CHAPTER IV

THE (m, N) POLICY FOR A REPAIRABLE BATCH ARRIVAL QUEUE WITH A SECOND OPTIONAL SERVICE CHANNEL UNDER DIFFERENT TYPES OF BERNOULLI SINGLE VACATIONS

INTRODUCTION

In queueing theory context, temporary periods of unavailability of service are referred to as server vacations. In addition to the arrival process, service process and queue discipline, the vacation queueing models include a vacation process governed by a vacation policy. One of the main aspects of vacation policy is vacation startup rule, which determines the starting time of a vacation. Exhaustive service discipline and non-exhaustive service discipline are the two major types of vacation starting rules. With an exhaustive service rule, the server cannot take a vacation until the system becomes empty. The vacation policies considered in Chapters II and III are of exhaustive service type.

Vacation policies with non-exhaustive discipline allow the server to take vacation when some customers are still in the system. The classical vacation scheme with Bernoulli service discipline, initiated and developed significantly by Keilson and Servi (1986), belongs to the non-exhaustive vacation policy. According to this policy, as soon as the service of a customer is completed, the server may go for a vacation with probability p (or) continue to stay in the system to provide service to a next customer, if any with probability (1 – p).

Queueing models exist in literature do not consider both types (exhaustive and non-exhaustive service discipline) of vacations in the same queue. In the present chapter, it is assumed that, if the system is empty at the end of a service, then the server takes a single vacation of length (VI) with probability **p′** and operates the bilevel threshold policy before starting a new busy period.

Since the purpose of taking vacations between services is manifold, it is also assumed that, the server is provided with M-types of vacations following different distributions $VB_i(t)$ $(1 \le i \le M)$ and the server, after completing a service to a customer may select the i^{th} type vacation with probability p_i (i = 1 to M) or continue to serve the next customer with probability $(1 - \sum_{i=1}^{M} p_i)$, during busy period.

4.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

4.1.1 Model Description

The customers arrive in batches according to the time-homogeneous Poisson process with group arrival rate λ as mentioned in Chapter II.

Idle Period

The server is deactivated whenever the system becomes empty. The deactivated server, may either takes a single vacation of random length (VI) with probability p' or stays idle in the system with probability $(1 - p')$. If the server, either on returning from the single vacation (or) while staying idle in the system, finds, atleast 'm' customers in the queue, then he starts a setup operation. The time taken for the setup operation is a random variable D whose distribution function and density function are D(t) and d(t) respectively. At the end of the setup period, if the queue length is greater than or equal to N, then the server begins to serve the customers exhaustively. Otherwise, the server remains dormant in the system waiting for the queue length to reach atleast N to start a service. Thus a busy period starts if the queue length reaches N or exceeds, either at the end of a setup period or at the end of a dormant period.

Busy Period

During busy period, the server provides two types of heterogeneous services of which the first one is essential and the second service is optional. The customers are served according to FCFS queue discipline. After the completion of First Essential Services (FES), the customer may leave the system with probability (1 – r) or may opt for the Second Optional Service (SOS) in an additional channel by the same server, with probability r ($0 \le r \le 1$). The service time random variables S_i, i = 1, 2 of two channels are assumed to be mutually independent of each other. The service times S_i follow general distributions and the distribution functions, density functions and the first two moments of these service times are denoted by $S_i(x)$, $s_i(x)$, $E(S_i^k)$, i = 1, 2 ; k = 1, 2 respectively.

Bernoulli Schedule Vacation Policy Between Services

The server may take a single vacation after completing each service to a customer. M different types of vacation facilities, each of random length VB_i ($1 \le i \le M$) are available. Whenever a service is completed and if a customer leaves the system, (either by completing the FES without opting SOS or completing the SOS), the server may take i^{th} type of vacation of random length VB_i with probability p_i ($1 \le i \le M$) or may continue to serve the next customer, with probability 1– p, where $p = \sum_{i=1}^{M} p_i$. Each VB_i ($1 \le i \le M$) follows general distribution $VB_i(t)$ with finite moments. This type of service continues until the system becomes empty again.

Breakdown Period

The server may breakdown at any time while serving customers. It is assumed that the server's life time follow exponential distribution with parameter $\mathbf{a_1}$ in the FES. In the second optional service, the service fails at an exponential rate $\mathbf{a_2}$. Whenever breakdown occurs, the server is sent for repair immediately and the customer just being served before server breaks down waits for the server in the service facility to complete the remaining service. The repair time distributions of both service phases (FES or SOS) R_1 and R_2 are arbitrarily distributed with probability distribution functions $R_1(y)$ and $R_2(y)$ respectively with density functions $r_1(y)$ and $r_2(y)$ with finite moments $E(R_i^k)$, i = 1, 2 ; k = 1, 2. Immediately after the server is fixed, it starts to serve the customer who is waiting to complete the remaining service. It is assumed that the service time for a customer is cumulative and after repair the server is considered as good as new.

In the present model, the server takes optional single vacation (VI) at the end of each cycle with probability **p′** and may take any of the single vacations VB_i $(1 \le i \le M)$ of type i with probability $\mathbf{p_i}$, $\left(\sum_{i=1}^{M} \mathbf{p_i} = \mathbf{p}\right)$ soon after completing a service to a customer.

A cycle begins, whenever the system becomes empty and the server either takes vacation or stays idle in the system and then the server does a setup operation when the queue length reaches atleast **m**. As the queue length reaches atleast **N**, the server becomes busy and continues with FES and SOS along with Bernoulli Schedule Vacations and Breakdowns. The cycle ends when the system becomes empty again. Thus a cycle is made up of server idle vacation, buildup period, setup period, busy period, breakdown period and vacation period between services.

We denote the model by $M^X_{(m,N)}/G_{SOS}/1/B_{VB_{i(1\le i\le M)}}$/Single vacation/ Breakdown, where B_{VB_i} denotes Bernoulli Schedule Vacation having M types of vacation. Various stochastic processes involved in the queueing system are assumed to be independent of each other. The customers continue to arrive and join the system independent of the system states following the compound Poisson process.

The following notations are used to write the partial differential equations using supplementary variable technique. The remaining times of random variables namely service times $(S_i^o(t))$, (i = 1, 2), setup time $(D^o(t))$, vacation time during idle period $(VI^o(t))$, repair times $(R_i^o(t))$, (i = 1, 2) and vacation time during busy period $(VB_j^o(t))$, $(1 \le j \le M)$ are introduced as the supplementary variables at time t. The notations $N_S(t)$, λ, X, g_k, $g_n^{(i)}$ $(i \ge 1, n \ge 1)$ X(z) are as same as Chapter **II** (Table 2.0).

The CDF, PDF and LST (the cumulative distribution function, probability density function and Laplace Stieltjes transform) of various random variables are listed here.

	RV	CDF	PDF	LST	k^{th} moments k = 1, 2
Vacation time during idle period	VI	VI(x)	vI(x)	$VI^*(\theta)$	$E(VI^k)$
Vacation time during busy period	VB_i	$VB_i(x)$	$vB_i(x)$	$VB_i^*(\theta)$	$E(VB_i^k)$, i = 1 to M
Setup time	D	D(x)	d(x)	$D^*(\theta)$	$E(D^k)$
FES time (i = 1) and SOS time (i = 2)	S_i	$S_i(x)$	$s_i(x)$	$S_i^*(\theta)$	$E(S_i^k)$, i = 1, 2
Repair time in FES (i = 1) and SOS (i = 2)	R_i	$R_i(y)$	$r_i(y)$	$R_i^{*1}(\theta_1)$	$E(R_i^k)$, i = 1, 2

At time t, let Y(t) = 0, 1, 2, 3, 4, 5, 6, 7 and 8 respectively denote the server is on vacation (idle), in buildup, setup, dormant states, busy with FES, busy with SOS, under repair in FES, repair in SOS and on vacation between services.

Then the state of the system at time t can be described by the Markov process {N(t), δ(t)}, where $\delta(t) = (VI^o(t), 0, D^o(t), 0, S_1^o(t), S_2^o(t), R_1^o(t), R_2^o(t), VB_i^o(t))$ according as Y(t) = 0, 1, 2, 3, 4, 5, 6, 7 and 8 respectively. The transient system state probabilities are defined by

$QI_n(x, t)\,dt$ = Pr $\{N_S(t) = n, x < VI^o(t) \le x + dt, Y(t) = 0\}$, $n \ge 0$ (during idle-vacation period)

$PI_n(t)$ = Pr $\{N_S(t) = n, Y(t) = 1\}$, $0 \le n \le m-1$ (buildup period)

$SE_n(x, t)\,dt$ = Pr $\{N_S(t) = n, x < D^o(t) \le x + dt, Y(t) = 2\}$, $n \ge m$ (setup period)

$U_n(t)$ = Pr $\{N_S(t) = n, Y(t) = 3\}$, $m \le n \le N-1$ (dormant period)

$P_{1,n}(x, t)\,dt$ = Pr $\{N_S(t) = n, x < S_1^o(t) \le x + dt, Y(t) = 4\}$, $n \ge 1$ (FES service)

$P_{2,n}(x, t)\,dt$ = Pr $\{N_S(t) = n, x < S_2^o(t) \le x + dt, Y(t) = 5\}$, $n \ge 1$ (SOS service)

$BR_{1,n}(x, y, t)dt$ = Pr $\{N_S(t) = n, S_1^o(t) = x, y < R_1^o(t) \le y + dt, Y(t) = 6\}$, $n \ge 1$ (server in repair facility due to breakdown in FES)

$BR_{2,n}(x, y, t)\,dt$ = Pr $\{N_S(t) = n, S_2^o(t) = x, y < R_2^o(t) \le y + dt, Y(t) = 7\}$, $n \ge 1$ (server is in repair facility due to breakdown in SOS)

$QB_{i,n}(x, t)\,dt = \Pr\{N_S(t) = n,\ x < VB_i^{o}(t) \le x + dt,\ Y(t) = 8\},\ n \ge 1;\ i = 1 \text{ to } M$

(i^{th} vacation during vacation between services)

The interpretation of the joint probability distribution corresponding to different states are similar to Chapter II.

4.1.2 The Steady State System Size Equations

Following the arguments of Lee et al. (1994a) and observing the changes of states during the interval (t, t + Δt) for any time t, the steady-state system size equations are given by :

Vacation During Idle Period

$$-\frac{d}{dx}QI_0(x) = -\lambda\, QI_0(x) + p'\, P_{2,1}(0)\, vI(x) + p'\,(1 - r)\, P_{1,1}(0)\, vI(x)$$

$$-\frac{d}{dx}QI_n(x) = -\lambda\, QI_n(x) + \lambda \sum_{k=1}^{n} QI_{n-k}(x)\, g_k,\ n \ge 1$$

Buildup State

$$\lambda\, PI_0 = QI_0(0) + (1 - p')\, P_{2,1}(0) + (1 - p')\,(1 - r)\, P_{1,1}(0)$$

$$\lambda\, PI_n = QI_n(0) + \lambda \sum_{k=1}^{n} PI_{n-k}\, g_k, \qquad 1 \le n \le m-1$$

Setup State

$$-\frac{d}{dx}SE_n(x) = -\lambda\, SE_n(x) + \lambda \sum_{k=n-m+1}^{n} PI_{n-k}\, g_k\, d(x) + QI_n(0)\, d(x) + (1 - \delta_{m,n})\ \lambda \sum_{k=1}^{n-m} SE_{n-k}(x)\, g_k, \qquad n \ge m$$

Dormant / Standby State

$$\lambda\, U_n = SE_n(0) + (1 - \delta_{m,n})\, \lambda \sum_{k=1}^{n-m} U_{n-k}\, g_k, \qquad m \le n \le N-1$$

Busy States

Busy with FES

$$-\frac{d}{dx}P_{1,n}(x) = -(\lambda + a_1)\,P_{1,n}(x) + (1-p)\,P_{2,n+1}(0)\,s_1(x) + \sum_{i=1}^{M} QB_{i,n}(0)\,s_1(x)$$
$$+ (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P_{1,n-k}(x)\,g_k + (1-p)\,(1-r)\,P_{1,n+1}(0)\,s_1(x)$$
$$+ BR_{1,n}(x, 0) \qquad \text{where } p = \sum_{i=1}^{M} p_i\,, \qquad 1 \le n \le N-1$$

$$-\frac{d}{dx}P_{1,n}(x) = -(\lambda + a_1)\,P_{1,n}(x) + (1-p)\,P_{2,n+1}(0)\,s_1(x) + \sum_{i=1}^{M} QB_{i,n}(0)\,s_1(x)$$
$$+ \lambda \sum_{k=1}^{n-1} P_{1,n-k}(x)\,g_k + (1-p)\,(1-r)\,P_{1,n+1}(0)\,s_1(x)$$
$$+ BR_{1,n}(x, 0) + SE_n(0)\,s_1(x) + \lambda \sum_{k=n-N+1}^{n-m} U_{n-k}\,g_k\,s_1(x), \quad n \ge N$$

Busy with SOS

$$-\frac{d}{dx}P_{2,n}(x) = -(\lambda + a_2)\,P_{2,n}(x) + r\,P_{1,n}(0)\,s_2(x)$$
$$+ (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P_{2,n-k}(x)\,g_k + BR_{2,n}(x, 0), \; n \ge 1$$

Vacation During Busy Period

$$-\frac{d}{dx}QB_{i,n}(x) = -\lambda\,QB_{i,n}(x) + p_i\,P_{2,n+1}(0)\,VB_i(x) + p_i\,(1-r)\,P_{1,n+1}(0)\,VB_i(x) +$$
$$+ (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} QB_{i,n-k}(x)\,g_k, \; i = 1 \text{ to } M \qquad n \ge 1$$

Breakdown Period

Breakdown in FES

$$-\frac{\partial}{\partial y}BR_{1,n}(x, y) = -\lambda\,BR_{1,n}(x, y) + a_1\,P_{1,n}(x)\,r_1(y)$$
$$+ (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} BR_{1,n-k}(x, y)\,g_k, \qquad n \ge 1$$

Breakdown in SOS

$$-\frac{\partial}{\partial y} BR_{2,n}(x, y) = -\lambda\, BR_{2,n}(x, y) + a_2\, P_{2,n}(x)\, r_2(y) + (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} BR_{2,n-k}(x, y)\, g_k, \qquad n \geq 1$$

Taking the LST of the steady-state equations, we have

$$\theta\, QI_0^*(\theta) - QI_0(0) = \lambda\, QI_0^*(\theta) - p'\, P(0)\, VI^*(\theta) \tag{4.1}$$

where $P(0) = P_{2,1}(0) + (1-r)\, P_{1,1}(0)$

$$\theta\, QI_n^*(\theta) - QI_n(0) = \lambda\, QI_n^*(\theta) - \lambda \sum_{k=1}^{n} QI_{n-k}^*(\theta)\, g_k, \qquad n \geq 1 \tag{4.2}$$

$$\lambda\, PI_0 = QI_0(0) + (1-p')\, P(0) \tag{4.3}$$

$$\lambda\, PI_n = QI_n(0) + \lambda \sum_{k=1}^{n} PI_{n-k}\, g_k, \qquad 1 \leq n \leq m-1 \tag{4.4}$$

$$\theta\, SE_n^*(\theta) - SE_n(0) = \lambda\, SE_n^*(\theta) - \lambda \sum_{k=n-m+1}^{n} PI_{n-k}\, g_k\, D^*(\theta) - QI_n(0)\, D^*(\theta) - (1-\delta_{m,n})\,\lambda \sum_{k=1}^{n-m} SE_{n-k}^*(\theta)\, g_k, \qquad n \geq m \tag{4.5}$$

$$\lambda\, U_n = SE_n(0) + (1-\delta_{m,n})\,\lambda \sum_{k=1}^{n-m} U_{n-k}\, g_k \qquad m \leq n \leq N-1 \tag{4.6}$$

$$\theta\, P_{1,n}^*(\theta) - P_{1,n}(0) = (\lambda + a_1)\, P_{1,n}^*(\theta) - (1-p)\, P_{2,n+1}(0)\, S_1^*(\theta) - \sum_{i=1}^{M} QB_{i,n}(0)\, S_1^*(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P_{1,n-k}^*(\theta)\, g_k - (1-p)(1-r)\, P_{1,n+1}(0)\, S_1^*(\theta) - BR_{1,n}^*(\theta, 0) \tag{4.7}$$

$$1 \leq n \leq N-1$$

$$\theta\, P_{1,n}^*(\theta) - P_{1,n}(0) = (\lambda + a_1)\, P_{1,n}^*(\theta) - (1-p)\, P_{2,n+1}(0)\, S_1^*(\theta) - \sum_{i=1}^{m} QB_{i,n}(0)\, S_1^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{1,n-k}^*(\theta)\, g_k - (1-p)(1-r)\, P_{1,n+1}(0)\, S_1^*(\theta) - BR_{1,n}^*(\theta, 0) - SE_n(0)\, S_1^*(\theta) - \lambda \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k\, S_1^*(\theta), \quad n \geq N \tag{4.8}$$

$$\theta\, P_{2,n}^*(\theta) - P_{2,n}(0) = (\lambda + a_2)\, P_{2,n}^*(\theta) - r\, P_{1,n}(0)\, S_2^*(\theta) - \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{2,n-k}^*(\theta)\, g_k - BR_{2,n}^*(\theta, 0), \quad n \geq 1 \tag{4.9}$$

$$\theta\ QB^{*}_{i,n}(\theta) - QB_{i,n}(0) = \lambda\ QB^{*}_{i,n}(\theta) - p_i\ P_{2,n+1}(0)\ VB^{*}_{i}(\theta)$$
$$- p_i\ (1-r)\ P_{1,n+1}(0)\, VB^{*}_{i}(\theta)$$
$$- (1-\delta_{1,n})\ \lambda \sum_{k=1}^{n-1} QB^{*}_{i,n-k}(\theta) g_k, \quad i = 1 \text{ to } M,\ n \geq 1 \quad (4.10)$$

LST of equations due to breakdown (with respect to x)

$$-\frac{\partial}{\partial y} BR^{*}_{1,n}(\theta, y) = -\lambda\ BR^{*}_{1,n}(\theta, y) + a_1\ P^{*}_{1,n}(\theta) r_1(y)$$
$$+ (1-\delta_{1,n})\ \lambda \sum_{k=1}^{n-1} BR^{*}_{1,n-k}(\theta, y) g_k, \qquad n \geq 1 \qquad (4.11)$$
$$-\frac{\partial}{\partial y} BR^{*}_{2,n}(\theta, y) = -\lambda\ BR^{*}_{2,n}(\theta, y) + a_2\ P^{*}_{2,n}(\theta) r_2(y)$$
$$+ (1-\delta_{1,n})\ \lambda \sum_{k=1}^{n-1} BR^{*}_{2,n-k}(\theta, y) g_k, \qquad n \geq 1 \qquad (4.12)$$

Taking the LST w.r. to y for equations (4.11) and (4.12), we get

$$\theta_1\ BR^{**1}_{1,n}(\theta, \theta_1) - BR^{*}_{1,n}(\theta, 0) = \lambda\ BR^{**1}_{1,n}(\theta, \theta_1) - a_1\ P^{*}_{1,n}(\theta)\, R^{*1}_{1}(\theta_1)$$
$$- \lambda\ (1-\delta_{1,n}) \sum_{k=1}^{n-1} BR^{**1}_{1,n-k}(\theta, \theta_1) g_k, \quad n \geq 1 \quad (4.13)$$
$$\theta_1\ BR^{**1}_{2,n}(\theta, \theta_1) - BR^{*}_{2,n}(\theta, 0) = \lambda\ BR^{**1}_{2,n}(\theta, \theta_1) - a_2\ P^{*}_{2,n}(\theta)\, R^{*1}_{2}(\theta_1)$$
$$- \lambda\ (1-\delta_{1,n}) \sum_{k=1}^{n-1} BR^{**1}_{2,n-k}(\theta, \theta_1) g_k, \quad n \geq 1 \quad (4.14)$$

4.1.3 Probability Generating Functions

By considering the following partial PGFs, steady-state distributions are obtained.

$PI(z) = \sum_{n=0}^{m-1} PI_n\ z^n$	$U(z) = \sum_{n=m}^{N-1} U_n\ z^n$
$QI^{*}(z,\theta) = \sum_{n=0}^{\infty} QI^{*}_{n}(\theta) z^n$	$QI(z, 0) = \sum_{n=0}^{\infty} QI_n(0) z^n$
$QB^{*}_{i}(z,\theta) = \sum_{n=1}^{\infty} QB^{*}_{i,n}(\theta) z^n$	$QB_i(z, 0) = \sum_{n=0}^{\infty} QB_{i,n}(0) z^n$, i = 1 to M
$SE^{*}(z,\theta) = \sum_{n=m}^{\infty} SE^{*}_{n}(\theta) z^n$	$SE(z, 0) = \sum_{n=m}^{\infty} SE_n(0) z^n$
$P^{*}_{i}(z,\theta) = \sum_{n=1}^{\infty} P^{*}_{i,n}(\theta) z^n$	$P_i(z, 0) = \sum_{n=1}^{\infty} P_{i,n}(0) z^n$, i = 1, 2
$BR^{**1}_{i}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} BR^{**1}_{i,n}(\theta,\theta_1) z^n$	$BR^{*}_{i}(z,\theta,0) = \sum_{n=1}^{\infty} BR^{*}_{i,n}(\theta,0) z^n$, i = 1,2

Multiplying the transformed equations by suitable powers of z and adding the corresponding equations the partial PGFs are obtained.

The partial PGFs of the system size, when the server is on vacation, during idle period, is obtained by using equations (4.1) and (4.2).

Equations (4.1) and (4.2) imply

$$(\theta - w_X(z))\ QI^*(z,\theta) = QI(z, 0) - p'\ P(0)\ VI^*(\theta) \qquad (4.15)$$

At $\theta = w_X(z)$,

$$QI(z, 0) = p'\ P(0)\ VI^*(w_X(z)) \qquad (4.16)$$

Substituting the value of $QI(z, 0)$ in equation (4.15), we get

$$QI^*(z,\theta) = p'\ P(0)\ \frac{(VI^*(w_X(z) - VI^*(\theta))}{(\theta - w_X(z))} \qquad (4.17)$$

If αI_n denotes the probability that n customers arrive during the vacation time (VI), then $VI^*(w_X(z)) = \sum_{n=0}^{\infty} \alpha I_n\ z^n$ (4.18)

(Gross and Harris, 1985)

Thus equations (4.18) and (4.16) imply,

$$\sum_{n=0}^{\infty} QI_n(0)\, z^n = p'\ P(0) \sum_{n=0}^{\infty} \alpha I_n\ z^n$$

$$\text{i.e., } QI_n(0) = p'\ P(0)\ \alpha I_n,\ n \geq 0 \qquad (4.19)$$

To calculate the generating function corresponding to the buildup state, the following result is used.

Theorem : 4.1

Let $\pi_0 = 1$, $\pi_n = \sum_{k=1}^{n} \pi_{n-k}\, g_k$ and $\psi_0 = \alpha I_0$, $\psi_n = \sum_{i=0}^{n} \alpha I_i\ \pi_{n-i}$ $(1 \leq n \leq m-1)$

(4.19.1)

then $PI(z) = (p'\ \psi(z) + (1 - p')\ \pi(z))\ P(0)$

where $\psi(z) = (\frac{1}{\lambda}) \sum_{n=0}^{m-1} \psi_n\ z^n$ and $\pi(z) = (\frac{1}{\lambda}) \sum_{n=0}^{m-1} \pi_n\ z^n$

Proof

Equations (4.3) and (4.4) recursively imply, for $0 \leq n \leq m-1$

$$\lambda\ PI_n = \sum_{k=0}^{n} QI_{n-k}(0)\ \pi_k + (1 - p')\ \pi_n\ P(0)$$

Substituting for $QI_n(0)$ from the equation (4.19),

$$\lambda\, PI_n = P(0)\left(p'\sum_{k=0}^{n}\alpha I_{n-k}\,\pi_k + (1-p')\,\pi_n\right),\ 0 \le n \le m-1$$

Adding the equation over n = 0 to m–1 and using the definitions of ψ_n,

$$\lambda\, PI(z) = \left(p'\sum_{n=0}^{m-1}\psi_n\, z^n + (1-p')\sum_{n=0}^{m-1}\pi_n\, z^n\right)P(0)$$

i.e., $$PI(z) = (p'\,\psi(z) + (1-p')\,\pi(z))\,P(0) \tag{4.20}$$

Corollary

For $0 \le n \le m-1$

(i) $$\sum_{i=0}^{n-1}\psi_i\, g_{n-i} = \sum_{i=0}^{n-1}\alpha I_i\,\pi_{n-i} \tag{4.20.1}$$

(ii) $$\alpha I_n + \sum_{i=0}^{n-1}\psi_i\, g_{n-i} = \psi_n \tag{4.20.2}$$

The proof of the corollary is analogues to corollary of Theorem 2.2 of Chapter II, with J = 1.

Equation (4.5) is used to obtain the PGF of the system size corresponding to the setup period. Multiplying equation (4.5) by appropriate powers of z and summing over m to ∞, it is found that

$$(\theta - w_X(z))\, SE^*(z,\theta) = SE(z,0) - \lambda\sum_{n=m}^{\infty} z^n \sum_{k=n-m+1}^{n} PI_{n-k}\, g_k\, D^*(\theta) - \sum_{n=m}^{\infty} QI_n(0)\, z^n\, D^*(\theta) \tag{4.21}$$

Adding, equations (4.3) and (4.4) over n = 0 to m–1, we have

$$\lambda\, PI(z) = (1-p')\,P(0) + \lambda\sum_{n=1}^{m-1} z^n \sum_{k=1}^{n} PI_{n-k}\, g_k + \sum_{n=0}^{m-1} QI_n(0)\, z^n \tag{4.22}$$

Multiplying the equation (4.22) by $(-D^*(\theta))$ and then adding with equation (4.21), the following equation is obtained.

$$(\theta - w_X(z))\, SE^*(z,\theta) = SE(z,0) + \lambda D^*(\theta)\, PI(z) - \lambda\, PI(z)\, X(z)\, D^*(\theta) - D^*(\theta)\, QI(z,0) - D^*(\theta)\,(1-p')\,P(0) \tag{4.23}$$

Substituting for PI(z) and QI(z, 0) (from equations (4.20), (4.16)) in (4.23) and putting $\theta = w_X(z)$, we have

$$SE(z, 0) = P(0)\,[p'(D^*(w_X(z))\,(VI^*(w_X(z)) - w_X(z)\,\psi(z)) + (1 - p')\,D^*(w_X(z))\,(1 - w_X(z)\,\pi(z))] \quad (4.24)$$

Using equation (4.24) in equation (4.21) we have,

$$SE^*(z, \theta) = P(0)\,\frac{(D^*(w_X(z)) - D^*(\theta))}{(\theta - w_X(z))}\,[p'(VI^*(w_X(z)) - w_X(z)\,\psi(z)) + (1 - p')\,(1 - w_X(z)\,\pi(z))] \quad (4.25)$$

The partial probability generating functions of the system size, when the server is in vacation state (during busy period) is obtained by using equation (4.10),

Equation (4.10) implies for $1 \le i \le M$,

$$(\theta - w_X(z))QB_i^*(z,\theta) = QB_i(z, 0) - \frac{p_i\,VB_i^*(\theta)}{z}(P_2(z, 0) + (1 - r)\,P_1(z, 0)) + p_i\,VB_i^*(\theta)(P_{2,1}(0) + (1 - r)\,P_{1,1}(0)) \quad (4.26)$$

At $\theta = w_X(z)$

$$QB_i(z, 0) = p_i\,VB_i^*(w_X(z))\left(\frac{P_2(z,0) + (1-r)P_1(z,0)}{z} - P(0)\right) \quad (4.27)$$

Substituting the value of $QB_i(z, 0)$ in equation (4.26), we get

$$QB_i^*(z,\theta) = p_i\left(\frac{P_2(z,0) + (1-r)P_1(z,0)}{z} - P(0)\right)\frac{(VB_i^*(w_X(z)) - VB_i^*(\theta))}{(\theta - w_X(z))} \quad (4.28)$$

Adding equations (4.27) and (4.28) over i = 1 to M, we have

$$QB(z, 0) = VB^*(w_X(z))\left(\frac{P_2(z,0) + (1-r)P_1(z,0)}{z} - P(0)\right) \quad (4.29)$$

$$QB^*(z,\theta) = \left(\frac{P_2(z,0) + (1-r)P_1(z,0)}{z} - P(0)\right)\left(\frac{VB^*(w_X(z)) - VB^*(\theta)}{\theta - w_X(z)}\right) \quad (4.30)$$

where

$$QB(z, 0) = \sum_{i=1}^{M} QB_i(z,0),\quad VB^*(w_X(z)) = \sum_{i=1}^{M} p_i\,VB_i^*(w_X(z)) \quad (4.31)$$

$$QB^*(z,\theta) = \sum_{i=1}^{M} QB_i^*(z,\theta) \text{ and } VB^*(\theta) = \sum_{i=1}^{M} p_i\,VB_i^*(\theta)$$

The equations (4.11) and (4.12) are used to calculate the generating functions of the system size corresponding to the breakdown states in FES and SOS at service completion epoch and at arbitrary epoch. The generating functions are given by :

$$BR_i^*(z,\theta,0) \quad = \quad a_i\, R_i^{*1}(w_X(z))\, P_i^*(z,\theta) \tag{4.32}$$

$$BR_i^{**1}(z,\theta,\theta_1) \;=\; \frac{a_i\, P_i^*(z,\theta)\,(R_i^{*1}(w_X(z)) - R_i^{*1}(\theta_1))}{\theta_1 - w_X(z)},\ i = 1, 2 \tag{4.33}$$

The equation (4.9) is used to calculate the generating functions of the system size when the server is in SOS, at termination epoch and arbitrary epoch.

The equations (4.9) and (4.32) at i = 2, give,

$$(\theta - h_{a_2}(w_X(z)))\, P_2^*(z,\theta) = P_2(z,0) - r\ S_2^*(\theta)\, P_1(z,0) \tag{4.34}$$

where $h_{a_2}(w_X(z)) \;=\; w_X(z) + a_2\,(1 - R_2^{*1}(w_X(z)))$

At $\theta = h_{a_2}(w_X(z))$,

$$P_2(z,0) \;=\; r\ S_2^*(h_{a_2}(w_X(z)))\, P_1(z,0) \tag{4.35}$$

Substituting the value of $P_2(z, 0)$ in equation (4.32), we get

$$P_2^*(z,\theta) \;=\; \frac{r\,P_1(z,0)\,(S_2^*(h_{a_2}(w_X(z))) - S_2^*(\theta))}{(\theta - h_{a_2}(w_X(z)))} \tag{4.36}$$

Next to calculate the partial generating functions corresponding to the FES state, the equations (4.7) and (4.8) are used.

Substituting for $B_1^*(z,\theta,0)$ from (4.32), the equations (4.7) and (4.8) imply,

$$\begin{aligned}(\theta - h_{a_1}(w_X(z)))\, P_1^*(z,\theta) \;=\;& P_1(z,0) - \frac{(1-p)\,S_1^*(\theta)}{z}(P_2(z,0) + (1-r)\,P_1(z,0)) \\ &+ (1-p)\ S_1^*(\theta)\,P(0) - S_1^*(\theta)\,QB(z,0) \\ &- S_1^*(\theta)\,\Big(\sum_{n=N}^{\infty} SB_n(0)\,z^n + \lambda \sum_{n=N}^{\infty} z^n \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k\Big)\end{aligned} \tag{4.37}$$

where $h_{a_1}(w_X(z)) = w_X(z) + a_1\,(1 - R_1^{*1}(w_X(z)))$

Adding equation (4.6) over n = m to N–1,

$$\lambda\, U(z) \;=\; \sum_{n=m}^{N-1} SE_n(0)\, z^n + \lambda \sum_{n=m+1}^{N-1} \sum_{k=1}^{n-m} U_{n-k}\, g_k \tag{4.38}$$

Multiplying the equation (4.38) by $-\,S_1^*(\theta)$ and adding with (4.37) we have

$$\begin{aligned}(\theta - h_{a_1}(w_X(z)))\, P_1^*(z,\theta) \;=\; & P_1(z,0) - \frac{(1-p)\, S_1^*(\theta)}{z}(P_2(z,0) + (1-r)\, P_1(z,0)) \\ & + (1-p)\, S_1^*(\theta)\, P(0) - S_1^*(\theta)\, QB(z,0) \\ & - S_1^*(\theta)\, (SE(z,0) - \lambda\, U(z) + \lambda\, X(z)\, U(z)) \qquad (4.39)\end{aligned}$$

Substituting for $P_2(z, 0)$ and $QB(z, 0)$ from equations (4.35) and (4.29), equation (4.39) gives,

$$(\theta - h_{a_1}(w_X(z)))\, P_1^*(z,\theta)$$

$$\begin{aligned}= \; & \frac{P_1(z,\theta)}{z}\, [z - S_1^*(\theta)\, (r\; S_2^*(h_{a_2}(w_X(z))) + 1 - r)\, (1 - p + VB^*(w_X(z)))] \\ & + S_1^*(\theta)\, P(0)\, (1 - p + VB^*(w_X(z))) - S_1^*(\theta)\, (SE(z,0) - U(z)\, w_X(z)) \qquad (4.40)\end{aligned}$$

At $\theta = h_{a_1}(w_X(z))$

$P_1(z, 0)$

$$= z\; S_1^*(h_{a_1}(w_X(z)))\, \frac{(SE(z,0) - U(z)\, w_X(z) - P(0)\, (1 - p + VB^*(w_X(z))))}{(z - S_{BV}^*(w_X(z)))} \tag{4.41}$$

where $S_{BV}^*(w_X(z)) \;=\; S_{SOS}^*(h(w_X(z)))((1-p) + VB^*(w_X(z)))$ (4.41.1)

and $S_{SOS}^*(h(w_X(z))) \;=\; S_1^*(h_{a_1}(w_X(z)))\, (1 - r + r\; S_2^*(h_{a_2}(w_X(z))))$ (4.41.2)

Substituting the value of $P_1(z, 0)$ in (4.40) we have,

$P_1^*(z,\theta)$

$$= \frac{z\, (S_1^*(h_{a_1}(w_X(z))) - S_1^*(\theta))\, [SE(z,0) - U(z)\, w_X(z) - P(0)\, (1 - p + VB^*(w_X(z)))]}{(\theta - h_{a_1}(w_X(z)))\, (z - S_{BV}^*(w_X(z)))} \tag{4.42}$$

To obtain the PGF of the system size, when the server is in dormant period, the equations in (4.6) are recursively used. Thus,

$$\lambda\, U_n \;=\; \sum_{k=m}^{n} SE_k(0)\, \pi_{n-k} \tag{4.43}$$

where $SE_k(0)$ = Coefficient of z^k in SE(z, 0)

For further simplification, the following theorem is proved.

Theorem : 4.2

(i) $$D^*(w_X(z))\,(VI^*(w_X(z)) - w_X(z)\,\psi(z)) = \sum_{k=m}^{\infty} z^k \sum_{i=m}^{k} \xi_i\, h_{k-I} \quad (4.43.1)$$

(ii) $$D^*(w_X(z))\,(1 - w_X(z)\,\pi(z)) = \sum_{k=m}^{\infty} z^k \sum_{i=m}^{k} S_i\, h_{k-I} \quad (4.43.2)$$

where $\xi_n = \alpha I_n + \sum_{i=0}^{m-1} \psi_i\, g_{n-i}$, $\quad S_n = \sum_{i=0}^{m-1} \pi_i\, g_{n-i}$,

and ψ_n's are given by the equation (4.19.1).

αI_n and h_k respectively denote the probability that n customers arrive in a vacation time (VI) and setup period D.

Proof

(i) $$VI^*(w_X(z)) - w_X(z)\,\psi(z) = \left(\sum_{n=0}^{\infty} \alpha I_n\, z^n - \sum_{n=0}^{m-1} \psi_n\, z^n\right) + \left(\sum_{k=1}^{\infty} g_k\, z^k\right)\left(\sum_{n=0}^{m-1} \psi_n\, z^n\right)$$

Through algebraic manipulations and using corollary (ii) of Theorem 4.1,

$$VI^*(w_X(z)) - w_X(z)\,\psi(z) = \sum_{n=m}^{\infty} z^n \left(\alpha I_n + \sum_{i=0}^{m-1} \psi_i\, g_{n-i}\right) = \sum_{n=m}^{\infty} \xi_n\, z^n$$

Thus

$$D^*(w_X(z))\,(VI^*(w_X(z)) - w_X(z)\psi(z)) = \left(\sum_{k=0}^{\infty} h_k\, z^k\right)\left(\sum_{n=m}^{\infty} \xi_n\, z^n\right) = \sum_{k=m}^{\infty} z^k \sum_{i=m}^{k} \xi_i\, h_{k-i}$$

This proves (4.43.1)

To prove (ii) consider,

$$(1 - w_X(z)\,\pi(z)) = \left(\sum_{k=1}^{\infty} g_k\, z^k\right)\left(\sum_{n=0}^{m-1} \pi_n\, z^n\right) - \sum_{n=1}^{m-1} \pi_n\, z^n = \sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \pi_i\, g_{n-i}$$

(using the definition of π_n)

$$= \sum_{n=m}^{\infty} S_n\, z^n$$

Thus

$$D^*(w_X(z))\,(1 - w_X(z)\,\pi(z)) = \left(\sum_{k=0}^{\infty} h_k\, z^k\right)\left(\sum_{n=m}^{\infty} S_n\, z^n\right) = \sum_{k=m}^{\infty} z^k \sum_{i=m}^{k} S_i\, h_{k-i}$$

This proves (4.43.2).

Now to obtain U_n's, we use equations (4.43.1) and (4.43.2) in (4.24) and collect the coefficients of z^k on both sides. Then,

$$SE_k(0) = \left(p' \sum_{i=m}^{k} \xi_i h_{k-i} + (1-p') \sum_{i=m}^{k} S_i h_{k-i}\right) P(0), \qquad k \geq m$$

Therefore equation (4.43) implies

$$\lambda U_n = \sum_{k=m}^{n} \left(\sum_{i=m}^{k} (p' \xi_i + (1-p') S_i) h_{k-i} \right) \pi_{n-k} P(0) \tag{4.44}$$

$$= (\phi_{(n,1)} p' + \phi_{(n,2)} (1 - p')) P(0) \tag{4.44.1}$$

where

$$\phi_{(n,1)} = \sum_{k=m}^{n} \pi_{n-k} \sum_{i=m}^{k} \xi_i h_{k-i} = \sum_{k=m}^{n} \xi_k \sum_{i=0}^{n-k} h_i \pi_{n-k-i}$$ (By rearranging the summations)

$$\phi_{(n,2)} = \sum_{k=m}^{n} \pi_{n-k} \sum_{i=m}^{k} S_i h_{k-i} = \sum_{k=m}^{n} S_k \sum_{i=0}^{n-k} h_i \pi_{n-k-i}$$

Summing equation (4.44) over n = m to N – 1 we have

$$U(z) = \phi(z) P(0) \tag{4.45}$$

where $$\phi(z) = \sum_{n=m}^{N-1} \frac{\phi_n z^n}{\lambda} \text{ and} \tag{4.45.1}$$

$$\phi_n = p' \phi_{(n,1)} + (1 - p') \phi_{(n,2)} \tag{4.45.2}$$

Substituting for SE(z, 0) and U(z) from (4.24) and (4.45), one can find that,

$$SE(z, 0) - U(z) w_X(z) - P(0) (1 - P + VB^*(w_X(z))) = - w_X(z) P(0) \tag{4.46}$$

where

$$I^{BV}_{(m,N)}(z) = p' \left[\frac{1 - VI^*(w_X(z)) D^*(w_X(z))}{w_X(z)} + D^*(w_X(z)) \psi(z)\right] + (1-p') \left[\frac{1 - D^*(w_X(z))}{w_X(z)} + D^*(w_X(z)) \pi(z)\right] + \phi(z) + \frac{VB^*(w_X(z)) - p}{w_X(z)} \tag{4.47}$$

Using (4.46) in (4.41) and (4.42),

$$P_1(z, 0) = \frac{z S_1^*(h_{a_1}(w_X(z)))(- w_X(z) P(0) I^{BV}_{(m,N)}(z))}{z - S^*_{BV}(w_X(z))} \tag{4.48}$$

$$P_1^*(z, \theta) = \frac{z (S_1^*(h_{a_1}(w_X(z))) - S_1^*(\theta))(- w_X(z) P(0) I^{BV}_{(m,N)}(z))}{(z - S^*_{BV}(w_X(z)))\ (\theta - h_{a_1}(w_X(z)))} \tag{4.49}$$

Thus the partial generating functions of the system size, corresponding to different states obtained at arbitrary epoch are listed below :

$$QI^*(z,0) = \frac{P(0)\,p'\,(1 - VI^*(w_X(z)))}{w_X(z)} \quad (4.50)$$

$$PI(z) = P(0)\,(p'\,\psi(z) + (1 - p')\,\pi(z)) \quad (4.51.1)$$

$$SE^*(z,0) = P(0) \quad [p'\,(VI^*(w_X(z)) - w_X(z)\,\psi(z)) + (1 - p')\,(1 - w_X(z)\,\pi(z))] \quad (4.51.2)$$

$$U(z) = P(0)\,\phi(z) \quad (4.51.3)$$

$$QB^*(z,0) = P(0)\left[\frac{S^*_{SOS}(h(w_X(z)))\,(w_X(z)\,I^{BV}_{(m,N)}(z))}{(z - S^*_{BV}(w_X(z)))} + 1\right]\left[\frac{VB^*(w_X(z)) - p}{w_X(z)}\right] \quad (4.51.4)$$

$$P_1^*(z,0) = \frac{z\,(S_1^*(h_{a_1}(w_X(z))) - 1)\,P(0)\,w_X(z)\,I^{BV}_{(m,N)}(z)}{h_{a_1}(w_X(z))\,(z - S^*_{BV}(w_X(z)))} \quad (4.51.5)$$

$$P_2^*(z,0) = \frac{P(0)\,z\,r\,w_X(z)\,I^{BV}_{(m,N)}(z)\,(S_2^*(h_{a_2}(w_X(z))) - 1)\,S_1^*(h_{a_1}(w_X(z)))}{h_{a_2}(w_X(z))\,(z - S^*_{BV}(w_X(z)))} \quad (4.51.6)$$

$$BR_i^{**1}(z,0,0) = \frac{a_i\,P_i^*(z,0)\,(1 - R_i^{*1}(w_X(z)))}{w_X(z)} \qquad i = 1, 2 \quad (4.51.7)$$

To derive the total PGF of the system size distribution the following generating functions are considered.

$P_{Idle}(z)$ = Probability generating function of the system when the server is idle (buildup + setup + dormant + vacation) state

$$= PI(z) + SE^*(z,0) + U(z) + (QI^*(z,0) + QB^*(z,0))$$

$$= \frac{P(0)\,(z - S^*_{SOS}(h(w_X(z))))\,I^{BV}_{(m,N)}(z)}{(z - S^*_{BV}(w_X(z)))} \quad (4.52)$$

where $I^{BV}_{(m,N)}(z)$ is given by equation (4.47)

$P_{Comp}(z)$ = Probability generating function of the system when the server is busy or in breakdown state

$$= \sum_{i=1}^{2} P_i^*(z,0) + \sum_{i=1}^{2} BR_i^{**1}(z,0,0)$$

$$= \frac{P(0)\,z\,(S^*_{SOS}(h(w_X(z))) - 1)\,I^{BV}_{(m,N)}(z)}{(z - S^*_{BV}(w_X(z)))} \quad (4.53)$$

Thus the total PGF of the system size distribution is given by

$$P_{(m,N)}^{BV}(z) = P_{Idle}(z) + P_{Comp}(z)$$

$$= \frac{P(0)\, I_{(m,N)}^{BV}(z)\,(z-1)\, S_{SOS}^{*}\,(h(w_X(z)))}{(z - S_{BV}^{*}(w_X(z)))} \quad (4.54)$$

where $S_{BV}^{*}(w_X(z))$ and $S_{SOS}^{*}(h(w_X(z)))$ are given by equations (4.41.1) and (4.41.2). The value of P(0) can be calculated by using the normalizing condition $P_{(m,N)}^{BV}(1) = 1$. Thus $P(0) = \frac{1-\rho_{BV}}{I_{(m,N)}^{BV}(1)}$, where

$$I_{(m,N)}^{BV}(1) = p'\,\psi(1) + (1 - p')\,\pi(1) + (p'\,E(VI) - E(VB)) + E(D) + \phi(1) \text{ and}$$

$$\psi(1) = \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda},\ \pi(1) = \sum_{n=0}^{m-1} \frac{\pi_n}{\lambda},\ \phi(1) = \sum_{n=m}^{N-1} \frac{\phi_n}{\lambda} \quad (4.54.1)$$

$$\rho_{BV} = \lambda\, E(X)\,(E(H) + E(VB)) \quad (4.54.2)$$

$$\rho_{SOS} = \lambda\, E(X)\, E(H) \quad (4.54.3)$$

$$E(H) = E(S_1)\,(1 + a_1\, E(R_1)) + r\, E(S_2)\,(1 + a_2\, E(R_2)) \quad (4.54.4)$$

Hence, $$P_{(m,N)}^{BV}(z) = \frac{(1-\rho_{BV})\,(z-1)\, S_{SOS}^{*}\,(h(w_X(z)))\, I_{(m,N)}^{BV}(z)}{(z - S_{BV}^{*}(w_X(z)))\, I_{(m,N)}^{BV}(1)} \quad (4.55)$$

Using equation (4.52), equation (4.55) can be re-written as

$$P_{(m,N)}^{BV}(z) = \left(\frac{(1-\rho_{SOS})\,(z-1)\, S_{SOS}^{*}\,(h(w_X(z)))}{z - S_{SOS}^{*}(h(w_X(z)))}\right)\left(\frac{P_{Idle}(z)}{P_{Idle}(1)}\right)$$

This proves the following decomposition property.

4.1.4 Decomposition Property

The steady state condition $\rho_{BV} < 1$, implies $\rho_{SOS} < 1$. Under this condition the PGF of the system size of the model under consideration is decomposed into the product of two probability generating functions one of which is the PGF of the classical $M^X/G_{SOS}/1$/Breakdown (without (m, N) – policy and without vacation) and the other is $P_{Idle}(z)/P_{Idle}(1)$ which gives the PGF of the conditional system size distribution during the server idle period.

The total PGF of the queue size distribution for the model is given by

$$P^{BV}_{Q(m,N)}(z) = \frac{P_{Comp}(z)}{z} + P_{Idle}(z) = \frac{(1-\rho_{BV})(z-1)\,I^{BV}_{(m,N)}(z)}{(z - S^{*}_{BV}(w_X(z)))\,I^{BV}_{(m,N)}(1)} \qquad (4.55.1)$$

4.1.5 Performance Measures

In this section, the steady-state system size probabilities and the mean number of customers in the system at different states are calculated.

The Server in Idle State

Let P_{set}, P_{build}, P_{dor}, P_{VI} and P_{VB} denote the steady-state system size probabilities and L_{set}, L_{build}, L_{dor}, L_{VI} and L_{VB} denote the mean number of customers when the system is in setup, buildup, dormant states, vacation during idle period and vacation during busy period respectively. Then using equations (4.50) and (4.51.1) to (4.51.4), we have,

$$P_{set} = \lim_{z\to 1} SE^{*}(z,0) = P(0)\,E(D) \qquad (4.56)$$

$$L_{set} = \left[\frac{d}{dz} SE^{*}(z,0)\right]_{z=1}$$

$$= P(0)\,\lambda\,E(X)\,\{\frac{E(D^2)}{2} + E(D)\,[p'\,(E(VI) + \psi(1)) + (1-p')\,\pi(1)]\}$$

$$P_{build} = \lim_{z\to 1} PI(z) = P(0)\,(p'\,\psi(1) + (1-p')\,\pi(1)) \qquad (4.57)$$

$$L_{build} = \left[\frac{d}{dz} PI(z)\right]_{z=1} = P(0)\,(p'\,\psi'(1) + (1-p')\,\pi'(1))$$

$$P_{dor} = \lim_{z\to 1} U(z) = P(0)\,\phi(1) \qquad (4.58)$$

$$L_{dor} = \left[\frac{d}{dz} U(z)\right]_{z=1} = P(0)\,\phi'(1)$$

$$P_{VI} = \lim_{z\to 1} QI^{*}(z,0) = P(0)\,p'\,E(VI) \qquad (4.59)$$

$$L_{VI} = \left[\frac{d}{dz} QI^{*}(z,0)\right]_{z=1} = P(0)\,p'\,\lambda\,E(X)\,\frac{E(VI^2)}{2}$$

where $\pi(1)$, $\psi(1)$, $\phi(1)$ are given by the equation (4.54.1),

$$\pi'(1) = \sum_{n=0}^{m-1} \frac{n\,\pi_n}{\lambda},\ \ \psi'(1) = \sum_{n=0}^{m-1} \frac{n\,\psi_n}{\lambda},\ \ \phi'(1) = \sum_{n=m}^{N-1} \frac{n\,\phi_n}{\lambda} \text{ and } P(0) = \frac{1-\rho_{BV}}{I^{BV}_{(m,N)}(1)}$$

$$P_{VB} = \lim_{z\to 1} QB^*(z,0) = P(0)\,E(VB)\left(\frac{\lambda\,E(X)\,I^{BV}_{(m,N)}(1)}{1-\rho_{BV}} - 1\right) \tag{4.60}$$

$$L_{VB} = \left[\frac{d}{dz} QB^*(z,0)\right]_{z=1}$$

$$= E(VB)\,F_{BV} + (\lambda\,E(X))^2\left[\frac{E(VB^2)}{2} + E(VB)\,E(H)\right] - P(0)\,\lambda\,E(X)\,\frac{E(VB^2)}{2}$$

where $$F_{BV} = \frac{\lambda\,E(X(X-1)) + (\lambda\,E(X))^3\,E(S^2_{BV})}{2(1-\rho_{BV})} + \frac{I^{BV\,\prime}_{(m,N)}(1)}{I^{BV}_{(m,N)}(1)}\,\lambda E(X) \tag{4.60.1}$$

$$E(S^2_{BV}) = E(H^2) + \sum_{i=1}^{M} p_i\,E(VB_i^2) + 2\,E(H)\sum_{i=1}^{M} p_i\,E(VB_i)$$

$$E(H^2) = E(S_1^2)\,(1 + a_1\,E(R_1))^2 + r\,E(S_2^2)\,(1 + a_2\,E(R_2))^2 + a_1\,E(S_1)\,E(R_1^2)$$
$$+ r\,a_2\,E(S_2)\,E(R_2^2) + 2\,r\,(E(S_1)\,E(S_2)\,(1 + a_1\,E(R_1))\,(1 + a_2\,E(R_2)))$$

$$E(H) = E(S_1)\,(1 + a_1\,E(R_1)) + r\,E(S_2)\,(1 + a_2\,E(R_2))$$

$$E(S_{BV}) = E(H) + \sum_{i=1}^{M} p_i\,E(VB_i) \text{ and}$$

$$I^{BV\,\prime}_{(m,N)}(1) = \left[\frac{d}{dz} I^{BV}_{(m,N)}(z)\right]_{z=1}$$

$$= p'\left[\lambda\,E(X)\left(\frac{E(D^2)}{2} + \frac{E(VI^2)}{2} + E(VI)\,E(D)\right) + \lambda E(X)\,E(D)\,\psi(1) + \psi'(1)\right]$$
$$+ (1-p')\left[\lambda\,E(X)\,\frac{E(D^2)}{2} + \lambda\,E(X)\,E(D)\,\pi(1) + \pi'(1)\right] + \phi'(1)$$
$$- \sum_{i=1}^{M} p_i\,\lambda\,E(X)\,\frac{E(VB_i^2)}{2}$$

The Server in Busy State

The probability that the server is busy (P_{busy}) and the expected number of customers in the system in busy state (L_{busy}) are calculated by using the equations (4.51.6) and (4.51.7).

$$P_{busy} = \lim_{z\to 1}\left(P_1^*(z,0) + P_2^*(z,0)\right) = \lambda\,E(X)\,[E(S_1) + r\,E(S_2)] = \rho_{SOS} \tag{4.61}$$

$$L_{busy} = \left[\frac{d}{dz}\left(P_1^*(z,0) + P_2^*(z,0)\right)\right]_{z=1} = L_{FES} + L_{SOS},$$ where,

$$L_{FES} = \left[\frac{d}{dz}(P_1^*(z,0)\right]_{z=1} = P_{FES} + (\lambda E(X))^2 \frac{E(S_1^2)}{2}(1 + a_1 E(R_1)) + E(S_1) F_{BV}$$

where F_{BV} is given by the equation (4.60.1)

$$P_{FES} = \lim_{z\to 1} P_1^*(z,0) = \lambda E(X) E(S_1)$$

$$L_{SOS} = \left[\frac{d}{dz}(P_2^*(z,0)\right]_{z=1} = P_{SOS} + r\{(\lambda E(X))^2 [\frac{E(S_2^2)}{2}(1 + a_2 E(R_2)) + E(S_2) E(S_1) (1 + a_1 E(R_1))] + E(S_2) F_{BV}\}$$

$$P_{SOS} = \lim_{z\to 1} P_2^*(z,0) = r \lambda E(X) E(S_2)$$

The Server in Breakdown State

The probability that the server is in breakdown state (P_{br}) and the expected number of customers in the system (L_{br}) are calculated by using the equations (4.51.7)

$$P_{br} = \lim_{z\to 1} [BR_1^{**1}(z,0,0) + BR_2^{**1}(z,0,0)]$$

$$= \lambda E(X) (a_1 E(R_1) E(S_1) + r a_2 E(R_2) E(S_2)) \tag{4.62}$$

$$L_{br} = \left[\frac{d}{dz}(BR_1^{**1}(z,0,0) + BR_2^{**1}(z,0,0))\right]_{z=1}$$

$$= a_1 E(R_1) L_{FES} + a_2 E(R_2) L_{SOS} + \lambda E(X) (P_{FES} \frac{E(R_1^2)}{2} + P_{SOS} \frac{E(R_2^2)}{2})$$

Mean System Size

The expected system size $L_{(m,N)}^{BV}$ of the model is given by

$$L_{(m,N)}^{BV} = \left[\frac{d}{dz} P_{m,N}^{BV}(z)\right]_{z=1} = \frac{I_{(m,N)}^{BV\,'}(1)}{I_{(m,N)}^{BV}(1)} + L_1^{BV} \tag{4.63}$$

where $$L_1^{BV} = \rho_{SOS} + \frac{(\lambda E(X))^2 E(S_{BV}^2) + \lambda E(X(X-1)) E(S_{BV})}{2(1-\rho_{BV})} \tag{4.64}$$

gives the mean system size of the SOS queueing model $M^X/G_{SOS}/1/S_{BV}(M)$/Breakdown (without threshold policy).

4.1.6 Other System Characteristics

Let E(Cycle), E(Busy), E(I) and E(w_S) denote the expected length of the cycle, busy period, idle period and expected waiting time of the system. Then,

(i) $$E(\text{Cycle}) = \frac{I^{BV}_{(m,N)}(1)}{1-\rho_{BV}}$$

(ii) $$E(\text{Busy}) = \frac{\rho_{SOS}}{1-\rho_{BV}} I^{BV}_{(m,N)}(1)$$

(iii) $$E(I) = (1 - \lambda E(X) E(H)) \frac{I^{BV}_{(m,N)}(1)}{1-\rho_{BV}}$$

(iv) $$E(w_S) = \frac{L^{BV}_{(m,N)}}{\lambda E(X)} \text{ (Using Little's formula)}$$

4.1.7 Queue Size Distribution at Departure Epoch

The PGF $\pi^+(z)$ of the queue size distribution $\{\pi^+_n ; n \geq 0\}$ at departure epoch is given by

$\pi^+(z) = \sum_{n=0}^{\infty} \pi^+_n z^n = \sum_{n=0}^{\infty} D_1 [(1 - r) P_{1,n+1}(0) + P_{2,n+1}(0)] z^n$ where D_1 is the normalizing constant.

(i.e.,) $\pi^+(z) = \frac{D_1}{z}[(1 - r) P_1(z, 0) + P_2(z, 0)]$.

Substituting for $P_1(z, 0)$ and $P_2(z, 0)$ from equations (4.48) and (4.35) respectively and evaluating the constant D_1 using normalizing condition,

$$\pi^+(z) = \frac{(X(z)-1)}{E(X)(z-1)} P^{BV}_{(m,N)}(z).$$

4.2 OPTIMAL MANAGEMENT POLICY

By following the procedure of Chapter II, the optimal threshold values (m^*, N^*) can be obtained. Recalling the cost structure as C_y (startup cost per cycle), C_{set} (setup cost), C_{build} (buildup cost), C_{dor} (standby cost), C_{br} (breakdown cost), C_{busy} (operating cost), C_{VB} (cost due to VB), C_h (holding cost per customer) and C_{VI} (reward cost) per unit time, the total average cost per unit time of the system is given by

$$T_C^{BV}(m,N) = \frac{C_y}{E_{cycle}} + C_{set} P_{set} + C_{dor} P_{dor} + C_{busy} P_{busy} + C_{build} P_{build} + C_{br} P_{br}$$
$$+ C_h L_{(m,N)}^{BV} + (C_{VB} P_{VB} - C_{VI} P_{VI}).$$

By substituting the corresponding measures, the above equation can be re-written as,

$$T_C^{BV}(m,N) = \overline{A}^M + \frac{1}{I_{(m,N)}^{BV}(1)} [A^M + z^{BV}(m) + C_h(p' \sum_{n=m}^{N-1} \frac{n\,\phi_{(n,1)}}{\lambda} + (1-p') \sum_{n=m}^{N-1} \frac{n\,\phi_{(n,2)}}{\lambda})$$
$$+ (\frac{1}{\lambda}) C_{dor} (1 - \rho_{BV}) (p' \sum_{n=m}^{N-1} \phi_{(n,1)} + (1 - p') \sum_{n=m}^{N-1} \phi_{(n,2)})]$$

where

$$\overline{A}^M = C_h + L_1^{BV} + C_{busy}\, \rho_{SOS} + C_{br} P_{br} + \lambda\, E(X)\, E(VB)\, C_{VB}$$

where L_1^{BV} is given by equation (4.64),

$$A^M = C_h\, \ell_1 + (1 - \rho_{BV})\, (C_{set}\, E(D) - p'\, E(VI)\, C_{VI} + C_y - C_{VB}\, E(VB)),$$

$$\ell_1 = \lambda\, E(X)\, [p' \, (\frac{E(VI^2)}{2} + E(VI)\, E(D)) + \frac{E(D^2)}{2} - \sum_{i=1}^{M} p_i \frac{E(VB_i^2)}{2}]$$

$$z^{BV}(m) = C_h\, \ell_2 + \frac{1-\rho_{BV}}{\lambda} (C_{build}\, (p' \sum_{n=0}^{m-1} \psi_n + (1 - p') \sum_{n=0}^{m-1} \pi_n)) \text{ and}$$

$$\ell_2 = p'\, (\lambda\, E(X)\, E(D) \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda} + \sum_{n=0}^{m-1} \frac{n\,\psi_n}{\lambda}) + (1-p')(\lambda\, E(X)\, E(D) \sum_{n=0}^{m-1} \frac{\pi_n}{\lambda} + \sum_{n=0}^{m-1} \frac{n\,\pi_n}{\lambda})$$

By calculation,

$$T_C^{BV}(m,k+1) - T_C^{BV}(m,k) = \frac{\phi_k}{\lambda\, I_{(m,k+1)}^{BV}(1)\, I_{(m,k)}^{BV}(1)} H^{BV}(m,k)$$

where

$$H^{BV}(m,k) = C_{dor}\, (1 - \rho_{BV})\, d_{BV} + C_h\, (k\, d_{BV} + \sum_{n=m}^{k-1} \frac{(k-n)}{\lambda} \phi_n) - (A^M + z^{BV}(m))\, ;$$

$$\phi_k = p'\, \phi_{(k,1)} + (1 - p')\, \phi_{(k,2)} \text{ and}$$

$$d_{BV} = p' \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda} + (1 - p') \sum_{n=0}^{m-1} \frac{\pi_n}{\lambda} + E(D) + (p'\, E(VI) - E(VB))$$

By proceeding as in Chapter II, it is found that the condition under which $T_C^{BV}(m,k+1) - T_C^{BV}(m,k) > 0$ for the first value of k is given by

$$N^*(m) = \min \{k / H^{BV}(m, k) > 0\}.$$

The optimal threshold values $(m^*, N^*(m))$ can be obtained by following the algorithm given in Chapter II.

4.3 PARTICULAR CASES

With the selection of appropriate parameters, the results of various queueing systems with or without (m, N) policy can be obtained. In this section the total PGF P(z) of the system size for some queueing models are mentioned as special cases.

Case 1

PGF (P(z)) for $M^X_{(m,N)}/G_{SOS}/1$/Breakdown model under single vacation threshold policy can be obtained with the selection of parameters $p' = 1$; $p_j = 0 \ \forall$ j = 1 to M in the equation (4.55) and given by

$$P(z) = \frac{(1-\rho_{SOS})(z-1)\,S^*_{SOS}(h(w_X(z)))}{z - S^*_{SOS}(w_X(z))}\,\frac{I_{(m,N)}(z)}{I_{(m,N)}(1)}$$

where $$I_{(m,N)}(z) = \frac{1 - VI^*(w_X(z))\,D^*(w_X(z))}{w_X(z)} + D^*(w_X(z))\,\psi(z) + \phi(z)$$

These results coincide with the special case (i) of Chapter II with C = 1 (ref : (2.75) and (2.76)).

Case 2

The results of Lee et al. (2003) for non vacation (system I) and single vacation (system II) queue can be deduced with the following substitutions in equation (4.55) for j = 1 to M ; i = 1, 2,

$(p', p_j, r, a_i) = 0$ (System I)

$(p_j, r, a_i) = 0$ and $p' = 1$ (System III)

By taking m = N, the PGF of the system size of the corresponding N policy models can be deduced and the results of classical 1-policy queueing models with or without setup or vacation can also be derived by taking m = N = 1.

For example, the results of N-policy $M^X/G/1$ queueing model with single vacation policy of Lee et al. (1995) can be deduced with the following substitutions in (4.55) and (4.47).

Case 3

$p' = 1$, $(p_j, r, a_i) = 0$, j = 1 to M ; i = 1, 2 ; J = 1,

P_r (D = 0) = 1 and m = N give,

$$P(z) \quad = \quad \frac{(1-\rho)(z-1)\,S^*(w_X(z))}{z-S^*(w_X(z))}\,\frac{I_{N,N}(z)}{I_{N,N}(1)}$$

where $I_{N,N}(z) = \dfrac{1-VI^*(w_X(z))}{w_X(z)} + \psi(z)$ (Lee et al. (1995))

Case 4

In addition if N = 1,

$I_{(1,1)}(z) \quad = \quad \dfrac{1-VI^*(w_X(z))}{w_X(z)} + \dfrac{\alpha\, I_0}{\lambda}$ gives the PGF of the conditional system size distribution during vacation time for the classical $M^X/G/1$ single vacation mode.

4.4 NUMERICAL ANALYSIS

In order to study the effects of system parameters ; the probability (r) with which the customers opt the SOS service, arrival rate (λ) and Bernoulli vacation probabilities (p_i) (i = 1, 2, 3), numerical values are presented in Tables 4.1 to 4.5. Three types of (VB_i) vacations following different distributions are considered for the numerical computation. The distributions assumed are presented in the following table.

Random variables (Y)		**Distribution F(Y)**	**Mean E(Y)**	**Second order moments $E(Y^2)$**
FES (S_1)		Two-stage hyper-exponential	$\frac{b_1}{\mu_1}+\frac{1-b_1}{\mu_2}$ $0 \le b_1 \le 1$	$2\left(\frac{b_1}{\mu_1^2}+\frac{1-b_1}{\mu_2^2}\right)$
SOS (S_2)		Erlang-2 type	$1 / \mu_{21}$	$3 / 2\ \mu_{21}^2$
Setup (D)		Erlang-3 type	$1 / \gamma$	$4 / 3\,\gamma^2$
Vacation during busy period	VB_1	Deterministic	$1 / \eta_1$	$1 / \eta_1^2$
	VB_2	Gamma-3 type	$3 / \eta_2$	$12 / \eta_2^2$
	VB_3	Exponential	$1 / \eta_3$	$2 / \eta_3^2$
Vacation during idle period (VI)		Gamma-3 type	$3 / \eta$	$12 / \eta^2$
Repair in FES (R_1)		Erlang-2 type	$1 / rp_1$	$3 / 2rp_1^2$
Repair in SOS (R_2)		Deterministic	$1 / rp_2$	$1 / rp_2^2$
Batch size (X)		Geometric (Geo(q))	1 /1 - q	$(q + 1) / (1 - q)^2$

The expected cost values of $T_C(m, N)$ for different values of m and N are listed in Table 4.1 and plotted in Figure 4.1 and the optimal threshold values (m^*, N^*) and $T_C\ (m^*, N^*(m))$ are respectively observed at (8, 10) and 667.720. This satisfies the solution procedure in the theory. Table 4.1 gives the data for the Figure 4.1.

The parametric values used to generate the Tables 4.1 to 4.5 are taken from Table 4.1, except the parameters mentioned in the corresponding tables.

Table 4.1 The Expected Cost $T_C(m, N)$ for Different Values m and N

$(C_h, C_{dor}, C_{busy}, C_{set}, C_y, C_{VI}, C_{VB}, C_{build}, C_{br}) = (20, 30, 10, 1000, 18000, 8, 115, 8, 20)$

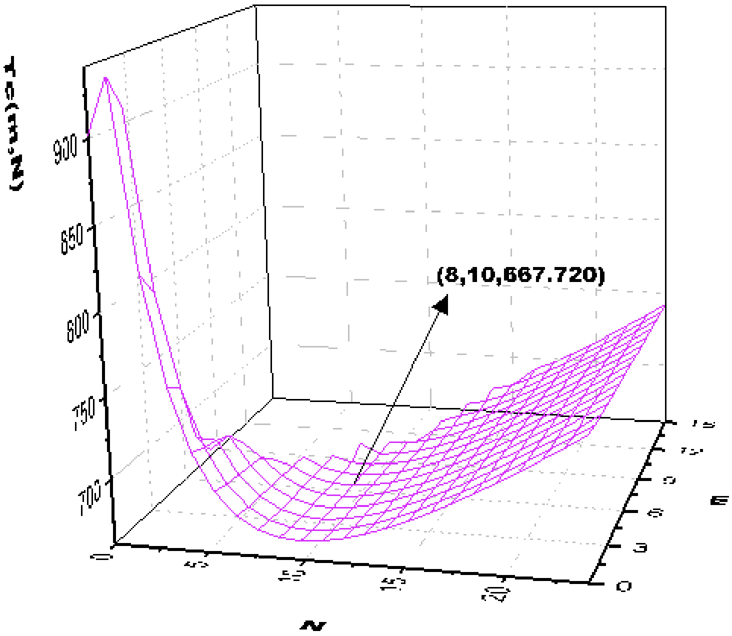
$(q, \lambda, \gamma, r, a_1, a_2, rp_1, rp_2) = (0.1, 0.5, 1, 0.2, 5, 6, 2, 3)$ M = 3

$(p', \eta, \eta_1, \eta_2, \eta_3, p_1, p_2, p_3) = (0.3, 2, 5.4, 6.8, 7.0, 0.80, 0.05, 0.1)$

$(b_1, \mu_1, \mu_2, \mu_{21}) = (0.1, 0.5, 5, 5)$

m \ N	1	4	6	9	10	14	17	21
1	814.013	758.073	697.126	672.291	671.298	683.001	700.503	729.386
3		744.432	693.014	669.939	669.190	681.387	699.046	728.037
5			689.601	669.057	668.400	680.776	698.490	727.518
7				668.444	667.867	680.341	698.086	727.130
8				668.224	667.720	680.173	697.924	726.969
9				673.849	667.799	680.037	697.787	726.828
12						680.051	697.534	726.530

Figure 4.1. The Total Expected Cost $T_C(m, N)$ for Different Values of m and N



The mean system size corresponding to different states of the server are presented in Table 4.2. The values of m and N for Tables 4.2 to 4.5 are respectively 5 and 16.

Table 4.2 Expected System Size at Different Stages Vs Arrival Rate λ

λ	ρ	L_{dor}	L_{bu}	L_{set}	L_{br}	L_{busy}	L_{vI}	L_{VB}
0.25	0.45	3.684	0.340	0.050	3.063	1.236	0.0011	0.460
0.30	0.54	3.061	0.283	0.051	3.988	1.607	0.0014	0.599
0.35	0.64	2.442	0.227	0.048	5.193	2.092	0.0016	0.785
0.40	0.73	1.826	0.171	0.041	6.957	2.804	0.0015	1.069
0.45	0.82	1.215	0.114	0.031	10.109	4.080	0.0013	1.598
0.50	0.91	0.608	0.057	0.018	18.755	7.596	0.0008	3.108

The system size probabilities corresponding to different states of the server are presented in Table 4.3. The Table shows that the server is busy for more percentage of time as the arrival rate λ increases.

Table 4.3. System Size Probabilities with Respect to λ

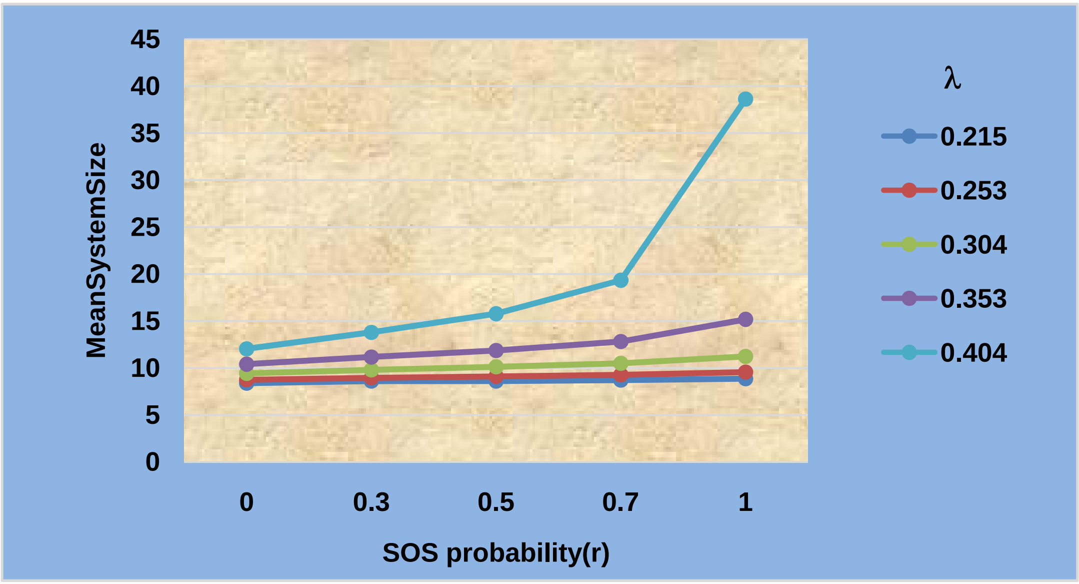
λ	P_{dor}	P_{build}	P_{set}	P_{br}	P_{busy}	P_{VI}	P_{VB}
0.215	0.408	0.190	0.009	0.246	0.100	0.004	0.042
0.253	0.360	0.168	0.009	0.290	0.118	0.004	0.050
0.304	0.297	0.139	0.009	0.348	0.142	0.004	0.061
0.353	0.237	0.111	0.009	0.404	0.165	0.004	0.071
0.404	0.174	0.082	0.008	0.462	0.189	0.003	0.081
0.452	0.117	0.055	0.006	0.517	0.211	0.003	0.092
0.503	0.056	0.027	0.003	0.576	0.235	0.001	0.103

The values of Table 4.4 and Figure 4.4 show that the waiting line increases if the arrival λ increases or more customers offer the SOS facility.

Table 4.4. Mean System Size as the SOS Probability r Changes

λ \ r	0	0.3	0.5	0.7	1.0
0.215	8.377	8. 504	8.596	8.695	8.858
0.253	8.740	8.941	9.091	9.261	9.555
0.304	9.421	9.804	10.117	10.492	11.234
0.353	10.404	11.167	11.864	12.814	15.175
0.404	12.042	13.799	15.782	19.343	38.624

Figure 4.4 Mean System Size Vs. r for Different Values of λ



It is important to study the effects of Bernoulli schedule control probabilities (during busy period) p_i (i = 1, 2, 3) on the mean system size. This is because if the server takes vacations often between services then the expected system size will increase. The Table 4.5 confirms this. The optimal system size for different values of p_i (i = 1, 2, 3) and λ are presented in Table 4.5 and depicted graphically in Figure 4.5.

Table 4.5. Optimal System Size $L(m^*, N^*)$ for Different Values of λ and p_i (i = 1, 2, 3)

(p_1, p_2, p_3)	p \ λ	0.40	0.42	0.45	0.47	0.49
(0.5, 0.5, 0)	1	13.171	14.867	20.169	28.233	53.924
(0.3, 0.1, 0.5)	0.9	12.518	13.284	15.424	17.933	22.317
(0.6, 0.1, 0.1)	0.8	12.502	13.261	15.380	17.860	22.182
(0.1, 0.1, 0.3)	0.5	12.487	13.018	14.690	16.324	18.738

Figure 4.5 Expected System Size Vs λ for Different Values of p