1

Preliminaries

In this chapter, the basic definitions of intuitionistic fuzzy sets, intuitionistic fuzzy continuous mappings, intuitionistic fuzzy closed mappings, intuitionistic fuzzy connectedness and some results in intuitionistic fuzzy topological spaces that are used to accomplish the present study are given in detail.

1.1 Intuitionistic Fuzzy Sets

Definition 1.1.1 : [Zadeh, 1965] Let X be a non empty set. A **fuzzy set** A in X can be described in the form

$$A = \{\langle x, \mu_A(x)\rangle \ / \ x \in X\}$$

where the function $\mu_A : X \to [0, 1]$ is called the membership function and $\mu_A(X)$ denotes the degree to which $x \in A$ and $0 \leq \mu_A(x) \leq 1$ for each $x \in X$.

Definition 1.1.2 : [Atanassov, 1986] An **intuitionistic fuzzy set** (IFS) A is an object having the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in A$. Denote by IFS(X), the set of all intuitionistic fuzzy sets in X.

An intuitionistic fuzzy set A in X is simply denoted by $A = \langle x, \mu_A, \nu_A\rangle$ instead of denoting $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$.

Definition 1.1.3 : [Atanassov, 1986] Let A and B be two IFSs of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$. Then,

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$,

(ii) $A = B$ if and only if $A \subseteq B$ and $A \supseteq B$,

(iii) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$,

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}$,

(v) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}$.

The intuitionistic fuzzy sets $0_\sim = \langle x, 0, 1\rangle$ and $1_\sim = \langle x, 1, 0\rangle$ are respectively the empty set and the whole set of X.

Definition 1.1.4 : [Coker, 1997] Let A, B and C be intuitionistic fuzzy sets in X. Then

(i) $A \subseteq B$ and $C \subseteq D \Rightarrow (A \cup C) \subseteq (B \cup D)$ and $(A \cap C) \subseteq (B \cap D)$

(ii) $A \subseteq B$ and $A \subseteq C \Rightarrow A \subseteq (B \cap C)$

(iii) $A \subseteq C$ and $B \subseteq C \Rightarrow (A \cup B) \subseteq C$

(iv) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

(v) $(A \cup B)^c = A^c \cap B^c$

(vi) $(A \cap B)^c = A^c \cup B^c$

(vii) $A \subseteq B \Rightarrow B^c \subseteq A^c$

(viii) $(A^c)^c = A$

(ix) $(0_\sim)^c = 1_\sim$

(x) $(1_\sim)^c = 0_\sim$

Definition 1.1.5 : [Coker, 1997] An **intuitionistic fuzzy topology** (IFT) on X is a family τ of IFSs in X satisfying the following axioms :

(i) $0_\sim, 1_\sim \in \tau$,

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

(iii) $\bigcup G_i \in \tau$ for any family $\{G_i : i \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called intuitionistic fuzzy topological space (IFTS) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS) in X. The complement A^c of an IFOS A in an IFTS (X, τ) is called an intuitionistic fuzzy closed set (IFCS) in X.

Definition 1.1.6 : [Coker, 1997] Let (X, τ) be an IFTS and A, B be intuitionistic fuzzy sets in X. Then the following properties hold :

(i) $int(A) \subseteq A$

(ii) $A \subseteq cl(A)$

(iii) $A \subseteq B \Rightarrow int(A) \subseteq int(B)$

(iv) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$

(v) $int(int(A)) = int(A)$

(vi) $cl(cl(A)) = cl(A)$

(vii) $int(A \cap B) = int(A) \cap int(B)$

(viii) $cl(A \cup B) = cl(A) \cup cl(B)$

(ix) $int(1_\sim) = 1_\sim$

(x) $cl(0_\sim) = 0_\sim$

Definition 1.1.7 : [Coker, 1997] Let A be an IFS in an IFTS (X, τ). Then the **interior** and **closure** of A are defined as

$int(A) = \bigcup \{G \mid G$ is an IFOS in X and $G \subseteq A\}$,

$cl(A) = \bigcap \{K \mid K$ is an IFCS in X and $A \subseteq K\}$.

It is to be noted that for any IFS A in (X, τ), we have $cl(A^c) = (int(A))^c$ and $int(A^c) = (cl(A))^c$.

Definition 1.1.8 : [Joung Kon Jeon, Young Bae Jun and Jin Han Park, 2005] An IFS $A = \langle x, \mu_A, \nu_A\rangle$ in an IFTS (X, τ) is said to be an

(i) **intuitionistic fuzzy semi closed set** (IFSCS) if $int(cl(A)) \subseteq A$,

(ii) **intuitionistic fuzzy pre closed set** (IFPCS) if $cl(int(A)) \subseteq A$,

(iii) **intuitionistic fuzzy regular closed set** (IFRCS) if $cl(int(A)) = A$,

(iv) **intuitionistic fuzzy α closed set** (IFαCS) if $cl(int(cl(A))) \subseteq A$,

(v) **intuitionistic fuzzy β closed set** (IFβCS) if $int(cl(int(A))) \subseteq A$.

Definition 1.1.9 : [Joung Kon Jeon, 2005] An IFS $A = \langle x, \mu_A, \nu_A \rangle$ in an IFTS (X, τ) is said to be an

(i) **intuitionistic fuzzy semi open set** (IFSOS) if $A \subseteq cl(int(A))$,

(ii) **intuitionistic fuzzy pre open set** (IFPOS) if $A \subseteq int(cl(A))$,

(iii) **intuitionistic fuzzy regular open set** (IFROS) if $int(cl(A)) = A$,

(iv) **intuitionistic fuzzy α open set** (IFαOS) if $A \subseteq int(cl(int(A)))$,

(v) **intuitionistic fuzzy β open set** (IFβOS) if $A \subseteq cl(int(cl(A)))$.

Definition 1.1.10 : [Hanafy, 2009] An IFS $A = \langle x, \mu_A, \nu_A \rangle$ in an IFTS (X, τ) is said to be an

(i) intuitionistic fuzzy γ closed set (IFγCS) if $int(cl(A)) \cap cl(int(A)) \subseteq A$,

(ii) intuitionistic fuzzy γ open set (IFγOS) if $A \subseteq int(cl(A)) \cup cl(int(A))$.

Definition 1.1.11 : [Thakur and Rekha Chaturvedi, 2008] An IFS A is an IFTS (X, τ) is said to be an **intuitionistic fuzzy generalized closed set** (IFGCS) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an IFOS in (X, τ).

Definition 1.1.12 : [Jayanthi, 2014_a] Let A be an IFS in an IFTS (X, τ). Then the β **interior** and β **closure** of A are defined as

$\beta\, int(A) = \bigcup \{G / G \text{ is an IF}\beta\text{OS in } X \text{ and } G \subseteq A\}$,

$\beta\, cl(A) = \bigcap \{K / K \text{ is an IF}\beta\text{CS in } X \text{ and } A \subseteq K\}$.

It is to be noted that for any IFS A in (X, τ), we have $\beta cl(A^c) = (\beta int(A))^c = (\beta int(A))^c$ and $\beta int(A^c) = (\beta cl(A))^c$.

Result 1.1.13 : [Saranya and Jayanthi, 2016_a] Let A be an IFS in (X, τ). Then

(i) $\beta\, cl(A) \supseteq A \cup int(cl(int(A)))$,

(ii) $\beta\, int(A) \subseteq A \cap cl(int(cl(A)))$.

Definition 1.1.14 : [Young Bae Jun and Seok-Zun Song, 2005] Let A be an IFS of an IFTS (X, τ). Then the **semi closure** of A and **semi interior** of A are defined as

$sint(A) = \bigcup \{G \,/\, G$ is an IFSOS in X and $G \subseteq A\}$,

$scl(A) = \bigcap \{K \,/\, K$ is an IFSCS in X and $A \subseteq K\}$.

Result 1.1.15 : [Gurcay, Coker and Haydar, 1997] Let A be an IFS in (X, τ), then

$scl(A) = A \bigcup int(cl(A))$,

$sint(A) = A \bigcap cl(int(A))$.

Definition 1.1.16 : [Krsteska and Abbas, 2006] Let (X, τ) be an IFTS and $A = \langle x, \mu_A, \nu_A \rangle$ be an IFS in X. The pre interior of A is denoted by pint(A) and is defined by the union of all fuzzy pre open sets of X which are contained in A. The intersection of all fuzzy pre-closed sets containing A is called the pre-closure of A and is denoted by pcl(A). That is

$pint(A) = \bigcup \{G \,/\, G$ is an IFPOS in X and $G \subseteq A\}$,

$pcl(A) = \bigcap \{K \,/\, K$ is an IFPCS in X and $A \subseteq K\}$.

Definition 1.1.17 : [Thakur and Rekha Chaturvedi, 2006_a] Two IFSs A and B are said to be **q-coincident** ($A_q B$) if and only if there exists an element $x \in X$ such that $\mu_A(x) > \nu_B(x)$ or $\nu_A(x) > \mu_B(x)$.

Definition 1.1.18 : [Thakur and Rekha Chaturvedi, 2006_a] Two IFSs A and B are said to be **not q-coincident** ($A_{\bar{q}} B$) if and only if $A \subseteq B^c$.

Definition 1.1.19 : [Coker and Demirci, 1995] An **intuitionistic fuzzy point** (IFP), written as $p_{(\alpha,\beta)}$ is defined to be an intuitionistic fuzzy set of X given by

$$p_{(\alpha,\beta)}(x) = \begin{cases} (\alpha,\beta) & \text{if } x = p, \\ (0,1) & \text{otherwise.} \end{cases}$$

An intuitionistic fuzzy point $p_{(\alpha,\beta)}$ is said to belong to a set A if $\alpha \le \mu_A$ and $\beta \ge \nu_A$.

Definition 1.1.20 : [Santhi and Jayanthi, 2012] An IFS A in (X, τ) is an **IFQ-set** if int(cl(A)) = cl(int(A)).

Definition 1.1.21 : [Dhavaseelan, Roja and Uma, 2011] An IFS A is said to be an **intuitionistic fuzzy dense** (IFD) in another IFS B in an IFTS (X, τ), if cl(A) = B.

Definition 1.1.22 : [Thakur and Dhavaseelan, 2015] An IFS A in (X, τ) is an **intuitionistic fuzzy nowhere dense set** if there exists no IFOS U such that U $\subseteq$ cl(A), that is int(cl(A)) = $0_\sim$.

Proposition 1.1.23 : [Thakur and Dhavaseelan, 2015] Let A be an IFS in X. If A is an **intuitionistic fuzzy nowhere dense set** in X, the int(A) = $0_\sim$.

Definition 1.1.24 : [Seok Jong Lee and Eun Pyo Lee, 2000] Let $p_{(\alpha,\beta)}$ be an IFP in (X, τ). An IFS A of X is called an **intuitionistic fuzzy neighbourhood** (IFN) of $p_{(\alpha,\beta)}$ if there exists an IFOS B in X such that $p_{(\alpha,\beta)} \in B \subseteq A$.

Definition 1.1.25 : [Jayanthi, 2014_a] Let $p_{(\alpha,\beta)}$ be an IFP in (X, τ). An IFS A of X is called an **intuitionistic fuzzy β neighbourhood** (IFβN) of $p_{(\alpha,\beta)}$ if there exists an IFβOS B in X such that $p_{(\alpha,\beta)} \in B \subseteq A$.

Corollary 1.1.26 : [Coker, 1997] Let A, A_i ($i \in J$) be intuitionistic fuzzy sets in X and B, B_j ($j \in K$) be intuitionistic fuzzy sets in Y and f : X → Y be a mapping. Then

(i) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$,

(ii) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

(iii) $A \subseteq f^{-1}(f(A))$ [If f is injective, then $A = f^{-1}(f(A))$],

(iv) $f(f^{-1}(B)) \subseteq B$ [If f is surjective, then $B = f(f^{-1}(B))$],

(v) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$,

(vi) $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$,

(vii) $f^{-1}(0_\sim) = 0_\sim$,

(viii) $f^{-1}(1_\sim) = 1_\sim$,

(ix) $f^{-1}(B^c) = (f^{-1}(B))^c$.

Definition 1.1.27 : [Coker and Demirci, 1995] Let (X, τ) be an IFTS and $A = \langle x, \mu_A, \nu_A\rangle$ be an IFS in X. Then **intuitionistic fuzzy kernel** of A is the intersection of all IFOSs containing A.

Definition 1.1.28 : [Thakur and Rekha Chaturvedi, 2008] An IFTS (X, τ) is said to be an **intuitionistic fuzzy $T_{1/2}$ space** if every IFGCS is an IFCS in (X, τ).

Definition 1.1.29 : [Coker, 1997] Let X and Y be two non empty sets and $f : X \to Y$ be a mapping. If $A = \{\langle x, (\mu_A(x), \nu_A(x)) / x \in X\rangle\}$ is an IFS in X, then the **image** of A under f, denoted by f(A), is the IFS in Y defined by

$$f(A) = \{\langle y, f(\mu_A)(y), f_-(\nu_A)(y) / y \in Y\rangle\}, \text{ where } f_-(\nu_A) = 1 - f(1 - \nu_A).$$

Definition 1.1.30 : [Coker, 1997] Let X and Y be two non empty sets and $f : X \to Y$ be a mapping. If $B = \{\langle y, (\mu_B(y), \nu_B(y)) / y \in Y\rangle\}$ is an IFS in Y, then the **pre image** of B under f is denoted as $f^{-1}(B)$ and defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x) / x \in X\rangle\}$$

where $f^{-1}(\mu_B(x) = \mu_B(f(x))$ for every $x \in X$.

1.2 Intuitionistic Fuzzy Continuous Mappings

Definition 1.2.1 : [Gurcay, Coker and Haydar, 1997] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an **intuitionistic fuzzy continuous** (IF continuous) **mapping** if $f^{-1}(V)$ is an IFCS in (X, τ) for every IFCS V of (Y, σ).

Definition 1.2.2 : [Joung Kon Jeon, Young Bae Jun and Jin Han Park, 2005] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an

(i) **intuitionistic fuzzy semi continuous** (IFS continuous) **mapping** if $f^{-1}(V)$ is an IFSCS in (X, τ) for every IFCS V of (Y, σ),

(ii) **intuitionistic fuzzy α continuous** (IFα continuous) **mapping** if $f^{-1}(V)$ is an IFαCS in (X, τ) for every IFCS V of (Y, σ),

(iii) **intuitionistic fuzzy pre continuous** (IFP continuous) **mapping** if $f^{-1}(V)$ is an IFPCS in (X, τ) for every IFCS V of (Y, σ).

(iv) **Intuitionistic fuzzy β continuous** (IFβ continuous) **mapping** if $f^{-1}(V)$ is an IFβCS in (X, τ) for every IFCS V of (Y, σ).

Definition 1.2.3 : [Krsteska and Ekici, 2007] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an

(i) **intuitionistic fuzzy contra continuous** (IF contra continuous) **mapping** if $f^{-1}(V)$ is an IFCS in X for every IFOS V in Y,

(ii) **intuitionistic fuzzy contra α continuous** (IF contra α continuous) **mapping** if $f^{-1}(V)$ is an IFαCS in X for every IFOS V in Y,

(iii) **intuitionistic fuzzy contra pre continuous** (IF contra P continuous) **mapping** if $f^{-1}(V)$ is an IFPCS in X for every IFOS V in Y,

(iv) **Intuitionistic fuzzy contra semi continuous** (IF contra S continuous) **mapping** if $f^{-1}(V)$ is an IFSCS in X for every IFOS V in Y,

(v) **Intuitionistic fuzzy contra β continuous** (IF contra β continuous) **mapping** if $f^{-1}(V)$ is an IFβCS in X for every IFOS V in Y.

Definition 1.2.4 : [Roja, Uma, Dhavaseelan, 2012] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an **intuitionistic fuzzy contra generalized continuous** (IF contra G continuous) **mapping** if $f^{-1}(B) \in$ IFGC(X) for each IFOS B in Y.

Definition 1.2.5 : [Hanafy and El-Arish, 2003] Let f : (X, τ) → (Y, σ) be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). The mapping f is called an **intuitionistic fuzzy completely continuous mapping** if $f^{-1}(B)$ is an IFROS in X for each IFOS B in Y.

1.3 Intuitionistic Fuzzy Closed Mappings

Definition 1.3.1 : [Joung Kon Jeon, Young Bae Jun and Jin Han Park, 2005] A mapping f : X → Y is called an **intuitionistic fuzzy closed mapping** (IFCM) if f(A) is an IFCS in Y for each IFCS A in X.

Definition 1.3.2 : [Joung Kon Jeon, Young Bae Jun and Jin Han Park, 2005] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an

(i) **intuitionistic fuzzy pre closed mapping** (IFPCM) if f(A) is an IFPCS in Y for each IFCS A in X,

(ii) **intuitionistic fuzzy α closed mapping** (IFαCM) if f(A) is an IFαCS in Y for each IFCS A in X,

(iii) **intuitionistic fuzzy semi closed mapping** (IFSCM) if f(A) is an IFSCS in Y for each IFCS A in X,

(iv) **intuitionistic fuzzy β closed mapping** (IFβCM) if f(A) is an IFβCS in Y for each IFCS A in X.

Definition 1.3.3 : [Thakur and Jyoti Pandey Bajpai, 2007] A map f : X → Y is called an **intuitionistic fuzzy generalized** (IFG) **closed mapping** if f(A) is an IFGCS in Y for each IFCS A in X.

1.4 Intuitionistic Fuzzy Connectedness

Definition 1.4.1 : [Coker, 1997] An IFTS (X, τ) is said to be an **intuitionistic fuzzy C_5-connected space** if the only IFSs which are both IFOS and IFCS are $0_{\sim}$ and $1_{\sim}$.

Definition 1.4.2 : [Thakur and Rekha Chaturvedi, 2008] An IFTS (X, τ) is said to be an **intuitionistic fuzzy GO-connected space** if the only IFSs which are both IFGOS and IFGCS are $0_{\sim}$ and $1_{\sim}$.

Definition 1.4.3 : [Thakur and Mahima Thakur, 2010] An IFTS (X, τ) is an **intuitionistic fuzzy C_5-connected between two IFSs A and B** if there is no IFOS E in (X, τ) such that A ⊆ E and $E_{q}{}^{c}$ B.