Chapter II

CHAPTER II

REGULAR MATRICES OVER SEMIRINGS

Definition: 2.1

Let A be a matrix in $M_{m,n}(S)$. Consider a matrix $X \in M_{m,n}(S)$ in the equation.

$$A \times A = A \qquad (*)$$

If $(*)$ has a solution X, then X is called a **generalised inverse of A**. Furthermore A is called **regular** if there is a solution of $(*)$.

Example: 2.2

Clearly, J and O are regular in $M_{m,n}(S)$ because JGJ = J and OGO = O, where G is any cell in $M_{m,n}(S)$.

Thus in general, a solution of $(*)$ although it exists, is not necessarily unique.

Example: 2.3

All idempotent matrices in $M_n(S)$ are regular.

The following Theorem is an immediate consequence of definitions of regular matrix and invertible matrix.

Theorem: 2.4

Let A be a matrix in $M_{m,n}(S)$.If $U \in M_m(S)$ and $V \in M_n(S)$ are invertible, then the following are equivalent:

(i) A is regular in $M_{m,n}(S)$;

(ii) UAV is regular in $M_{m,n}(S)$;

(iii) A^t is regular in $M_{n,m}(S)$.

Theorem: 2.5

A matrix A $\in M_{m,n}(S)$ is regular if and only if $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is regular for all matrices B $\in M_{p,q}(S)$.

Definition: 2.6

For matrices A, B $\in M_{m,n}(S)$, we say **A dominates B** (written $B \subseteq A$ or $A \supseteq B$) if $a_{i,j} = 0$ implies $b_{i,j} = 0$ for all i and j.

Definition: 2.7

If $A, B \in M_{m,n}(S)$ and $A \supseteq B$, we define A\B to be the matrix C where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0 \\ a_{i,j} & \text{otherwise.} \end{cases}$$

Definition: 2.8

Define an **upper triangular matrix** Λ_n in $M_n(S)$ by

$$\Lambda_n = [\lambda_{i,j}] = \Big(\sum_{i \le j}^{n} E_{i,j}\Big) \setminus E_{1,n} = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 \\ & 1 & \dots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}$$

Theorem: 2.9

Λ_n is regular in $M_n(S)$ if and only if $n \le 2$.

Proof:

Clearly, Λ_n is regular for $n \le 2$ because $\Lambda_n I_n \Lambda_n = \Lambda_n$. Conversely, assume that Λ_n is regular for some $n \ge 3$. Then there is a nonzero $B \in M_n(S)$ such that $\Lambda_n = \Lambda_n B \Lambda_n$. From $0 = \lambda_{1,n} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} b_{i,j}$, all entries of the 2th column of

B are zero except for $b_{n,2}$. From $0 = \lambda_{2,1} = \sum_{i=2}^{n} b_{i,1}$ all entries of 1st column of B are zero except for $b_{1,1}$.Also, from $0 = \lambda_{3,2} = \sum_{i=3}^{n}\sum_{j=1}^{2} b_{i,j}$, we have $b_{n,2} = 0$. If we combine these three results, we conclude that all entries of the first two columns are zero except for b_{11}. But then $1 = \lambda_{2,2} = \sum_{i=2}^{n}\sum_{j=1}^{2} b_{i,j} = 0$, a contradiction. Hence Λ_n is not regular for all $n \geq 3$.

Note: 2.10

In particular, $\Lambda_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is not regular. Let $\phi_{m,n} = \begin{bmatrix} \Lambda_3 & O \\ O & O \end{bmatrix}$ for all $\min\{m,n\} \geq 3$. Then $\phi_{m,n}$ is not regular by Theorem 2.5.

Theorem: 2.11

Let $\min\{m,n\} \geq 3$. For every cell E in $M_{m,n}(S)$,there is a regular matrix A such that E+A is not regular.

Proof:

Consider the matrix $\phi_{m,n}$ in Note 2.10. Let P and Q be permutation matrices such that $PEQ = E_{1,1}$. Consider a matrix A satisfying $PAQ = E_{1,2} + E_{2,2} + E_{2,3} + E_{3,3}$. Then $(PAQ)\,(G_{2,1}+G_{3,3})\,(PAQ) = PAQ$ and $P(E+A)Q = \phi_{m,n}$, where $G_{i,j}$ are cells in $M_{n,m}(S)$. Thus E+A is not regular, while A is regular by Theorem 2.4.

Definition: 2.12

The **pattern**, A^*, of a matrix $A \in M_{m,n}(S)$ is the matrix in $M_{m,n}(B)$ whose $(i,j)^{th}$ entry is 0 if and only if $a_{i,j} = 0$.

Remark: 2.13

By the definition, we have $(AB)^* = A^* B^*$ and $(B+C)^* = B^* + C^*$, for all A $M_{m,n}(S)$ and for all $B,C \in M_{n,q}(S)$. If follows that if A is regular in, $M_{m,n}(S)$ then A^* is regular in $M_{m,n}(\mathbb{B})$.

Notation: 2.14

Let $\mathcal{R}(S)$ be the set of all regular matrices in $M_{m,n}(S)$ that is

$\mathcal{R}(S) = \{X \in M_{m,n}(S) \mid X \text{ is regular}\}$ and let

$\mathcal{R}(S)^* = \{Y \in M_{m,n}(\mathbb{B}) \mid Y = X^* \text{ for some } X \in R(S)\}$.

Note: 2.15

In general, $\mathcal{R}(S)^* \neq \mathcal{R}(\mathbb{B})$ as shown in the following example.

Example: 2.16

Consider the matrix $Y = E_{1,1}+E_{1,2}+E_{2,2}$.Then we can easily check that $Y \in \mathcal{R}(\mathbb{B})$ but $Y \notin \mathcal{R}(\mathbb{R}_+)^*$.

Definition: 2.17

The weight of a matrix A in $M_n(\mathbb{B}_1)$ is the number of nonzero entries of A and will be denoted by w(A).

Theorem: 2.18

Let A be a matrix in $M_{m,n}(S)$ with w(A) = 5 such that A has a row or a column that has atleast 3 nonzero entries. If $E_{1,1}+E_{1,2}+E_{2,2} \in \mathcal{R}(S)^*$, then $A^* \in \mathcal{R}(S)^*$.

Proof:

Assume that A has a row that has atleast 3 nonzero entries. By Theorem 2.4, without loss of generality we assume that

$A^* = E_{1,1} + E_{1,2} + E_{1,3} + C$, where $C \in \Xi_1 \cup \Xi_2$,

$\Xi_1 = \{E_{1,4} + E_{1,5}, E_{1,4} + E_{2,5}, E_{2,4} + E_{2,5}, E_{2,3} + E_{2,4}, E_{2,4} + E_{3,5}, E_{2,4} + E_{3,4}\}$ and

$\Xi_2 = \{E_{1,4} + E_{2,4}, E_{2,2} + E_{2,3}, E_{2,3} + E_{3,4}, E_{2,3} + E_{3,3}, E_{2,2} + E_{3,3}\}$.

If $C \in \Xi_1$, then we can easily show that $A^* \in \mathcal{R}(S)^*$ and hence $A^* \in \mathcal{R}(S)^*$.

Let $C \in \Xi_2$, since $E_{1,1} + E_{1,2} + E_{2,2} \in \mathcal{R}(S)^*$, there are nonzero a, b, $c \in S$ such that $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is a regular matrix with a generalized inverse $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ (in fact, $z = 0$). Let

$$X = \begin{cases} a(E_{1,1} + E_{1,2} + E_{1,3}) + bE_{1,4} + cE_{2,4} & \text{if } C = E_{1,4} + E_{2,4}, \\ a(E_{1,1} + b(E_{1,2} + E_{1,3}) + c(E_{2,.2} + E_{2,3}) & \text{if } C = E_{2,2} + E_{2,3}, \\ a(E_{1,1} + E_{1,2}) + bE_{1,3} + cE_{2,3} + E_{3,4} & \text{if } C = E_{2,3} + E_{3,4}, \\ a(E_{1,1} + E_{1,2}) + bE_{1,3} + c(E_{2,3} + E_{3,.3}) & \text{if } C = E_{2,3} + E_{3,3}, \\ aE_{1,1} + b(E_{1,2} + E_{1,3}) + c(E_{2,2} + E_{3.,3}) & \text{if } C = E_{2,2} + E_{3,3}, \end{cases}$$

and

$$Y = \begin{cases} xG_{3,1} + yG_{3,2} + wG_{4,2} & \text{if } C = E_{1,4} + E_{2,4}, \\ xG_{1,1} + yG_{1,2} + wG_{2,2} & \text{if } C = E_{2,2} + E_{2,3}, \\ xG_{2,1} + yG_{2,2} + wG_{2,3} + G_{4,3} & \text{if } C = E_{2,3} + E_{3,4}, \\ xG_{2,1} + yG_{2,2} + wG_{3,2} & \text{if } C = E_{2,3} + E_{3,3}, \\ xG_{1,1} + y(G_{1,2} + G_{1,3}) + w(G_{2,2} + G_{3.3}) & \text{if } C = E_{2,2} + E_{3,3}, \end{cases}$$

Where $G_{i,j}$ are cells in $M_{n,m}(S)$. Then we have $XYX = X$ so that $X \in \mathcal{R}(S)$ and hence $X^* = E_{1,1} + E_{1,2} + E_{1,3} + C = A^* \in \mathcal{R}(S)^*$.

Definition: 2.19

The **(factor) rank or semiring rank**, fr(A), of a nonzero matrix $A \in M_{m,n}(S)$ is defined as the least integer r for which there are $B \in M_{m,r}(S)$ and $C \in M_{r,n}(S)$ such that A=BC.

Note: 2.20

The rank of a zero matrix is zero. Also we can easily obtain

$0 \leq fr(A) \leq \min\{m,n\}$ and $fr(AB) \leq \min\{fr(A), fr(B)\}$ for all $A \in M_{m,n}(S)$ and for all $B \in M_{n,q}(S)$.

Theorem: 2.21

Let $\min\{m,n\} \geq 3$. If A is a matrix in $M_{m,n}(S)$ with w(A) = 3 and fr(A) = 2 or 3, then there is a matrix B with w(B) = 2 such that $(A+B)^* \notin \mathcal{R}(S)^*$.

Proof:

Since w(A) = 3 and fr(A) = 2 or 3, there are permutations P and Q such that $PAQ \subseteq \phi_{m,n}$. Let C be a matrix in $M_{m,n}(S)$ with w(C) = 2 such that $(PAQ+C)^* = \phi_{m,n}$. If we take $B = P^TCQ^T$, then $(A+B)^* = P^T \phi_{m,n} Q^T \notin \mathcal{R}(\mathbb{B})$ by Theorem 2.4 and hence $(A+B)^* \notin \mathcal{R}(S)^*$.