CHAPTER - 4

DIFFERENT VERSIONS OF HAUSDORFF SEPARATION AXIOM IN SECOND ORDER BIPOLAR FUZZY TOPOLOGICAL SPACES

Different versions of Hausdorff separation axiom have been defined and studied in the case of fuzzy topological spaces by many authors. Three different Hausdorff separation axioms in fuzzy topological spaces introduced by Gantner et al. (1978), Katsaras (1981) and Srivastava et al. (1981) were extended to second order fuzzy topological spaces by Kalaichelvi (2000, 2011b, 2013) and were denoted as W – Hausdorff, K – Hausdorff and S – Hausdorff axioms respectively.

In the first section, W – Hausdorff, K – Hausdorff and S – Hausdorff separation axioms of Warren, Katsaras and Srivastava are extended to first order bipolar fuzzy topological spaces.

In the second section, W – Hausdorff, K – Hausdorff and S – Hausdorff axioms are extended to second order bipolar fuzzy topological spaces.

Third section is devoted to study the relations between the first order bipolar fuzzy and second order bipolar fuzzy Hausdorff axioms with special reference to R_1, R_3 and R_5. The behaviour of these concepts with regard to $i(\hat{\tau}_{\mathfrak{B}})$, $i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}})$, $i^{*}(\hat{\tau}_{\mathfrak{B}})$, $\widehat{\omega(\tau)}$, $\widehat{\omega_{\varepsilon}(\tau)}$, $\widehat{\omega_{*}(\tau)}$ are also analysed.

SECTION 4.1

W-HAUSDORFF, K-HAUSDORFF AND S-HAUSDORFF SEPARATION AXIOMS IN FIRST ORDER BIPOLAR FUZZY TOPOLOGICAL SPACES

Definition:4.1.1

Let $(X, \tau_{\mathfrak{B}})$ be a bipolar fuzzy topological space. Let $Y \subseteq X$.
If $A_{bp} = (A_{bp}^{+}, A_{bp}^{-}) \in \tau_{\mathfrak{B}}$ and $A_{bp}/Y = (A_{bp}^{+}/Y, A_{bp}^{-}/Y)$ is the restriction function where $(A_{bp}^{+}/Y)(z) = A_{bp}^{+}(z)$ and $(A_{bp}^{-}/Y)(z) = A_{bp}^{-}(z)$, for every $z \in Y$, then the collection $(\tau_{\mathfrak{B}}/Y) = \{(A_{bp}/Y) \,/\, A_{bp} \in \tau_{\mathfrak{B}}\}$ is called the **bipolar fuzzy topology** on Y and $(Y, \tau_{\mathfrak{B}}/Y)$ is called a **bipolar fuzzy subspace** of $(X, \tau_{\mathfrak{B}})$

Definition:4.1.2

A bipolar fuzzy topological space $(X, \tau_{\mathfrak{B}})$ is said to be a **bipolar fuzzy W-Hausdorff or bipolar fuzzy W-T₂**, if for any two distinct points $x, y \in X$, there exist two bipolar fuzzy open sets $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}}$ such that $A_{bp}^{+}(x) = 1, A_{bp}^{-}(x) = -1$, $B_{bp}^{+}(y) = 1$, $B_{bp}^{-}(y) = -1$ and $A_{bp} \cap B_{bp} = 0_{bp}$.

Theorem:4.1.3

Subspace of a bipolar fuzzy W-Hausdorff space is bipolar fuzzy W-Hausdorff.

Proof:

Let $(X, \tau_{\mathfrak{B}})$ be a bipolar fuzzy W-Hausdorff space and Y be a non-empty subset of X. Then $(Y, \tau_{\mathfrak{B}}/Y)$ be a bipolar fuzzy subspace of $(X, \tau_{\mathfrak{B}})$.

Consider $y_1, y_2 \in Y$, $y_1 \neq y_2$. Therefore $y_1, y_2 \in X$ since $(X, \tau_{\mathfrak{B}})$ be a bipolar fuzzy W-T₂, there exist two bipolar fuzzy open sets $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}}$ such that $A_{bp}^{+}(y_1) = 1$,

$A_{bp}^{-}(y_1) = -1$, $B_{bp}^{+}(y_2) = 1$, $B_{bp}^{-}(y_2) = -1$ and $A_{bp} \cap B_{bp} = 0_{bp}$.

Since Y is a subset of X, A_{bp}/Y, $B_{bp}/Y \in \tau_{\mathfrak{B}}/Y$ where

$(A_{bp}^{+}/Y)(y_1) = A_{bp}^{+}(y_1) = 1$, $(A_{bp}^{-}/Y)(y_1) = A_{bp}^{-}(y_1) = -1$ and

$(B_{bp}{}^{+}/Y)(y_2) = B_{bp}{}^{+}(y_2) = 1, (B_{bp}{}^{-}/Y)(y_2) = B_{bp}{}^{-}(y_2) = -1.$

Consider

$(A_{bp}/Y) \cap (B_{bp}/Y) = \big((A_{bp}{}^{+}/Y) \wedge (B_{bp}{}^{+}/Y), (A_{bp}{}^{-}/Y) \vee (B_{bp}{}^{-}/Y)\big)$

$$\begin{aligned}((A_{bp}{}^{+}/Y) \wedge (B_{bp}{}^{+}/Y))(y) &= (A_{bp}{}^{+}/Y)(y) \wedge (B_{bp}{}^{+}/Y)(y), \text{ for every } y \in Y \subseteq X\\ &= A_{bp}{}^{+}(y) \wedge B_{bp}{}^{+}(y), \text{ for every } y \in Y \subseteq X\\ &= (A_{bp}{}^{+} \wedge B_{bp}{}^{+})(y), \text{ for every } y \in Y \subseteq X\\ &= 0^{+}(y), \text{ for every } y \in Y \subseteq X.\end{aligned}$$

$(A_{bp}{}^{+}/Y) \wedge (B_{bp}{}^{+}/Y) = 0^{+}.$

$$\begin{aligned}\big((A_{bp}{}^{-}/Y) \vee (B_{bp}{}^{-}/Y)\big)(y) &= (A_{bp}{}^{-}/Y)(y) \vee (B_{bp}{}^{-}/Y)(y), \text{ for every } y \in Y \subseteq X\\ &= A_{bp}{}^{-}(y) \vee B_{bp}{}^{-}(y), \text{ for every } y \in Y \subseteq X\\ &= (A_{bp}{}^{-} \vee B_{bp}{}^{-})(y), \text{ for every } y \in Y \subseteq X\\ &= 0^{-}(y), \text{ for every } y \in Y \subseteq X.\end{aligned}$$

$(A_{bp}{}^{-}/Y) \wedge (B_{bp}{}^{-}/Y) = 0^{-}.$

Thus $(A_{bp}/Y) \cap (B_{bp}/Y) = (0^{+}, 0^{-}) = 0_{bp}$.

Hence subspace of a bipolar fuzzy W-Hausdorff space is bipolar fuzzy W-Hausdorff.

Theorem:4.1.4

Product of two bipolar fuzzy W-Hausdorff spaces is a bipolar fuzzy W-Hausdorff space.

Proof:

Let $(X, \tau_{\mathfrak{B}_1})$ and $(Y, \tau_{\mathfrak{B}_2})$ be two bipolar fuzzy W-Hausdorff spaces.

To prove : $(X \times Y, \tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2})$ is a bipolar fuzzy W-Hausdorff space

Consider two distinct points $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Either $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume $x_1 \neq x_2$, since $(X, \tau_{\mathfrak{B}_1})$ be a bipolar fuzzy W-T2, there exist two bipolar fuzzy open sets $A_{bp} = (A_{bp}{}^{+}, A_{bp}{}^{-})$ and $B_{bp} = (B_{bp}{}^{+}, B_{bp}{}^{-})$ such that $A_{bp}{}^{+}(x_1) = 1, A_{bp}{}^{-}(x_1) = -1$, $B_{bp}{}^{+}(x_2) = 1, B_{bp}{}^{-}(x_2) = -1$ and $A_{bp} \cap B_{bp} = 0_{bp}$ where 0_{bp} is a bipolar fuzzy null set in X. Therefore $A_{bp} \times 1_{bp}$, $B_{bp} \times 1_{bp} \in \tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2}$ where $A_{bp} \times 1_{bp} = (A_{bp}{}^{+} \times 1_{bp}{}^{+}, A_{bp}{}^{-} \times 1_{bp}{}^{-})$ and $B_{bp} \times 1_{bp} = (B_{bp}{}^{+} \times 1_{bp}{}^{+}, B_{bp}{}^{-} \times 1_{bp}{}^{-})$.

Also $(A_{bp}^{+} \times 1_{bp}^{+})(x_1, y_1) = \min\{A_{bp}^{+}(x_1), 1_{bp}^{+}(y_1)\}$

$= \min\{1,1\}$

$= 1.$

$(A_{bp}^{-} \times 1_{bp}^{-})(x_1, y_1) = \max\{A_{bp}^{-}(x_1), 1_{bp}^{-}(y_1)\}$

$= \max\{-1, -1\}$

$= -1.$

Similarly, $(B_{bp}^{+} \times 1_{bp}^{+})(x_2, y_2) = 1$ and $(B_{bp}^{-} \times 1_{bp}^{-})(x_2, y_2) = -1$.

So, $A_{bp} \cap B_{bp} = 0_{bp}$ implies $\left(A_{bp}^{+} \wedge B_{bp}^{+},\ A_{bp}^{-} \vee B_{bp}^{-}\right) = 0_{bp}$

implies $(A_{bp}^{+} \wedge B_{bp}^{+})(x) = 0^{+}(x)$ and $(A_{bp}^{-} \vee B_{bp}^{-})(x) = 0^{-}(x)$, for every $x \in X$

implies $A_{bp}^{+}(x) \wedge B_{bp}^{+}(x) = 0$ and $A_{bp}^{-}(x) \vee B_{bp}^{-}(x) = 0$, for every $x \in X$

implies $A_{bp}^{+}(x) = 0$ or $B_{bp}^{+}(x) = 0$ and

$A_{bp}^{-}(x) = 0$ or $B_{bp}^{-}(x) = 0$, for every $x \in X$

implies $A_{bp}^{+}(x) \wedge 1_{bp}^{+}(y) = 0$ or $B_{bp}^{+}(x) \wedge 1_{bp}^{+}(y) = 0$ and

$A_{bp}^{-}(x) \vee 1_{bp}^{-}(y) = 0$ or $B_{bp}^{-}(x) \vee 1_{bp}^{-}(y) = 0$, for every $x \in X$ and for every $y \in Y$

implies $(A_{bp}^{+} \times 1_{bp}^{+})(x, y) = 0$ or $(B_{bp}^{+} \times 1_{bp}^{+})(x, y) = 0$ and

$(A_{bp}^{-} \times 1_{bp}^{-})(x, y) = 0$ or $(B_{bp}^{-} \times 1_{bp}^{-})(x, y) = 0$, for every $(x, y) \in X \times Y$

implies $\left((A_{bp}^{+} \times 1_{bp}^{+}) \wedge (B_{bp}^{+} \times 1_{bp}^{+})\right)(x, y) = 0$ and

$\left((A_{bp}^{-} \times 1_{bp}^{-}) \vee \left(B_{bp}^{-} \times 1_{bp}^{-}\right)\right)(x, y) = 0$, for every $(x, y) \in X \times Y$

implies $(A_{bp} \times 1_{bp}) \cap (B_{bp} \times 1_{bp}) = 0_{bp}$.

The proof in the other case is similar.

Therefore product of two bipolar fuzzy W-Hausdorff spaces is a bipolar fuzzy W-Hausdorff space.

Theorem:4.1.5

Arbitrary product of bipolar fuzzy W-Hausdorff spaces is bipolar fuzzy W-Hausdorff.

Proof:

Let $\{(X_{\lambda}, \tau_{\mathfrak{B}_{\lambda}}) / \lambda \in \Lambda\}$ be a collection of bipolar fuzzy W-Hausdorff spaces and $X = \prod_{\lambda \in \Lambda} X_{\lambda}$. Consider two distinct points $(x_{\lambda})_{\lambda \in \Lambda}, (y_{\lambda})_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} X_{\lambda}$. Assume $x_{\mu} \neq y_{\mu}$ for some $\mu \in \Lambda$, there exist two bipolar fuzzy open sets, $\left(A_{bp}\right)_{\mu}, \left(B_{bp}\right)_{\mu} \in \tau_{\mathfrak{B}_{\mu}}$ such that

$\left(A_{bp}^{+}\right)_{\mu}(x_{\mu}) = 1$, $\left(A_{bp}^{-}\right)_{\mu}(x_{\mu}) = -1$, $\left(B_{bp}^{+}\right)_{\mu}(y_{\mu}) = 1$, $\left(B_{bp}^{-}\right)_{\mu}(y_{\mu}) = -1$ and $\left(A_{bp}\right)_{\mu} \cap \left(B_{bp}\right)_{\mu} = \left(0_{bp}\right)_{\mu}$.

Let $A_{bp} = \prod_{\lambda \in \Lambda}\left(A_{bp}\right)_{\lambda}$, where $\left(A_{bp}\right)_{\lambda} = \left(1_{bp}\right)_{\lambda}$ for $\lambda \neq \mu$

and $B_{bp} = \prod_{\lambda \in \Lambda}\left(B_{bp}\right)_{\lambda}$, where $\left(B_{bp}\right)_{\lambda} = \left(1_{bp}\right)_{\lambda}$ for $\lambda \neq \mu$.

Then $A_{bp}, B_{bp} \in \prod_{\lambda \in \Lambda} \tau_{\mathfrak{B}_{\lambda}}$ where $A_{bp} = \prod_{\lambda \in \Lambda}\left(A_{bp}\right)_{\lambda} = \left(\bigwedge_{\lambda \in \Lambda}\left(A_{bp}^{+}\right)_{\lambda},\ \bigvee_{\lambda \in \Lambda}\left(A_{bp}^{-}\right)_{\lambda}\right)$

and $B_{bp} = \prod_{\lambda \in \Lambda}\left(B_{bp}\right)_{\lambda} = \left(\bigwedge_{\lambda \in \Lambda}\left(B_{bp}^{+}\right)_{\lambda},\ \bigvee_{\lambda \in \Lambda}\left(B_{bp}^{-}\right)_{\lambda}\right)$.

Consider $\bigwedge_{\lambda \in \Lambda}\left(A_{bp}^{+}\right)_{\lambda} (x_{\lambda})_{\lambda \in \Lambda} = \min\{\left(A_{bp}\right)_{\lambda}^{+}(x_{\lambda})_{\lambda \in \Lambda}\}$

$= \left(A_{bp}\right)_{\mu}^{+}(x_{\mu})$, for some $\mu \in \Lambda$

$= 1$

$\bigvee_{\lambda \in \Lambda}\left(A_{bp}^{-}\right)_{\lambda} (x_{\lambda})_{\lambda \in \Lambda} = \max\{\left(A_{bp}\right)_{\lambda}^{-}(x_{\lambda})_{\lambda \in \Lambda}\}$

$= \left(A_{bp}\right)_{\mu}^{-}(x_{\mu})$, for some $\mu \in \Lambda$

$= -1$.

Similarly, $\bigwedge_{\lambda \in \Lambda}\left(B_{bp}^{+}\right)_{\lambda} (y_{\lambda})_{\lambda \in \Lambda} = 1$ and $\bigvee_{\lambda \in \Lambda}\left(B_{bp}^{-}\right)_{\lambda} (y_{\lambda})_{\lambda \in \Lambda} = -1$.

Consider $A_{bp} \cap B_{bp} = \prod_{\lambda \in \Lambda}\left(A_{bp}\right)_{\lambda} \cap \prod_{\lambda \in \Lambda}\left(B_{bp}\right)_{\lambda}$

$= \left(\left(\bigwedge_{\lambda \in \Lambda}\left(A_{bp}^{+}\right)_{\lambda}\right) \wedge \left(\bigwedge_{\lambda \in \Lambda}\left(B_{bp}^{+}\right)_{\lambda}\right), \left(\bigvee_{\lambda \in \Lambda}\left(A_{bp}^{-}\right)_{\lambda}\right) \vee \left(\bigvee_{\lambda \in \Lambda}\left(B_{bp}^{-}\right)_{\lambda}\right)\right)$.

Then $\left(\left(\bigwedge_{\lambda \in \Lambda}\left(A_{bp}^{+}\right)_{\lambda}\right) \wedge \left(\bigwedge_{\lambda \in \Lambda}\left(B_{bp}^{+}\right)_{\lambda}\right)\right)(x_{\lambda})_{\lambda \in \Lambda}$

$= \left(\bigwedge_{\lambda \in \Lambda}\left(A_{bp}^{+}\right)_{\lambda} (x_{\lambda})_{\lambda \in \Lambda}\right) \wedge \left(\bigwedge_{\lambda \in \Lambda}\left(B_{bp}^{+}\right)_{\lambda} (x_{\lambda})_{\lambda \in \Lambda}\right)$

$= \min\{\left(A_{bp}^{+}\right)_{\lambda}(x_{\lambda})_{\lambda \in \Lambda}\} \wedge \min\{\left(B_{bp}^{+}\right)_{\lambda}(x_{\lambda})_{\lambda \in \Lambda}\}$

$= \left(A_{bp}^{+}\right)_{\mu}(x_{\mu}) \wedge \left(B_{bp}^{+}\right)_{\mu}(x_{\mu})$

$= \left(\left(A_{bp}^{+}\right)_{\mu} \wedge \left(B_{bp}^{+}\right)_{\mu}\right)(x_{\mu})$

$= 0$.

Similarly, $\left(\left(\bigvee_{\lambda \in \Lambda}\left(A_{bp}^{-}\right)_{\lambda}\right) \vee \left(\bigvee_{\lambda \in \Lambda}\left(B_{bp}^{-}\right)_{\lambda}\right)\right)(y_{\lambda})_{\lambda \in \Lambda}$

$= \left(\bigvee_{\lambda \in \Lambda}\left(A_{bp}^{-}\right)_{\lambda} (y_{\lambda})_{\lambda \in \Lambda}\right) \vee \left(\bigvee_{\lambda \in \Lambda}\left(B_{bp}^{-}\right)_{\lambda} (y_{\lambda})_{\lambda \in \Lambda}\right)$

$= \max\{\left(A_{bp}^{-}\right)_{\lambda}(y_{\lambda})_{\lambda \in \Lambda}\} \vee \max\{\left(B_{bp}^{-}\right)_{\lambda}(y_{\lambda})_{\lambda \in \Lambda}\}$

$= \left({A_{bp}}^{-}\right)_{\mu}(y_{\mu}) \vee \left({B_{bp}}^{-}\right)_{\mu}(y_{\mu})$

$=\left(\left({A_{bp}}^{-}\right)_{\mu} \vee \left({B_{bp}}^{-}\right)_{\mu}\right)(y_{\mu})$

$= 0.$

Thus $A_{bp} \cap B_{bp} = 0_{bp}$.

Hence arbitrary product of bipolar fuzzy W-Hausdorff spaces is a bipolar fuzzy W-Hausdorff space.

Definition:4.1.6

A bipolar fuzzy topological space $(X, \tau_{\mathfrak{B}})$ is said to be **bipolar fuzzy K-Hausdorff or bipolar fuzzy K-T₂**, if for any two distinct points $x, y \in X$, there exists two bipolar fuzzy open sets $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}}$ such that

${A_{bp}}^{+}(x) > 0,\ {A_{bp}}^{-}(x) < 0,\ {B_{bp}}^{+}(y) > 0, {B_{bp}}^{-}(y) < 0$ and $A_{bp} \cap B_{bp} = 0_{bp}$.

Note:

From 4.1.2 and 4.1.6, it is clear that bipolar fuzzy W- hausdorff ⇒ bipolar fuzzy K – Hausdorff.

The following example shows that the converse of the above implication is not true.

Example:4.1.7

Let X be a non-empty set, For any two distinct points $x, y \in X$, A_{bp}, B_{bp} where ${A_{bp}}^{+}: X \rightarrow I$, ${A_{bp}}^{-}: X \rightarrow [-1,0]$ and ${B_{bp}}^{+}: X \rightarrow I$, ${B_{bp}}^{-}: X \rightarrow [-1,0]$ are defined as follows:

${A_{bp}}^{+}(x) = 0.2,\ {A_{bp}}^{-}(x) = -0.3, {A_{bp}}^{+}(y) = 0, {A_{bp}}^{-}(y) = 0$ and

${B_{bp}}^{+}(x) = 0, {B_{bp}}^{-}(x) = 0,\ {B_{bp}}^{+}(y) = 0.5, {B_{bp}}^{-}(y) = -0.6.$

Then $\tau_{\mathfrak{B}} = \{0_{bp}, 1_{bp}, A_{bp}, B_{bp}, A_{bp} \cup B_{bp}\}$ is a first order bipolar fuzzy topology on X.

Here $x \neq y$, ${A_{bp}}^{+}(x) > 0, {A_{bp}}^{-}(x) < 0,\ {B_{bp}}^{+}(y) > 0, {B_{bp}}^{-}(y) < 0$ and $A_{bp} \cap B_{bp} = 0_{bp}$.

Therefore $(X, \tau_{\mathfrak{B}})$ is bipolar fuzzy K-Hausdorff. But it is not bipolar fuzzy W-Hausdorff as ${A_{bp}}^{+}(x) \neq 1, {A_{bp}}^{-}(x) \neq -1,\ {B_{bp}}^{+}(y) \neq 1, {B_{bp}}^{-}(y) \neq -1.$

Theorem:4.1.8

Subspace of a bipolar fuzzy K-Hausdorff space is bipolar fuzzy K-Hausdorff.

Proof:

Let $(X, \tau_{\mathfrak{B}})$ be a bipolar fuzzy K-Hausdorff space and Y be a non-empty subset of X. Then $(Y, \tau_{\mathfrak{B}}/Y)$ is a subspace of $(X, \tau_{\mathfrak{B}})$. Consider $y_1, y_2 \in Y$, $y_1 \neq y_2$. Then $y_1, y_2 \in X$. Since $(X, \tau_{\mathfrak{B}})$ is bipolar fuzzy K – Hausdorff, there exist two bipolar fuzzy open sets A_{bp}, $B_{bp} \in \tau_{\mathfrak{B}}$ such that ${A_{bp}}^{+}(y_1) > 0, {A_{bp}}^{-}(y_1) < 0$, ${B_{bp}}^{+}(y_2) > 0, {B_{bp}}^{-}(y_2) < 0$ and $A_{bp} \cap B_{bp} = 0_{bp}$.

Since Y is a subset of X, A_{bp}/Y, $B_{bp}/Y \in \tau_{\mathfrak{B}}/Y$ where

$A_{bp}/Y = \left({A_{bp}}^{+}/Y, {A_{bp}}^{-}/Y\right)$ and $B_{bp}/Y = \left({B_{bp}}^{+}/Y, {B_{bp}}^{-}/Y\right)$.

Therefore $({A_{bp}}^{+}/Y)(y_1) = {A_{bp}}^{+}(y_1) > 0$ and $({A_{bp}}^{-}/Y)(y_1) = {A_{bp}}^{-}(y_1) < 0$

$({B_{bp}}^{+}/Y)(y_2) = {B_{bp}}^{+}(y_2) > 0$ and $({B_{bp}}^{-}/Y)(y_2) = {B_{bp}}^{-}(y_2) < 0$.

Consider

$(A_{bp}/Y) \cap (B_{bp}/Y) = \left(({A_{bp}}^{+}/Y) \wedge \left({B_{bp}}^{+}/Y\right), ({A_{bp}}^{-}/Y) \vee ({B_{bp}}^{-}/Y)\right)$

$$\begin{aligned}\left(({A_{bp}}^{+}/Y) \wedge \left({B_{bp}}^{+}/Y\right)\right)(y) &= ({A_{bp}}^{+}/Y)(y) \wedge \left({B_{bp}}^{+}/Y\right)(y), \text{ for every } y \in Y \subseteq X\\ &= {A_{bp}}^{+}(y) \wedge {B_{bp}}^{+}(y), \text{ for every } y \in Y \subseteq X\\ &= ({A_{bp}}^{+} \wedge {B_{bp}}^{+})(y), \text{ for every } y \in Y \subseteq X\\ &= 0^{+}(y), \text{ for every } y \in Y \subseteq X.\end{aligned}$$

Then $({A_{bp}}^{+}/Y) \wedge \left({B_{bp}}^{+}/Y\right) = 0^{+}$ and

$$\begin{aligned}\left(({A_{bp}}^{-}/Y) \vee \left({B_{bp}}^{-}/Y\right)\right)(y) &= ({A_{bp}}^{-}/Y)(y) \vee \left({B_{bp}}^{-}/Y\right)(y), \text{ for every } y \in Y \subseteq X\\ &= {A_{bp}}^{-}(y) \vee {B_{bp}}^{-}(y), \text{ for every } y \in Y \subseteq X\\ &= ({A_{bp}}^{-} \vee {B_{bp}}^{-})(y), \text{ for every } y \in Y \subseteq X\\ &= 0^{-}(y), \text{ for every } y \in Y \subseteq X.\end{aligned}$$

Then $({A_{bp}}^{-}/Y) \wedge \left({B_{bp}}^{-}/Y\right) = 0^{-}$.

Thus $(A_{bp}/Y) \cap \left(B_{bp}/Y\right) = (0^{+}, 0^{-}) = 0_{bp}$.

Hence subspace of a bipolar fuzzy K-Hausdorff space is a bipolar fuzzy K-Hausdorff space.

Theorem:4.1.9

Product of two bipolar fuzzy K-Hausdorff spaces is a bipolar fuzzy K-Hausdorff space.

Proof:

Let $(X, \tau_{\mathfrak{B}_1})$ and $(Y, \tau_{\mathfrak{B}_2})$ be two bipolar fuzzy Hausdorff spaces.

To prove: $(X \times Y, \tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2})$ is a bipolar fuzzy K-Hausdorff space

Consider two distinct points $(x_1, y_1), (x_2, y_2) \in X \times Y$. Either $x_1 \neq x_2$ or $y_1 \neq y_2$.

Assume $x_1 \neq x_2$, there exist two bipolar fuzzy open sets A_{bp}, $B_{bp} \in \tau_{\mathfrak{B}}$ such that

${A_{bp}}^{+}(x_1) > 0, {A_{bp}}^{-}(x_1) < 0$, ${B_{bp}}^{+}(x_2) > 0, {B_{bp}}^{-}(x_2) < 0$ and

$A_{bp} \cap B_{bp} = 0_{bp}$ where 0_{bp} is a bipolar fuzzy null set in X.

Therefore $A_{bp} \times 1_{bp}$, $B_{bp} \times 1_{bp} \in \tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2}$ where $A_{bp} \times 1_{bp} = ({A_{bp}}^{+} \times {1_{bp}}^{+}, {A_{bp}}^{-} \times {1_{bp}}^{-})$ and $B_{bp} \times 1_{bp} = ({B_{bp}}^{+} \times {1_{bp}}^{+}, {B_{bp}}^{-} \times {1_{bp}}^{-})$.

Consider $({A_{bp}}^{+} \times {1_{bp}}^{+})(x_1, y_1) = \min\{ {A_{bp}}^{+}(x_1), {1_{bp}}^{+}(y_1)\} = {A_{bp}}^{+}(x_1) > 0$

$$({A_{bp}}^{-} \times {1_{bp}}^{-})(x_1, y_1) = \max\{ {A_{bp}}^{-}(x_1), {1_{bp}}^{-}(y_1)\} = {A_{bp}}^{-}(x_1) < 0.$$

Similarly $({B_{bp}}^{+} \times {1_{bp}}^{+})(x_2, y_2) > 0$ and $({B_{bp}}^{-} \times {1_{bp}}^{-})(x_2, y_2) < 0$.

Also, $A_{bp} \cap B_{bp} = 0_{bp}$ implies $(A_{bp} \times 1_{bp}) \cap (B_{bp} \times 1_{bp}) = 0_{bp}$.

The proof for the other case is similar.

Hence product of two bipolar fuzzy K-Hausdorff spaces is a bipolar fuzzy K-Hausdorff space.

Theorem:4.1.10

Arbitrary product of bipolar fuzzy K-Hausdorff spaces is a bipolar fuzzy K-Hausdorff space.

Proof:

Let $\{(X_\lambda, \tau_{\mathfrak{B}_\lambda})/\lambda \in \Lambda\}$ be a collection of bipolar fuzzy K-Hausdorff spaces. Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ and $\tau_{\mathfrak{B}} = \prod_{\lambda \in \Lambda} \tau_{\mathfrak{B}_\lambda}$. Consider two distinct points $(x_\lambda)_{\lambda \in \Lambda}$, $(y_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} X_\lambda$.

Assume $x_\mu \neq y_\mu$ for some $\mu \in \Lambda$, then there exist two bipolar fuzzy open sets, $(A_{bp})_\mu, (B_{bp})_\mu \in \tau_{\mathfrak{B}_\mu}$ such that $({A_{bp}}^{+})_\mu(x_\mu) > 0$,$({A_{bp}}^{-})_\mu(x_\mu) < 0$, $({B_{bp}}^{+})_\mu(y_\mu) > 0$, $({B_{bp}}^{-})_\mu(y_\mu) < 0$ and $(A_{bp})_\mu \cap (B_{bp})_\mu = (0_{bp})_\mu$.

Let $A_{bp} = \prod_{\lambda \in \Lambda}(A_{bp})_\lambda$, where $(A_{bp})_\lambda = (1_{bp})_\lambda$ for $\lambda \neq \mu$ and

$B_{bp} = \prod_{\lambda \in \Lambda}(B_{bp})_\lambda$, where $(B_{bp})_\lambda = (1_{bp})_\lambda$ for $\lambda \neq \mu$.

Then $A_{bp}, B_{bp} \in \prod_{\lambda \in \Lambda} \tau_{\mathfrak{B}_\lambda}$, where $A_{bp} = \prod_{\lambda \in \Lambda}(A_{bp})_{\lambda} = \left(\wedge_{\lambda \in \Lambda}({A_{bp}}^{+})_{\lambda}\ ,\ \vee_{\lambda \in \Lambda}({A_{bp}}^{-})_{\lambda}\right)$ and

$$B_{bp} = \prod_{\lambda \in \Lambda}(B_{bp})_{\lambda} = \left(\wedge_{\lambda \in \Lambda}({B_{bp}}^{+})_{\lambda},\ \vee_{\lambda \in \Lambda}({B_{bp}}^{-})_{\lambda}\right)$$

$\wedge_{\lambda \in \Lambda}({A_{bp}}^{+})_{\lambda}\ (x_{\lambda})_{\lambda \in \Lambda} = \min\{({A_{bp}}^{+})_{\lambda}(x_{\lambda})_{\lambda \in \Lambda}\}$

$$= ({A_{bp}}^{+})_{\mu}(x_{\mu}) > 0, \text{ for some } \mu \in \Lambda$$

$\wedge_{\lambda \in \Lambda}({A_{bp}}^{+})_{\lambda}\ (x_{\lambda})_{\lambda \in \Lambda} > 0$

$\vee_{\lambda \in \Lambda}({A_{bp}}^{-})_{\lambda}\ (x_{\lambda})_{\lambda \in \Lambda} = \max\{({A_{bp}}^{-})_{\lambda}(x_{\lambda})_{\lambda \in \Lambda}\}$

$$= ({A_{bp}}^{-})_{\mu}(x_{\mu}) < 0, \text{ for some } \mu \in \Lambda$$

$\vee_{\lambda \in \Lambda}({A_{bp}}^{-})_{\lambda}\ (x_{\lambda})_{\lambda \in \Lambda} < 0.$

Similarly, $\wedge_{\lambda \in \Lambda}({B_{bp}}^{+})_{\lambda}\ (y_{\lambda})_{\lambda \in \Lambda} > 0$ and $\vee_{\lambda \in \Lambda}({B_{bp}}^{-})_{\lambda}\ (y_{\lambda})_{\lambda \in \Lambda} < 0.$

Also $A_{bp} \cap B_{bp} = 0_{bp}$.

Hence arbitrary product of bipolar fuzzy K-Hausdorff spaces is a bipolar fuzzy K-Hausdorff space.

Definition:4.1.11

A bipolar fuzzy topological space $(X, \tau_{\mathfrak{B}})$ is said to be **bipolar fuzzy S-Hausdorff or bipolar fuzzy S-T₂**, if for any pair of distinct bipolar fuzzy points $x_{(\alpha,\beta)}, y_{(\gamma,\delta)}$ in X, there exist two bipolar fuzzy open sets $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}}$ such that $x_{(\alpha,\beta)} \in A_{bp}, y_{(\gamma,\delta)} \in B_{bp}$ that is, ${A_{bp}}^{+}(x) \geq \alpha, {A_{bp}}^{-}(x) \leq \beta$, ${B_{bp}}^{+}(y) \geq \gamma, {B_{bp}}^{-}(y) \leq \delta$ and $A_{bp} \cap B_{bp} = 0_{bp}$.

Theorem:4.1.12

Subspace of a bipolar fuzzy S-Hausdorff space is bipolar fuzzy S-Hausdorff.

Proof:

Let $(X, \tau_{\mathfrak{B}})$ be a bipolar fuzzy S-Hausdorff space and Y is a non-empty subset of X. Then $(Y, \tau_{\mathfrak{B}}/Y)$ be a subspace of $(X, \tau_{\mathfrak{B}})$.

For $x, y \in Y$, $x \neq y$, consider a pair of distinct bipolar fuzzy points $x_{(\alpha,\beta)}, y_{(\gamma,\delta)}$ in Y

Therefore $x_{(\alpha,\beta)}, y_{(\gamma,\delta)}$ are the distinct bipolar fuzzy points in X, then there exist two bipolar fuzzy open sets $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}}$ such that ${A_{bp}}^{+}(x) \geq \alpha, {A_{bp}}^{-}(x) \leq \beta$, ${B_{bp}}^{+}(y) \geq \gamma$, ${B_{bp}}^{-}(y) \leq \delta$ and $A_{bp} \cap B_{bp} = 0_{bp}$.

Since Y is a subset of X, A_{bp}/Y, $B_{bp}/Y \in \tau_{\mathfrak{B}}/Y$ where $A_{bp}/Y = \left({A_{bp}}^{+}/Y, {A_{bp}}^{-}/Y\right)$ and $B_{bp}/Y = \left({B_{bp}}^{+}/Y, {B_{bp}}^{-}/Y\right)$.

Then $({A_{bp}}^{+}/Y)(x) = {A_{bp}}^{+}(x) \geq \alpha$ and $({A_{bp}}^{-}/Y)(x) = {A_{bp}}^{-}(x) \leq \beta$

$({B_{bp}}^{+}/Y)(y) = {B_{bp}}^{+}(y) \geq \gamma$ and $({B_{bp}}^{-}/Y)(y) = {B_{bp}}^{-}(y) \leq \delta$.

Also $A_{bp} \cap B_{bp} = 0_{bp}$ implies $(A_{bp}/Y) \cap (B_{bp}/Y) = (0^{+}, 0^{-}) = 0_{bp}$.

Hence subspace of a bipolar fuzzy S-Hausdorff space is a bipolar fuzzy S-Hausdorff space.

Theorem: 4.1.13

Product of two bipolar fuzzy S-Hausdorff spaces is a bipolar fuzzy S-Hausdorff space.

Proof:

Let $\left(X, \tau_{\mathfrak{B}_1}\right)$ and $\left(Y, \tau_{\mathfrak{B}_2}\right)$ be two bipolar fuzzy S-Hausdorff spaces.

To prove : $(X \times Y, \tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2})$ is a bipolar fuzzy S-Hausdorff space

Consider two distinct bipolar fuzzy points $y_{(\alpha,\beta)}, z_{(\gamma,\delta)} \in X \times Y$ where $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Either $y_1 \neq z_1$ or $y_2 \neq z_2$.

Assume $y_1 \neq z_1$ implies $(y_1)_{(\alpha,\beta)} \neq (z_1)_{(\gamma,\delta)}$.

Then there exists two bipolar fuzzy open sets $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}_1}$ such that

${A_{bp}}^{+}(y_1) \geq \alpha, {A_{bp}}^{-}(y_1) \leq \beta, {B_{bp}}^{+}(z_1) \geq \gamma, {B_{bp}}^{-}(z_1) \leq \delta$.

Therefore $A_{bp} \times 1_{bp}$ and $B_{bp} \times 1_{bp} \in \tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2}$ where

$A_{bp} \times 1_{bp} = \left({A_{bp}}^{+} \times {1_{bp}}^{+}, {A_{bp}}^{-} \times {1_{bp}}^{-}\right)$ and $B_{bp} \times 1_{bp} = \left({B_{bp}}^{+} \times {1_{bp}}^{+}, {B_{bp}}^{-} \times {1_{bp}}^{-}\right)$

$({A_{bp}}^{+} \times {1_{bp}}^{+})(y_1, z_1) = \min\{{A_{bp}}^{+}(y_1), {1_{bp}}^{+}(z_1)\} \geq \alpha$

$({A_{bp}}^{-} \times {1_{bp}}^{-})(y_1, z_1) = \max\{{A_{bp}}^{-}(y_1), {1_{bp}}^{-}(z_1)\} \leq \beta$

$({B_{bp}}^{+} \times {1_{bp}}^{+})(y_2, z_2) = \min\{{B_{bp}}^{+}(y_2), {1_{bp}}^{+}(z_2)\} \geq \gamma$

$({B_{bp}}^{-} \times {1_{bp}}^{-})(y_2, z_2) = \max\{{B_{bp}}^{-}(y_2), {1_{bp}}^{-}(z_2)\} \leq \delta$.

Also $A_{bp} \cap B_{bp} = 0_{bp}$

The proof in the other case is similar.

Hence product of two bipolar fuzzy S-Hausdorff spaces is a bipolar fuzzy S-Hausdorff space.

Theorem:4.1.14

Arbitrary product of bipolar fuzzy S-Hausdorff spaces is a bipolar fuzzy S-Hausdorff space.

Proof:

Let $\{(X_\lambda, \tau_{\mathfrak{B}_\lambda}) \,/\, \lambda \in \Lambda\}$ be a collection of bipolar fuzzy S-Hausdorff spaces

Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ and $\tau_{\mathfrak{B}} = \prod_{\lambda\in\Lambda}\tau_{\mathfrak{B}_\lambda}$. Consider two distinct bipolar fuzzy points $((x_\lambda)_{\lambda\in\Lambda})_{(\alpha,\beta)}, ((y_\lambda)_{\lambda\in\Lambda})_{(\gamma,\delta)} \in \prod_{\lambda\in\Lambda} X_\lambda$. Then $(x_\lambda)_{\lambda\in\Lambda} \neq (y_\lambda)_{\lambda\in\Lambda}$. Assume $x_\mu \neq y_\mu$ for some $\mu \in \Lambda$, there exist two bipolar fuzzy open sets, $(A_{bp})_\mu, (B_{bp})_\mu \in \tau_{\mathfrak{B}_\mu}$ such that $({A_{bp}}^{+})_\mu(x_\mu) \geq \alpha$, $({A_{bp}}^{-})_\mu(x_\mu) \leq \beta$, $({B_{bp}}^{+})_\mu(y_\mu) \geq \gamma$, $({B_{bp}}^{-})_\mu(y_\mu) \leq \delta$ and $(A_{bp})_\mu \cap (B_{bp})_\mu = (0_{bp})_\mu$.

Let $A_{bp} = \prod_{\lambda\in\Lambda}(A_{bp})_\lambda$, where $(A_{bp})_\lambda = (1_{bp})_\lambda$ for $\lambda \neq \mu$ and

$B_{bp} = \prod_{\lambda\in\Lambda}(B_{bp})_\lambda$, where $(B_{bp})_\lambda = (1_{bp})_\lambda$ for $\lambda \neq \mu$.

Then $A_{bp}, B_{bp} \in \prod_{\lambda\in\Lambda}\tau_{\mathfrak{B}_\lambda}$ where $A_{bp} = \prod_{\lambda\in\Lambda}(A_{bp})_\lambda = \left(\prod_{\lambda\in\Lambda}({A_{bp}}^{+})_\lambda\,,\ \prod_{\lambda\in\Lambda}({A_{bp}}^{-})_\lambda\right)$

and $B_{bp} = \prod_{\lambda\in\Lambda}(B_{bp})_\lambda = \left(\prod_{\lambda\in\Lambda}({B_{bp}}^{+})_\lambda\,,\ \prod_{\lambda\in\Lambda}({B_{bp}}^{-})_\lambda\right)$.

$$\prod_{\lambda\in\Lambda}({A_{bp}}^{+})_\lambda\,(x_\lambda)_{\lambda\in\Lambda} = \min\{({A_{bp}}^{+})_\lambda(x_\lambda)_{\lambda\in\Lambda}\}$$

$$= ({A_{bp}}^{+})_\mu(x_\mu) \geq \alpha\text{ , for some } \mu \in \Lambda$$

$$\prod_{\lambda\in\Lambda}({A_{bp}}^{+})_\lambda\,(x_\lambda)_{\lambda\in\Lambda} \geq \alpha$$

$$\prod_{\lambda\in\Lambda}({A_{bp}}^{-})_\lambda\,(x_\lambda)_{\lambda\in\Lambda} = \max\{({A_{bp}}^{-})_\lambda(x_\lambda)_{\lambda\in\Lambda}\}$$

$$= ({A_{bp}}^{-})_\mu(x_\mu) \leq \beta\text{, for some } \mu \in \Lambda$$

$$\prod_{\lambda\in\Lambda}({A_{bp}}^{-})_\lambda\,(x_\lambda)_{\lambda\in\Lambda} \leq \beta.$$

Similarly, $\prod_{\lambda\in\Lambda}({B_{bp}}^{+})_\lambda\,(y_\lambda)_{\lambda\in\Lambda} \geq \gamma$ and $\prod_{\lambda\in\Lambda}({B_{bp}}^{-})_\lambda\,(y_\lambda)_{\lambda\in\Lambda} \leq \delta$.

Also $A_{bp} \cap B_{bp} = 0_{bp}$.

Hence arbitrary product of bipolar fuzzy S-Hausdorff spaces is a bipolar fuzzy S-Hausdorff space.

SECTION 4.2

W-HAUSDORFF, S-HAUSDORFF AND K-HAUSDORFF SEPARATION AXIOMS IN SECOND ORDER BIPOLAR FUZZY TOPOLOGICAL SPACES

Definition:4.2.1

Let $(X, \hat{\tau}_{\mathfrak{B}})$ be a second order bipolar fuzzy topological space and $Y \subseteq X$. If $\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ and $\hat{A}_{bp}/Y$ is the restriction function where $\left(\hat{A}_{bp}{}^{+}/Y\right)(z)(\alpha) = \hat{A}_{bp}{}^{+}(z)(\alpha)$ and $\left(\hat{A}_{bp}{}^{-}/Y\right)(z)(\alpha) = \hat{A}_{bp}{}^{-}(z)(\alpha)$, for all $z \in Y$, for all $\alpha \in I$, then $(\hat{\tau}_{\mathfrak{B}}/Y) = \{(\hat{A}_{bp}/Y)$ such that $\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\}$ is called the **second order bipolar fuzzy topology on Y** and $(Y, \hat{\tau}_{\mathfrak{B}}/Y)$ is called a **second order bipolar fuzzy subspace of** $(X, \hat{\tau}_{\mathfrak{B}})$.

Definition:4.2.2

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be **second order bipolar fuzzy W-Hausdorff space of type 1, denoted by $(\mathbf{SBPFW-H})_{\mathbf{1}}$**, if for every $x, y \in X$, $x \neq y$, there exist two second order bipolar fuzzy open sets $\hat{A}_{bp}, \hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ where $\hat{A}_{bp} = \left(\hat{A}_{bp}{}^{+}, \hat{A}_{bp}{}^{-}\right)$ and $\hat{B}_{bp} = \left(\hat{B}_{bp}{}^{+}, \hat{B}_{bp}{}^{-}\right)$ such that $\hat{A}_{bp}{}^{+}(x) = \mathbf{1}$, $\hat{A}_{bp}{}^{-}(x) = -\mathbf{1}$, $\hat{B}_{bp}{}^{+}(y) = \mathbf{1}$, $\hat{B}_{bp}{}^{-}(y) = -\mathbf{1}$ and $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$.

Definition:4.2.3

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be **second order bipolar fuzzy W-Hausdorff space of type 2, denoted by $(\mathbf{SBPFW-H})_{\mathbf{2}}$**, is defined by replacing the condition $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$ in the above definition by $\hat{A}_{bp} \cap_2 \hat{B}_{bp} = \hat{0}_{bp}$.

Example:4.2.4

Let X be a non-empty set. For any two distinct points $x, y \in X$, $x \neq y$, there exists two second order bipolar fuzzy open sets $\hat{A}_{bp}, \hat{B}_{bp}$ where $\hat{A}_{bp}{}^{+}: X \rightarrow [0,1]^{[0,1]}$,

$\widehat{A}_{bp}{}^{-}: X \to [-1,0]^{[0,1]}$ and $\widehat{B}_{bp}{}^{+}: X \to [0,1]^{[0,1]}$, $\widehat{B}_{bp}{}^{-}: X \to [-1,0]^{[0,1]}$ are defined as follows:

$\widehat{A}_{bp}{}^{+}(x)(\alpha) = 1, \widehat{A}_{bp}{}^{-}(x)(\alpha) = -1, \widehat{A}_{bp}{}^{+}(y)(\alpha) = 0, \widehat{A}_{bp}{}^{-}(y)(\alpha) = 0$

$\widehat{B}_{bp}{}^{+}(x)(\alpha) = 0, \widehat{B}_{bp}{}^{-}(x)(\alpha) = 0, \widehat{B}_{bp}{}^{+}(y)(\alpha) = 1, \widehat{B}_{bp}{}^{-}(y)(\alpha) = -1$

Then $\widehat{\tau}_{\mathfrak{B}} = \{\widehat{0}_{bp}, \widehat{1}_{bp}, \widehat{A}_{bp}, \widehat{B}_{bp}, \widehat{A}_{bp} \cup \widehat{B}_{bp}\}$ is a second order bipolar fuzzy topology on X.

For $x \in X$, since $\widehat{B}_{bp}{}^{+}(x)(\alpha) = 0, \widehat{B}_{bp}{}^{-}(x)(\alpha) = 0$, for every $\alpha \in I$,

$\widehat{A}_{bp}{}^{+}(x)(\alpha) \wedge \widehat{B}_{bp}{}^{+}(x)(\alpha) = 0$ and $\widehat{A}_{bp}{}^{-}(x)(\alpha) \vee \widehat{B}_{bp}{}^{-}(x)(\alpha) = 0$.

Similarly, for $y \in X$, since $\widehat{A}_{bp}{}^{+}(y)(\alpha) = 0, \widehat{A}_{bp}{}^{-}(y)(\alpha) = 0$, for every $\alpha \in I$,

$\widehat{A}_{bp}{}^{+}(y)(\alpha) \wedge \widehat{B}_{bp}{}^{+}(y)(\alpha) = 0$ and $\widehat{A}_{bp}{}^{-}(y)(\alpha) \vee \widehat{B}_{bp}{}^{-}(y)(\alpha) = 0$.

Hence $\widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \widehat{0}_{bp}$.

Note:4.2.5

$(X, \widehat{\tau}_{\mathfrak{B}})$ is $(SBPFW-H)_1 \Rightarrow (X, \widehat{\tau}_{\mathfrak{B}})$ is $(SBPFW-H)_2$. Since $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \widehat{0}_{bp} \Rightarrow \widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \widehat{0}_{bp}$. The converse of the above implication is not true since $\widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \widehat{0}_{bp}$ need not imply that $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \widehat{0}_{bp}$.

Theorem: 4.2.6

1. Second order bipolar fuzzy subspace of a $(SBPFW-H)_1$ space is $(SBPFW-H)_1$.
2. Second order bipolar fuzzy subspace of a $(SBPFW-H)_2$ space is $(SBPFW-H)_2$.

Proof

1. Let $(X, \widehat{\tau}_{\mathfrak{B}})$ be a $(SBPFW-H)_1$ space and $Y \subseteq X$. Then $(Y, \widehat{\tau}_{\mathfrak{B}}/Y)$ be a second order bipolar fuzzy subspace of $(X, \widehat{\tau}_{\mathfrak{B}})$. Consider $y_1, y_2 \in Y, y_1 \neq y_2$. Then $y_1, y_2 \in X$. Since $(X, \widehat{\tau}_{\mathfrak{B}})$ be a $(SBPFW-H)_1$, there exist $\widehat{A}_{bp}, \widehat{B}_{bp} \in \widehat{\tau}_{\mathfrak{B}}$ such that $\widehat{A}_{bp}{}^{+}(y_1) = \mathbf{1}$, $\widehat{A}_{bp}{}^{-}(y_1) = -\mathbf{1}$, $\widehat{B}_{bp}{}^{+}(y_2) = \mathbf{1}, \widehat{B}_{bp}{}^{-}(y_2) = -\mathbf{1}$ and $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \widehat{0}_{bp}$.

Therefore $\widehat{A}_{bp}/Y,\ \widehat{B}_{bp}/Y \in \hat{\tau}_{\mathfrak{B}}/Y$ where $\widehat{A}_{bp}/Y = \left(\widehat{A}_{bp}{}^{+}/Y, \widehat{A}_{bp}{}^{-}/Y\right)$ and $\widehat{B}_{bp}/Y = \left(\widehat{B}_{bp}{}^{+}/Y, \widehat{B}_{bp}{}^{-}/Y\right)$.

Then $\left(\widehat{A}_{bp}{}^{+}/Y\right)(y_1) = \widehat{A}_{bp}{}^{+}(y_1) = \mathbf{1}$ and $\left(\widehat{A}_{bp}{}^{-}/Y\right)(y_1) = \widehat{A}_{bp}{}^{-}(y_1) = -\mathbf{1}$,

$$\left(\widehat{B}_{bp}{}^{+}/Y\right)(y_2) = \widehat{B}_{bp}{}^{+}(y_2) = \mathbf{1} \text{ and } \left(\widehat{B}_{bp}{}^{-}/Y\right)(y_2) = \widehat{B}_{bp}{}^{-}(y_2) = -\mathbf{1}.$$

Consider $\left(\widehat{A}_{bp}/Y\right) \cap_1 \left(\widehat{B}_{bp}/Y\right) = \left(\left(\widehat{A}_{bp}{}^{+}/Y\right) \wedge \left(\widehat{B}_{bp}{}^{+}/Y\right), \left(\widehat{A}_{bp}{}^{-}/Y\right) \vee \left(\widehat{B}_{bp}{}^{-}/Y\right)\right)$

$$\begin{aligned}\left(\left(\widehat{A}_{bp}{}^{+}/Y\right) \wedge \left(\widehat{B}_{bp}{}^{+}/Y\right)\right)(y) &= \left(\widehat{A}_{bp}{}^{+}/Y\right)(y) \wedge \left(\widehat{B}_{bp}{}^{+}/Y\right)(y), \text{ for every } y \in Y \subseteq X\\ &= \widehat{A}_{bp}{}^{+}(y) \wedge \widehat{B}_{bp}{}^{+}(y), \text{ for every } y \in Y \subseteq X\\ &= \left(\widehat{A}_{bp}{}^{+} \wedge \widehat{B}_{bp}{}^{+}\right)(y), \text{ for every } y \in Y \subseteq X\\ &= \hat{0}_{bp}{}^{+}(y), \text{ for every } y \in Y \subseteq X.\end{aligned}$$

$$\begin{aligned}\left(\left(\widehat{A}_{bp}{}^{-}/Y\right) \vee \left(\widehat{B}_{bp}{}^{-}/Y\right)\right)(y) &= \left(\widehat{A}_{bp}{}^{-}/Y\right)(y) \vee \left(\widehat{B}_{bp}{}^{-}/Y\right)(y), \text{ for every } y \in Y \subseteq X\\ &= \widehat{A}_{bp}{}^{-}(y) \vee \widehat{B}_{bp}{}^{-}(y), \text{ for every } y \in Y \subseteq X\\ &= \left(\widehat{A}_{bp}{}^{-} \vee \widehat{B}_{bp}{}^{-}\right)(y), \text{ for every } y \in Y \subseteq X\\ &= \hat{0}_{bp}{}^{-}(y), \text{ for every } y \in Y \subseteq X.\end{aligned}$$

Thus $\left(\widehat{A}_{bp}/Y\right) \cap_1 \left(\widehat{B}_{bp}/Y\right) = \hat{0}_{bp}$.

Hence second order bipolar fuzzy subspace of $(SBPFW - H)_1$ is $(SBPFW - H)_1$.

The proof of 2 is obvious.

Theorem:4.2.7

1. Product of two $(SBPFW - H)_1$ spaces is $(SBPFW - H)_1$.
2. Product of two $(SBPFW - H)_2$ spaces is $(SBPFW - H)_2$.

Proof:

1. Let $\left(X, \hat{\tau}_{\mathfrak{B}_1}\right)$ and $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ be two $(SBPFW - H)_1$ spaces.

Consider two points $(x_1, y_1), (x_2, y_2) \in X \times Y$. Either $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume $x_1 \neq x_2$. Since $\left(X, \hat{\tau}_{\mathfrak{B}_1}\right)$ is $(SBPFW - H)_1$, there exist two second order bipolar fuzzy open sets $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}$ such that $\widehat{A}_{bp}{}^{+}(x_1) = \mathbf{1}$, $\widehat{A}_{bp}{}^{-}(x_1) = -\mathbf{1}$, $\widehat{B}_{bp}{}^{+}(x_2) = \mathbf{1}$, $\widehat{B}_{bp}{}^{-}(x_2) = -\mathbf{1}$ and $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

$\widehat{A}_{bp} \times \hat{1}_{bp} \in \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$ and $\widehat{B}_{bp} \times \hat{1}_{bp} \in \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$, where

$\widehat{A}_{bp} \times \hat{1}_{bp} = \left(\widehat{A}_{bp}{}^{+} \times \hat{1}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-} \times \hat{1}_{bp}{}^{-}\right)$ and $\widehat{B}_{bp} \times \hat{1}_{bp} = \left(\widehat{B}_{bp}{}^{+} \times \hat{1}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-} \times \hat{1}_{bp}{}^{-}\right)$

$\left(\widehat{A}_{bp}{}^{+} \times \hat{1}_{bp}{}^{+}\right)(x_1, y_1)(\alpha) = \widehat{A}_{bp}{}^{+}(x_1)(\alpha) \wedge \hat{1}_{bp}{}^{+}(y_1)(\alpha)$, for every $(x_1, y_1) \in X \times Y$
and for every $\alpha \in I$
$= \mathbf{1}(\alpha) \wedge \mathbf{1}(\alpha)$, for every $\alpha \in I$
$= \mathbf{1}(\alpha)$, for every $\alpha \in I$

$(\widehat{A}_{bp}{}^{-} \times \hat{1}_{bp}{}^{-})(x_1, y_1)(\alpha) = \widehat{A}_{bp}{}^{-}(x_1)(\alpha) \vee \hat{1}_{bp}{}^{-}(y_1)(\alpha)$, for every $(x_1, y_1) \in X \times Y$
and for every $\alpha \in I$
$= -\mathbf{1}(\alpha) \wedge -\mathbf{1}(\alpha)$, for every $\alpha \in I$
$= -\mathbf{1}(\alpha)$, for every $\alpha \in I$.

Similarly, $\left(\widehat{B}_{bp}{}^{+} \times \hat{1}_{bp}{}^{+}\right)(x_2, y_2) = \mathbf{1}$ and $(\widehat{B}_{bp}{}^{-} \times \hat{1}_{bp}{}^{-})(x_2, y_2) = -\mathbf{1}$**.**

For any $(x, y) \in X \times Y$, consider $\widehat{A}_{bp} \times \hat{1}_{bp} \neq \hat{0}_{bp}$

implies $\left(\widehat{A}_{bp}{}^{+} \times \hat{1}_{bp}{}^{+}\right)(x, y) \neq \mathbf{0}$, $(\widehat{A}_{bp}{}^{-} \times \hat{1}_{bp}{}^{-})(x, y) \neq \mathbf{0}$

implies $\widehat{A}_{bp}{}^{+}(x)(\alpha) \neq 0$ **,** $\widehat{A}_{bp}{}^{-}(x)(\alpha) \neq 0$, for every $x \in X$ and for some $\alpha \in I$

implies $\widehat{A}_{bp}{}^{+}(x) \neq \mathbf{0}$ **,** $\widehat{A}_{bp}{}^{-}(x) \neq \mathbf{0}$, for every $x \in X$

implies $\widehat{B}_{bp}{}^{+}(x) = \mathbf{0}$ **,** $\widehat{B}_{bp}{}^{-}(x) = \mathbf{0}$**,** for every $x \in X$ (since $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$)

implies $\widehat{B}_{bp}{}^{+}(x) \wedge \hat{1}_{bp}{}^{+}(y) = \mathbf{0}$**,** $\widehat{B}_{bp}{}^{-}(x) \vee \hat{1}_{bp}{}^{-}(y) = \mathbf{0}$

implies $\left(\widehat{B}_{bp}{}^{+} \times \hat{1}_{bp}{}^{+}\right)(x, y) = \mathbf{0}$, $(\widehat{B}_{bp}{}^{-} \times \hat{1}_{bp}{}^{-})(x, y) = \mathbf{0}$

implies $\widehat{B}_{bp} \times \hat{1}_{bp} = \mathbf{0}$

Thus $(\widehat{A}_{bp} \times \hat{1}_{bp}) \cap_1 (\widehat{B}_{bp} \times \hat{1}_{bp}) = \hat{0}_{bp}$.

The proof in the other case is the same.

Hence $(X{\times}Y, \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$ is $(\mathrm{SBPFW-H})_1$.

Since $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp} \Rightarrow \widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \hat{0}_{bp}$, $(X{\times}Y, \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$ is a $(\mathrm{SBPFW-H})_2$.

Theorem:4.2.8

1. Arbitrary product of $(\mathrm{SBPFW-H})_1$ spaces is $(\mathrm{SBPFW-H})_1$.
2. Arbitrary product of $(\mathrm{SBPFW-H})_2$ spaces is $(\mathrm{SBPFW-H})_2$.

Proof:

Let $\left\{\left(X_{\lambda}, \hat{\tau}_{\mathfrak{B}_{\lambda}}\right) / \lambda \in \Lambda\right\}$ be a collection of $(\mathrm{SBPFW}-\mathrm{H})_1$ spaces. Let $X = \prod_{\lambda \in \Lambda} X_{\lambda}$ and $\hat{\tau}_{\mathfrak{B}} = \prod_{\lambda \in \Lambda} \hat{\tau}_{\mathfrak{B}_{\lambda}}$. Consider two distinct points $(x_{\lambda})_{\lambda \in \Lambda}, (y_{\lambda})_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} X_{\lambda}$.

Assume $x_{\mu} \neq y_{\mu}$ for some $\mu \in \Lambda$. Since $\left(X_{\mu}, \hat{\tau}_{\mathfrak{B}_{\mu}}\right)$ is $(\mathrm{SBPFW}-\mathrm{H})_1$, there exist two second order bipolar fuzzy open sets, $\left(\hat{A}_{bp}\right)_{\mu} = \left(\left(\hat{A}_{bp}{}^{+}\right)_{\mu}, \left(\hat{A}_{bp}{}^{-}\right)_{\mu}\right)$ and $\left(\hat{B}_{bp}\right)_{\mu} = \left(\left(\hat{B}_{bp}{}^{+}\right)_{\mu}, \left(\hat{B}_{bp}{}^{-}\right)_{\mu}\right) \in \hat{\tau}_{\mathfrak{B}_{\mu}}$ such that $\left(\hat{A}_{bp}{}^{+}\right)_{\mu}(x_{\mu}) = \mathbf{1}$, $\left(\hat{A}_{bp}{}^{-}\right)_{\mu}(x_{\mu}) = \mathbf{-1}$, $\left(\hat{B}_{bp}{}^{+}\right)_{\mu}(y_{\mu}) = \mathbf{1}$, $\left(\hat{B}_{bp}{}^{-}\right)_{\mu}(y_{\mu}) = \mathbf{-1}$ and $\left(\hat{A}_{bp}\right)_{\mu} \cap_1 \left(\hat{B}_{bp}\right)_{\mu} = \left(\hat{0}_{bp}\right)_{\mu}$.

Let $\hat{A}_{bp} = \prod_{\lambda \in \Lambda}\left(\hat{A}_{bp}\right)_{\lambda}$, where $\left(\hat{A}_{bp}\right)_{\lambda} = \left(\hat{1}_{bp}\right)_{\lambda}$ for $\lambda \neq \mu$ and

$\hat{B}_{bp} = \prod_{\lambda \in \Lambda}\left(\hat{B}_{bp}\right)_{\lambda}$, where $\left(\hat{B}_{bp}\right)_{\lambda} = \left(\hat{1}_{bp}\right)_{\lambda}$ for $\lambda \neq \mu$.

Then $\hat{A}_{bp}, \hat{B}_{bp} \in \prod_{\lambda \in \Lambda} \hat{\tau}_{\mathfrak{B}_{\lambda}}$

$\hat{A}_{bp} = \prod_{\lambda \in \Lambda}\left(\hat{A}_{bp}\right)_{\lambda} = \left(\wedge_{\lambda \in \Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}, \vee_{\lambda \in \Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)$ and

$\hat{B}_{bp} = \prod_{\lambda \in \Lambda}\left(\hat{B}_{bp}\right)_{\lambda} = \left(\wedge_{\lambda \in \Lambda}\left(\hat{B}_{bp}{}^{+}\right)_{\lambda}, \vee_{\lambda \in \Lambda}\left(\hat{B}_{bp}{}^{-}\right)_{\lambda}\right)$.

$$\begin{aligned}
\left(\wedge_{\lambda \in \Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x_{\lambda})_{\lambda \in \Lambda} &= \wedge\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x_{\lambda})_{\lambda \in \Lambda}\right) \\
&= \left(\hat{A}_{bp}{}^{+}\right)_{\mu}(x_{\mu}), \text{ for some } \mu \in \Lambda \\
&= \mathbf{1}
\end{aligned}$$

$$\begin{aligned}
\left(\vee_{\lambda \in \Lambda}\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x_{\lambda})_{\lambda \in \Lambda} &= \vee\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x_{\lambda})_{\lambda \in \Lambda}\right) \\
&= \left(\hat{A}_{bp}{}^{-}\right)_{\mu}(x_{\mu}), \text{ for some } \mu \in \Lambda \\
&= \mathbf{-1.}
\end{aligned}$$

Similarly, $\left(\wedge_{\lambda \in \Lambda}\left(\hat{B}_{bp}{}^{+}\right)_{\lambda}\right)(y_{\lambda})_{\lambda \in \Lambda} = \mathbf{1}$ and $\left(\vee_{\lambda \in \Lambda}\left(\hat{B}_{bp}{}^{-}\right)_{\lambda}\right)(y_{\lambda})_{\lambda \in \Lambda} = \mathbf{-1.}$

To prove $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$ where $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \left(\hat{A}_{bp}{}^{+} \wedge \hat{B}_{bp}{}^{+}, \hat{A}_{bp}{}^{-} \vee \hat{B}_{bp}{}^{-}\right)$

$$\begin{aligned}
\left(\hat{A}_{bp}{}^{+} \wedge \hat{B}_{bp}{}^{+}\right)(x_{\lambda})_{\lambda \in \Lambda} &= \left(\left(\wedge_{\lambda \in \Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right) \wedge \left(\wedge_{\lambda \in \Lambda}\left(\hat{B}_{bp}{}^{+}\right)_{\lambda}\right)\right)(x_{\lambda})_{\lambda \in \Lambda} \\
&= \left(\left(\wedge_{\lambda \in \Lambda}\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x_{\lambda})_{\lambda \in \Lambda}\right) \wedge \left(\left(\wedge_{\lambda \in \Lambda}\left(\hat{B}_{bp}{}^{+}\right)_{\lambda}\right)(x_{\lambda})_{\lambda \in \Lambda}\right)
\end{aligned}$$

$$= \left(\wedge_{\lambda\in\Lambda}\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x_\lambda)_{\lambda\in\Lambda}\right) \wedge \left(\wedge_{\lambda\in\Lambda}\left(\widehat{B}_{bp}{}^{+}\right)_{\lambda}(x_\lambda)_{\lambda\in\Lambda}\right)$$

$$= \left(\widehat{A}_{bp}{}^{+}\right)_{\mu}(x_\mu) \wedge \left(\widehat{B}_{bp}{}^{+}\right)_{\mu}(x_\mu), \text{ for some } \mu \in \Lambda$$

$$= \left(\left(\widehat{A}_{bp}{}^{+}\right)_{\mu} \wedge \left(\widehat{B}_{bp}{}^{+}\right)_{\mu}\right)(x_\mu), \text{ for some } \mu \in \Lambda$$

$$= \mathbf{0}$$

$$\left(\widehat{A}_{bp}{}^{-} \vee \widehat{B}_{bp}{}^{+}\right)(x_\lambda)_{\lambda\in\Lambda} = \left(\left(\vee_{\lambda\in\Lambda}\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right) \vee \left(\vee_{\lambda\in\Lambda}\left(\widehat{B}_{bp}{}^{-}\right)_{\lambda}\right)\right)(x_\lambda)_{\lambda\in\Lambda}$$

$$= \left(\left(\vee_{\lambda\in\Lambda}\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x_\lambda)_{\lambda\in\Lambda}\right) \vee \left(\left(\vee_{\lambda\in\Lambda}\left(\widehat{B}_{bp}{}^{-}\right)_{\lambda}\right)(x_\lambda)_{\lambda\in\Lambda}\right)$$

$$= \left(\vee_{\lambda\in\Lambda}\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x_\lambda)_{\lambda\in\Lambda}\right) \vee \left(\vee_{\lambda\in\Lambda}\left(\widehat{B}_{bp}{}^{-}\right)_{\lambda}(x_\lambda)_{\lambda\in\Lambda}\right)$$

$$= \left(\widehat{A}_{bp}{}^{-}\right)_{\mu}(x_\mu) \vee \left(\widehat{B}_{bp}{}^{-}\right)_{\mu}(x_\mu), \text{ for some } \mu \in \Lambda$$

$$= \left(\left(\widehat{A}_{bp}{}^{-}\right)_{\mu} \vee \left(\widehat{B}_{bp}{}^{-}\right)_{\mu}\right)(x_\mu), \text{ for some } \mu \in \Lambda$$

$$= \mathbf{0}.$$

Thus $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \widehat{0}_{bp}$.

Hence $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFW - H)_1$.

$(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFW - H)_2$ as $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \widehat{0}_{bp}$ implies $\widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \widehat{0}_{bp}$.

Definition:4.2.9

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be **second order bipolar fuzzy K-Hausdorff space of type 1, denoted by** $(\mathbf{SBPFK - H})_{\mathbf{1}}$, if for every $x, y \in X$, $x \neq y$, there exist two bipolar fuzzy open sets $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ such that $\widehat{A}_{bp}{}^{+}(x) > \mathbf{0}$, $\widehat{A}_{bp}{}^{-}(x) < \mathbf{0}$, $\widehat{B}_{bp}{}^{+}(y) > \mathbf{0}$, $\widehat{B}_{bp}{}^{-}(y) < \mathbf{0}$ and $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \widehat{0}_{bp}$.

Definition:4.2.10

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be **second order bipolar fuzzy K-Hausdorff space of type 2, denoted by** $(\mathbf{SBPFK - H})_{\mathbf{2}}$, is defined by replacing the condition $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \widehat{0}_{bp}$ in the above definition by $\widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \widehat{0}_{bp}$.

Example:4.2.11

Let X be a non-empty set. For any two distinct points $x, y \in X$, $x \neq y$, there exists two second order bipolar fuzzy open sets $\widehat{A}_{bp}, \widehat{B}_{bp}$ where ${\widehat{A}_{bp}}^{+}: X \to [0,1]^{[0,1]}$, ${\widehat{A}_{bp}}^{-}: X \to [-1,0]^{[0,1]}$ and ${\widehat{B}_{bp}}^{+}: X \to [0,1]^{[0,1]}$, ${\widehat{B}_{bp}}^{-}: X \to [-1,0]^{[0,1]}$ are defined as follows:

${\widehat{A}_{bp}}^{+}(x)(\alpha) = 0.5 > 0, {\widehat{A}_{bp}}^{-}(x)(\alpha) = -0.4 < 0,$

${\widehat{B}_{bp}}^{+}(y)(\alpha) = 0.2 > 0, {\widehat{B}_{bp}}^{-}(y)(\alpha) = -0.9 < 0$

Then $\hat{\tau}_{\mathfrak{B}} = \{\hat{0}_{bp}, \hat{1}_{bp}, \widehat{A}_{bp}, \widehat{B}_{bp}, \widehat{A}_{bp} \cup \widehat{B}_{bp}\}$ is a second order bipolar fuzzy topology on X

For $x \in X$, since ${\widehat{B}_{bp}}^{+}(x)(\alpha) = 0, {\widehat{B}_{bp}}^{-}(x)(\alpha) = 0$, for every $\alpha \in I$,

${\widehat{A}_{bp}}^{+}(x)(\alpha) \wedge {\widehat{B}_{bp}}^{+}(x)(\alpha) = 0$ and ${\widehat{A}_{bp}}^{-}(x)(\alpha) \vee {\widehat{B}_{bp}}^{-}(x)(\alpha) = 0.$

Similarly, for $y \in X$, since ${\widehat{A}_{bp}}^{+}(y)(\alpha) = 0, {\widehat{A}_{bp}}^{-}(y)(\alpha) = 0$, for every $\alpha \in I$,

${\widehat{A}_{bp}}^{+}(y)(\alpha) \wedge {\widehat{B}_{bp}}^{+}(y)(\alpha) = 0$ and ${\widehat{A}_{bp}}^{-}(y)(\alpha) \vee {\widehat{B}_{bp}}^{-}(y)(\alpha) = 0.$

Hence $\widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \hat{0}_{bp}$.

Note:4.2.12

1. $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFK - H)_1 \Rightarrow (X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFK - H)_2$. Since $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$ implies $\widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \hat{0}_{bp}$. The converse of the above implication is not true since $\widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \hat{0}_{bp}$ need not imply that $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

2. Theorem 4.2.6, 4.2.7 and 4.2.8 proved for separation axioms $(SBPFW - H)_1$ and $(SBPFW - H)_2$ have exact parallels for the separation axioms $(SBPFK - H)_1$ and $(SBPFK - H)_2$.

Definition:4.2.13

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be **second order bipolar fuzzy S-Hausdorff space of type 1, denoted by $(\mathbf{SBPFS - H})_1$**, if for any pair of distinct second order bipolar fuzzy points $\hat{x}_{(r,s)}$, $\hat{y}_{(m,n)}$ in X, there exist two second order bipolar fuzzy open sets $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ such that $\hat{x}_{(r,s)} \in \widehat{A}_{bp}$, $\hat{y}_{(m,n)} \in \widehat{B}_{bp}$ and $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

Definition:4.2.14

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be **second order bipolar fuzzy S-Hausdorff space of type 2, denoted by $(\mathbf{SBPFS - H})_2$**, is defined by replacing the condition $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$ in the above definition by $\widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \hat{0}_{bp}$.

Note:4.2.15

$(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS - H)_1 \Rightarrow (X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS - H)_2$.

Theorem: 4.2.16

1. Second order bipolar fuzzy subspace of a $(SBPFS - H)_1$ space is $(SBPFS - H)_1$.
2. Second order bipolar fuzzy subspace of a $(SBPFS - H)_2$ space is $(SBPFS - H)_2$.

Proof

1. Let $(X, \hat{\tau}_{\mathfrak{B}})$ be $(SBPFS - H)_1$ and Y be a nonempty subset of X. Consider a pair of distinct second order bipolar fuzzy points $\hat{y}_{(r,s)}, \hat{z}_{(m,n)}$ in Y. Therefore $\hat{y}_{(r,s)}$, $\hat{z}_{(m,n)}$ are the distinct second order bipolar fuzzy points in X. Then there exist two second order bipolar fuzzy open sets $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ such that $\hat{y}_{(r,s)} \in \widehat{A}_{bp}$, $\hat{z}_{(m,n)} \in \widehat{B}_{bp}$ and $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$. Then $\widehat{A}_{bp}/Y$, $\widehat{B}_{bp}/Y \in \hat{\tau}_{\mathfrak{B}}/Y$ where

   $\widehat{A}_{bp}/Y = \left(\widehat{A}_{bp}{}^{+}/Y, \widehat{A}_{bp}{}^{-}/Y\right)$ and $\widehat{B}_{bp}/Y = \left(\widehat{B}_{bp}{}^{+}/Y, \widehat{B}_{bp}{}^{-}/Y\right)$.

   $\left(\widehat{A}_{bp}{}^{+}/Y\right)(y)(\alpha) = \widehat{A}_{bp}{}^{+}(y)(\alpha) \geq r$, for every $\alpha \in (0,1]$

   $\left(\widehat{A}_{bp}{}^{-}/Y\right)(y)(\alpha) = \widehat{A}_{bp}{}^{-}(y)(\alpha) \leq s$, for every $\alpha \in [-1,0)$

   $\left(\widehat{B}_{bp}{}^{+}/Y\right)(z)(\alpha) = \widehat{B}_{bp}{}^{+}(z)(\alpha) \geq m$, for every $\alpha \in (0,1]$

   $\left(\widehat{B}_{bp}{}^{-}/Y\right)(z)(\alpha) = \widehat{B}_{bp}{}^{-}(z)(\alpha) \leq n$, for every $\alpha \in [-1,0)$

   implies $\hat{y}_{(r,s)} \in \widehat{A}_{bp}/Y$, $\hat{z}_{(m,n)} \in \widehat{B}_{bp}/Y$.

   Also $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$ implies $\left(\widehat{A}_{bp}/Y\right) \cap_1 \left(\widehat{B}_{bp}/Y\right) = \hat{0}_{bp}$.

   Hence second order bipolar fuzzy subspace of a $(SBPFS - H)_1$ is $(SBPFS - H)_1$.

Proof of (2) is obvious.

Theorem:4.2.17

1. Product of two $(\mathrm{SBPFS}-\mathrm{H})_1$ spaces is $(\mathrm{SBPFS}-\mathrm{H})_1$.
2. Product of two $(\mathrm{SBPFS}-\mathrm{H})_2$ spaces is $(\mathrm{SBPFS}-\mathrm{H})_2$.

Proof:

1. Let $\left(X,\hat{\tau}_{\mathfrak{B}_1}\right)$ and $\left(Y,\hat{\tau}_{\mathfrak{B}_2}\right)$ be two $(\mathrm{SBPFS}-\mathrm{H})_1$ spaces.

Consider two distinct second order bipolar fuzzy points $\hat{z}_{(r,s)}$, $\hat{u}_{(m,n)} \in X\times Y$, where $z=(x,y)$ and $u=(p,q)$. Either $x\neq p$ or $y\neq q$. Assume $x\neq p$, then $x_{(r,s)}\neq p_{(m,n)}$. Then there exist two second order bipolar fuzzy open sets $\widehat{A}_{bp},\widehat{B}_{bp}\in\hat{\tau}_{\mathfrak{B}}$ such that $\hat{x}_{(r,s)}\in\widehat{A}_{bp}$, $\hat{p}_{(m,n)}\in\widehat{B}_{bp}$ and $\widehat{A}_{bp}\cap_1\widehat{B}_{bp}=\hat{0}_{bp}$. Therefore $\widehat{A}_{bp}\times\hat{1}_{bp}\in\hat{\tau}_{\mathfrak{B}_1}\times\hat{\tau}_{\mathfrak{B}_2}$ and $\widehat{B}_{bp}\times\hat{1}_{bp}\in\hat{\tau}_{\mathfrak{B}_1}\times\hat{\tau}_{\mathfrak{B}_2}$, where $\widehat{A}_{bp}\times\hat{1}_{bp}=\left(\widehat{A}_{bp}{}^{+}\times\hat{1}_{bp}{}^{+},\widehat{A}_{bp}{}^{-}\times\hat{1}_{bp}{}^{-}\right)$ and

$\widehat{B}_{bp}\times\hat{1}_{bp}=\left(\widehat{B}_{bp}{}^{+}\times\hat{1}_{bp}{}^{+},\ \widehat{B}_{bp}{}^{-}\times\hat{1}_{bp}{}^{-}\right)$.

$\left(\widehat{A}_{bp}{}^{+}\times\hat{1}_{bp}{}^{+}\right)(x,y)(\alpha)=\widehat{A}_{bp}{}^{+}(x)(\alpha)\wedge\hat{1}_{bp}{}^{+}(y)(\alpha)$, for every $(x,y)\in X\times Y$,

for every $\alpha\in I$

$=\widehat{A}_{bp}{}^{+}(x)(\alpha)\geq r$ (since $\hat{x}_{(r,s)}\in\widehat{A}_{bp}$)

$\left(\widehat{A}_{bp}{}^{-}\times\hat{1}_{bp}{}^{-}\right)(x,y)(\alpha)=\widehat{A}_{bp}{}^{-}(x)(\alpha)\vee\hat{1}_{bp}{}^{-}(y)(\alpha)$, for every $(x,y)\in X\times Y$,

for every $\alpha\in I$

$=\widehat{A}_{bp}{}^{-}(x)(\alpha)\leq s$ (since $\hat{x}_{(r,s)}\in\widehat{A}_{bp}$).

Therefore $\hat{z}_{(r,s)}\in\widehat{A}_{bp}\times\hat{1}_{bp}$.

Similarly, $\left(\widehat{B}_{bp}{}^{+}\times\hat{1}_{bp}{}^{+}\right)(p,q)(\alpha)=\widehat{B}_{bp}{}^{+}(p)(\alpha)\geq m$ (since $\hat{p}_{(m,n)}\in\widehat{B}_{bp}$)

$\left(\widehat{B}_{bp}{}^{-}\times\hat{1}_{bp}{}^{-}\right)(p,q)(\alpha)=\widehat{B}_{bp}{}^{-}(p)(\alpha)\leq n$ (since $\hat{p}_{(m,n)}\in\widehat{B}_{bp}$).

Therefore $\hat{u}_{(m,n)}\in\widehat{B}_{bp}\times\hat{1}_{bp}$.

Also $\widehat{A}_{bp}\cap_1\widehat{B}_{bp}=\hat{0}_{bp}$ implies $\left(\widehat{A}_{bp}\times\hat{1}_{bp}\right)\cap_1\left(\widehat{B}_{bp}\times\hat{1}_{bp}\right)=\hat{0}_{bp}$.

Proof for $y\neq q$ is similar.

Hence $(X\times Y,\hat{\tau}_{\mathfrak{B}_1}\times\hat{\tau}_{\mathfrak{B}_2})$ is $(\mathrm{SBPFS}-\mathrm{H})_1$.

2. Since $\widehat{A}_{bp}\cap_1\widehat{B}_{bp}=\hat{0}_{bp}\Rightarrow\widehat{A}_{bp}\cap_2\widehat{B}_{bp}=\hat{0}_{bp}$, $(X\times Y,\hat{\tau}_{\mathfrak{B}_1}\times\hat{\tau}_{\mathfrak{B}_2})$ is $(\mathrm{SBPFS}-\mathrm{H})_2$.

Theorem:4.2.18

1. An Arbitrary product of $(\mathrm{SBPFS}-\mathrm{H})_1$ spaces is $(\mathrm{SBPFS}-\mathrm{H})_1$.
2. An Arbitrary product of $(\mathrm{SBPFS}-\mathrm{H})_2$ spaces is $(\mathrm{SBPFS}-\mathrm{H})_2$.

Proof:

1. Let $\{(X_\lambda, \hat{\tau}_{\mathfrak{B}_\lambda})/\lambda \in \Lambda\}$ be a collection of $(SBPFS-\mathrm{H})_1$ spaces.

Let $X=\prod_{\lambda\in\Lambda} X_\lambda$ and $\hat{\tau}_{\mathfrak{B}}=\prod_{\lambda\in\Lambda}\hat{\tau}_{\mathfrak{B}_\lambda}$. Consider two distinct second order bipolar fuzzy points $((x_\lambda)_{\lambda\in\Lambda})_{(r,s)}, ((y_\lambda)_{\lambda\in\Lambda})_{(m,n)} \in \prod_{\lambda\in\Lambda} X_\lambda$. Then $(x_\lambda)_{\lambda\in\Lambda} \neq (y_\lambda)_{\lambda\in\Lambda}$.

Assume $x_\mu \neq y_\mu$ for some $\mu \in \Lambda$. Since $\left(X_\mu, \hat{\tau}_{\mathfrak{B}_\mu}\right)$ is $(\mathrm{SBPFS}-\mathrm{H})_1$, there exist two second order bipolar fuzzy open sets, $\left(\hat{A}_{bp}\right)_\mu=\left(\left(\hat{A}_{bp}{}^{+}\right)_\mu, \left(\hat{A}_{bp}{}^{-}\right)_\mu\right)$ and $\left(\hat{B}_{bp}\right)_\mu=\left(\left(\hat{B}_{bp}{}^{+}\right)_\mu, \left(\hat{B}_{bp}{}^{-}\right)_\mu\right) \in \hat{\tau}_{\mathfrak{B}_\mu}$ such that $\left(\hat{x}_\mu\right)_{(\alpha,\beta)} \in \left(\hat{A}_{bp}\right)_\mu$, $\left(\hat{y}_\mu\right)_{(\gamma,\delta)} \in \left(\hat{B}_{bp}\right)_\mu$ and $\left(\hat{A}_{bp}\right)_\mu \cap_1 \left(\hat{B}_{bp}\right)_\mu=\left(\hat{0}_{bp}\right)_\mu$.

Let $\hat{A}_{bp}=\prod_{\lambda\in\Lambda}\left(\hat{A}_{bp}\right)_\lambda$, where $\left(\hat{A}_{bp}\right)_\lambda=\left(\hat{1}_{bp}\right)_\lambda$ for $\lambda \neq \mu$ and

$\hat{B}_{bp}=\prod_{\lambda\in\Lambda}\left(\hat{B}_{bp}\right)_\lambda$, where $\left(\hat{B}_{bp}\right)_\lambda=\left(\hat{1}_{bp}\right)_\lambda$ for $\lambda \neq \mu$.

Then $\hat{A}_{bp}, \hat{B}_{bp} \in \prod_{\lambda\in\Lambda}\hat{\tau}_{\mathfrak{B}_\lambda}$.

$\hat{A}_{bp}=\prod_{\lambda\in\Lambda}\left(\hat{A}_{bp}\right)_\lambda=\left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_\lambda, \vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_\lambda\right)$ and

$\hat{B}_{bp}=\prod_{\lambda\in\Lambda}\left(\hat{B}_{bp}\right)_\lambda=\left(\wedge_{\lambda\in\Lambda}\left(\hat{B}_{bp}{}^{+}\right)_\lambda, \vee_{\lambda\in\Lambda}\left(\hat{B}_{bp}{}^{-}\right)_\lambda\right)$.

$$\begin{aligned}\left(\hat{A}_{bp}{}^{+}\right)((x_\lambda)_{\lambda\in\Lambda})(\alpha) &= \left(\wedge_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{+}\right)_\lambda\right)((x_\lambda)_{\lambda\in\Lambda})(\alpha)\\ &= \wedge_{\lambda\in\Lambda}\left(\left(\hat{A}_{bp}{}^{+}\right)_\lambda (x_\lambda)_{\lambda\in\Lambda}(\alpha)\right)\\ &= \left(\hat{A}_{bp}{}^{+}\right)_\mu (x_\mu)(\alpha) \geq r, \text{ for some } \mu \in \Lambda\end{aligned}$$

$$\begin{aligned}\left(\hat{A}_{bp}{}^{-}\right)((x_\lambda)_{\lambda\in\Lambda})(\alpha) &= \left(\vee_{\lambda\in\Lambda}\left(\hat{A}_{bp}{}^{-}\right)_\lambda\right)((x_\lambda)_{\lambda\in\Lambda})(\alpha)\\ &= \vee_{\lambda\in\Lambda}\left(\left(\hat{A}_{bp}{}^{-}\right)_\lambda (x_\lambda)_{\lambda\in\Lambda}(\alpha)\right)\\ &= \left(\hat{A}_{bp}{}^{-}\right)_\mu (x_\mu)(\alpha) \leq s, \text{ for some } \mu \in \Lambda.\end{aligned}$$

Therefore $(\hat{x}_\lambda)_{(\alpha,\beta)} \in \hat{A}_{bp}$.

Similarly, $\left(\hat{B}_{bp}{}^{+}\right)((y_\lambda)_{\lambda\in\Lambda})(\alpha)=\left(\hat{B}_{bp}{}^{+}\right)_\mu (y_\mu)(\alpha) \geq m$, for some $\mu \in \Lambda$.

$\left(\widehat{B}_{bp}{}^{-}\right)\left((y_\lambda)_{\lambda\in\Lambda}\right)(\alpha) = \left(\widehat{B}_{bp}{}^{-}\right)_{\mu}\left(y_\mu\right)(\alpha) \le n$, for some $\mu \in \Lambda$.

Therefore $(\hat{y}_\lambda)_{(\gamma,\delta)} \in \widehat{B}_{bp}$.

Hence $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS - H)_1$.

Proof of (2) is similar as (1) since $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$ implies $\widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \hat{0}_{bp}$.

SECTION - 4.3

RELATIONS BETWEEN FIRST AND SECOND ORDER BIPOLAR FUZZY HAUSDORFF SEPARATION AXIOMS

Theorem:4.3.1

$(X, \tau_{\mathfrak{B}})$ is $(\mathrm{BPFW}-\mathrm{H})$ if and only if $(X, \hat{\tau}_{\mathfrak{B}})$ is $(\mathrm{SBPFW}-\mathrm{H})_1$, where $(X, \hat{\tau}_{\mathfrak{B}})$ is from $(X, \tau_{\mathfrak{B}})$ through the relation R_1.

Proof :

Assume $(X, \tau_{\mathfrak{B}})$ is $(\mathrm{BPFW}-\mathrm{H})$. Given $\tau_{\mathfrak{B}}$, $\hat{\tau}_{\mathfrak{B}} = \{\widehat{A}_{bp} \,/\, A_{bp} \in \tau_{\mathfrak{B}}\}$ where $\widehat{A}_{bp}{}^{+}(x)(\alpha) = A_{bp}{}^{+}(x)$, $\widehat{A}_{bp}{}^{-}(x)(\alpha) = A_{bp}{}^{-}(x)$, for every $x \in X$ and for every $\alpha \in I$. Consider $x, y \in X$, $x \neq y$. Since $(X, \tau_{\mathfrak{B}})$ is $(\mathrm{BPFW}-\mathrm{H})$, there exist two bipolar fuzzy open sets $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}}$ such that $A_{bp}{}^{+}(x) = 1, A_{bp}{}^{-}(x) = -1$, $B_{bp}{}^{+}(y) = 1, B_{bp}{}^{-}(y) = -1$ and $A_{bp} \cap B_{bp} = 0_{bp}$. Then $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$.

$\widehat{A}_{bp}{}^{+}(x)(\alpha) = A_{bp}{}^{+}(x) = 1 \Rightarrow \widehat{A}_{bp}{}^{+}(x) = \mathbf{1}$ and

$\widehat{A}_{bp}{}^{-}(x)(\alpha) = A_{bp}{}^{-}(x) = -1 \Rightarrow \widehat{A}_{bp}{}^{-}(x) = \mathbf{-1}.$

$\widehat{B}_{bp}{}^{+}(y)(\alpha) = B_{bp}{}^{+}(y) = 1 \quad \Rightarrow \widehat{B}_{bp}{}^{+}(y) = \mathbf{1}$ and

$\widehat{B}_{bp}{}^{-}(y)(\alpha) = B_{bp}{}^{-}(y) = -1 \Rightarrow \widehat{B}_{bp}{}^{-}(y) = \mathbf{-1}.$

Claim: $A_{bp} \cap B_{bp} = 0_{bp}$ implies $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

For $z \in X$, $\widehat{A}_{bp}{}^{+}(z) \neq \mathbf{0}$, $\widehat{A}_{bp}{}^{-}(z) \neq \mathbf{0}$ implies $\widehat{A}_{bp}{}^{+}(z)(\alpha) \neq 0$, $\widehat{A}_{bp}{}^{-}(z)(\alpha) \neq 0$, for some $\alpha \in I$ implies $A_{bp}{}^{+}(z) \neq 0$, $A_{bp}{}^{-}(z) \neq 0$

implies $B_{bp}{}^{+}(z) = 0$, $B_{bp}{}^{-}(z) = 0$ (since $A_{bp} \cap B_{bp} = 0_{bp}$) implies $\widehat{B}_{bp}{}^{+}(z)(\alpha) = 0$, $\widehat{B}_{bp}{}^{-}(z)(\alpha) = 0$, for every $\alpha \in I$ implies $\widehat{B}_{bp}{}^{+}(z) = \mathbf{0}$, $\widehat{B}_{bp}{}^{-}(z) = \mathbf{0}$

implies $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

Hence $(X, \hat{\tau}_{\mathfrak{B}})$ is $(\mathrm{SBPFW}-\mathrm{H})_1$.

To prove the converse, it is enough to observe that $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$ implies $A_{bp} \cap B_{bp} = 0_{bp}$.

$\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$ implies $\hat{A}_{bp}{}^{+}(x) = \mathbf{0}$ and $\hat{A}_{bp}{}^{-}(x) = \mathbf{0}$ or $\hat{B}_{bp}{}^{+}(x) = \mathbf{0}$ and $\hat{B}_{bp}{}^{-}(x) = \mathbf{0}$, for every $x \in X$. $\hat{A}_{bp}{}^{+}(x)(\alpha) = 0$ and $\hat{A}_{bp}{}^{-}(x)(\alpha) = 0$ or $\hat{B}_{bp}{}^{+}(x)(\alpha) = 0$ and $\hat{B}_{bp}{}^{-}(x)(\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$. $A_{bp}{}^{+}(x) = 0$ and $A_{bp}{}^{-}(x) = 0$ or $B_{bp}{}^{+}(x) = 0$ and $B_{bp}{}^{-}(x) = 0$, for every $x \in X$.

Hence $A_{bp} \cap B_{bp} = 0_{bp}$.

Theorem:4.3.2

If $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFW - H)_1$ then for $\alpha \in I$, $(X, (\hat{\tau}_{\mathfrak{B}})_{\alpha})$ is bipolar fuzzy W-Hausdorff where $(X, (\hat{\tau}_{\mathfrak{B}})_{\alpha})$ is from $(X, \hat{\tau}_{\mathfrak{B}})$ through the relation R_3.

Proof :

Assume $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFW - H)_1$. Given $\hat{\tau}_{\mathfrak{B}}$, for $\alpha \in I$, $(\hat{\tau}_{\mathfrak{B}})_{\alpha} = \left\{ \left(\hat{A}_{bp}\right)_{\alpha} / \hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}} \right\}$ where $\left(\hat{A}_{bp}\right)_{\alpha} = \left(\left(\hat{A}_{bp}{}^{+}\right)_{\alpha}, \left(\hat{A}_{bp}{}^{-}\right)_{\alpha} \right)$ such that $\left(\hat{A}_{bp}{}^{+}\right)_{\alpha}(x) = \hat{A}_{bp}{}^{+}(x)(\alpha)$, $\left(\hat{A}_{bp}{}^{-}\right)_{\alpha}(x) = \hat{A}_{bp}{}^{-}(x)(\alpha)$, for every $x \in X$.

Consider $x, y \in X$, $x \neq y$. Since $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFW - H)_1$, there exist two second order bipolar fuzzy open sets $\hat{A}_{bp}$, $\hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ such that $\hat{A}_{bp}{}^{+}(x) = \mathbf{1}$, $\hat{A}_{bp}{}^{-}(x) = -\mathbf{1}$, $\hat{B}_{bp}{}^{+}(y) = \mathbf{1}$, $\hat{B}_{bp}{}^{-}(y) = -\mathbf{1}$ and $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$.

Then $\left(\hat{A}_{bp}\right)_{\alpha}, \left(\hat{B}_{bp}\right)_{\alpha} \in (\hat{\tau}_{\mathfrak{B}})_{\alpha}$.

$\left(\hat{A}_{bp}{}^{+}\right)_{\alpha}(x) = \hat{A}_{bp}{}^{+}(x)(\alpha) = \mathbf{1}(\alpha) = 1$ and

$\left(\hat{A}_{bp}{}^{-}\right)_{\alpha}(x) = \hat{A}_{bp}{}^{-}(x)(\alpha) = -\mathbf{1}(\alpha) = -1$.

$\left(\hat{B}_{bp}{}^{+}\right)_{\alpha}(y) = \hat{B}_{bp}{}^{+}(y)(\alpha) = \mathbf{1}(\alpha) = 1$ and

$\left(\hat{B}_{bp}{}^{-}\right)_{\alpha}(y) = \hat{B}_{bp}{}^{-}(y)(\alpha) = -\mathbf{1}(\alpha) = -1$.

Let $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$. Then $\hat{A}_{bp}{}^{+}(x) = \mathbf{0}$ and $\hat{A}_{bp}{}^{-}(x) = \mathbf{0}$ or $\hat{B}_{bp}{}^{+}(x) = \mathbf{0}$ and $\hat{B}_{bp}{}^{-}(x) = \mathbf{0}$, for every $x \in X$. $\hat{A}_{bp}{}^{+}(x)(\alpha) = 0$ and $\hat{A}_{bp}{}^{-}(x)(\alpha) = 0$ or $\hat{B}_{bp}{}^{+}(x)(\alpha) = 0$ and $\hat{B}_{bp}{}^{-}(x)(\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$. $\left(\hat{A}_{bp}{}^{+}\right)_{\alpha}(x) = 0$ and $\left(\hat{A}_{bp}{}^{-}\right)_{\alpha}(x) = 0$ or $\left(\hat{B}_{bp}{}^{+}\right)_{\alpha}(x) = 0$ and $\left(\hat{B}_{bp}{}^{-}\right)_{\alpha}(x) = 0$, for every $x \in X$.

Thus $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\alpha} \cap \left(\widehat{\mathrm{B}}_{\mathrm{bp}}\right)_{\alpha} = \left(\widehat{0}_{\mathrm{bp}}\right)_{\alpha}$.

Hence $(\mathrm{X}, (\hat{\tau}_{\mathfrak{B}})_{\alpha})$ is bipolar fuzzy W-Hausdorff.

Theorem:4.3.3

If $(\mathrm{X}, \hat{\tau}_{\mathfrak{B}})$ is $(\mathrm{SBPFW}-\mathrm{H})_1$ if and only if $(\mathrm{X}, (\hat{\tau}_{\mathfrak{B}})_{c})$ is $(\mathrm{SBPFW}-\mathrm{H})_1$, where $(\mathrm{X}, (\hat{\tau}_{\mathfrak{B}})_{c})$ is from $(\mathrm{X}, \hat{\tau}_{\mathfrak{B}})$ through the relation R_5.

Proof :

Assume $(\mathrm{X}, \hat{\tau}_{\mathfrak{B}})$ is $(\mathrm{SBPFW}-\mathrm{H})_1$. Given $\hat{\tau}_{\mathfrak{B}}$, $(\hat{\tau}_{\mathfrak{B}})_{c} = \left\{\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{c} / \widehat{\mathrm{A}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}}\right\}$ where $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{c} = \left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{c}, \left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{c}\right)$ such that $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{c}(\mathrm{x})(\alpha) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}(\mathrm{x})(1-\alpha)$, $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{c}(\mathrm{x})(\alpha) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}(\mathrm{x})(1-\alpha)$, for every $\mathrm{x} \in \mathrm{X}$ and for every $\alpha \in \mathrm{I}$.

Consider $\mathrm{x}, \mathrm{y} \in \mathrm{X}$, $\mathrm{x} \neq \mathrm{y}$. Since $(\mathrm{X}, \hat{\tau}_{\mathfrak{B}})$ is $(\mathrm{SBPFW}-\mathrm{H})_1$, there exist two second order bipolar fuzzy open sets $\widehat{\mathrm{A}}_{\mathrm{bp}}$, $\widehat{\mathrm{B}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}}$ such that $\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}(\mathrm{x}) = \mathbf{1}, \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}(\mathrm{x}) = -\mathbf{1}$, $\widehat{\mathrm{B}}_{\mathrm{bp}}{}^{+}(\mathrm{y}) = \mathbf{1}, \widehat{\mathrm{B}}_{\mathrm{bp}}{}^{-}(\mathrm{y}) = -\mathbf{1}$ and $\widehat{\mathrm{A}}_{\mathrm{bp}} \cap_1 \widehat{\mathrm{B}}_{\mathrm{bp}} = \widehat{0}_{\mathrm{bp}}$. Then $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{c}, \left(\widehat{\mathrm{B}}_{\mathrm{bp}}\right)_{c} \in (\hat{\tau}_{\mathfrak{B}})_{c}$.

$\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{c}(\mathrm{x})(\alpha) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}(\mathrm{x})(1-\alpha)$, for every $\mathrm{x} \in \mathrm{X}$ and for every $\alpha \in \mathrm{I}$

$= \mathbf{1}(1-\alpha)$, for every $\alpha \in \mathrm{I}$

$= 1 = \mathbf{1}(\alpha)$, for every $\alpha \in \mathrm{I}$.

Then $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{c}(\mathrm{x}) = \mathbf{1}$.

$\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{c}(\mathrm{x})(\alpha) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}(\mathrm{x})(1-\alpha)$, for every $\mathrm{x} \in \mathrm{X}$ and for every $\alpha \in \mathrm{I}$

$= -\mathbf{1}(1-\alpha)$, for every $\alpha \in \mathrm{I}$

$= -1 = -\mathbf{1}(\alpha)$, for every $\alpha \in \mathrm{I}$.

Then $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{c}(\mathrm{x}) = -\mathbf{1}$.

Similarly, $\left(\widehat{\mathrm{B}}_{\mathrm{bp}}{}^{+}\right)_{c}(\mathrm{y}) = \mathbf{1}$, $\left(\widehat{\mathrm{B}}_{\mathrm{bp}}{}^{-}\right)_{c}(\mathrm{y}) = -\mathbf{1}$.

Also $\widehat{\mathrm{A}}_{\mathrm{bp}} \cap_1 \widehat{\mathrm{B}}_{\mathrm{bp}} = \widehat{0}_{\mathrm{bp}}$.

implies $\widehat{A}_{bp}^{+}(x) = \mathbf{0}$ and $\widehat{A}_{bp}^{-}(x) = \mathbf{0}$ or $\widehat{B}_{bp}^{+}(x) = \mathbf{0}$ and $\widehat{B}_{bp}^{-}(x) = \mathbf{0}$, for every $x \in X$.

implies $\widehat{A}_{bp}^{+}(x)(\alpha) = 0$ and $\widehat{A}_{bp}^{-}(x)(\alpha) = 0$ or $\widehat{B}_{bp}^{+}(x)(\alpha) = 0$ and $\widehat{B}_{bp}^{-}(x)(\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$

implies $\left(\widehat{A}_{bp}^{+}\right)(x)(1-\alpha) = 0$ and $\left(\widehat{A}_{bp}^{-}\right)(x)(1-\alpha) = 0$ or $\left(\widehat{B}_{bp}^{+}\right)(x)(1-\alpha) = 0$ and $\left(\widehat{B}_{bp}^{-}\right)(x)(1-\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$

implies $\left(\widehat{A}_{bp}^{+}\right)_c(x)(\alpha) = 0$ and $\left(\widehat{A}_{bp}^{-}\right)_c(x)(\alpha) = 0$ or $\left(\widehat{B}_{bp}^{+}\right)_c(x)(\alpha) = 0$ and $\left(\widehat{B}_{bp}^{-}\right)_c(x)(\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$

implies $\left(\widehat{A}_{bp}^{+}\right)_c(x) = \mathbf{0}$ and $\left(\widehat{A}_{bp}^{-}\right)_c(x) = \mathbf{0}$ or $\left(\widehat{B}_{bp}^{+}\right)_c(x) = \mathbf{0}$ and $\left(\widehat{B}_{bp}^{-}\right)_c(x) = \mathbf{0}$, for every $x \in X$

Thus $\left(\widehat{A}_{bp}\right)_c \cap_1 \left(\widehat{B}_{bp}\right)_c = \hat{0}_{bp}$.

Hence $(X, (\hat{\tau}_{\mathfrak{B}})_c)$ is $(SBPFW - H)_1$.

To prove the converse part it is enough to observe that $\left(\widehat{A}_{bp}\right)_c \cap_1 \left(\widehat{B}_{bp}\right)_c = \hat{0}_{bp}$ implies $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

$\left(\widehat{A}_{bp}\right)_c \cap_1 \left(\widehat{B}_{bp}\right)_c = \hat{0}_{bp}$ implies $\left(\widehat{A}_{bp}^{+}\right)_c(x) = \mathbf{0}$ and $\left(\widehat{A}_{bp}^{-}\right)_c(x) = \mathbf{0}$ or $\left(\widehat{B}_{bp}^{+}\right)_c(x) = \mathbf{0}$ and $\left(\widehat{B}_{bp}^{-}\right)_c(x) = \mathbf{0}$, for every $x \in X$.

$\left(\widehat{A}_{bp}^{+}\right)_c(x)(\alpha) = 0$ and $\left(\widehat{A}_{bp}^{-}\right)_c(x)(\alpha) = 0$ or $\left(\widehat{B}_{bp}^{+}\right)_c(x)(\alpha) = 0$ and $\left(\widehat{B}_{bp}^{-}\right)_c(x)(\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$.

$\left(\widehat{A}_{bp}^{+}\right)(x)(1-\alpha) = 0$ and $\left(\widehat{A}_{bp}^{-}\right)(x)(1-\alpha) = 0$ or $\left(\widehat{B}_{bp}^{+}\right)(x)(1-\alpha) = 0$ and $\left(\widehat{B}_{bp}^{-}\right)(x)(1-\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$.

$\widehat{A}_{bp}^{+}(x) = \mathbf{0}$ and $\widehat{A}_{bp}^{-}(x) = \mathbf{0}$ or $\widehat{B}_{bp}^{+}(x) = \mathbf{0}$ and $\widehat{B}_{bp}^{-}(x) = \mathbf{0}$, for every $x \in X$.

Thus $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

Theorem:4.3.4

If $(X, \hat{\tau}_{\mathfrak{B}})$ is $(\mathrm{SBPFW}-\mathrm{H})_1$ then

(i) For $\varepsilon \in (0,1)$, $\left(X, i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.

(ii) $\left(X, i^{*}(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.

(iii) $\left(X, i(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.

Proof:

(i) Consider $x, y \in X$, $x \neq y$, then there exist $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ such that ${\widehat{A}_{bp}}^{+}(x) = \mathbf{1}$, ${\widehat{A}_{bp}}^{-}(x) = -\mathbf{1}$, ${\widehat{B}_{bp}}^{+}(y) = \mathbf{1}$, ${\widehat{B}_{bp}}^{-}(y) = -\mathbf{1}$ and $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

${\widehat{A}_{bp}}^{+}(x) = \mathbf{1} \Rightarrow \left({\widehat{A}_{bp}}^{+}(x)\right)^{-1}(\varepsilon, 1] = I$ and

${\widehat{A}_{bp}}^{-}(x) = -\mathbf{1} \Rightarrow \left({\widehat{A}_{bp}}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I$ implies $x \in \left(L_{\widehat{A}_{bp}}\right)_{\varepsilon}$.

Similarly, ${\widehat{B}_{bp}}^{+}(y) = \mathbf{1} \Rightarrow \left({\widehat{B}_{bp}}^{+}(y)\right)^{-1}(\varepsilon, 1] = I$ and

$${\widehat{B}_{bp}}^{-}(y) = -\mathbf{1} \Rightarrow \left({\widehat{B}_{bp}}^{-}(y)\right)^{-1}[-1, -\varepsilon) = I$$

implies $y \in \left(L_{\widehat{B}_{bp}}\right)_{\varepsilon}$.

Therefore $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}} \Rightarrow \left(L_{\widehat{A}_{bp}}\right)_{\varepsilon}, \left(L_{\widehat{B}_{bp}}\right)_{\varepsilon} \in i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}})$.

Let $z \in \left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} \cap \left(L_{\widehat{B}_{bp}}\right)_{\varepsilon}$ implies $\left(\left({\widehat{A}_{bp}}^{+}(z)\right)^{-1}(\varepsilon, 1] = I, \left({\widehat{A}_{bp}}^{-}(z)\right)^{-1}[-1, -\varepsilon) = I\right)$ and

$$\left(\left({\widehat{B}_{bp}}^{+}(z)\right)^{-1}(\varepsilon, 1] = I, \left({\widehat{B}_{bp}}^{-}(z)\right)^{-1}[-1, -\varepsilon) = I\right).$$

${\widehat{A}_{bp}}^{+}(z) \neq \mathbf{0}$, ${\widehat{A}_{bp}}^{-}(z) \neq \mathbf{0}$ and ${\widehat{B}_{bp}}^{+}(z) \neq \mathbf{0}$, ${\widehat{B}_{bp}}^{-}(z) \neq \mathbf{0}$ which is a contradiction since $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

Thus $\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} \cap \left(L_{\widehat{B}_{bp}}\right)_{\varepsilon} = \emptyset$.

Hence $\left(X, i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.

Proofs of (ii) and (iii) are similar.

Theorem:4.3.5

If (X, τ) is Hausdorff, then

(i) For $\varepsilon \in (0,1)$, $\left(X, \widehat{\omega_\varepsilon(\tau)}\right)$ is $(\text{SBPFW} - H)_1$.

(ii) $\left(X, \widehat{\omega(\tau)}\right)$ is $(\text{SBPFW} - \ H)_1$.

(iii)$\left(X, \widehat{\omega_*(\tau)}\right)$ is $(\text{SBPFW} - H)_1$.

Proof:

(i) For $x, y \in X$, $x \neq y$, there exist $A, B \in \tau$ such that $x \in A$, $y \in B$ and $A \cap B = \emptyset$.

Then $\left(\hat{\chi}_{bp}{}^{+}\right)_A (x) = \mathbf{1}$, $\left(\hat{\chi}_{bp}{}^{-}\right)_A (x) = \mathbf{-1}$ and $\left(\hat{\chi}_{bp}{}^{+}\right)_B (y) = \mathbf{1}$, $\left(\hat{\chi}_{bp}{}^{-}\right)_B (y) = \mathbf{-1.}$

Now to prove $\left(\hat{\chi}_{bp}\right)_A \cap_1 \left(\hat{\chi}_{bp}\right)_B = 0_{bp}$.

For $z \in X$, $\left(\hat{\chi}_{bp}{}^{+}\right)_A (z) \neq \mathbf{0}$, $\left(\hat{\chi}_{bp}{}^{-}\right)_A (z) \neq \mathbf{0}$ implies $\left(\hat{\chi}_{bp}{}^{+}\right)_A (z) = \mathbf{1}$, $\left(\hat{\chi}_{bp}{}^{-}\right)_A (z) = \mathbf{-1}$ implies $z \in A$. Then $z \notin B$ implies $\left(\hat{\chi}_{bp}{}^{+}\right)_B (z) = \mathbf{0}$, $\left(\hat{\chi}_{bp}{}^{-}\right)_B (z) = \mathbf{0.}$

Therefore $\left(\hat{\chi}_{bp}\right)_A \cap_1 \left(\hat{\chi}_{bp}\right)_B = \hat{0}_{bp}$.

Since $\left(L_{\left(\hat{\chi}_{bp}\right)_A}\right)_\varepsilon = A \in \tau$, $\left(\hat{\chi}_{bp}\right)_A \in \left(\widehat{K}_{bp}\right)_\varepsilon$ which is a basis of $\widehat{\omega_\varepsilon(\tau)}$.

Similarly, $\left(L_{\left(\hat{\chi}_{bp}\right)_B}\right)_\varepsilon = B \in \tau$, $\left(\hat{\chi}_{bp}\right)_B \in \left(\widehat{K}_{bp}\right)_\varepsilon$.

Hence $\left(X, \widehat{\omega_\varepsilon(\tau)}\right)$ is $(\text{SBPFW} - H)_1$.

Proofs of (ii) and (iii) are similar.

Theorem:4.3.6

If $(X, \hat{\tau}_{\mathfrak{B}})$ is $(\text{SBPFW} - H)_1$, then $\left(X, S_2(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.

Proof:

Let $(X, \hat{\tau}_{\mathfrak{B}})$ is $(\text{SBPFW} - H)_1$.

Consider $x, y \in X$, $x \neq y$, then there exist two second order bipolar fuzzy open sets

$\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ such that $\widehat{A}_{bp}{}^{+}(x) = \mathbf{1}, \widehat{A}_{bp}{}^{-}(x) = -\mathbf{1}$, $\widehat{B}_{bp}{}^{+}(y) = \mathbf{1}, \widehat{B}_{bp}{}^{-}(y) = -\mathbf{1}$ and $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

$$\begin{aligned} \widehat{A}_{bp}{}^{+}(x) = \mathbf{1} \Rightarrow \widehat{A}_{bp}{}^{+}(x)(\alpha) &= \mathbf{1}(\alpha), \text{ for every } x \in X \text{ and for every } \alpha \in I \\ &= 1 \\ \widehat{A}_{bp}{}^{-}(x) = \mathbf{1} \Rightarrow \widehat{A}_{bp}{}^{-}(x)(\alpha) &= -\mathbf{1}(\alpha), \text{ for every } x \in X \text{ and for every } \alpha \in I \\ &= -1 \end{aligned}$$

implies $\widehat{A}_{bp}{}^{+}(x)(\alpha) > 0$, $\widehat{A}_{bp}{}^{-}(x)(\alpha) < 0$, for every $x \in X$, for every $\alpha \in I$ implies $x \in S_2(\widehat{A}_{bp})$.

Similarly, $\widehat{B}_{bp}{}^{+}(y) = \mathbf{1}$, $\widehat{B}_{bp}{}^{-}(y) = -\mathbf{1}$ implies $\widehat{B}_{bp}{}^{+}(y)(\alpha) > 0$, $\widehat{B}_{bp}{}^{-}(y)(\alpha) < 0$, for every $x \in X$ and for every $\alpha \in I$ implies $y \in S_2(\widehat{B}_{bp})$.
Also, $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$
implies $\widehat{A}_{bp}{}^{+}(x) = \mathbf{0}, \widehat{A}_{bp}{}^{-}(x) = \mathbf{0}$ or $\widehat{B}_{bp}{}^{+}(x) = \mathbf{0}$, $\widehat{B}_{bp}{}^{-}(x) = \mathbf{0}$, for every $x \in X$
implies $\widehat{A}_{bp}{}^{+}(x)(\alpha) = 0, \widehat{A}_{bp}{}^{-}(x)(\alpha) = 0$ or $\widehat{B}_{bp}{}^{+}(x)(\alpha) = 0, \widehat{B}_{bp}{}^{-}(x)(\alpha) = 0$,
for every $x \in X$ and for every $\alpha \in I$ implies $x \notin S_2(\widehat{A}_{bp})$ or $x \notin S_2(\widehat{B}_{bp})$.
Therefore $S_2(\widehat{A}_{bp}) \cap S_2(\widehat{B}_{bp}) = \emptyset$.
Hence $(X, S_2(\hat{\tau}_{\mathfrak{B}}))$ is Hausdorff.

Theorem:4.3.7

$(X, \tau_{\mathfrak{B}})$ is $(\text{BPFS} - \text{H})$ iff $(X, \hat{\tau}_{\mathfrak{B}})$ is $(\text{SBPFS} - \text{H})_1$, where $(X, \hat{\tau}_{\mathfrak{B}})$ is from $(X, \tau_{\mathfrak{B}})$ through the relation R_1.

Proof :

Assume $(X, \tau_{\mathfrak{B}})$ is $(\text{BPFS} - \text{H})$. Given $\tau_{\mathfrak{B}}$, $\hat{\tau}_{\mathfrak{B}} = \{\widehat{A}_{bp} \,/\, A_{bp} \in \tau_{\mathfrak{B}}\}$ where $\widehat{A}_{bp}{}^{+}(x)(\alpha) = A_{bp}{}^{+}(x)$, $\widehat{A}_{bp}{}^{-}(x)(\alpha) = A_{bp}{}^{-}(x)$, for every $x \in X$ and for every $\alpha \in I$. Consider two distinct bipolar fuzzy points $x_{(r,s)}, y_{(m,n)}$ in X. Since $(X, \tau_{\mathfrak{B}})$ is $(\text{BPFS} - \text{H})$, there exist two bipolar fuzzy open sets $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}}$ such that $x_{(r,s)} \in A_{bp}, y_{(m,n)} \in B_{bp}$ and $A_{bp} \cap B_{bp} = 0_{bp}$.

Then $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$

$x_{(r,s)} \in A_{bp}$ implies $A_{bp}{}^{+}(x) \geq \mathbf{r}, A_{bp}{}^{-}(x) \leq \mathbf{s}$ implies $\widehat{A}_{bp}{}^{+}(x)(\alpha) \geq r, \widehat{A}_{bp}{}^{-}(x)(\alpha) \leq s$, for every $\alpha \in I$ implies $\hat{x}_{(r,s)} \in \widehat{A}_{bp}$

$y_{(m,n)} \in B_{bp}$ implies $B_{bp}{}^{+}(y) \geq \mathbf{m}, B_{bp}{}^{-}(y) \leq \mathbf{n}$ implies $\widehat{B}_{bp}{}^{+}(y)(\alpha) \geq m, \widehat{B}_{bp}{}^{-}(y)(\alpha) \leq n$, for every $\alpha \in I$ implies $\hat{y}_{(m,n)} \in \widehat{B}_{bp}$.

Claim: $A_{bp} \cap B_{bp} = 0_{bp}$ implies $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

Consider, for $z \in X$, $\widehat{A}_{bp}{}^{+}(z) \neq \mathbf{0}, \widehat{A}_{bp}{}^{-}(z) \neq \mathbf{0}$

implies $\widehat{A}_{bp}{}^{+}(z)(\alpha) \neq 0, \widehat{A}_{bp}{}^{-}(z)(\alpha) \neq 0$, for some $\alpha \in I$

implies $A_{bp}{}^{+}(z) \neq 0, A_{bp}{}^{-}(z) \neq 0$

implies $B_{bp}{}^{+}(z) = 0, B_{bp}{}^{-}(z) = 0$ (since $A_{bp} \cap B_{bp} = 0_{bp}$)

implies $\widehat{B}_{bp}{}^{+}(z)(\alpha) = 0, \widehat{B}_{bp}{}^{-}(z)(\alpha) = 0$, for every $\alpha \in I$

implies $\widehat{B}_{bp}{}^{+}(z) = \mathbf{0}, \widehat{B}_{bp}{}^{-}(z) = \mathbf{0}$

implies $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$

Hence $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS - H)_1$.

To prove the converse, it is enough to observe that $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$ implies $A_{bp} \cap B_{bp} = 0_{bp}$.

Theorem:4.3.8

If $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS - H)_1$, then for $\alpha \in I$, $(X, (\hat{\tau}_{\mathfrak{B}})_{\alpha})$ is $(BPFS - H)$ where $(X, (\hat{\tau}_{\mathfrak{B}})_{\alpha})$ is from $(X, \hat{\tau}_{\mathfrak{B}})$ through the relation R_3.

Proof :

Assume $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS - H)_1$. Given $\hat{\tau}_{\mathfrak{B}}$, for $\alpha \in I$,

$(\hat{\tau}_{\mathfrak{B}})_{\alpha} = \left\{ \left(\widehat{A}_{bp}\right)_{\alpha} / \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}} \right\}$ where $\left(\widehat{A}_{bp}\right)_{\alpha} = \left(\left(\widehat{A}_{bp}{}^{+}\right)_{\alpha}, \left(\widehat{A}_{bp}{}^{-}\right)_{\alpha} \right)$ such that

$\left(\widehat{A}_{bp}{}^{+}\right)_{\alpha}(x) = \widehat{A}_{bp}{}^{+}(x)(\alpha), \left(\widehat{A}_{bp}{}^{-}\right)_{\alpha}(x) = \widehat{A}_{bp}{}^{-}(x)(\alpha)$, for every $x \in X$.

Consider two distinct bipolar fuzzy points $x_{(m,n)}, y_{(p,q)}$ in X, then $\hat{x}_{(m,n)}, \hat{y}_{(p,q)}$ are distinct.

Since $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS - H)_1$, there exist two second order bipolar fuzzy open sets

$\hat{A}_{bp}, \hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ such that $\hat{x}_{(m,n)} \in \hat{A}_{bp}, \hat{y}_{(p,q)} \in \hat{B}_{bp}$ and $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$.

Then $\left(\hat{A}_{bp}\right)_\alpha, \left(\hat{B}_{bp}\right)_\alpha \in (\hat{\tau}_{\mathfrak{B}})_\alpha$

$\hat{x}_{(m,n)} \in \hat{A}_{bp}$ implies $\hat{A}_{bp}{}^{+}(x)(\alpha) \geq m, \hat{A}_{bp}{}^{-}(x)(\alpha) \leq n$, for every $\alpha \in I$

implies $\left(\hat{A}_{bp}{}^{+}\right)_\alpha (x) \geq m, \left(\hat{A}_{bp}{}^{-}\right)_\alpha (x) \leq n$

implies $x_{(m,n)} \in \left(\hat{A}_{bp}\right)_\alpha$

$\hat{y}_{(p,q)} \in \hat{B}_{bp}$ implies $\hat{B}_{bp}{}^{+}(y)(\alpha) \geq p, \hat{B}_{bp}{}^{-}(y)(\alpha) \leq q$, for every $\alpha \in I$

implies $\left(\hat{B}_{bp}{}^{+}\right)_\alpha (y) \geq p, \left(\hat{B}_{bp}{}^{-}\right)_\alpha (y) \leq q$

implies $y_{(p,q)} \in \left(\hat{B}_{bp}\right)_\alpha$.

Also $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$ implies $\left(\hat{A}_{bp}\right)_\alpha \cap \left(\hat{B}_{bp}\right)_\alpha = \left(\hat{0}_{bp}\right)_\alpha$.

Hence $\left(X, \hat{\tau}_{\mathfrak{B}_\alpha}\right)$ is $(BPFS - H)$.

Theorem:4.3.9

$(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS - H)_1$ iff $(X, (\hat{\tau}_{\mathfrak{B}})_c)$ is $(SBPFS - H)_1$, where $(X, (\hat{\tau}_{\mathfrak{B}})_c)$ is from $(X, \hat{\tau}_{\mathfrak{B}})$ through the relation R_5.

Proof :

Assume $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS - H)_1$. Given $\hat{\tau}_{\mathfrak{B}}$, $(\hat{\tau}_{\mathfrak{B}})_c = \left\{\left(\hat{A}_{bp}\right)_c / \hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$ where $\left(\hat{A}_{bp}\right)_c = \left(\left(\hat{A}_{bp}{}^{+}\right)_c, \left(\hat{A}_{bp}{}^{-}\right)_c\right)$ such that $\left(\hat{A}_{bp}{}^{+}\right)_c (x)(\alpha) = \hat{A}_{bp}{}^{+}(x)(1 - \alpha)$, $\left(\hat{A}_{bp}{}^{-}\right)_c (x)(\alpha) = \hat{A}_{bp}{}^{-}(x)(1 - \alpha)$, for every $x \in X$ and for every $\alpha \in I$.

Consider two distinct second order bipolar fuzzy points $\hat{x}_{(m,n)}, \hat{y}_{(p,q)}$ in X. Since $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS - H)_1$, there exist two second order bipolar fuzzy open sets $\hat{A}_{bp}, \hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ such that $\hat{x}_{(m,n)} \in \hat{A}_{bp}, \hat{y}_{(p,q)} \in \hat{B}_{bp}$ and $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$.

Then $\left(\hat{A}_{bp}\right)_c, \left(\hat{B}_{bp}\right)_c \in (\hat{\tau}_{\mathfrak{B}})_c$.

$\hat{x}_{(m,n)} \in \hat{A}_{bp}$ implies $\hat{A}_{bp}{}^{+}(x)(\alpha) \geq m, \hat{A}_{bp}{}^{-}(x)(\alpha) \leq n$, for every $\alpha \in I$

implies $\hat{A}_{bp}{}^{+}(x)(1-\alpha) \geq m, \hat{A}_{bp}{}^{-}(x)(1-\alpha) \leq n$, for every $\alpha \in I$

implies $\left(\hat{A}_{bp}{}^{+}\right)_c(x)(\alpha) \geq m, \left(\hat{A}_{bp}{}^{-}\right)_c(x)(\alpha) \leq n$, for every $\alpha \in I$

implies $\hat{x}_{(m,n)} \in \left(\hat{A}_{bp}\right)_c$.

Similarly, $\hat{y}_{(p,q)} \in \hat{B}_{bp} \Rightarrow \hat{y}_{(p,q)} \in \left(\hat{B}_{bp}\right)_c$.

Also $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$ implies $\left(\hat{A}_{bp}\right)_c \cap_1 \left(\hat{B}_{bp}\right)_c = \hat{0}_{bp}$.

Hence $(X, (\hat{\tau}_{\mathfrak{B}})_c)$ is $(SBPFS-H)_1$.

To prove the converse part, it is enough to observe that $\left(\hat{A}_{bp}\right)_c \cap_1 \left(\hat{B}_{bp}\right)_c = \hat{0}_{bp}$ implies $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$.

Theorem : 4.3.10

If $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS-H)_1$ then

(i) For $\varepsilon \in (0,1), \left(X, i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.

(ii) $\left(X, i^{*}(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.

(iii) $\left(X, i(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.

Proof:

(i) Let $x, y \in X, x \neq y$ and $(m.n), (p,q) \in (0,1] \times [-1,0)$ such that for $m > \varepsilon$, $n < -\varepsilon, p > \varepsilon, q < -\varepsilon$, there exist two distinct second order bipolar fuzzy points $\hat{x}_{(m,n)}, \hat{y}_{(p,q)}$ in X. Since $(X, \hat{\tau}_{\mathfrak{B}})$ is $(SBPFS-H)_1$, there exist $\hat{A}_{bp}, \hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ such that $\hat{x}_{(m,n)} \in \hat{A}_{bp}, \hat{y}_{(p,q)} \in \hat{B}_{bp}$ and $\hat{A}_{bp} \cap_1 \hat{B}_{bp} = \hat{0}_{bp}$.

$\hat{x}_{(m,n)} \in \hat{A}_{bp}$ implies $\hat{A}_{bp}{}^{+}(x)(\alpha) \geq m, \hat{A}_{bp}{}^{-}(x)(\alpha) \leq n$, for every $\alpha \in I$

implies $\left(\hat{A}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon, 1] = I, \left(\hat{A}_{bp}{}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I$

implies $x \in \left(L_{\hat{A}_{bp}}\right)_{\varepsilon}$.

$\hat{y}_{(p,q)} \in \hat{B}_{bp}$ implies $\hat{B}_{bp}{}^{+}(y)(\alpha) \geq p, \hat{B}_{bp}{}^{-}(y)(\alpha) \leq q$, for every $\alpha \in I$

implies $\left(\hat{B}_{bp}{}^{+}(y)\right)^{-1}(\varepsilon, 1] = I, \left(\hat{B}_{bp}{}^{-}(y)\right)^{-1}[-1, -\varepsilon) = I$

implies $y \in \left(L_{\hat{A}_{bp}}\right)_{\varepsilon}$.

Then $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ implies $\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon}, \left(L_{\widehat{B}_{bp}}\right)_{\varepsilon} \in i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}})$.

Also $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$ implies $\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} \cap \left(L_{\widehat{B}_{bp}}\right)_{\varepsilon} = \emptyset$.

Hence $\left(X, i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.

Proofs of (ii) and (iii) are similar.

Theorem:4.3.11

If (X, τ) is Hausdorff, then

(i) For $\varepsilon \in (0,1)$, $\left(X, \widehat{\omega_{\varepsilon}(\tau)}\right)$ is $(SBPFW - H)_1$.

(ii) $\left(X, \widehat{\omega(\tau)}\right)$ is $(SBPFW - H)_1$.

(iii) $\left(X, \widehat{\omega_{*}(\tau)}\right)$ is $(SBPFW - H)_1$.

Proof:

(i) Consider two distinct second order bipolar fuzzy points $\hat{x}_{(m,n)}, \hat{y}_{(p,q)}$ in X, where $x \neq y$. Then there exists $A, B \in \tau$ such that $x \in A, y \in B$ and $A \cap B = \emptyset$.

$x \in A$ implies $\left(\hat{\chi}_{bp}{}^{+}\right)_{A}(x) = \mathbf{1}, \left(\hat{\chi}_{bp}{}^{-}\right)_{A}(x) = -\mathbf{1}$

implies $\left(\hat{\chi}_{bp}{}^{+}\right)_{A}(x)(\alpha) \geq m, \left(\hat{\chi}_{bp}{}^{-}\right)_{A}(x)(\alpha) \leq n$, for every $\alpha \in I$

implies $\hat{x}_{(m,n)} \in \left(\hat{\chi}_{bp}\right)_{A}$.

$y \in B$ implies $\left(\hat{\chi}_{bp}{}^{+}\right)_{B}(y) = \mathbf{1}, \left(\hat{\chi}_{bp}{}^{-}\right)_{B}(y) = -\mathbf{1}$

implies $\left(\hat{\chi}_{bp}{}^{+}\right)_{B}(y)(\alpha) \geq p, \left(\hat{\chi}_{bp}{}^{-}\right)_{B}(y)(\alpha) \leq q$, for every $\alpha \in I$

implies $\hat{y}_{(p,q)} \in \left(\hat{\chi}_{bp}\right)_{B}$.

Also $A \cap B = \emptyset$ implies $\left(\hat{\chi}_{bp}\right)_{A} \cap_1 \left(\hat{\chi}_{bp}\right)_{B} = \hat{0}_{bp}$.

Since $\left(L_{\left(\hat{\chi}_{bp}\right)_{A}}\right)_{\varepsilon} = A, \left(L_{\left(\hat{\chi}_{bp}\right)_{B}}\right)_{\varepsilon} = B$ belong to τ, $\left(\hat{\chi}_{bp}\right)_{A}$ and $\left(\hat{\chi}_{bp}\right)_{B}$ belong to $\left(\widehat{K}_{bp}\right)_{\varepsilon}$ where $\left(\widehat{K}_{bp}\right)_{\varepsilon}$ is a basis of $\widehat{\omega_{\varepsilon}(\tau)}$.

Hence $\left(X, \widehat{\omega_{\varepsilon}(\tau)}\right)$ is $(\mathrm{SBPFS} - \mathrm{H})_1$.

Proofs of (ii) and (iii) are similar.

Theorem:4.3.12

If $(X, \hat{\tau}_{\mathfrak{B}})$ is $(\mathrm{SBPFS} - \mathrm{H})_1$ then $\left(X, S_2(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.

Proof:

Let $(X, \hat{\tau}_{\mathfrak{B}})$ is $(\mathrm{SBPFS} - \mathrm{H})_1$. Given $\hat{\tau}_{\mathfrak{B}}$, a base for $S_2(\hat{\tau}_{\mathfrak{B}})$ is the collection $\left\{S_2\left(\widehat{A}_{bp}\right) / \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$

Consider $x, y \in X$, $x \neq y$ and $(m, n), (p, q) \in [-1,0) \times (0,1]$, then $\hat{x}_{(m,n)}, \hat{y}_{(p,q)}$ in X are distinct. Since $(X, \hat{\tau}_{\mathfrak{B}})$ is $(\mathrm{SBPFS} - \mathrm{H})_1$, there exist two second order bipolar fuzzy open sets $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ such that $\hat{x}_{(m,n)} \in \widehat{A}_{bp}, \hat{y}_{(p,q)} \in \widehat{B}_{bp}$ and $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$.

$\hat{x}_{(m,n)} \in \widehat{A}_{bp}$ implies ${\widehat{A}_{bp}}^{+}(x)(\alpha) \geq m$, ${\widehat{A}_{bp}}^{-}(x)(\alpha) \leq n$, for every $x \in X$ and for every $\alpha \in I$

implies ${\widehat{A}_{bp}}^{+}(x)(\alpha) > 0$, ${\widehat{A}_{bp}}^{-}(x)(\alpha) < 0$, for every $x \in X$ and for every $\alpha \in I$

implies $x \in S_2\left(\widehat{A}_{bp}\right)$.

$\hat{y}_{(p,q)} \in \widehat{B}_{bp}$ implies ${\widehat{B}_{bp}}^{+}(y)(\alpha) \geq p$, ${\widehat{B}_{bp}}^{-}(y)(\alpha) \leq q$, for every $x \in X$ and for every $\alpha \in I$

implies ${\widehat{B}_{bp}}^{+}(y)(\alpha) > 0$, ${\widehat{B}_{bp}}^{-}(y)(\alpha) < 0$, for every $y \in X$ and for every $\alpha \in I$

implies $y \in S_2\left(\widehat{B}_{bp}\right)$.

Also, $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$ implies $S_2\left(\widehat{A}_{bp}\right) \cap S_2\left(\widehat{B}_{bp}\right) = \emptyset$.

Hence $\left(X, S_2(\hat{\tau}_{\mathfrak{B}})\right)$ is Hausdorff.