# Chapter VI

# CHAPTER VI

# REGULARITY PRESERVERS FOR MATRICES OVER SEMIRINGS

**Definition: 6.1**

A mapping T : $M_{m,n}(S) \rightarrow M_{m,n}(S)$ is called a **linear operator** if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all A, $B \in M_{m,n}(S)$ and for all $\alpha, \beta \in S$.

**Definition: 6.2**

Let T be a linear operator $M_{m,n}(S)$. We say that

(∗) T **preserves regularity** (or **T preserves R(S)**) if $T(A) \in R(S)$ whenever $A \in R(S)$;

(∗) T **strongly preserves regularity** (or **T strongly preserves R(S)**) when $T(A) \in R(S)$ if and only if $A \in R(s)$ for all $A \in M_{m,n}(S)$.

(∗) T is **singular** if T(X) = 0 for some nonzero X; otherwise T is **nonsingular**.

**Example: 6.3**

Let A be any nonzero regular matrix in $M_{m,n}(S)$, where $S = \mathbb{B}$ or $\mathbb{C}$, where $\mathbb{B}$ is a Boolean algebra {0, 1} and $\mathbb{C}$ is a chain semiring. Define a linear operator T on $M_{m,n}(S)$ by

$$T(X) = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} \right) A$$

for all $X \in M_{m,n}(S)$. Then we can easily show that T is a nonsingular linear operator that preserves regularity. But T does not preserve nonregular matrices. Hence T does not strongly preserve regularity.

**Lemma: 6.4**

Let min{m, n} $\leq$ 3. If T strongly preserves regularity on $M_{m,n}(S)$, then T is nonsingular.

**Proof:**

If T(X) = O for some nonzero $X \in M_{m,n}(S)$, then T(E) = O for all cells $E \subseteq X$. For each E, there is a matrix A such that $A \in \mathcal{R}(S)$ and $E+A \notin \mathcal{R}(S)$ by Theorem 2.11. Nevertheless, T(E+A) = T(A), a contradiction to the fact that T strongly preserves regularity. Hence $T(X) \neq 0$ for all nonzero X. Therefore T is nonsingular.

**Definition: 6.5**

Let A and B be matrices in $M_{m,n}(S)$. Then the matrix $A \circ B$ denotes the **Hadamard product** (or schur product). That is the $(i, j)^{th}$ entry of $A \circ B$ is $a_{i,j}b_{i,j}$.

**Lemma: 6.6**

Let min{m, n} $\geq$ 3 and $B \in M_{m,n}(S)$. Suppose that T is a linear operator on $M_{m,n}(S)$ defined by $T(X) = X \circ B$ for all $X \in M_{m,n}(S)$. If T strongly preserves regularity, then all entries of B are nonzero and regular. In particular if fr(B) = 1, then there are diagonal matrices D and E such that T(X) = DXE for all $X \in M_{m,n}(S)$.

**Definition: 6.7**

For $A \in M_{m,n}(\mathbb{B})$ with fr(A) = k, A is said to be **space decomposable** if there are matrices $B \in M_{m,k}(\mathbb{B})$ and $C \in M_{k,n}(\mathbb{B})$ such that A = BC, C(A) = C(B) and R(A) = R(B).

**Theorem: 6.8**

Let A be a matrix in $M_{m,n}(\mathbb{B})$. Then A is regular if and only if A is space decomposable.

**Lemma: 6.9**

Let A is a matrix in $M_{m,n}(\mathbb{B})$ with fr(A) $\leq$ 2, then A is regular.

**Lemma: 6.10**

Let min{m, n} $\leq$ 2. If T is an operator (that need not be linear) on $M_{m,n}(\mathbb{B})$, then T strongly preserves regularity.

**Definition: 6.11**

If T is a linear operator on $M_{m,n}(s)$, let $T^*$, **its pattern** be the linear operator on $M_{m,n}(\mathbb{B})$ defined by $T^*(E_{i,j}) = [T(E_{i,j})]^*$ for all cells $E_{i,j}$.

**Note: 6.12**

Since S is a semiring that is commutative, antinegative and free of zero-divisors, we have $T^*(A) = [T(A)]^*$ for all $A \in M_{m,n}(S)$.

**Theorem: 6.13**

Let $A \in M_{m,n}(S)$ be a sum of k cells with fr(A) = k, where min{m, n} $\geq$ 3 and 2 $\leq$ k $\leq$ min{m, n}. Then $J \backslash A \in R(S)^*$ if and only if k = 2. In particular, $J \backslash A \notin R(s)$ for k $\geq$ 3.

**Proof:**

Without loss of generality, we assume that min{m, n} = m $\geq$ 3 and

$A = \sum_{t=1}^{K} E_{t,t}$. If k = 2, consider a matrix $X = \begin{bmatrix} X' & J_{2,n-2} \\ J_{m-2,2} & 2j_{m-2,n-2} \end{bmatrix} \in M_{m,n}(S)$, where $X' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $X(G_{1,2} + G_{2,1})X = X$, where $G_{i,j}$ are cells in $M_{m,n}(S)$; that is , $X \in \mathcal{R}(S)$. Hence $j \setminus A (=X^*) \in \mathcal{R}(S)^*$.

Let $k \geq 3$. Now we will show that $Y = J \setminus A \notin \mathcal{R}(\mathbb{B})$. If not, there is a nonzero $B \in M_{m,n}(\mathbb{B})$ such that $Y = YBY$. Then the $(t, t)^{th}$ entry of YBY becomes

$$\sum_{i=I}\sum_{j=J} b_{i,j} \qquad (*)$$

for all $t = 1, \ldots, k$, where $I = \{1, \ldots, n\} \setminus \{t\}$ and $J = \{1, \ldots, m\} \setminus \{t\}$. From $y_{1,1} = 0$ and $(*)$ we have $b_{i,j} = 0$ for all $i = 2, \ldots, n$ ; $j = 2, \ldots, m$. $(*)$ consider the $1^{st}$ row and the $1^{st}$ column of B. It follows from $y_{2,2} = 0$ and $(*)$ that $b_{i,1} = 0 = b_{1,j}$ for all $i = 1, 3, 4, \ldots, n$; $j = 1, 3, 4, \ldots, m$. $(*)$

Also, from $y_{3,3} = 0$, we obtain $b_{1,2} = b_{2,1} = 0$, and hence $B = 0$ by $(*_1)$ and $(*_2)$, a contradiction. Thus $J \setminus A \notin \mathcal{R}(\mathbb{B})$, equivalently $Z \notin \mathcal{R}(S)$ for all $Z \in M_{m,n}(S)$ with $Z^* = J \setminus A$. Hence $J \setminus A \notin \mathcal{R}(S)^*$.

**Note: 6.14**

As shown in Example 6.3, if $\min\{m, n\} \geq 3$, there is a linear operator on $M_{m,n}(\mathbb{B})$ such that T preserves regularity, while T does not strongly preserve regularity.

**Theorem: 6.15**

Assume that $\min\{m, n\} = m \geq 3$ and T is a linear operator on $M_{m,n}(S)$ that strongly preserves $\mathcal{R}(S)$. Then $T^*$ strongly preserves $\mathcal{R}(S)^*$.

Let $k_{max} = \max\{w(N) : N \in M_{m,n}(\mathbb{B})$ and $N \notin \mathcal{R}(S)^*\}$ and

$\mathcal{N} = \{N \in M_{m,n}(\mathbb{B}) : N \notin \mathcal{R}(S)^*$ with $w(N) = k_{max}\}$.

Since $J \in \mathcal{R}(S)$, $k_{max} < mn$, and so by Theorem 6.13, $mn - 3 \leq k_{max} \leq mn - 1$.

**Theorem: 6.16**

For distinct cells, E and F, $T^*(E) \not\subseteq T^*(F)$. In particular if $w(T^*(E)) = w(T^*(F)) = 1$, then $T^*(E) \neq T^*(F)$.

**Proof:**

Suppose $T^*(E) \subseteq T^*(F)$ for some distinct cells E and F. Then there are cells $E_1$ and $E_2$ different from F such that $fr(E+E_1+E_2) = 3$. Since $F \subseteq J\backslash(E+E_1+E_2)$, we have $T^*(E) \subseteq T^*(F) \subseteq T^*(J\backslash(E+E_1+E_2))$.

So that $T^*(J\backslash( E_1+E_2)) = T^*(E)+ T^*(J\backslash(E+E_1+E_2)) = T^*(J\backslash(E+E_1+E_2))$

But this is impossible as $J\backslash(E+E_1+E_2) \notin \mathcal{R}(S)^*$ while $J\backslash(E_1+E_2) \in \mathcal{R}(S)^*$ by Theorem 6.13.

**Lemma: 6.17**

If $k_{max} = mn - 1$, then $T^*(\mathcal{N}) \subseteq \mathcal{N}$.

**Proof:**

If $k_{max} = mn-1$, then $\mathcal{N} = \{J \backslash E \in M_{m,n}(\mathbb{B}) : E \text{ is a cell}\}$. It suffices to show that $w(T^*(J\backslash E)) = mn-1$ for all cells E. If $w(T^*(J\backslash E)) = mn$ for some cell E, then $T^*(J\backslash E) = J \in \mathcal{R}(S)^*$ which contradicts $J\backslash E \notin \mathcal{R}(S)^*$. Next suppose $w(T^*(J\backslash E)) < mn-1$ for some cell E. Then there is a matrix C with $C \subseteq J\backslash E$ and $w(C) < mn-1$ such that $T^*(C) = T^*(J\backslash E)$. Take a cell F different from E such that $F \not\subseteq C$. If follows from $(J\backslash E) + (J\backslash(C+F)) = J$ that

$$T^*(J) = T^*(J\backslash E) + T^*(J \backslash(C+F)) = T^*(C) + T^*(J \backslash (C+F)) = T^*(J\backslash F)$$

But this is impossible as $J\backslash F \notin \mathcal{R}(S)^*$ while $J \in \mathcal{R}(S)^*$.

**Lemma: 6.18**

If m = n = 3 and $k_{max}$ = 7, then $T^*(\mathcal{N}) \subseteq \mathcal{N}$.

**Proof:**

In this case, $\mathcal{N} = \{\mathcal{N} \in M_3(\mathbb{B}) : N \notin \mathcal{R}(S)^*$ and $w(N) = 7\}$.

Let $N \in \mathcal{N}$ be arbitary. It suffices to show $w(T^*(N)) = 7$. It follows form $k_{max} = 7$ that $w(T^*(N)) \leq 7$. Suppose $w(T^*(N)) \leq 6$. Write $N = \sum_{i=1}^{7} E_i$ and $J = \sum_{i=1}^{9} E_i$ for cells $E_1, \ldots\ldots\ldots, E_9$. By Lemma 6.4 $w(T^*(E_i)) \leq 1$ for all i. $w(T^*(E_i)) = 1$ for all i = 1, ……, 7, then by Theorem 6.16, $T^*(E_1)$, ……., $T^*(E_7)$ are distinct cells. But then $7 = w(T^*(E_1) + \ldots\ldots + T^*(E_7)) = w(T^*(E_1) + \ldots\ldots + T^*(E_7)) = w(T^*(N)) \leq 6$, a contradiction. Hence one of $T^*(E_i)$, say $T^*(E_1)$, has at least two nonzero entries. Since $w(T^*(N)) \leq 6$, we can find four cells in $\{E_2, \ldots\ldots\ldots, E_7\}$, say $E_2, \ldots\ldots\ldots, E_7$ such that $T^*(E_1 + \ldots\ldots + E_n) = T^*(N)$.

Notice that by Theorem 6.13, $fr(J \backslash N) = 1$. By Theorem 2.4, we have $X \in \mathcal{N}$ for all $X \in M_3(\mathbb{B})$ with $w(X) = 7$ and $fr(J \backslash X) = 1$. If $fr(E_6+E_7) = 1$, then

$T^*(J) = T^*(N) + T^*(E_8+E_9) = T^*(E_1 + \ldots\ldots + E_n) + T^*(E_8+E_9) = T^*(J \backslash (E_6+E_7))$,

which is impossible as $J \backslash (E_6+E_7) \notin R(S)^*$ while $J \in R(S)^*$. Thus $fr(E_6+E_7) = 2$. Since m = n = 3, there is a cell in $\{E_6+E_7\}$, say $E_6$, and a cell in $\{E_8+E_9\}$, say $E_8$ such that $fr(E_6+E_8) = 1$. Since $T^*(N) = T^*(E_1 + \ldots\ldots + E_5) \subseteq T^*(E_1 + \ldots\ldots + E_5 + E_7) \subseteq T^*(N)$, we have $T^*(N) = T^*(E_1 + \ldots\ldots + E_5 + E_7)$. But then

$$T^*(J \backslash E_8) = T^*(N) + T^*(E_9) = T^*(E_1 + \ldots\ldots + E_5 + E_7) + T^*(E_9)$$

$$= T^*(J \backslash (E_6+E_8)),$$

which is impossible because $J \backslash E_8 \in R(S)^*$ while $J \backslash (E_6+E_8) \notin \mathcal{R}(S)^*$. Thus, $w(T^*(N)) = 7$.

**Lemma: 6.19**

If m = n = 3 and $k_{max}$ = 6, then $T^*(\mathcal{N}) \subseteq \mathcal{N}$.

**Proof:**

In this case, $\mathcal{N}$ = $\{N \in M_3(\mathbb{B}) : N \notin \mathcal{R}(S)^*$ and $w(N) = 6\}$. Let $N \in \mathcal{N}$ be arbitary. If suffices to show $w(T^*(N)) = 6$. It follows from $k_{max} = 6$ that $w(T^*(N)) \leq 6$. Suppose $w(T^*(N)) \leq 5$. Write $N = \sum_{i=1}^{6} E_i$ and $J = \sum_{i=1}^{9} E_i$ for cells $E_1, \ldots, E_9$. By Lemma 6.4, $w(T^*(E_i)) \geq 1$ for all i. We will claim that there are distinct cells $E_i$, $E_j$, $E_k$ in $\{E_1, \ldots, E_6\}$ such that

$$T^*(E_i) + T^*(E_j) + T^*(E_k) = T^*(E_i + E_j + E_k) = T^*(N).$$

Clearly the claim holds if there is a cell $E_i$ in $\{E_1, \ldots, E_6\}$ such that $w(T^*(E_i)) \geq 3$. Suppose $w(T^*(E_i)) \leq 2$ for i = 1, …., 6. Let $F_i = T^*(E_i)$ for i = 1, ……, 6. Without loss of generality, we may assume that

$$w(F_1) = \ldots = w(F_r) = 2 \text{ and } w(F_{r+1}) = \ldots = w(F_6) = 1 \text{ for some r.}$$

If r = 0 or r = 1, the by Theorem 6.16, we see that $F_1, \ldots, F_6$ are all disjoint and hence

$$w(T^*(N)) = w(F_1 + \ldots + F_6) = w(F_1) + \ldots + w(F_6) \geq 6,$$

which is impossible. Thus $r \geq 2$. Now suppose $w(F_i+F_j) \leq 3$ for all $1 < i < j \leq r$. Then there is a cell G such that $G \subseteq F_i$ for all i = 1,………,r. It follows from Theorem 6.16 that the six cells $F_1|G, \ldots, F_r|G, F_{r+1}, \ldots, F_6$ are distinct and so

$$w(T^*(N)) = w(F_1 + \ldots + F_6) \geq w(F_1|G) + \ldots + w(F_r|G) + w(F_{r+1}) + \ldots + w(F_6) = 6$$

which is impossible. Thus there are two cells in $\{F_1, \ldots\ldots, F_r\}$, say $F_1$ and $F_2$, such that $w(F_1+F_2) = 4$. In this case, we can always find another cell $F_K$ in $\{F_3, \ldots\ldots\ldots, F_6\}$ such that $F_1 + F_2 + F_k = T^*(N)$. So our claim holds.

Without loss of generality, we may assume $T^*(E_1 + E_2 + E_3) = T^*(N)$. Since $N \notin \mathcal{R}(S)^*$, we must have $E_1 + E_2 + E_3 \notin \mathcal{R}(S)^*$ and it can be easily checked that this is possible only if $E_1 + E_2 + E_3 = P(E_{11} + E_{12} + E_{21})Q$ for some permutation matrices P and Q. If $PE_{22}Q \in \{E_4, E_5, E_6\}$, then

$$T^*(N) = T^*(P(E_{11} + E_{12} + E_{21})Q) \subseteq T^*(P(E_{11} + E_{12} + E_{21} + E_{22})Q) \subseteq T^*(N),$$

and so $T^*(P(E_{11} + E_{12} + E_{21} + E_{22})Q) = T^*(N)$ which is impossible as $P(E_{11} + E_{12} + E_{21} + E_{22})Q \in R(S)^*$. Thus, $PE_{22}Q \notin \{E_4, E_5, E_6\}$. Similarly, we can check $PE_{23}Q, PE_{32}Q \notin \{E_4, E_5, E_6\}$.

Therefore, $\{E_4, E_5, E_6\} = \{PE_{13}Q, PE_{31}Q, PE_{33}Q\}$. In particular, we have $T^*(P(E_{11} + E_{12} + E_{21} + E_{13})Q) = T^*(P(E_{11} + E_{12} + E_{21})Q)$. But then

$$\begin{aligned} T^*(P(J \backslash E_{22} + E_{31})Q) &= T^*(P(E_{11} + E_{12} + E_{21} + E_{13})Q) + T^*(P(E_{23} + E_{32} + E_{33})Q) \\ &= T^*(P(E_{11} + E_{12} + E_{21})Q) + T^*(P(E_{23} + E_{32} + E_{33})Q) \\ &= T^*(P(J \backslash (E_{13} + E_{22} + E_{31}))Q), \end{aligned}$$

which is impossible by Theorem 6.13. Therefore $w(T^*(N)) = 6$ and the result follows.

**Lemma: 6.20**

Suppose $n \geq 4$ and $k_{max} \leq mn - 2$. For any matrix A with $w(A) \leq mn - 2$, $w(T^*(A)) \geq w(A)$. Consequently, we have $T^*(\mathcal{N}) \subseteq \mathcal{N}$.

**Proof:**

Let $w(A) = p$, we will prove the first part of the result by induction on p. By lemma 6.4 the result holds for $p = 1$. Assume that the result holds for all B with $w(B) < p$. Let A be an arbitary matrix with $w(A) = p$.

Suppose at least two rows and two colums of A contains zero entries. Since $n \geq 4$, there are cells $E_1, E_2 \not\subseteq A$ and $E_3 \subseteq A$ such that $fr(E_1+E_2+E_3) = 3$. If $w(T^*(A)) = w(T^*(A \backslash E_3))$, then $T^*(A) = T^*(A \backslash E_3)$ and so

$$\begin{aligned} T^*(J\backslash(E_1+E_2)) &= T^*(A) + T^*(J\backslash(A+E_1+E_2)) \\ &= T^*(A\backslash E_3) + T^*(J\backslash(A+E_1+E_2)) \\ &= T^*(J\backslash (E_1+E_2+E_3)), \end{aligned}$$

a contradiction by Theorem 6.13. Then by assumption, we have $w(T^*(A)) > w(T^*(A\backslash E_3)) \geq w(A\backslash E_3) = p-1$. Thus, the result follows.

Now suppose A has zero entries in one row only. Since $w(A) \leq mn-2$, without loss of generality we may assume that A has zero entries in the first row with zero $(1,1)^{th}$ and $(1,2)^{th}$ entries.

By assumption, $w(T^*(A)) \geq w(T^*(A\backslash E_{21})) \geq w(A\backslash E_{21}) = p-1$.

Suppose $w(T^*(A)) = p-1$. Take

$G_1 = E_{21}+E_{33}$, $G_2 = E_{21}+E_{34}$, $G_3 = E_{22}+E_{33}$ and $G_4 = E_{22}+E_{34}$. We claim that there is an index i in {1, 2, 3, 4} such that $T^*(A\backslash G_i) = T^*(A)$.

If the claim holds, we take $F = \begin{cases} E_{12} & \text{if } i \in \{1,2\}; \\ E_{11} & \text{if } i \in \{3,4\}. \end{cases}$

Then $fr(F+G_i) = 3$ and

$$\begin{aligned} T^*(J\backslash F) = T^*(A) + T^*(J\backslash(A+F)) &= T^*(A\backslash G_i) + T^*(J\backslash A+F)) \\ &= T^*(J\backslash(F+G_i)), \end{aligned}$$

a contradiction by Theorem 6.13. Thus, $w(T^*(A)) \geq p$ and the result follows.

It remains to prove our claim. If the claim does not hold, then for each $i \in \{1, 2, 3, 4\}$. $p-1 = w(T^*(A)) > w(T^*(A\backslash G_i)) \geq w(A\backslash G_i) = p-2$.

That is, $w(T^*(A\backslash G_i)) = p-2$ and so $T^*(A\backslash G_i) = T^*(A)\backslash H_i$ for some cell $H_i \subseteq T^*(A)$. Notice that for any $i \neq j$, $(A\backslash G_i) + (A\backslash G_j)$ equals either A or $A\backslash E$ for some

$E \in \{E_{21}, E_{22}, E_{33}, E_{34}\}$. Since $p-1 = w(T^*(A)) \geq w(T^*(A \backslash E)) \geq p-1$, we have $T^*(A \backslash E) = T^*(A)$ and so $T^*(A \backslash G_i)) + (A \backslash G_j) = T^*(A)$ in both cases. Then

$$T^*(A) = T^*((A \backslash G_i) + (A \backslash G_j)) = T^*(A \backslash G_i) + T^*(A \backslash G_j) =$$
$$= (T^*(A) \backslash H_i) + (T^*(A) \backslash H_j).$$

Thus, we must have $H_i \neq H_j$. Hence $w(H_1+H_2+H_3+H_4) = 4$.

Now let $B = A \backslash (E_{21}+E_{22}+E_{33}+E_{34})$. Since $T^*(B) \subseteq T^*(A \backslash G_i)$ and $H_i \not\subseteq T^*(B)$ for all i = 1, .........., 4 and hence $H_1+H_2+H_3+H_4 \not\subseteq T^*(B)$. Then $T^*(B) \subseteq T^*(A) \backslash (H_1+H_2+H_3+H_4)$ and hence by assumption.

$$p-4 = w(B) \leq w(T^*(B)) \leq w(T^*(A) \backslash (H_1+H_2+H_3+H_4))$$
$$= (p-1) - 4 = p - 5.$$

which is impossible. Therefore, our claim holds.

Finally suppose A has zero entries in one column only. Without loss of generality we may assume that A has zero entries in the first column with zero $(1, 1)^{th}$ and $(2, 1)^{th}$ entries. Then the result follows by a similar argument with $G_1 = E_{12} + E_{33}$, $G_2 = E_{12}+E_{34}$, $G_3 = E_{22}+E_{33}$ and $G_4 = E_{22}+E_{34}$.

**Corollary: 6.21**

The map $T^*|_{\mathcal{N}}$ is bijective from $\mathcal{N}$ onto $\mathcal{N}$.

**Proof:**

Suppose $T^*(N) = T^*(M)$ for some distinct $N, M \in \mathcal{N}$.

Then $T^*(N+M) = T^*(N) + T^*(M) = T^*(N) \notin \mathcal{R}(S)^*$.

But $w(N+M) > w(N) = k_{max}$ which contradicts the definition of $k_{max}$. Thus, $T^*$ is injective in $\mathcal{N}$. Also by Lemmas 6.17, 6.18, 6.19 and 6.20, $T^*(\mathcal{N}) \subseteq \mathcal{N}$. Since $\mathcal{N}$ is finite $T^*(\mathcal{N}) = \mathcal{N}$. Thus, the result follows.

**Lemma: 6.22**

For any cell E, $T^*(E)$ is a cell. Furthermore, $T^*$ is bijective on the set of cells.

**Lemma: 6.23**

$T^*$ preserves all line matrices.

**Proof:**

By Lemma 6.22, $T^*$ is bijective on the set of cells. If $T^*$ does not map some line matrix into a line matrix, without loss of generality, we assume that $T^*(E_{1,1}) = E_{1,1}$ and $T^*(E_{1,2}) = E_{2,2}$.

**Case 1:**

$E_{1,1}+ E_{1,2}+ E_{2,2} \notin \mathcal{R}(S)^*$: consider a matrix $X = E_{1,1}+ E_{1,2}+ E_{i,j}$, where $i \geq 2$ and $j \leq 2$. Then $X \notin \mathcal{R}(S)^*$. Since $T^*$ strongly preserves $\mathcal{R}(S)^*$, $T^*(X) \notin \mathcal{R}(S)^*$ and hence $T^*(E_{ij}) = E_{1,2}$ or $E_{2,1}$ for all $i \geq 2$ and $j \leq 2$. This contradicts Lemma 6.22.

**Case 2:**

$E_{1,1}+ E_{1,2}+ E_{2,2} \in \mathcal{R}(S)^*$: consider a matrix $X = E_{1,1}+ E_{1,2}+ E_{i,j},$. Then we have $fr(T(X)) = 2$ or $3$. By Theorem 2.19 there is a matrix B with $w(B) = 2$ such that $(T(X)+B)^* \notin \mathcal{R}(S)^*$. Furthermore we can write $B = T(C)$ for some matrix C with $w(C) = 2$ so that $T(X)+B = T(X+C)$. But then $(X+C)^* \in \mathcal{R}(S)^*$ by Theorem 2.16, contradicting that $T^*$ strongly preserves $\mathcal{R}(S)^*$.

Therefore $T^*$ preserves all line matrices.

**Definition: 6.24**

An operator T on $M_{m,n}(S)$ is called an **(P, Q, B)-operator** if there are permutation matrices P and Q, and a matrix B with $B^* = J$ such that $T(X)$ $P(X \circ B)Q$ for all $X \in M_{m,n}(S)$, or $m = n$ and $T(X) = P(X^t \circ B)Q$ for all $X \in M_n(S)$.

**Theorem: 6.25**

Let T be a linear operator on $M_{m,n}(S)$ with $\min\{m, n\} \geq 3$. If T strongly preserves regularity, then T is a (P, Q, B)-operator.

**Proof:**

Suppose that T strongly preserves regularity. Then $T^*$ is bijective on the set of cells by Lemma 6.22 and $T^*$ preserves all line matrices by Corollary 6.23. Since no combination of s row matrices and t column matrices can dominate $J_{m,n}$ where $s+t = \min\{m, n\}$ unless $s = 0$ or $t = 0$, we have that either

i) the image of $T^*$ of each row matrix is a row matrix and the image of $T^*$ of each column matrix is a column matrix, or

ii) the image of $T^*$ of each row matrix is a column matrix and the image of $T^*$ of each column matrix is a row matrix.

If (i) holds, then there are permutations $\sigma$ and T of {1, ….., m} and {1, ……., n}, respectively such that $T^*(R_i) = R_{\sigma(i)}$ and $T^*(C_j) = C_{T(j)}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let P and Q be permutation matrices corresponding to $\sigma$ and *T*, respectively. Then we have

$T(E_{i,j}) = b_{i,j}E_{\sigma(i),T(j)} = P(b_{i,j}E_{i,j})Q$, where $b_{i,j} \neq 0$ for all cells $E_{i,j}$. By the action of T on the cells, we have $T(X) = P(X \circ B)Q$. If (ii) holds, then $m = n$ and a parallel argument show that there are permutation matrices P and Q, and a matrix B with $B^* = J$ such that $T(X) = P(X^t \circ B)Q$ for all $X \in M_n(S)$.

**Corollary: 6.26**

The T be a linear operator on $M_{m,n}(\mathbb{B})$ with $\min\{m,n\} \geq 3$. Then T strongly preserves regularity if and only if there are permutation matrices P and Q such that $T(X) = PXQ$ for all $X \in M_{m,n}(\mathbb{B})$, or $m = n$ and $T(X) = PX^tQ$ for $X \in M_n(\mathbb{B})$.

**Proof:**

It follows from Theorems 6.25 and 6.24.

## CHARACTERIZATIONS OF LINEAR OPERATORS THAT STRONGLY PRESERVE REGULARITY OVER SEMIRINGS

**Notation: 6.27**

Let $\mathbb{P}_+$ be the nonnegative part of a subring $\mathbb{P}$ with identity of the reals. The nonnegative integers $\mathbb{Z}_+$, and the nonnegative reals $\mathbb{R}_+$ are good examples of $\mathbb{P}_+$.

**Theorem: 6.28**

Let a, b, c and d be units in $\mathbb{P}_+$. Then $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is regular over $\mathbb{P}_+$ if and only if ad = bc.

**Proof:**

If ad = bc, then we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Conversely assume that X is regular. Then there is a nonzero matrix $Y = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, say $x \neq 0$ such that XYX = X : that is

$$\begin{bmatrix} a(ax+by)+b(az+cw) & a(bx+dy)+b(bz+dw) \\ c(ax+by)+d(az+cw) & c(bx+dy)+d(bz+dw) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

From $(1, 2)^{th}$ and $(2, 2)^{th}$ entries of XYX and X, we have $ab^{-1}(bx+dy) + (bz+dw) = 1 = cd^{-1}(bx+dy) + (bz+dw)$, and hence $ab^{-1}(bx+dy) = cd^{-1}(bx+dy)$. Since $bx+dy \neq 0$, it follows by the cancellation property that $ab^{-1} = cd^{-1}$, equivalently $ad = bc$.

**Theorem: 6.29**

Let $\min\{m, n\} \geq 3$ and T be a linear operator on $M_{m,n}(\mathbb{P}_+)$. Then T strongly preserves regularity if and only if there are invertible matrices U and V such that $T(X) = UXV$ for all $X \in M_{m,n}(\mathbb{P}_+)$, or $m = n$ and $T(X) = UX^tV$ for all $X \in M_n(\mathbb{P}_+)$.

**Proof:**

By Theorem 2.4, the sufficient is obvious. To prove the necessity assume that T strongly preserves regularity. By Theorem 6.25, there are permutation matrices P and Q, a matrix B with $B^* = J$ such that $T(X) = P(X \circ B)Q$ for all $X \in M_{m,n}(\mathbb{P}_+)$ or $m = n$ and $T(X) = P(X^t \circ B)Q$ for all $X \in M_n(\mathbb{P}_+)$. For the case of $T(X) = P(X \circ B)Q$, we define the operator L on $M_{m,n}(\mathbb{P}_+)$ by $L(X) = P^tT(X)Q^t = X \circ B$. Since T strongly preserves regularity, so does L. By Lemma 6.6, all entries of B are regular and hence units because only units are nonzero regular elements over $\mathbb{P}_+$.

If $fr(B) \neq 1$, there is a $2\times2$ submatrix $\mathcal{C}$ of B such that $fr(\mathcal{C}) = 2$. Without loss of generality, we assume that $\mathcal{C} = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$. Then $b_{1,1}b_{2,2} \neq b_{1,2}b_{2,1}$ and hence $\mathcal{C}$ is not regular by Theorem 6.28. Consider a matrix $Y = E_{1,1} + E_{1,2} +$

$E_{2,1} + E_{2,2}$. Then clearly Y is regular, while $L(Y) = \begin{bmatrix} C & O \\ O & O \end{bmatrix}$ is not regular by theorem 2.5, a contradiction. Hence fr(B) = 1. By Lemma 6.6, there are diagonal matrices D and E such that L(X) = DXE for all $X \in M_{m,n}(\mathbb{P}_+)$. Since all entries of B are units, all diagonal entries of D and E are units. Since $L(X) = P^tT(X)\,Q^t = X \circ B$, we have T(X) = PDXEQ. If we let U = PD and V = EQ, then $U \in M_m(\mathbb{P}_+)$ and $V \in M_n(\mathbb{P}_+)$ are invertible. Thus we have T(X) = UXV for all $X \in M_{m,n}(\mathbb{P}_+)$.

If m = n and T is of the form $T(X) = P(X^t \circ B)Q$, then a parallel argument shows that there are invertible matrices U and V such that $T(X) = UX^tV$ for all $X \in M_n(\mathbb{P}_+)$.

**Corollary: 6.30**

Let min{m, n} $\geq$ 3 and T be a linear operator on $M_{m,n}(\mathbb{Z}_+)$. Then T strongly preserves regularity if and only if there are permutation matrices P and Q such that T(X) = PXQ for all $X \in M_n(\mathbb{Z}_+)$, or m = n and $T(X) = PX^tQ$ for all $X \in M_n(\mathbb{Z}_+)$.

**Theorem: 6.31**

Let S be only chain semiring. Let $A = \begin{bmatrix} p & q \\ q & 0 \end{bmatrix}$ be a matrix in $M_2(S)$ with $pq \neq 0$. Then A is regular if and only if pq = p.

**Proof:**

If pq = p, then $\begin{bmatrix} p & q \\ q & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ q & 0 \end{bmatrix} = \begin{bmatrix} p & q \\ q & 0 \end{bmatrix}$ and hence A is regular.

Conversely, assume that A is regular and $pq \neq b$. Then $p \neq q$, $pq = q$ (i.e., $q < p$) and there is a nonzero $G = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_2(S)$ such that AGA = A and

hence $AGA = \begin{bmatrix} px+q(y+z+w) & q(x+z) \\ q(x+y) & qx \end{bmatrix} = \begin{bmatrix} p & q \\ q & 0 \end{bmatrix} = A$.

From $(2, 2)^{th}$ entries of AGA and A, $x = 0$ since $q \neq 0$. Again from $(1, 1)^{th}$ entries of AGA and A, $q(y+z+w) = p$. But this is impossible because $q < p$. Therefore A is not regular for $pq \neq p$.

**Note: 6.32**

If A is a monomial matrix in $M_n(S)$, then A is invertible if and only if A is a permutation matrix because 1 is the only unit element in S.

**Theorem: 6.33**

Let $\min\{m, n\} \geq 3$ and T be a linear operator on $M_{m,n}(S)$. Then T strongly preserves regularity if and only if there are permutation matrices P and Q such that $T(X) = PXQ$ for all $X \in M_{m,n}(S)$, or m = n and $T(X) = PX^tQ$ for all $X \in M_n(S)$.

**Proof:**

The sufficient follows Theorem 2.4. For the necessary, assume that T strongly preserves regularity. By Theorem 6.25 there are permutation matrices P, Q and a matrix B with $B^* = J$, such that $T(X) = P(X \circ B)Q$ for all $X \in M_{m,n}(S)$, or m = n and $T(X) = P(X^t \circ B)Q$ for all $X \in M_n(S)$.

Let T be of the form $T(X) = P(X \circ B)Q$. Without loss of generality, we assume that $P = I_m$ and $Q = I_n$ so that $T(X) = X \circ B$ and $T(J) = B$. Now we will show that B = J, equivalently $b_{i,j} = 1$ for all i and j. It is sufficient to consider $b_{1,1}$ : for $b_{i,j}$ is any entry of T(J), let P' be the transposition matrix that exchanges $1^{st}$ and $i^{th}$ rows from identity matrix $I_m$, and Q' the transposition matrix that exchanges $1^{st}$ and $j^{th}$ rows from identity matrix $I_n$. Define a linear operator L on $M_{m,n}(S)$ by L(X) = P'T(X)Q' for all X. Since T strongly preserves regularity, so does L. Furthermore the $(1, 1)^{th}$ entry of L(J) is $b_{i,j}$.

If $b_{1,1} \neq 1$, let $\alpha = \min\{b_{1,1}, b_{1,2}, b_{2,1}\}$. Then $\alpha \neq 0, 1$. Consider a matrix $A = E_{1,1} + \alpha(E_{1,2} + E_{2,1})$. By Theorem 6.31, A is not regular and hence $T(A) = b_{1,1}E_{1,1} + \alpha(E_{1,2} + E_{2,1})$ is not regular so that $b_{1,1}\alpha = \alpha$ and $b_{1,1}\alpha \neq \alpha$. Thus $\alpha = b_{1,1}$ or $b_{2,1}$. If $\alpha = b_{1,2}$, consider a matrix $A_1 = b_{1,1}(E_{1,1} + E_{1,2}) + \alpha E_{2,1}$. Then $A_1$ is regular $A_2[b_{1,1}(G_{1,2}+G_{2,1})]\ A_1 = A_1$, where $G_{i,j}$ are cells in $M_{n,m}(S)$. But $T(A_1) = b_{1,1}E_{1,1} + \alpha(E_{1,2} + E_{2,1})$ is not regular by Theorem 6.31, a contradiction. For the case $\alpha = b_{2,1}$, if we consider a matrix $A_2 = b_{1,1}(E_{1,1} + E_{2,1}) + \alpha E_{1,2}$, then $A_2$ is regular while $T(A_2)$ is not regular, a contradiction. Therefore $b_{1,1} = 1$. Hence B = J. Therefore T(X) = PXQ for all $X \in M_{m,n}(S)$.

For the case of m = n and $T(X) = P(X^t \circ B)Q$, a parallel argument shows that B = J so that $T(X) = PX^tQ$ for all $X \in M_n(S)$.