Stone's Representation Theorem for a Boolean Algebra and Some Interesting Topological Theorems

By

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A Thesis Submitted to the
Avinashilingam Institute for Home Science
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In partial fulfilment of the requirements
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Certified as Bonafide Research Work

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INTRODUCTION

INTRODUCTION

During the period 1930-1950 the famous mathematician M.H.STONE published a number of papers on Boolean algebras and their applications to topology. One of the important result proved by him is known as The Stone's Representation Theorem of a Boolean algebra. Several mathematicians have used this representation theorem to study different problems on Boolean algebra and topological problems and problems connecting both. In this dissertation we shall present a brief study of the Stone's representation theorem and some simple topological theorems deduced from it.

In the I - Chapter we shall collect some preliminary results on Boolean algebras with special emphasis on Stone's Representation Theorem. In Section 1 we start with definition of a partially ordered set leading to the definition of a Boolean algebra, and then proceed to study some examples. In Section 2 Stone's Representation Theorem is discussed and some interesting examples connecting Boolean algebra and its Stone space are presented in Section 3.

In Chapter - II we shall examine some deeper connections between a given Boolean algebra and its Stone space. Section 1 deals with separability in Boolean algebras. In section 2 we shall discuss how the following 3 important topological theorems can be obtained as immediate consequences of study of 3 fundamental problems in the theory of Boolean algebras.

These results are published by WILLIAM HANF([10]) in 1957.

Theorem 1

There exists a zero-dimensional compact topological space S with two subspaces S_1 and S_2 such that (i) each of the subspaces S_1 and S_2 is both closed and open in S.

(ii) each of the subspaces S_1 and S_2 is homeomorphic to its complement (with respect to S) and

(iii) S_1 and S_2 are not homeomorphic.

Theorem 2

Every zero-dimensional, compact, and separable topological space, with infinitely many isolated points, is homeomorphic to any of its subspaces obtained by removing one isolated point.

Theorem 3

There exists a zero-dimensional compact topological space with infinitely many isolated points, which is homeomorphic to each of its subspaces obtained by removing n isolated points, but is not homeomorphic to any subspace obtained by removing k isolated points for k = 1, 2, ..., n-1.

CHAPTER - I

CHAPTER - I

SECTION 1 : DEFINITION OF BOOLEAN ALGEBRA AND SOME EXAMPLES

Definition 1.1.1

A partially ordered set is a set X together with a binary relation $\leq$ defined on X which satisfies the following conditions for arbitrary elements x, y and z of X

(1) $\leq$ is reflexive : $x \leq x$ for all x

(2) $\leq$ is transitive : $x \leq y$ and $y \leq z$ implies that $x \leq z$

(3) $\leq$ is antisymmetric : $x \leq y$ and $y \leq x$ implies that $x = y$

Definition 1.1.2

An upper bound for a subset A of X is an element x such that $a \leq x$ for all $a \in A$ and a lower bound for A is an element $z \ni z \leq a$ for all $a \in A$.

Definition 1.1.3

A least upper bound for the set A is an upper bound x such that for any other upper bound y, $x \leq y$.

A greatest lower bound for the set A is an lower bound x such that for any other lower bound y, $x \geq y$.

The least upper bound of a family $\{x_\alpha\}$ is written $\vee_\alpha x_\alpha$ and the greatest lower bound is written $\wedge_\alpha x_\alpha$. The terms join and supremum (respectively meet and infimum) are often used instead of least upper bound (respectively greatest lower bound).

Definition 1.1.4

A lattice is a partially ordered set in which every pair of elements has a supremum and an infimum.

Definition 1.1.5

A lattice is said to be complete if every set has a supremum and an infimum.

Definition 1.1.6

A lattice is called distributive if the operations meet and join satisfy the two identities.

$$x \wedge (Y \vee Z) = (x \wedge y) \vee (x \wedge Z)$$

and $$x \vee (Y \wedge Z) = (x \vee y) \wedge (x \vee Z)$$

Definition 1.1.7

A lattice is called complemented if it contains distinct elements 0 and 1 such that $0 \leq x \leq 1$ for all elements x and to each element x is assigned an element x' such that $x \vee x' = 1$ and $x \wedge x' = 0$. The element x' is called the complement of x and the elements 0 and 1 are often referred to as the zero element and the unit element respectively.

Definition 1.1.8

A subset F of a topological space X is called a regular closed set if F = cl(int F). Similarly a subset G of X is said to be a regular open set if G = int (cl G).

Definition 1.1.9

A Boolean algebra is a complemented distributive lattice.

Definition 1.1.10

A Boolean algebra is said to be complete if it is complete as a lattice.

Definition 1.1.11

The set of all subsets of a set X with set union, intersection and set complimentation as the Boolean operations.

Example 1.1.12

Denote the family of clopen (closed and open) subsets of a topological space X by CO(X). Since the complement of a clopen set is clopen, it is easy to see that the clopen subsets form a Boolean algebra with the operations of set theoretic complementation, union and intersection.

Example 1.1.13

The family R(X) of regular closed subsets of a space X is a complete Boolean algebra with the following operations :

(1) $A \leq B$ if and only if $A \subseteq B$

(2) $\vee_\alpha A_\alpha = Cl(\cup_\alpha \text{ int } A_\alpha)$

(3) $\wedge_\alpha A_\alpha = Cl(\text{int } (\cap_\alpha A_\alpha))$

cl(X∖A) (4) $A' = Cl(X - A)$.

Proof

Let $\{A_\alpha\}$ be any collection of regular closed subsets of X.

Set B = $cl(\cup_\alpha \text{ int } A_\alpha)$. Then B is the closure of an open set and is clearly in R(X). Since B contains $cl(\text{int } A_\alpha) = A_\alpha$ for each α. B is an upper bound of $\{A_\alpha\}$

Suppose that E is another upper bound.

Assume $E \geq A_\alpha = cl(\text{int } A_\alpha)$

To prove $E \geq B = cl(\cup_\alpha \text{ int } A_\alpha)$

$E \geq cl\ (\text{int } A_\alpha) \geq \text{int } A_\alpha$

Therefore

$E \geq \cup_\alpha(\text{int } A_\alpha)$

Therefore $\text{int } E \geq \cup_\alpha \text{ int } A_\alpha$

$E = cl(\text{int } E) \geq cl(\cup_\alpha \text{ int } A_\alpha) = B$

Hence $E \geq B = cl(\cup_\alpha \text{ int } A_\alpha)$

Hence B is the least upper bound of $\{A_\alpha\}$

Set $F = cl(\text{int } (\cap_\alpha A_\alpha))$

Since int($\cap_\alpha A_\alpha$) is contained in int A_α for all α . F is contained in A_α for every α and F is a lower bound for $\{ A_\alpha \}$.

Suppose that E is another lower bound. Then $E \subseteq \cap A_\alpha$ so that int E is $\subseteq$ int($\cap_\alpha A_\alpha$).

Therefore $E = cl(int\ E) \subseteq cl(int\ (\cap_\alpha A_\alpha)) = F$

Therefore F is the greatest lower bound.

Take the empty set and X to be the zero element and the unit element respectively.

For any A in R(X), put $A' = cl(X - A)$. Then $A \wedge A'$ is the closure of the interior of the boundary of A and is therefore empty.

Similarly $A \vee A'$ is the closure of a dense set and is therefore all of X.

It remains to show that the distributive law holds.

Let A, B and C belong to R(X).

$$cl(int\ B \cup int\ C) = B \vee C = cl(int\ (B \vee C)) \qquad (1)$$

In any topological space, if 2 sets S and T have the same closure and G is an open set, then

$$cl(G \cap S) = cl(G \cap T) \qquad (2)$$

From (1) and (2) we get

$$\begin{aligned} A \wedge (B \vee C) &= cl(\text{int } A \cap \text{int } (B \vee C)) \\ &= cl(\text{int } A \cap (\text{int } B \cup \text{int } C)) \\ &= cl((\text{int } A \cap \text{int } B) \cup (\text{int } A \cap \text{int } C)) \\ &= cl(\text{int } A \cap \text{int } B) \cup cl(\text{int } A \cap \text{int } c) \\ &= (A \wedge B) \cup (A \wedge C) \\ &= (A \wedge B) \vee (A \wedge C) \end{aligned}$$

By interchanging the roles the closure and interior operators, one can show that the Boolean algebra of regular open sets of a space is also a complete Boolean algebra.

Definition 1.1.14

A space is said to be totally disconnected if the only connected subsets are the singletons.

Example 1.1.15

Any discrete space and the space of rationals with usual topology are totally disconnected spaces.

Definition 1.1.16

A space is said to be zero-dimensional if it has a open base consisting of clopen sets.

Definition 1.1.17

A space is said to be extremally disconnected if the closure of every open subspace is open.

The following two theorems from topology are used in the construction and study of Stone's space of a Boolean algebra.

Theorem 1.1.18

A compact space is zero-dimensional if and only if it is totally disconnected.

Theorem 1.1.19

The Boolean algebra of clopen substs of a zero-dimensional space is complete if and only if the space is extremally disconnected.

SECTION 2 : THE STONE REPRESENTATION THEOREM

Definition 1.2.1

An ideal in a Boolean algebra L is a subset Δ of L satisfying the following conditions :

(1) $a \vee b$ belongs to Δ whenever both a and b belong to Δ

(2) If b belongs to Δ and $a \leq b$, then a belongs to Δ

It is clear that an ideal is not all of L only if the unit element is not in the ideal. Such an ideal is called a proper ideal.

For any subset A of L, there is a smallest ideal containing A, called the ideal generated by A.

The members of the ideal generated by A are all members of L which are dominated by the supremum of some finite subset of A.

A principal ideal is an ideal generated by a single element of L, and the principal ideal generated by a is denoted by L_a.

Definition 1.2.2

A filter in L is a subset F of L satisfying the following conditions :

(1) $a \wedge b$ belongs to F whenever both a and b belong to F.

(2) If b belongs to F and $b \leq a$, then a belongs to F.

A filter is proper, ie not all of L, exactly when it does not contain the zero element.

Definition 1.2.3

A field of sets is a family of subsets of a set X which is closed under finite unions, finite intersections, and complementation.

Definition 1.2.4

A field of subsets of X is called a reduced field if for every pair of distinct points of X, there is a member of the field containing one of the points but not the other.

Definition 1.2.5

A filter (respectively ideal) in a field is said to be determined by a point if it is the set of all members of the field containing (respectively not containing) the point.

Definition 1.2.6

A perfect field is a field in which every maximal ideal, or equivalently every maximal filter, is determined by a point.

Definition 1.2.7

A filter or ideal is called maximal if the only filter or ideal properly containing it is L itself.

Theorem 1.2.8

If ξ is a perfect reduced field of subsets of a set X, then a topology can be defined on X such that the space X is compact and totally disconnected and ξ is the Boolean algebra of clopen subsets of X.

Proof

Take ξ to be the basis for a topology on X that is a set $G \subseteq X$ is open if and only if G is a union of members of ξ. Since every set of ξ is open in this topology and ξ is a field, the sets of ξ are also closed.

Since ξ is a reduced field, for any 2 points of X, there is a clopen set containing one and missing the other so that X is totally disconnected.

To show that X is compact. That is to show that any open covering $\mathcal{U}$ of X has a finite sub cover.

Assume that $\mathcal{U}$ is made up of members of ξ. Let Δ be the ideal of ξ generated by $\mathcal{U}$. If X is not the union of finitely many elements of $\mathcal{U}$, then X does not belong to Δ and Δ is proper.

Using Zorn's lemma it can be proved that there exist a maximal ideal containing Δ.

Because ξ is perfect, there is some point of X which does not belong to any member of the maximal ideal and hence, does not belong to any member of $\mathcal{U}$.

Thus $\mathcal{U}$ cannot be a covering of X and the assumption that $\mathcal{U}$ does not admit a finite subcover leads to a contraduction.

It remains to prove that every clopen subset U of X belongs to ξ.

The clopen set U is the union of a family $\{V_\alpha\}$ of members of ξ. But then $\{V_\alpha\} \cup \{X - U\}$ is an open cover of X and therefore has a finite subcover.

Hence, U is the union of finitely many sets belonging to ξ so that U belongs to ξ.

Definition 1.2.9

A Boolean algebra homomorphism h from a Boolean algebra L to a Boolean algebra M is a function which preserves the Boolean operations,

that is (1) $h(a \wedge b) = h(a) \wedge h(b)$

(2) $h(a \vee b) = h(a) \vee h(b)$

(3) $h(a') = (h(a))'$

Definition 1.2.10

A Boolean algebra homomorphism is called a momomorphism if it is one-to-one.

An isomorphism is a monomorphism which is onto.

Theorem 1.2.11

A Boolean algebra homomorphism h is a monomorphism if and only if h(a) = 0 implies that a is 0.

Proof

Assume a Boolean algebra homomorphism h is a monomorphism.

this implies that it is one-to-one [From the definition]

implies that h(a) = 0

implies that a = 0

conversely assume that h(a) = h(b)

consider $h(a \wedge b') = h(a) \wedge h(b')$

$= h(a) \wedge (h(b))'$

$= 0$

Therefore $a \wedge b' = 0$

Similarly $a' \wedge b = 0$

This implies that $a \leq b$ and $b \leq a$ and therefore a = b.

Theorem 1.2.12

If S(L) is the set of maximal filters of a Boolean algebra L and if, for every element a of L, h(a) denotes the set of maximal filters containing a, then the family $\xi = \{h(a) : a \in L\}$ is a reduced field of subsets of S(L) and h is a homomorphism onto ξ. If in addition, every non zero element of L belongs to some maximal filter, then h is an isomorphism.

Proof

Let F be a maximal filter of L.
Then by definition of h, $a \in F$ if and only if $F \in h(a)$.

Since $a \wedge b$ is in F if and only if a is in F and b is in F.

$$h(a \wedge b) = h(a) \wedge h(b)$$

Because S(L) is made up of maximal filters,

$$h(a') = S(L) - h(a)$$

Thus, h is a homomorphism and $\xi = \{h(a) : a \in L\}$ is a field of subsets of S(L) since it is the image of L under h.

If F_1 and F_2 are distinct maximal filters of L then, there is an element a of L belonging to F_1 but not F_2. Hence F_1 belong to h(a) and F_2 does not, this imply ξ is a reduced field.

Finally if every non zero element of L belongs to some maximal filter, $h(a) = 0$ implies that $a = 0$, and h is therefore an isomorphism.

Theorem 1.2.13 : The Stone Representation Theorem

Every Boolean algebra is isomorphic to the Boolean algebra of clopen sets of a compact totally disconnected space.

Proof

Let L be a Boolean algebra and S(L) be the set of maximal filters of a Boolean algebra L and the family $\xi = \{ h(a) : a \in L \}$ is a reduced field of subsets of S(L) and h is a homomorphism onto ξ.

If a in L is not 0, then the principal filter generated by a is proper and therefore can be extended to a maximal filter.

It follows from the theorem 1.2.12 that h is an isomorphism of L onto the reduced field ξ.

Now To prove ξ is a perfect field.

To prove this we apply theorem 1.2.8. Let ζ be a maximal filter of ξ. Then the set F of all elements a of L such that h(a) belongs to ζ is a maximal filter of L by isomorphism so that F is also a point of S(L).

If 'b' belongs to L, we have h(b) ε ζ if and only if b ε F, and as in the proof of theorem 1.2.12. h(b) ε ζ if and only if F ε h(b).

Hence, the filter ζ is determined by the point F of S(L).

The proof is completed by appealing to theorem 1.2.8 to show that ξ is the Boolean algebra of clopen subsets of S(L) for the topology generated by ξ and that this topology is compact and totally disconnected.

SECTION 3 : SOME INTERESTING EXAMPLES AND RESULTS CONNECTING A BOOLEAN ALGEBRA AND ITS STONE SPACE

Definition 1.3.1

A compact totally disconnected space X is said to be the Stone space of a Boolean algebra L provided L is isomorphic to the (perfect reduced) field of all open-closed subsets of X.

Example 1.3.2

If $\mathcal{U}$ is the field of all open-closed subsets of a compact totally disconnected space X, then X is the Stone space of $\mathcal{U}$.

Example 1.3.3

If a Boolean algebra L is finite, then the Stone space X of L is a finite, Hausdorff space (and conversely). Then ξ is the class of all subsets of X. If X has n elements then L has 2^n elements. Therefore if two finite Boolean algebras have the same number of elements, they are isomorphic.

In particular, any one point space is the Stone space of the two element Boolean algebra. The empty set is the Stone space of the degenerate Boolean algebra.

Example 1.3.4

The Stone space of a Boolean algebra L is metrizable if and only if L is at most enumerable.

This follows from the following metrisation theorem.

"A compact Hausdorff space is metrizable if and only if it has an enumerable open basis".

Definition 1.3.5

Let Δ be an ideal of a Boolean algebra L. Given two elements A, B ε L. We say A $\sim$ B if and only if A-B ε Δ and B-A ε Δ we can show that this is an equivalence relation in L. We can also show that if $A_1 \sim A_2$ and $B_1 \sim B_2$ then $A_1 \cup B_1 \sim A_2 \cup B_2$ and $A_1 \cap B_1 \sim A_2 \cap B_2$ and $-A_1 \sim -A_2$. Consider the collection of equivalence class mod Δ we denote this set of equivalence class by L/Δ. We denote the equivalence class containing A by [A]. We define [A $\cup$ B] = [A] $\cup$ [B]. Similarly [A $\cap$ B] = [A] $\cap$ [B], [-A] = -[A]. We find that with this operation L/Δ is a Boolean algebra.

Example 1.3.6

To determine the Stone space of L/Δ interms of the Stone's space of L.

Let h be the isomorphism of L onto the field ξ of all open closed subsets of X defined as in theorem (1.2.12) and let D be the union of all sets h(A) where A ε Δ.

The set D being open, the set E = X-D is closed therefore it is a compact totally disconnected space and the set of elements of ξ which are contained in E, is the field of all open closed subsets of E, we denote this by ξ/E. Observe that $A \varepsilon \Delta$ if and only if $h(A) \subset D$, that is $h(A) \cap E = \Lambda$. Since the mapping h_0 defined by the formula

$$h_0([A]) = h(A) \cap E$$

is an isomorphism of L/Δ onto ξ/E. Then the space E is the Stone space of L/Δ.

Thus Stone spaces of quotient algebras L/Δ are (upto homeomorphisms) closed subsets of the Stone space of L, and conversely every homomorphic image of L being isomorphic to a suitable quotient algebra L/Δ we infer that Stone spaces of homomorphic images of L are homeomorphic images of closed subsets of the Stone's space of L, and conversely a Boolean algebra L' is a homomorphic image of L if and only if its Stone space is homeomorphic to a closed subset of the Stone space of L.

Example 1.3.7

If L and M are Boolean algebras, there is a natural one-to-one correspondence between homomorphisms of L into M and mappings of S(M) into S(L).

We will first show that a homomorphism f of L into M induces a homomorphism g of co(S(L)) into co(S(M)).

Let $h_L : L \longrightarrow CO(S(L))$ and $h_M : M \longrightarrow CO(S(M))$ be the isomorphisms provided by the Stone Representation Theorem and for U in CO(S(L)), define $g(U) = h_M \circ f \circ h_L^{-}(U)$

Then g is a homomorphism.

$$\begin{array}{ccc} CO(S(L)) & \xrightarrow{\quad g \quad} & CO(S(M)) \\ {\scriptstyle h_L}\uparrow & \# & \uparrow{\scriptstyle h_M} \\ L & \xrightarrow[\quad f \quad]{} & M \end{array}$$

Now we will use g to define a mapping $\phi : S(M) \longrightarrow S(L)$. For every $x \in S(M)$: Let $\mathcal{F}(x)$ denote the maximal filter of CO(S(M)) determined by x. Then

$$g.^{-}\, \mathcal{F}(x) = \{ U \in CO(S(L)) : g[U] \in \mathcal{F}(x) \}$$

is a maximal filter in CO(S(L)) and is therefore determined by a point y of S(L).

Define $\phi(x) = y$. The function ϕ is well defined because distinct points of S(L) determine different maximal filters. Since S(L) is zero-dimensional. If U is a clopen subset of S(L), the definition of ϕ shows that

$$\phi(x) \in U \text{ if and only if } x \in g[U] \longrightarrow (*)$$

Thus $\phi^{-}(U) = g[U]$ is clopen because g is a homomorphism.

Hence ϕ is continuous, since S(L) has a base of clopen sets. Note that the process of obtaining the mapping ϕ from the original homomorphism f has "reversed the arrows".

$$L \xrightarrow{f} M$$

$$S(L) \xrightarrow{\phi} S(M)$$

The inverse procedure to obtain a homomorphism sending L to M from a mapping of S(M) into S(L) is less complicated. The definition of the homomorphism is actually dictated by (*) above.

If $\phi : S(M) \longrightarrow S(L)$ is a mapping, the ϕ^{-} preserves the Boolean operations and sends clopen sets to clopen sets.

Thus, the definition

$$g[U] = \phi^{-}(U)$$

defines a homomorphism of CO(S(L)) into CO(S(M)). The homomorphism g then gives a homomorphism f of L into M by defining $f = h_M^{-}.g.h_L$.

Further, it is evident that if f is an isomorphism then ϕ is a homeomorphism and conversely.

CHAPTER - II

CHAPTER - II

SECTION 1 : SEPARABILITY IN BOOLEAN ALGEBRAS

Definition 2.1.1

A Boolean algebra is said to be dense in itself if whenever one element is properly less another, then a third element can be properly interposed between them, that is if $a < b$ then there exists c such that $a < c < b$.

Theorem 2.1.2

A compact zero-dimensional space Y has no isolated points if and only if the algebra CO(Y) is dense in itself.

Proof

If y is an isolated point of Y.

Put $A = \phi$ and $B = \{ y \}$

Then no member of CO(Y) can be interposed between A and B and CO(Y) is not dense in itself.

On the other hand, if CO(Y) fails to be dense in itself, there exist members A and B of CO(Y) such that A is contained in B and no proper clopen subset of B is a proper superset of A. But since Y is zero-dimensional and B-A is clopen, B-A must be a singlten and Y contains an isolated point.

Definition 2.1.3

A Boolean algebra is said to be Cantor separable if no strictly increasing sequence has a least upper bound.

that is if $a_1 < \dots < a_n < \dots < b$ then there exists an element C such that $a_n < c < b$ for every 'n'.

Definition 2.1.4

A Boolean algebra is said to be DuBois-Reymond separable if a strictly increasing sequence can be separated from a strictly decreasing sequence dominating the increasing one.

that is if $a_1 < \dots < a_n < \dots < b_n \dots < b_1$ then there exists h such that $a_n < h < b_n$ for all n.

Theorem 2.1.5

Every non empty G_δ in a zero dimensional space Y has non empty interior if CO(Y) is Cantor separable.

Proof

Let $\cap U_i$ be non empty G_δ in Y.

Choose a point y in $\cap U_i$ and Let V_i be a clopen neighbourhood of y contained in U_i. We can assume that $\{V_i\}$ is a decreasing sequence. Hence $Y - V_1 \subset \dots \subset Y - V_n \subset \dots \subset Y$ is an increasing sequence. Because CO(Y) is Cantor separable there exists a clopen set C such that $Y - V_n \subset C \subset Y$ for all n. Therefore $\phi \subset Y - C \subset V_n$ for all n. Hence Y - C is contained in $\cap V_n$ and $\cap U_n$ therefore has non empty interior.

Definition 2.1.6.

A point in a topological space is called a k-point if it is the limit of a sequence of distinct points of the space.

Theorem 2.1.7

A zero-dimensional space Y contains no k-points if CO(Y) is Du Bois - Reymond separable.

Proof

Suppose that $\{ y_n \}$ is a sequence of distinct points converging to y and that $y_n \neq y$ for any n. We will construct a neighbourhood of y which misses infinitely many points of the sequence, thus contradicting convergence and showing that no point of Y can be a k-point.

Choose U_1 to be a clopen set containing y_1 and missing y and the other points of the sequence.

Choose U_2 to be a clopen set containing y_2 and missing U_1, y and all other points of the sequence.

Continuing by induction we obtain a sequence $\{ U_i \}$ of pairwise disjoint clopen sets such that y_i is in U_i for each i and y belongs to none of the sets.

For each $n \geq 1$, put $A_n = \cup \{ U_{2i-1} : i \leq n \}$ and $B_n = Y - (\cup \{ U_{2i} : i \leq n \})$. Observe that A_n contains the first n points having odd indices and B_n excludes the first n points having even indices.

Further, the sequences $\{ A_n \}$ and $\{ B_n \}$ satisfy $A_1 \subset \ldots\ldots \subset A_n \subset \ldots\ldots \subset B_n \subset \ldots\ldots \subset B_1$.

DuBois-Reymond separability implies that there exists a clopen set H such that $A_n \subset H \subset B_n$ for every n. Thus H contains all points having even indices and excludes all points having odd indices.

Then y belongs to either H or Y - H.

Either set fails to contain infinitely many points of the sequence, which contradicts the assumption that it converges to y.

Theorem 2.1.8

The Boolean algebra of clopen subsets of a totally disconnected compact space without isolated points and in which every zero-set is regular closed in Cantor separable.

Proof

Let $\{ A_n \}$ be a strictly increasing sequence of clopen subsets of a space X satisfying the stated hypotheses and let B be a clopen subset properly containing each A_n. Because B is compact $\cup A_n$ is properly contained in B since $\{ A_n \}$ is an open covering of its union $\cup A_n$ which has no finite subcovering.

The set $\cup A_n$ is a cozero-set since it is a countable union of cozero-sets.

Therefore $B - \cup A_n = \cap (B - A_n)$ is a non empty G_δ which must contain a non empty zero-set. Since the zero-sets are regular closed, $B - \cup A_n$ contains a non-void open set and therefore $cl(\cup A_n)$ is a proper subset of B. Since X has no isolated points there exist distinct points x and y contained in $B - cl(\cup A_n)$. But then $cl(\cup A_n) \cup \{x\}$ and $(X - B) \cup \{y\}$ are disjoint compact subsets of the compact totally disconnected space X. Since X has a base of clopen sets by one theorem (1.1.18), there is a clopen subset C of X containing the first set and missing the second.

But then we have $A_n \subset C \subset B$ and both containments are proper.

Hence, the Boolean algebra of clopen sets is Cantor separable.

Theorem 2.1.9

The Boolean algebra of clopen subsets of a totally disconnected compact F-space is DuBois-Reymond separable.

Proof

Let X be such a space and let $\{A_n\}$ and $\{B_n\}$ be sequences of clopen subsets of X such that

$A_1 \subset \ldots\ldots \subset A_n \subset \ldots\ldots \subset B_n \ldots\ldots \subset B_1$.

Then $\cup A_n$ and $X - \cap B_n$ are disjoint cozero-sets and therefore have disjoint closures since X is an F-space. Hence, there exists a clopen subset H of X containing $cl(\cup A_n)$ and missing $cl(X - \cap A_n)$.

Thus $A_n \subset H \subset B_n$ for each 'n', and the Boolean algebra of clopen subsets of X is DuBois-Reymond separable.

SECTION 2

William Hanf in his paper "On some fundamental problems concerning isomorphism of Boolean algebras" has discussed the following three problems.

1. If $\mathcal{U}$, $\mathcal{B}$ and $\mathcal{C}$ are Boolean algebras, does $\mathcal{U} \cong \mathcal{U} \times \mathcal{B} \times \mathcal{C}$ imply $\mathcal{U} \cong \mathcal{U} \times \mathcal{B}$
2. Does $\mathcal{U}^2 \cong \mathcal{B}^2$ imply $\mathcal{U} \cong \mathcal{B}$ for any $\mathcal{U}$ and $\mathcal{B}$
3. Does $\mathcal{U} \cong \mathcal{U} \times \mathcal{B} \times \mathcal{B}$ imply $\mathcal{U} \cong \mathcal{U} \times \mathcal{B}$ for any $\mathcal{U}$ and $\mathcal{B}$

Answering these problems William Hanf has proved the following theorems.

Theorem 2.2.1

There exist denumerable Boolean algebras $\mathcal{U}$ and $\mathcal{C}$ such that $\mathcal{U}^2 \cong \mathcal{C}^2$ but $\mathcal{U}$ non $\cong$ $\mathcal{C}$. More generally given a positive integer n, there exist denumerable Boolean algebras $\mathcal{U}$ and $\mathcal{C}$ such that for every positive integer k, $\mathcal{U}^k \cong \mathcal{C}^k$ in case k is a multiple of n, and $\mathcal{U}^k$ non $\cong$ $\mathcal{C}^k$ otherwise.

Theorem 2.2.2

Let $\mathcal{U}$ be a denumerable Boolean algebra or more generally, a Boolean algebra with ordered basis

(i) If $\mathcal{U}$ has infinitely many atoms and $\mathcal{B}$ is any finite Boolean algebra, then $\mathcal{U} \cong \mathcal{U} \times \mathcal{B}$

(ii) If $\mathcal{B}$ and $\mathcal{C}$ are finite Boolean algebras and $\mathcal{U} \cong \mathcal{U} \times \mathcal{B} \times \mathcal{C}$ then $\mathcal{U} \cong \mathcal{U} \times \mathcal{B} \cong \mathcal{U} \times \mathcal{C}$

Theorem 2.2.3

There exists a Boolean algebra $\mathcal{U}$ such that $\mathcal{U} \cong \mathcal{U} \times \mathcal{I}^n$ but $\mathcal{U}$ non - $\cong \mathcal{U} \times \mathcal{I}^k$ for k = 1, 2, ... n-1 ($\mathcal{I}$ is a two element Boolean algebra).

We shall collect the definitions and results which we need for discussing the above theorems.

Definition 2.2.4

A subset A of B will be said to form an ordered basis of the Boolean algebra $\mathcal{B}$ if

(i) A contains the zero but not unit element of $\mathcal{B}$.

(ii) A generates $\mathcal{B}$, that is, every element of B is a finite union of finite intersections of elements of A and their complements and

(iii) A is simply ordered by inclusion relation of $\mathcal{B}$.

Let Γ be the set of all order types α of the form $1 + \beta$, that is, α is the type of a simply ordering relation which has a first element.

Definition 2.2.5

We shall write $\alpha \approx \beta$ if some Boolean algebra has both an ordered basis of type α and one of the type β.

If Boolean algebras $\mathcal{U}$ and $\mathcal{B}$ have ordered bases of type α and β respectively then the direct product $\mathcal{U} \times \mathcal{B}$ has an ordered basis of type $\alpha + \beta$.

From this we see that, for each $\alpha, \beta \in \Gamma$, $\alpha + \beta \approx \beta + \alpha$, we will also make use of the formula

$$\sum_{n \in w} \alpha_n + \delta \approx \sum_{n \in w} \beta_n + \delta$$

which holds whenever $\alpha_n, \beta_n \in \Gamma$ and $\alpha_n \approx \beta_n$ for each $n \in w$ and δ is an arbitrary order type.

Lemma 2.2.6

There exists denumerable Boolean algebras $\mathfrak{U}$ and $\mathfrak{B}$ such that $\mathfrak{U} \cong \mathfrak{U} \times \mathfrak{B}^2$ but $\mathfrak{U}$ non - $\cong$ $\mathfrak{U} \times \mathfrak{B}$. More generally given a positive integer n, there exist denumerable Boolean algebras $\mathfrak{U}$ and $\mathfrak{B}$ such that, for each positive integer m, $\mathfrak{U}^m \cong \mathfrak{U}^m \times \mathfrak{B}^k$ just in case k is a multiple of n.

Now we shall given a short proof of the above lemma.

Proof

Suppose n is given.

We set $\lambda_i = \omega^{i+1} + \eta$, $\sigma_i = \sum_{j \in \omega} \lambda_{i+j}$,

$\tau_i = \sum_{j \in \omega} (\lambda_{i+j} \cdot n)$ for each $i \in \omega$.

Let $\mathcal{U}$ be a Boolean algebra with ordered basis of type α where $\alpha = \tau_0 \cdot \omega + \sum_{i\varepsilon\omega} (\sigma_{i+1} \cdot n + \tau_{i+1})$ and let $\mathcal{B}$ be a Boolean algebra with ordered basis of type σ_0.

To show $\mathcal{U}^m \simeq \mathcal{U}^m \times \mathcal{B}^k$ whenever k is a multiple of n, it will clearly suffice to show that $\mathcal{U} \simeq \mathcal{U} \times \mathcal{B}^k$. Since $\omega^{i+1} = 1 + \omega^{i+1}$, all the other types defined above are elements of Γ.

Furthermore $\tau_i = \lambda_i \cdot n + \tau_{i+1}$ and $\sigma_i = \lambda_i + \sigma_{i+1}$. Thus we have

$$\begin{aligned} \tau_i + \sigma_{i+1} \cdot n &= \lambda_i \cdot n + \tau_{i+1} + \sigma_{i+1} \cdot n \\ &\approx (\lambda_i + \sigma_{i+1}) \cdot n + \tau_{i+1} \\ &= \sigma_i \cdot n + \tau_{i+1} \end{aligned}$$

and hence $\sum_{i \varepsilon \omega} (\tau_i + \sigma_{i+1} \cdot n) \approx \sum_{i \varepsilon \omega} (\sigma_i \cdot n + \tau_{i+1})$

Making use of the general associative law, it follows from this set

$$\begin{aligned} \alpha &= \tau_0 \cdot \omega + \sum_{i \varepsilon \omega} (\sigma_{i+1} \cdot n + \tau_{i+1}) \\ &= \tau_0(1 + \omega) + \sum_{i \varepsilon \omega} (\sigma_{i+1} \cdot n + \tau_{i+1}) \quad (\text{since } \omega = 1 + \omega) \end{aligned}$$

$$= (\tau_0 + \tau_0 \cdot \omega) + \sum_{i \in \omega} (\sigma_{i+1} \cdot n + \tau_{i+1})$$

$$\approx \tau_0 \cdot \omega + [\tau_0 + \sum_{i \in \omega} (\sigma_{i+1} \cdot n + \tau_{i+1})]$$

$$= \tau_0 \cdot \omega + \sum_{i \in \omega} (\tau_i + \sigma_{i+1} \cdot n)$$

$$\approx \tau_0 \cdot \omega + \sum_{i \in \omega} (\sigma_i \cdot n + \tau_{i+1})$$

$$= \tau_0 \cdot \omega + [\sigma_0 \cdot n + \sum_{i \in \omega} (\tau_{i+1} + \sigma_{i+1} \cdot n)]$$

$$\approx \tau_0 \cdot \omega + \sum_{i \in \omega} (\sigma_{i+1} \cdot n + \tau_{i+1}) + \sigma_0 \cdot n$$

$$= \alpha + \sigma_0 \cdot n$$

Using the idea of order type equivalence, we can show that both and $\mathcal{U} \times \mathcal{B}^n$ have ordered bases of equivalent order types and hence $\mathcal{U}$ and $\mathcal{U} \times \mathcal{B}^n$ are isomorphic.

Notation 2.2.7

For any natural numbers n and i, let $a_{n,i}$ be the set consisting of the natural numbers in, in+1, ... in + (n-1). For each positive integer n, let M_n be the family of all those subsets x of ω such that, for some $y \subseteq \omega$ and some finite $z \subseteq \omega$.

$$x = \bigcup_{i \in y} a_{n,i} \cup z$$

It is easily seen that the set M_n, under the operations of union, $\cup$, intersection, $\cap$, and complementation with respect to ω, $-$, forms a Boolean algebra. We denotes this by $\mathcal{M}_n$. $\mathcal{M}_n$ is easily seen to be infinite (the power of the continuum) and atomistic, the atoms of the algebra being $a_{1,i}$ (the set consisting of just the single integer i) for each $i \in \omega$.

We now state in a series of lemmas the properties of $\mathcal{M}_n$. In the following $\mathcal{I}$ will denote a two element Boolean algebra.

Lemma 2.2.8

$$\mathcal{M}_n \cong \mathcal{M}_n \times \mathcal{I}^n$$

Lemma 2.2.9

$$\mathcal{M}_n \text{ non} - \cong \mathcal{M}_n \times \mathcal{I}^k \text{ for } 0 < k < n$$

Proof of the Theorem 2.2.1

This is an immediate consequence of

Lemma 2.2.6

Let $\mathcal{C} \cong \mathcal{U} \times \mathcal{B}$

Consider
$$\begin{aligned} \mathcal{C}^2 &\cong \mathcal{U}^2 \times \mathcal{B}^2 \\ &= \mathcal{U} \times \mathcal{U} \times \mathcal{B}^2 \\ &\cong \mathcal{U} \times \mathcal{U} \\ &= \mathcal{U}^2 \end{aligned}$$

hence $C^2 \simeq U^2$

Take m = k in the second part of that lemma.

We get $U^k \simeq U^k \times B^k$

$= (U \times B)^k$

$\simeq C^k$

hence $U^k \simeq C^k$

Remark

The proof of the theorem 2.2.2 involves the properties of cardinal and ordinal numbers. The proof given in ([10]) can not simplified in any way.

Proof of the theorem 2.2.3

Taking $U = M_n$ in lemma 2.2.8 and 2.2.9. Then we get

$U \simeq U \times I^n$ but

U non - $\simeq U \times I^k$ for k = 1, 2, ... n-1

The following three important theorems on topology can be deduced from the above three theorems on Boolean algebra.

Theorem 2.2.10

There exists a zero dimensional compact topological space S with two subspaces S_1 and S_2 such that

(i) each of the subspaces S_1 and S_2 is both closed and open in S.

(ii) each of the subspaces S_1 and S_2 is homeomorphic to its complement (with respect to S) and

(iii) S_1 and S_2 are not homeomorphic.

Theorem 2.2.11

Every zero-dimensional compact and seprable topological space, with infinitely many isolated points, is homeomorphic to any of its subspaces obtained by removing one isolated points.

Theorem 2.2.12

There exists a zero-dimensional compact topological space with infinitely many isolated points, which is homeomorphic to each of its subspace obtained by removing n isolated points, but is not homeomorphic to any subspace obtained by removing k isolated points for k = 1, 2, ... n-1.

To deduce the theorems 2.2.10, 2.2.11 and 2.2.12 we have only to observe the following points.

The main result used in the proof is the Stone representation theorem.

If two Boolean algebras are isomorphic then their Stone spaces are homeomorphic. The atoms of a Boolean algebra correspond to the isolated points of its Stone space.

If $\mathcal{U}$ is an infinite Boolean algebra and $\mathcal{B}$ is a finite Boolean algebra with n-elements, the statement $\mathcal{U} \cong \mathcal{U} \times \mathcal{B}$ implies $\mathcal{U}$ is isomorphic to a subalgebra of $\mathcal{U}$ got by removing n atoms.

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