda, v, \eta, \mu, \alpha, \beta)$=(.75, 1, .3, .3, 2, .04,.4)

m \ N	1	2….	11	12	13	14	15	16	17	18…
1	1234.71	1196.57	1018.52	1015.33	1014.31	1015.2	1017.75	1021.77	1027.07	1033.5
	1286.26	1263.18	1095.43	1086.11	1078.56	1072.64	1068.22	1065.19	1063.4	1062.76
2		1177.82	1016.41	1013.39	1012.49	1013.46	1016.07	1020.14	1025.48	1031.94
: :		1245.73	1093.3	1084.3	1076.98	1071.23	1066.94	1063.99	1062.27	1061.68
7			1008.29	1006.11	1005.67	1006.84	1009.52	1013.58	1018.87	1025.27
			1081.82	1074.99	1069.24	1064.59	1061.08	1058.66	1057.29	1056.93
8			1007.5	1005.48	1005.08	1006.24	1008.86	1012.84	1018.05	1024.38
			1079.5	1073.23	1067.87	1063.5	1060.17	1057.87	1056.57	1056.25
9			1007.11	1005.25	**1004.88**	1006	1008.53	1012.4	1017.49	1023.7
			1077.2	1071.54	1066.61	1062.55	1059.41	1057.23	1056	1055.71
10			1007.19	1005.48	1005.13	1006.17	1008.58	1012.3	1017.24	1023.31
			1074.96	1069.95	1065.49	1061.74	1058.81	1056.76	1055.6	1055.33
11			1007.78	1006.24	1005.89	1006.83	1009.09	1012.62	1017.36	1023.23
			1072.79	1068.48	1064.52	1061.12	1058.41	1056.49	1055.4	**1055.14**
12				1007.59	1007.23	1008.05	1010.11	1013.4	1017.9	1023.54
				1067.14	1063.73	1060.7	1058.23	1056.45	1055.42	1055.18

□ $(r_1, = r_2 = r_3 = .2)$ ■ (yellow) $(r_1, r_2, r_3) = (.1,.2,.3)$

Fig. (2.2) The expected cost Tc^R Vs (m,N) for multiple vacation

Fig. (2.2a) when $r_1 = r_2 = r_3 = .2$

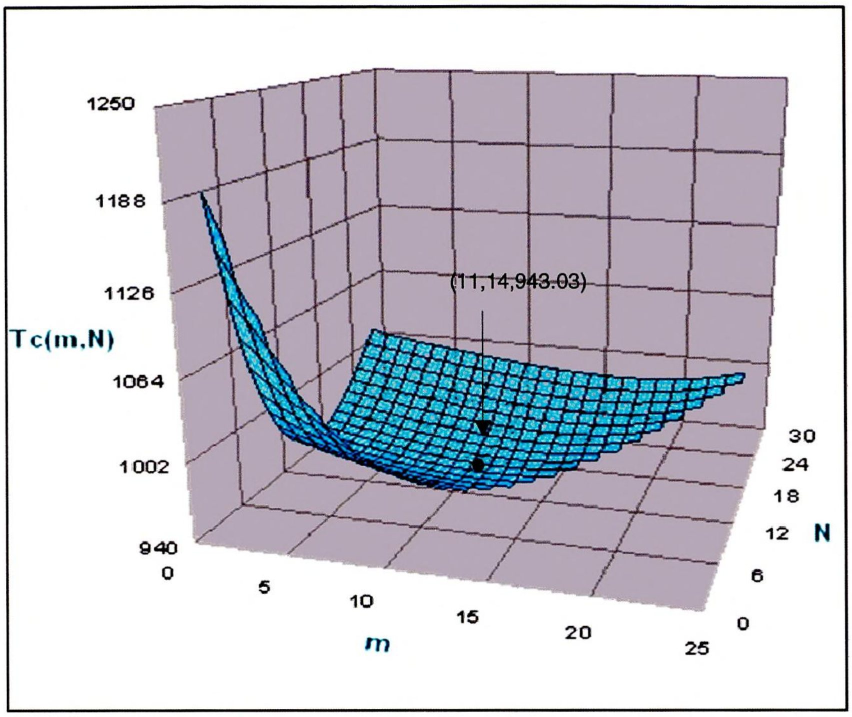

Fig. (2.2b) when $(r_1, r_2, r_3) = (.3, .2, .1)$

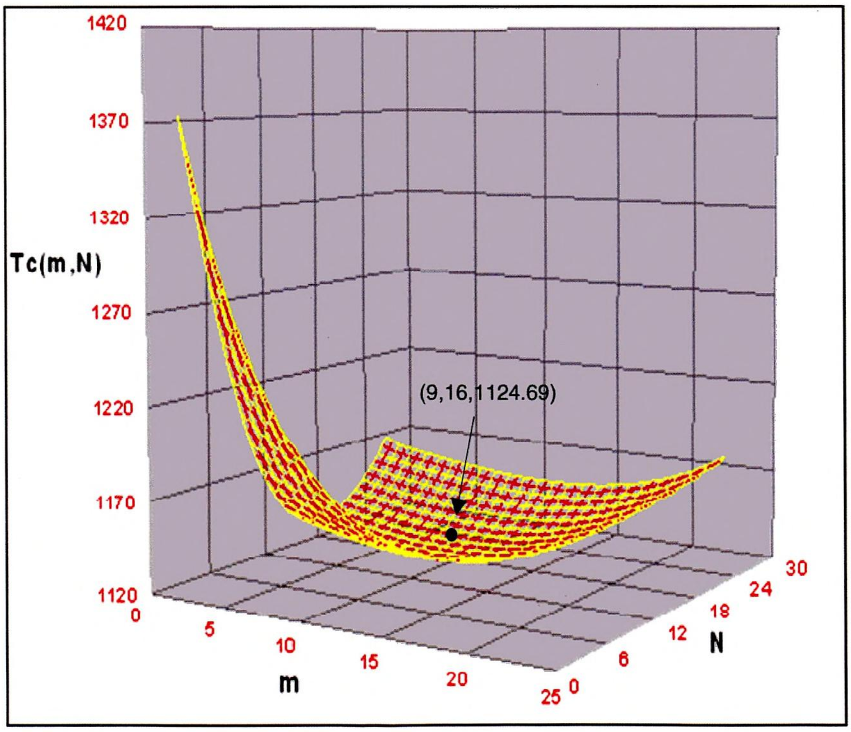

Table (2.2): The total expected cost $Tc^R(m,N)$ Vs m and N for Multiple vacation model

$(C_h, C_{set}\ C_{dor}, C_{busy}, C_y, C_v\ C_{br})$=(30, 100, 100 ,1000 , 10000, 8, 10); $(p, \lambda, v, \eta, \mu, \alpha, \beta)$ = (.75, 1, .5, .5, 2, .05,2.5)

m \ N	1	2….	11	12	13	14	15	16	17	18…
1	1235.5	1191.6	974.92	968.91	965.16	963.38	963.31	964.74	967.47	971.33
	1404.04	1373.08	1162.22	1152.43	1144.94	1139.52	1135.98	1134.1	1133.72	1134.67
2		1158.51	971	965.35	961.88	960.3	960.39	961.95	964.78	968.73
⋮		1338.97	1158.84	1149.59	1142.48	1137.33	1133.97	1132.22	1131.93	1132.95
7			954.31	950.57	948.36	947.62	948.27	950.2	953.31	957.48
			1144.17	1137.99	1132.99	1129.24	1126.78	1125.57	1125.57	1126.72
8			951.83	948.48	946.49	945.87	946.57	948.51	951.6	955.74
			1141.85	1136.36	1131.83	1128.38	1126.09	1124.97	1124.99	1126.12
9			949.74	946.78	945.01	944.49	945.21	947.12	950.17	954.25
			1139.81	1135.03	1130.99	1127.85	1125.73	**1124.69**	1124.72	1125.81
10			948.09	945.53	943.97	943.52	944.24	946.1	949.06	953.05
			1138.07	1134.03	1130.49	1127.67	1125.73	1124.76	1124.79	1125.83
11			946.94	944.78	943.42	**943.03**	943.72	945.48	948.32	952.18
			1136.66	1133.38	1130.37	1127.88	1126.12	1125.21	1125.23	1126.19
12				944.59	943.43	943.09	943.7	945.34	948.02	951.7
				1133.11	1130.64	1128.51	1126.94	1126.09	1126.08	1126.96

□ $(r_1, = r_2 = r_3 = .2)$ ■ (yellow) $(r_1, r_2, r_3) = (.1,.2,.3)$

The optimal mean system size $L^s(m^*,N^*)$ for single vacation model and $L^R(m^*,N^*)$ for multiple vacation model are presented in tables and figures 2.3 and 2.4 for various values of r_1,η and v.

From the figures it is observed that the mean system size decreases as η or v increases (i.e.,) the system size can be reduced by reducing the mean vacation time E(V) = 1/η, or mean setup time E(D)=1/v . The corresponding graphical representations are given in figures (2.3) and (2.4). The cost structure and the parameters considered in Tables (2.3) and (2.4) are same.

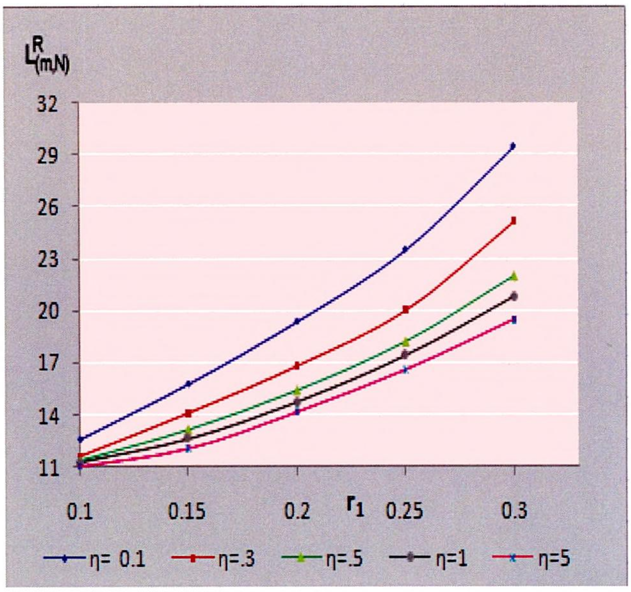

Fig. (2.3) $L^R_{(m^*,N^*)}$ with respect to (r_1) for various η of multiple vacation.

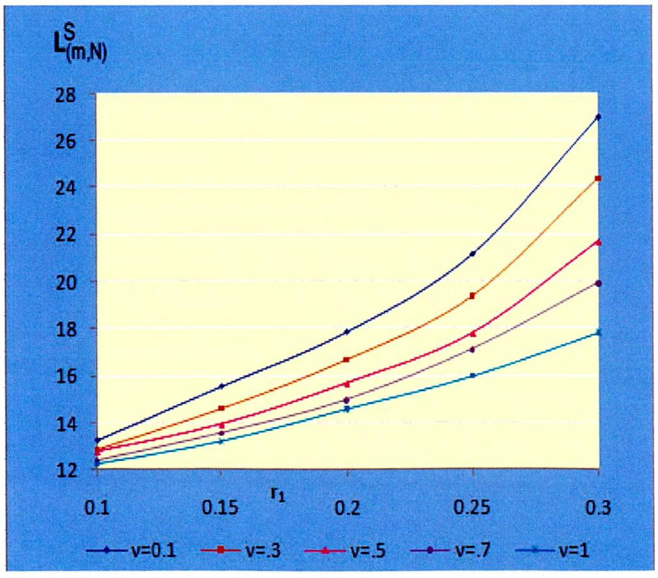

Fig. (2.4) $L^S_{(m^*,N^*)}$ with respect to (r_1) for various v of single vacation.

Table (2.3) Mean system size for various (r_1) with respect to η.

(C_h, C_{build}, C_{set}, $C_{dor,}$, C_{busy}, C_y, C_v, C_{br}) = (60, 8, 100, 200, 500,10000, 8,10)

(p, λ, v , η ,μ, α, β) = (.75, 1,.5, 1, 2, .05,.2)and (r_2, r_3) = (.3,.2)

r_1 \ η	.1	.3	.5	1	5
.1	13.82	12.83	12.77	12.35	12.33
	12.55	11.58	11.41	11.31	11.03
.15	17.62	15.39	15.23	15.17	15.15
	15.73	14.07	13.16	12.65	12.05
.2	20.79	17.31	17.03	16.92	16.89
	19.33	16.79	15.4	14.75	14.15
.25	23.63	18.79	18.09	17.89	17.83
	23.47	17.99	17.19	17.43	16.58
.3	29.43	25.12	21.95	20.8	19.46
	26.22	19.97	19.22	18.91	18.82

■ Single vacation ■ Multiple vacation

Table (2.4) Mean system size for various (r_1) with respect to v

The cost structures and parameters are as in table (2.3) with η = .35

r_1	.1	.3	.5	.7	1
.1	13.24	12.85	12.76	12.37	12.23
	13.15	12.36	12.18	11.99	11.85
.15	15.54	14.59	13.93	13.54	13.2
	14.91	14.42	13.63	13.51	13.15
.2	17.84	16.66	15.68	14.96	14.58
	17.33	15.99	15.5	14.43	14.41
.25	21.14	19.35	17.79	17.09	15.96
	20.50	19.13	17.37	16.85	15.65
.3	26.96	24.34	21.68	19.89	17.82
	26.46	20.95	19.82	19.46	19.23

■ Single vacation ■ Multiple vacation

In Tables (2.5) and (2.6) the mean system size is compared with (i) the breakdown rate (α) for different probabilities r_1 and (ii) repair rate β for different values of r_3. The table values show that the mean system size of both the models, increases with α and decreases as β increases. (i.e.,) more breakdowns will increase the system size and the mean system size can be reduced by reducing the repair time. The cost structure and the parameters of Table (2.3) are used to construct Tables (2.5) and (2.6).

Table (2.5) Mean system size for various (r_1) with respect to α

The values of $(\lambda,\eta) = (.5,.5)$

r_1 \ α	.05	.07	.09	.1
.1	12.77	19.14	32.76	46.32
	12.17	18.28	30.82	43.65
.15	15.23	22.29	36.68	50.73
	14.63	21.35	35.25	48.79
.2	17.03	24.29	39.04	53.69
	16.50	23.80	38.37	52.50
.25	18.10	25.62	40.84	55.54
	18.37	25.99	41.01	55.37
.3	19.21	27.06	42.41	57.18
	19.82	27.74	43.17	57.93

Table (2.6) Mean system size for various (r_3) with respect to β

The values of $(\lambda,\eta,r_1) = (.5,.5,2)$

r_3 \ β	1	1.5	2	2.5
.1	11.05	10.93	10.87	10.83
	10.40	10.29	10.23	10.21
.15	11.18	11.01	10.93	10.88
	10.68	10.52	10.45	10.40
.2	11.18	11.09	10.99	10.93
	10.97	10.76	10.66	10.6
.25	11.20	11.23	11.05	10.97
	11.10	11.13	10.97	10.89

Single vacation (blue)   Multiple vacation (red)

The joint optimum threshold values (m^*,N^*) and the minimum expected cost $Tc(m^*,N^*)$ for two cost structures are summarized in Table (2.7) for different values of (r_1, r_2). From Table (2.7) it can be observed that (m*,N*) decreases and Tc(m*,N*) increases as r_1 or r_2 increases.

Case 1:$(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v, C_{br})$ = (30, 8, 100, 100, 1000, 10000, 8, 10)

Case 2 : $(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v, C_{br})$=(60, 8, 100, 200, 500, 10000, 8, 10)

Table (2.7) Optimum threshold values for various (r_1, r_2)
$(p, \lambda, v, \eta, \mu, \alpha, \beta)$ = (.75, .5, 1, 2, .05,.2)

r_3 = .2	(r_1, r_2)	(.2,.1)	(.2,.3)	(.2,.4)	(.15,.2)	(.2,.2)
Case 1	N*	14	11	6	12	13
	Tc(N*)	877.34	1143.68	2008.51	845.17	961.11
	(m*,N*)	(14,14)	(9,11)	(1,7)	(12,12)	(12,13)
	Tc(m*,N*)	877.34	1142.97	2002.97	845.17	961.09
Case 2		9	7	4	8	8
		953.43	1302.61	2873.89	919.49	1045.65
		(9,9)	(6,7)	(2,5)	(8,8)	(8,9)
		953.43	1301.42	2862.25	829.34	953.43
Case 1		13	10	6	11	12
		884.93	1121.81	1862.34	838.75	959.14
		(13,14)	(8,11)	(1,7)	(11,11)	(11,13)
		884.87	1121.76	1854.82	838.75	959.04
Case 2		8	6	3	7	7
		984.98	1266.72	2588.71	917.38	1051.9
		(8,9)	(4,7)	(1,5)	(7,7)	(6,8)
		980.058	1266.18	2572.32	917.38	1051.81

The conditional optimal values N*(m) of N and Tc(m,N*(m)) for a given m are presented in Table (2.8) and in figures (2.5a)and (2.5b) for both single and multiple vacation models.

Table 2.8:The Conditional Optimum values(m ,N*(m)) and Tc(m,N*(m))
$(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v, C_{br})$=(35, 8, 100, 100,1000, 10000, 8,10);
$(p, \lambda, v, \eta, \mu, \alpha, \beta)$ =(.75, 1,.3, .3, 2, .04,.4); $(r_1 = r_2 = r_3 = .2)$

(m,N*)	(1,13)	(2,13)	**(3,13)**	(4,13)	(5,13)	(6,13)
Tc(m,N*)	1014.31	1012.49	**1010.77**	1009.18	1007.77	1006.58
(m,N*)	(1,18)	(2,18)	**(3,18)**	(4,18)	(5,18)	(6,18)
Tc(m,N*)	963.38	960.3	**957.37**	954.6	952.03	949.69
(m,N*)	(7,13)	(8,13)	**(9,13)**	(10,13)	(11,13)	(12,13)
Tc(m,N*)	1005.67	1005.08	**1004.88**	1005.13	1005.89	1007.23
(m,N*)	(7,18)	(8,18)	**(9,18)**	(10,18)	(11,18)	(12,18)
Tc(m,N*)	947.62	945.87	**944.49**	943.52	943.04	943.09

■ Single vacation ■ Multiple vacation

The Conditional Optimum values (m ,N*(m)) and Tc(m ,N*(m))

Fig. (2.5a) Single vacation Model

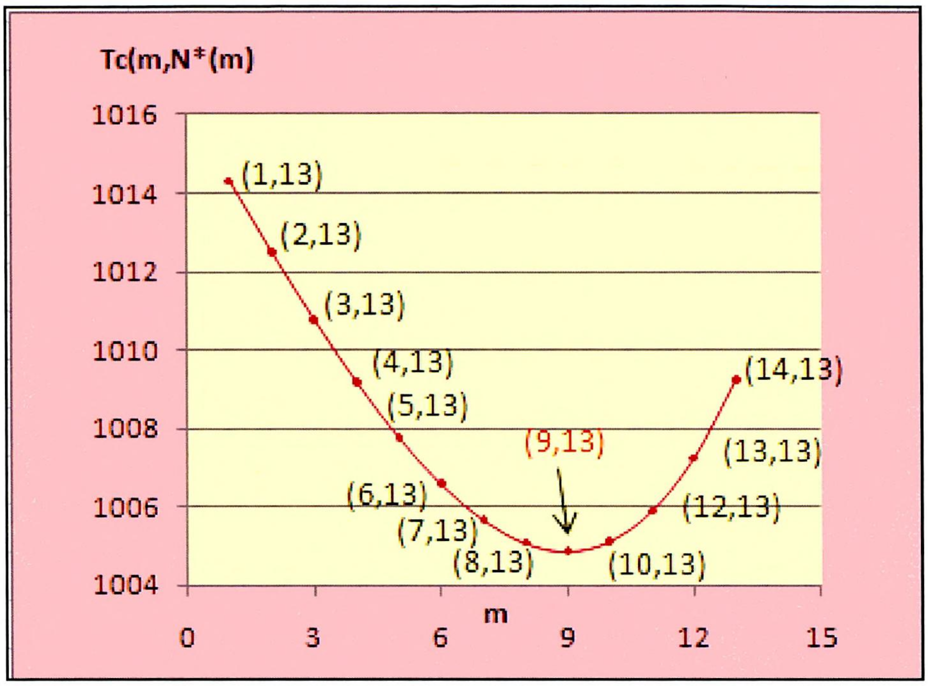

Fig. (2.5b) Multiple vacation Model