Chapter 2

J,J*and J**-Closed Sets in Topological Spaces

§ 2.1. Introduction

In 1937, Stone introduced regular open sets and used it to define the semi-regularization of a topological space. In 1968, Velicko proposed δ-open sets which are stronger than open sets. Levine has brought generalized closed sets in 1970. Dunham has established a generalized closure using Levine's generalized closed sets as Cl*.In 2016, Annalakshmi has introduced regular*-open sets (r*-open sets) using Cl*.In this Chapter, a class of new sets namely η*-open sets,a union of r*-open sets, which is placed between the classes of δ-open set and open set is introduced.Its basic properties are procured and the concepts of η*-cluster point, η*-adherent point and a η*-derived set are introduced and studied. In this Chapter, a new class of sets namely J-closed sets is initiated using η*-open sets in topological spaces. This new concept is weaker than closedness and infact it is weaker than g-closedness but stronger than gδ-closedness which are essential to characterize almost weakly Hausdorff spaces and thus the digital line. The properties and relationships with other g-closed sets are analysed. The class of J-closed sets is placed between that of generalized closed (g-closed) sets and generalized δ-closed sets (gδ-closed). Changing the position of η*-Closure operator and η*-openness another two concepts are introduced namely J*-closed sets and J**-closed sets and their status are analysed and properties are obtained.Many interesting chararcterizations are established.Finally in a semi-regular space,J,J* and J**-closed sets coincide which has to be proved by Example.

§ 2.2. η* -Open Sets

Definition 2.2.1. A subset D of a topological space (Y,ζ) is called **η*-open set** if it is a union of regular*-open sets (r*-open sets).We denote the set of all η*-open sets in (Y,ζ) by η*O(Y,ζ) or η*O(Y).

Example 2.2.2. Let $Y=\{p,q,r\}$, $\zeta=\{\phi, Y,\{p\},\{q\},\{p,q\}\}$.Then $r^*O(Y)=\{\phi, Y, \{p\},\{q\}\}$, $\eta^*O(Y)=\{\phi, Y, \{p\},\{q\},\{p,q\}\}$.

Theorem 2.2.3. Every δ-open set is a η^*-open set but not conversely.

Proof Let D be a δ-open set.By definition of a δ-open set,D is a union of regular open sets. Take $x\in D$,then $x\in\cup B_{\mu}$,where B_{μ} is regular open. Since every regular open set is regular*-open, [Theorem 3.12 (Annalakshmi,2016)] D is a union of regular*-open sets.Thus D is a η^*-open set. Hence every δ-open set is η^*-open.

Counter Example 2.2.4. Let $Y=\{p,q,r\}$, $\zeta=\{\phi, Y,\{p\}\}$. Then the subset $\{p\}$ is a η^*-open set but it is not a δ-open set in(Y,ζ).

Theorem 2.2.5. Every η^*-open set is an open set but not conversely.

Proof Since every r*-open set is an open set [Theorem 3.15 (Annalakshmi,2016)],the proof follows as in the proof of **Theorem 2.2.3.**

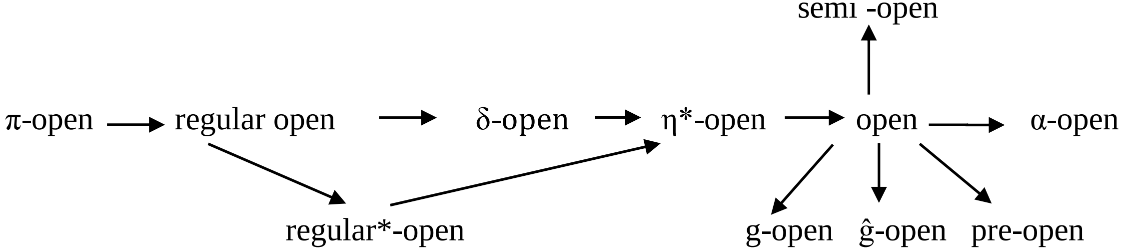
Counter Example 2.2.6. Let $Y=\{p,q,r\}$, $\zeta=\{\phi, Y, \{p,q\}\}$. Then the subset $\{p,q\}$ is a open set but it is not a η^*-open set in (Y,ζ).

Theorem 2.2.7. Every r*-open set is a η^*-open set but not conversely.

Proof Follows from **Definition 2.2.1.**

Counter Example 2.2.8. Consider the **Example 2.2.2.**,the subset $\{p,q\}$ is a η^*-open set but it is not a r*-open set in (Y,ζ).

Note 2.2.9. From the above Theorems we get the following implication diagram.



Theorem 2.2.10. (a) In any topological space (Y,ζ), Y and ϕ are η*-open sets.

(b) Any arbitrary union of η*-open sets is a η*-open set.

(c)The finite intersection of η*-open sets is a η*-open set.

Result 2.2.11. η*O(Y, ζ) forms a topology which is finer than ζ and coarser than ζ_δ.

η*-Interior Operator

Definition 2.2.12. For a subset D of a topological space (Y,ζ),**η*-Interior of D** is the union of all η*-open sets of Y contained in D.We denote by the symbol η*-Int(D).

Result 2.2.13. (i) η*-Int(D) is η*-open.

(ii) η*-Int(D) is the maximum η*-open set contained in D.

Theorem 2.2.14. D is η*-open iff D = η*-Int(D).

Proof (⟹) Let D be η*-open. As η*-Int(D) is the maximum η*-open set contained in D, η*-Int(D) = D.

(⟸) Let D = η*-Int(D).Since η*-Int(D) is the union of all η* -open sets of Y contained in D.Then D is η*-open.

Properties 2.2.15. In any topological space (Y,ζ), if D and E are subsets of Y then we get the following :

a. **η*-Int(ϕ) = ϕ**

b. **η*-Int(Y) = Y**

c. **η*-Int(D) ⊆ D**

d. **D ⊆ E ⇒ η*-Int(D) ⊆ η*-Int(E)**

e. **Int(η*-Int(D)) ⊆ Int(D)**

f. **δInt(D)⊆ η*-Int(D)⊆ Int(D) ⊆D**

g. $\eta^*\text{-Int}(\cup_{i\in\varepsilon}\{D_i\}) = \cup_{i\in\varepsilon}\eta^* - \mathrm{Int}(\{D_i\})$

h. **η*-Int(D∩E) = η*-Int(D) ∩ η*-Int(E).**

η^*-Closed Sets

Definition 2.2.16. The complement of a η^*-open set is called a **η^*-closed set**. We denote η^*-closed sets in (Y,ζ) by $\eta^*C(Y,\zeta)$ or $\eta^*C(Y)$.

Theorem 2.2.17. Every η^*-closed set is closed but not conversely.

Proof Let D be a η^*-closed set in (Y,ζ),then $Y-D$ is η^*-open. By **Theorem 2.2.5.**, $Y-D$ is open and hence D is closed. Hence every η^*-closed set is closed.

Counter Example 2.2.18. Let $Y=\{p,q,r\}$, $\zeta=\{\phi, Y,\{p\},\{p,q\}\}$. Then the subset $\{r\}$ is closed but not η^*-closed in (Y,ζ).

Theorem 2.2.19. Every δ-closed is η^*-closed but not conversely.

Proof Let D be a δ-closed set in (Y,ζ),then $X-D$ is δ-open. By **Theorem 2.2.3.**, $Y-D$ is η^*-open and hence D is η^*-closed. Hence every δ-closed is η^*-closed.

Counter Example 2.2.20. Let $Y=\{p,q,r\}$, $\zeta=\{\phi, Y, \{p\}\}$.Then the subset $\{q,r\}$ is a η^*-closed set but it is not a δ-closed set.

Note 2.2.21. From the above Theorems we get the following diagrammatic implications:

$$\begin{tikzcd}
 & & \text{semi-closed} \arrow[r] & \text{semi pre-closed} \\
\delta\text{-closed} \arrow[r] & \eta^*\text{-closed} \arrow[r] & \text{closed} \arrow[r] \arrow[u] \arrow[d] & \alpha\text{-closed} \\
 & & \text{pre-closed} &
\end{tikzcd}$$

Remark 2.2.22. The family of η^*-closed sets is properly placed between the family of δ-closed sets and closed sets.

Lemma 2.2.23. In a $T_{1/2}$-space, η^*-closed sets coincide with δ-closed sets.

Proof In a $T_{1/2}$-space,g-closed sets coincide with closed sets. Therefore generalized closure of a set coincides with closure of a set. Hence regular*-open sets are regular open sets in $T_{1/2}$-spaces and η^*-closed sets coincide with δ-closed sets.

Example 2.2.24. Let $Y=\{p,q,r\}$,$\zeta=\{\phi, Y, \{p\},\{q\},\{p,q\},\{p,r\}\}$.Here $\zeta^c = gC(Y,\zeta) = \{\phi, Y, \{q\},\{r\},\{q,r\},\{p,r\}\}$ represents a $T_{1/2}$-space and also $\eta^*C(Y,\zeta) = \delta C(Y,\zeta) = \{\phi, Y, \{q\}\{p,r\}\}$.

The converse of Lemma 2.2.23. is not true which is proved by the following Counter Example.

Counter Example 2.2.25. Let $Y=\{p,q,r\}$,$\zeta=\{\phi, Y, \{p,q\}\}$.Here $\eta^*C(Y,\zeta) = \delta C(Y,\zeta) =\{\phi, Y\}$ but it is not a $T_{1/2}$-space.

Lemma 2.2.26. If a space is a semi-regular space, then η*-closed sets coincide with closed sets.

Proof By **Note 2.2.21.**,δ closed $\longrightarrow$ η*- closed $\longrightarrow$ closed.In a semi-regular space, δ-closed sets coincide with closed sets**.**Hence η*-closed sets coincide with closed sets.

η*-Closure Operator

Definition 2.2.27. The intersection of all η*-closed sets of Y containing D is called as the **η*-Closure of D** (η*-Cl(D)).

Note 2.2.28. η*-Cl(D) is η*-closed.

Properties 2.2.29. In any topological space (Y,ζ),we get the following result hold:

- **a.** $\eta^*\text{-Cl}(\emptyset) = \emptyset$
- **b.** $\eta^*\text{-Cl}(Y) = Y$
- **c.** $D \subseteq \eta^*\text{-Cl}(D)$
- **d.** $D \subseteq E \Rightarrow \eta^*\text{-Cl}(D) \subseteq \eta^*\text{-Cl}(E)$**,if D and E are subsets of Y**
- **e.** $D \subseteq Cl(D) \subseteq \eta^*\text{-Cl}(D) \subseteq \delta Cl(D)$
- **f.** $\eta^*\text{-Cl}(D \cup E) = \eta^*\text{-Cl}(D) \cup \eta^*\text{-Cl}(E)$**,if D and E are subsets of Y**
- **g.** $\eta^*\text{-Cl}(\cap_{i\in\varepsilon} \{D_i\}) = \cap_{i\in\varepsilon} \eta^*\text{-Cl}(\{D_i\})$
- **h.** $Cl(D) \subseteq Cl(\eta^*\text{-Cl}(D))$.

Remark 2.2.30. In any topological space (Y, ζ) , (i) η*-(η*-Cl(D)) = η*-Cl(D)

(ii) η*-Cl(D) =D iff D is η*-closed.

Definition 2.2.31. Let C $\subseteq$Y. An element x$\in$Y is called a **η*-adherent point of C** if every η*-open set in Y containing x intersects C.

Definition 2.2.32. Let C be a subset of the topological space (Y,ζ).A point x∈Y is called a **η*-cluster point of C** if for every η*-open set V containing x intersects C in a point different from x.

Definition 2.2.33. We denote the set of all η*-cluster points of C by **η*-D(C)**.

Definition 2.2.34. Let C be a subset of a topological space (Y,ζ).Then the subset M of C is **η*-open in C** if C = M ∩ V where V is η*-open in Y.

Definition 2.2.35. Let C be a subset of a topological space (Y,ζ).For a subset M of C, **η*-Closure of M in C** denoted by η*-Cl_C(M) = η*-Cl(M) ∩ C.

Theorem 2.2.36.

(i) **$(\eta^*\text{-Cl}(D))^c = \eta^*\text{-Int}(D^c)$**

(ii) **$(\eta^*\text{-Int}(D))^c = \eta^*\text{-Cl}(D^c)$**

Proof (i) Let x∈$(\eta^*\text{-Cl}(D))^c$ which implies x∉ η*-Cl(D) iff there exists atleast one η*-open set V containing x such that D∩V= ∅ and V⊆D^c such that x∈V⊆D^c iff x∈ η*-Int(D^c).

(ii) Let x∈ η*-Cl(D^c) which implies every η*-open set V containing x intersects D^c ⟺ V⊈ D,that is, union of η*-open sets containing x is not contained in D ⟺ x∉ η*-Int(D),that is, x∈(η*-Int(D))^c.

§ 2.3. J-Closed Sets in Topological Spaces

In this section a new class of generalized closed sets, called J-closed sets are introduced. The relations between J-closed sets and various existing closed sets are analysed.

Definition 2.3.1. A subset D of a topological space (Y,ζ) is said to be **J-closed set** if Cl(D)$\subseteq$M whenever D$\subseteq$M, where M is η*-open in (Y,ζ). The class of all J-closed sets of (Y,ζ) is denoted by JC(Y,ζ).

Proposition 2.3.2. Every closed set is J-closed but not conversely.

Proof Let D be a closed set and M be any η^*-open set containing D. Since D is closed, Cl(D) =D. Therefore Cl(D) =D $\subseteq$ M. Hence D is J-closed.

Counter Example 2.3.3. Let Y = {p,q,r}, ζ = {ϕ,Y, {p},{p,q}}.$\eta^*O(Y,\zeta)$ = {ϕ,Y,{p}} and JC(Y,ζ) = P(Y) –{p}. The subset {q} is J-closed but not closed.

Proposition 2.3.4. Every δ-closed set is J-closed but not conversely.

Proof Let D be a δ- closed set and M be any η^*-open set containing D. Since D is δ-closed, δCl(D)=D. Therefore δCl(D) = D $\subseteq$ M.As Cl(D) $\subseteq$ δCl(D),D is J-closed.

Counter Example 2.3.5. Consider **Counter Example 2.3.3.**The subset {r} is J-closed but not δ-closed.

Proposition 2.3.6. Every δg^*-closed set is J-closed but not conversely.

Proof The proof follows from the facts that (i) η^*-open set $\longrightarrow$ g-open set [By **Note 2.2.9.].** (ii) Cl(D) $\subseteq$ δCl(D) and from the definitions of δg^*-closed sets and J-closed sets.

Counter Example 2.3.7. Let Y = {p,q,r},ζ = {ϕ, Y, {p}, {p,q}, {p,r}}. Then the subset {q} is J-closed but not δg^*-closed in (Y,ζ).

Proposition 2.3.8. Every δg-closed set is J-closed but not conversely.

Proof The proof follows from the facts that (i) η^*-open set $\longrightarrow$ open set [By **Theorem 2.2.5.].** (ii) Cl(D) $\subseteq$ δCl(D) and from the definitions of δg-closed sets and J-closed sets.

Counter Example 2.3.9. Let Y={p,q,r},ζ={ϕ, Y,{p},{p,q},{p,r}}. Then the subset {q} is J-closed but not δg-closed in (Y,ζ).

Proposition 2.3.10. Every g-closed set is J-closed but not conversely.

Proof The proof follows from the fact that η^*-open set $\longrightarrow$ open set [By **Theorem 2.2.5.]** and from the definitions of g-closed sets and J-closed sets.

Counter Example 2.3.11. Let Y = {p,q,r}, ζ = {ϕ, Y, {p},{p,q}}. The subset {q} is J-closed but not g-closed.

Proposition 2.3.12. Every J-closed set is gδ-closed but not conversely.

Proof The proof follows from the fact that η*-open set ⟶ open set [By **Theorem 2.2.5.]** and from the definitions of J-closed sets and gδ-closed sets.

Counter Example 2.3.13. Let Y = {p,q,r}, ζ = {ϕ, Y,{p}, {p,q}}.Then the subset {p} is gδ-closed but not J-closed in (Y,ζ).

Proposition 2.3.14. Every g*-closed set is J-closed but not conversely.

Proof The proof follows from the fact that η*-open set ⟶ g-open set [By **Note 2.2.9.]** and from the definitions of g*-closed sets and J-closed sets.

Counter Example 2.3.15. Let Y = {p,q,r}, ζ = {ϕ, Y, {p},{p,q}}. The subset {q} is J-closed but not g*-closed.

Proposition 2.3.16. Every ĝ-closed set is J-closed but not conversely.

Proof The proof follows from the fact that η*-open set ⟶ semi-open set [By **Note 2.2.9.]** and from the definitions of ĝ-closed sets and J-closed sets.

Counter Example 2.3.17. Let Y = {p,q,r}, ζ = {ϕ, Y, {p},{p,q}}. The subset {q} is J-closed but not ĝ -closed.

Proposition 2.3.18. Every *g-closed set is J-closed but not conversely.

Proof The proof follows from the fact that η*-open set ⟶ ĝ-open set [By **Note 2.2.9.]** and from the definitions of *g-closed sets and J-closed sets.

Counter Example 2.3.19. Let Y = {p,q,r}, ζ = {ϕ, Y, {p},{p,q}}. The subset {q} is J-closed but not *g-closed.

Remark 2.3.20.

$$\begin{tikzcd}
\text{g*-closed} \arrow[rr] & & \text{*g-closed} \arrow[drr] & & \text{closed} \arrow[d] & \\
\delta\text{ - closed} \arrow[r] & \delta g^{*}\text{- closed} \arrow[ul] \arrow[ur] \arrow[r] & \delta g\text{ - closed} \arrow[r] & g\text{ -closed} \arrow[r] & J\text{ - closed} \arrow[r] & g\delta\text{ -closed} \\
 & & & \hat{g}\text{-closed} \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

Proposition 2.3.21. Every J-closed set is rg-closed but not conversely.

Proof The proof follows from the fact that regular open set ⟶ η*-open set [By **Note 2.2.9.]** and from the definitions of J-closed sets and rg-closed sets.

Counter Example 2.3.22. Let Y={p,q,r},ζ ={ϕ, Y, {p},{p,q}}. Then the subset {p} is rg-closed but not J-closed in (Y,ζ).

Proposition 2.3.23. Every J-closed set is gpr-closed but not conversely.

Proof The proof follows from the facts that (i) regular open set ⟶ η*-open set [By **Note 2.2.9.]** (ii) pCl(D) $\subseteq$ Cl(D) and from the definitions of J-closed sets and gpr-closed sets.

Counter Example 2.3.24. Let Y={p,q,r},ζ ={ϕ, Y,{p},{q},{p,q}} Then the subset {p,q} is gpr-closed but not J-closed in (Y, ζ).

Proposition 2.3.25. Every J-closed set is rwg-closed but not conversely.

Proof Let D be J-closed and M be any regular open set containing D in Y. By **Note 2.2.9.**, every regular open set is η*-open and D is J-closed, Cl(D) $\subseteq$ M. As int(D) $\subseteq$ D. We have Cl(int(D)) $\subseteq$ Cl(D) $\subseteq$ M and hence D is rwg-closed.

Counter Example 2.3.26. Let Y={p,q,r},ζ ={ϕ, Y,{p},{q},{p,q}}. Then the subset {p,q} is rwg -closed but not J-closed in (Y,ζ).

Proposition 2.3.27. Every J-closed set is gspr-closed but not conversely.

Proof The proof follows from the facts that (i) regular open set ⟶ η*-open set [By **Note 2.2.9.]** (ii) spCl(D) $\subseteq$ Cl(D) and from the definitions of J-closed sets and gspr-closed sets.

Counter Example 2.3.28. Let Y = {p,q,r}, ζ = {ϕ, Y,{p},{q},{p,q}}. Then the subset {q} is gspr-closed but not J-closed in (Y,ζ).

Proposition 2.3.29. Every J-closed set is πg-closed but not conversely.

Proof The proof follows from the fact that π-open set ⟶ η*-open set [By **Note 2.2.9.]** and from the definitions of J-closed sets and πg-closed sets.

Counter Example 2.3.30. Let Y={p,q,r},ζ ={ϕ, Y,{p}, {p,q}}. Then the subset {p} is πg-closed but not J-closed in(Y,ζ).

Proposition 2.3.31. Every J-closed set is πgp-closed but not conversely.

Proof The proof follows from the facts that (i) π-open set ⟶ η*-open set [By **Note 2.2.9.]** (ii) pCl(D) ⊆ Cl(D) and from the definitions of J-closed sets and πgp-closed sets.

Counter Example 2.3.32. Let Y={p,q,r},ζ ={ϕ, Y,{p}, {p,q}}Then the subset {p} is πgp-closed but not J-closed in (Y,ζ).

Proposition 2.3.33. Every J-closed set is πgsp-closed but not conversely.

Proof The proof follows from the facts that (i) π-open set ⟶ η*-open set [By **Note 2.2.9.]** (ii) spCl(D) ⊆ Cl(D) and from the definitions of J-closed sets and πgsp-closed sets.

Counter Example 2.3.34. Let Y ={p,q,r},ζ ={ϕ, Y, {p}, {q}, {p,q}}. Then the subset {p} is πgsp-closed but not J-closed in (Y,ζ).

Proposition 2.3.35. Every J-closed set is πgs-closed but not conversely.

Proof The proof follows from the facts that (i) π-open set ⟶ η*-open set [By **Note 2.2.9.]** (ii) sCl(D) ⊆ Cl(D) and from the definitions of J-closed sets and πgs-closed sets.

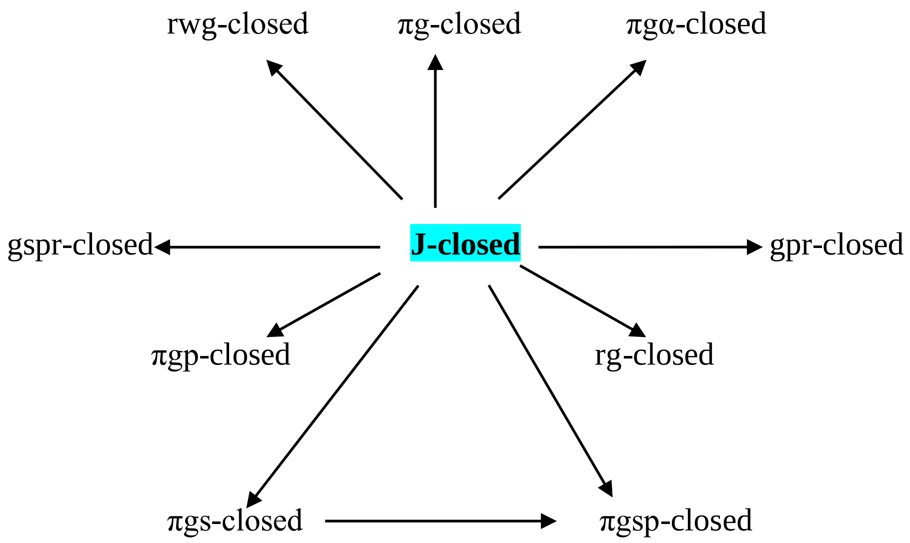
Counter Example 2.3.36. Let Y={p,q,r},ζ ={ϕ, Y, {p}, {q},{p,q}}. Then the subset {p} is πgs-closed but not J-closed in (Y,ζ).

Proposition 2.3.37. Every J-closed set is πgα-closed but not conversely.

Proof The proof follows from the facts that (i) π-open set ⟶ η*-open set [By **Note 2.2.9.]** (ii) αCl(D) ⊆ Cl(D) and from the definitions of J-closed sets and πgα-closed sets.

Counter Example 2.3.38. Let Y={p,q,r},ζ ={ϕ, Y, {p},{p,q}}. Then the subset {p} is πgα-closed but not J-closed in (Y,ζ).

Remark 2.3.39. From the above discussions, the class of J-closed sets is stronger than that of the given g-closed sets in the following figure.



Remark 2.3.40. The following counter examples show that J-closed set is independent from gs-closed, #gs-closed, g*s-closed, $\delta g^{\ddagger}$-closed set and αg-closed sets.

Counter Example 2.3.41. Let $Y=\{p,q,r\}$, $\zeta=\{\phi, Y, \{p\},\{q\},\{p,q\}\}$. The subset $\{p\}$ is gs-closed, #gs-closed, g*s-closed but not J-closed in (Y,ζ).

Counter Example 2.3.42. Let $Y=\{p,q,r\}$, $\zeta=\{\phi, Y, \{p,q\}\}$ The subset $\{p\}$ is J-closed but not gs-closed, #gs-closed, g*s-closed in (Y,ζ).

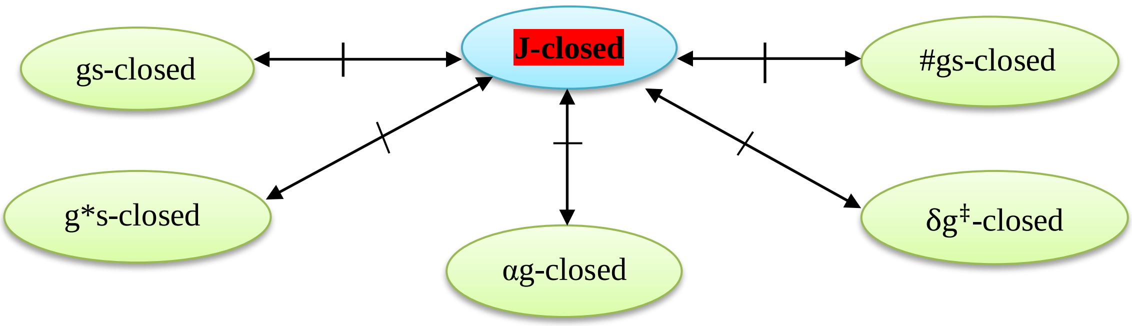
Counter Example 2.3.43. Let $Y=\{p,q,r,s\}$, $\zeta=\{\phi, Y, \{p\}\}$. The subset $\{p\}$ is not J-closed but it is $\delta g^{\ddagger}$-closed in (Y,ζ).

Counter Example 2.3.44. Let $Y=\{p,q,r,s\}$, $\zeta=\{\phi, Y, \{p\},\{r\},\{p,q\},\{p,r\},\{p,q,r\},\{p,r,s\}\}$ The subset $\{q\}$ is J-closed but not $\delta g^{\ddagger}$-closed in (Y,ζ).

Counter Example 2.3.45. Let $Y=\{p,q,r\}$, $\zeta=\{\phi, Y, \{p\},\{p,q\}\}$ The subset $\{p,q\}$ is J-closed but not αg-closed in (Y,ζ).

Counter Example 2.3.46. Let $Y=\{p,q,r,s\}$, $\zeta=\{\phi, Y, \{p\},\{p,q\}\}$. The subset $\{q\}$ is αg-closed but it is J-closed in (Y,ζ).

Remark 2.3.47. From the above discussions, we get J-closed sets are isolated from the following g-closed sets.



Properties of J-Closed Sets in Topological Spaces

Theorem 2.3.48. The finite union of J-closed sets is J-closed.

Proof Let $\{D_i\}_{i=1}^{n}$ be a finite class of J-closed sets of (Y,ζ). Let $D = \cup_{i=1}^{n} D_i$. Let M be a η*-open set containing D. This implies $D_i \subseteq$ M for every i. By assumption Cl(D_i) $\subseteq$ M for every i. This implies $\cup_{i=1}^{n} \mathrm{Cl}(D_i) \subseteq$ M. Then $\mathrm{Cl}(\cup_{i=1}^{n} D_i) \subseteq$ M. Thus Cl(D) $\subseteq$ M. Hence finite union of J-closed sets is J-closed in (Y,ζ).

Remark 2.3.49. The following Counter Example shows that the finite intersection of J-closed sets need not be J-closed.

Counter Example 2.3.50. Let Y={p,q,r},ζ={ϕ, Y, {p},{p,q}}. Here JC(Y,ζ) = P(Y) – {p}.Take A = {p.q} and B = {p,r} are J-closed sets. Then A ∩ B ={p} is not J-closed.

Theorem 2.3.51. The intersection of a J-closed set and a δ-closed set is always J-closed.

Proof Let D be J-closed and F be δ-closed. Let V = D ∩ F.Take M to be η*-open such that V ⊆ M. Then D ∩ F ⊆ M which implies D ⊆ M ∪ F^c. Here $F^c$is δ-open. So F^c is η*-open. Hence M ∪ F^c is η*-open (By **Theorem 2.2.10.(b)**) and by assumption D ⊆ M ∪ F^c which implies Cl(D) ⊆ M ∪ F^c and hence Cl(D) ∩ F ⊆ M. Now Cl(V) = Cl(D ∩ F) ⊆ Cl(D) ∩ Cl(F) ⊆ Cl(D) ∩ δ-Cl(F)= Cl(D) ∩ F ⊆ M. Thus Cl(V) ⊆ M. Hence D ∩ F is J-closed.

Remark 2.3.52. The following Counter Example shows that an infinite countable union of J-closed sets need not be J-closed.

Counter Example 2.3.53. Consider R, the real line with the usual topology ζ. Set $D = \bigcup_{n\in N}\left\{\frac{1}{n}\right\}$, N being the set of all positive integers. Clearly *D* is an infinite countable union of J-closed sets but *D* is not J-closed in (R,ζ), since $D \subseteq (0,2), 0 \in Cl(D) \Rightarrow Cl(D) \nsubseteq (0,2)$.

The following Counter Example shows that the difference of any two J-closed sets in Y need not be J-closed.

Counter Example 2.3.54. Let Y = {p,q,r,s }, ζ = {Y, ϕ,{p,q}}. Then the set {p,r} and {p,s} are J-closed but their difference {p} is not J- closed in (Y,ζ).

Theorem 2.3.55. Let D be a J-closed set of (Y,ζ). Then Cl(D) – D does not contain a non- empty η*-closed set.

Proof Suppose that D is J-closed, let M be a η*-closed set contained in Cl(D) – D. Now M^c is a η*-open set in Y such that D $\subseteq$ M^c. Since D is J-closed,Cl(D) $\subseteq$ M^c.Thus M $\subseteq$ $(Cl(D))^c$. Also M $\subseteq$ Cl(D)− D. Therefore M $\subseteq$ $(Cl(D))^c \cap Cl(D)$ = ϕ.Hence M = ϕ.

Theorem 2.3.56. Let D be a J-closed set of (Y,ζ).Then D is closed iff Cl(D) – D is η*-closed.

Proof (Necessity): Let D be a closed subset of (Y,ζ).Then Cl(D) = D and therefore Cl(D) – D = ϕ which is η*-closed.

(**Suffficiency):** Let Cl(D) −D be η*-closed set. Since D is J-closed, by **Theorem 2.3.55.**, Cl(D)−D does not contain a non-empty η*-closed set which implies Cl(D) −D = ϕ. That is Cl(D) = D. Hence D is closed.

Proposition 2.3.57. If D is a η*-open set and a J-closed set of (Y,ζ), then D is a closed set of Y.

Proof Since D is η*-open and J-closed, Cl(D) $\subseteq$ D. Hence D is closed in (Y,ζ).

Theorem 2.3.58. If D is J-closed and η*-open and F is closed in (Y,ζ), then D $\cap$ F is closed.

Proof Since D is J-closed and η*-open, D is closed by **Proposition 2.3.57.**Since F is closed in Y, $D \cap F$ is closed in Y.

Proposition 2.3.59. If D is a J-closed set in a space (Y,ζ) and $D \subseteq B \subseteq Cl(D)$ then B is also a J-closed set.

Proof Let M be η*-open set of Y such that $B \subseteq M$. Then $D \subseteq M$. Since D is J-closed set, $Cl(D) \subseteq M$. Since $B \subseteq Cl(D)$, $Cl(B) \subseteq Cl(Cl(D)) = Cl(D)$.Hence $Cl(B) \subseteq M$. Therefore B is also a J-closed set.

Definition 2.3.60. Let $B \subseteq A \subseteq Y$. Then B is **J-closed relative to A** if $Cl_A(B) \subseteq M$, whenever $B \subseteq M$, M is η*-open in A as in **Definition 2.2.34.**

Theorem 2.3.61. Let $B \subseteq A \subseteq Y$ and suppose that B is J-closed in Y, then B is J-closed relative to A. The converse is true if A is closed in Y.

Proof Suppose that B is J-closed in Y. Let $B \subseteq M$, M is η*-open in A. Since M is η*-open in A,$M = V \cap A$ for some η*-open set V in Y. Hence $B \subseteq M \subseteq V$. Since B is J-closed in Y,$Cl(B) \subseteq V$. Hence $Cl(B) \cap A \subseteq V \cap A$ which in turn and hence $Cl_A(B) \subseteq V \cap A = M$. Therefore B is J-closed relative to A.

Now to prove the converse, assume that $B \subseteq A \subseteq Y$ where A is closed in Y and B is J-closed relative to A. Let $B \subseteq M$, M is η*-open in Y. Then $A \cap M$ is η*-open in A by the definition of subspace topology.Since $B \subseteq A$ and $B \subseteq M$, $B \subseteq A \cap M$. Since B is J-closed relative to A, $Cl_A(B) \subseteq A \cap M$.Since $B \subseteq A$, $Cl(B) \subseteq Cl(A)$. Hence $Cl(B) \subseteq A$. Therefore $Cl(B) \cap A = Cl(B)$ which implies $Cl_A(B) = Cl(B)$. Hence $Cl(B) \subseteq A \cap M = M$. Thus B is J-closed in Y.

Characterizations of J-Closed Sets

Theorem 2.3.62. In a $T_{1/2}$-space for a subset D the following are equivalent:

(i) D is J-closed

(ii) D is gδ-closed.

Proof (i) ⇒(ii) By **Proposition 2.3.12.,**a J-closed set is a gδ-closed set.

(ii) ⇒(i) Let D be gδ-closed, D$\subseteq$ M where M is η*-open. In a $T_{1/2}$-space, η*-open sets coincide with δ-open sets (By **Lemma 2.2.23.**).Since D is gδ-closed, Cl(D)$\subseteq$ M ⇒ D is J-closed.

Example 2.3.63. Consider Y={p,q,r},ζ ={ϕ, Y,{p},{q},{p,q}}.Here JC(Y, ζ) =gδC(Y, ζ) = {ϕ, Y,{r},{p,r},{q,r}}.(Y, ζ) is a $T_{1/2}$ -space.

Theorem 2.3.64. Let D be a subset of a semi-regular space (Y,ζ), then the following are equivalent:

(a) D is δg-closed
(b) D is g-closed
(c) D is J-closed
(d) D is gδ-closed.

Proof (a) ⟹ (b), (b) ⟹ (c) and **(c) ⟹(d)** are true for any subset D by Theorem 3.1(iii) of (Dontchev,1996), **Proposition 2.3.10.and Proposition 2.3.12.**In a semi regular space,δ-closed sets coincide with closed sets.Hence **(d) ⟹ (a).**

Corollary 2.3.65. Let D be a subset of a $T_{1/2}$ -space and a semi-regular space (Y,ζ), then D is J-closed if and only if D is closed.

Proof The result follows from **(b)** ⇔ **(c)** of **Theorem 2.3.64.** and the fact that in a $T_{1/2}$ -space,g-closed sets coincide with closed sets.

Remark 2.3.66. In general the concepts of J-closed sets and δg‡-closed sets are not equivalent. This concept is characterized in following two ways.

Corollary 2.3.67. J-closed sets are equivalent to δg‡-closed sets in semi-regular spaces.

Proof By **Theorem 2.3.64.**,J-closed sets are equivalent to δg-closed sets and gδ-closed sets.But δg-closed ⟶ δg‡-closed ⟶ gδ-closed (Dontchev,1996).Hence δg‡-closed sets equivalent to J-closed sets also.

But for a compact subset D of a space which is $T_{1/2}$ and R_1, the concepts of J-closed sets and δg‡-closed sets coincide. This can be seen in the following Theorem.

A topological space (Y,ζ) is called R_1-space if every two different points with distinct closures have disjoint neighbourhoods. (Davis,1961).

Theorem 2.3.68. For a compact subset D of a R_1 -topological space (Y,ζ), every J-closed set is a δ$g^{‡}$-closed set. The converse is true if (Y,ζ) is $T_{1/2}$.

Proof Let D be J-closed and D be a compact subset of R_1 -topological space (Y,ζ). Let D ⊆ M, where M is δ-open. By **Theorem 2.2.3.**, every δ-open set is η*-open -----(1). In R_1-spaces the concepts of closure and δ-closure coincide for compact sets (By Theorem 3.6 (Jankovic,1980)). Hence for a compact subset D,δCl(D) =Cl(D) ----- (2).From (1) and (2) and since D is J-closed, we get D is a δ$g^{‡}$-closed set.

Conversely let (Y,ζ) be $T_{1/2}$ and D be δ$g^{‡}$-closed.Let D⊆ M, where M is η*-open. By **Lemma 2.2.23.**, η*-open sets coincide with δ-open sets in $T_{1/2}$-spaces.By assumption, δCl(D) ⊆ M which implies Cl(D) ⊆δCl(D) ⊆ M. Then D becomes J-closed.

Corollary 2.3.69. In Hausdroff spaces, also a finite set is J-closed if and only if it is δ$g^{‡}$-closed.

In T_δ-spaces the equivalence in Theorem 2.3.64. becomes true.

Proposition 2.3.70. If a topological space (Y,ζ) is a T_δ-space, then the following conditions are equivalent:

(a) D is δg-closed
(b) D is g-closed
(c) D is J-closed
(d) D is gδ-closed.

Proof **(a) ⟹ (b), (b) ⟹ (c)** and **(c) ⟹(d)** are true for any subset D by Theorem 3.1(iii) of (Dontchev,1996), **Proposition 2.3.10.and Proposition 2.3.12.**In T_δ-space, a gδ-closed set is a δ-closed set and by Theorem 3.1(i) of (Dontchev,2000), a δ-closed set is a δg-closed set. Hence **(d) ⇒ (a).**

Theorem 2.3.71. The following conditions (a) and (b) hold good.

(a) For a topological space (Y,ζ),if Y is a partition space, then every subset of Y is J-closed.

(b) The converse is true if Y is a $T_{1/2}$-space and a semi -regular space.

Proof (a) Let D be an arbitrary subset of Y and M be a η*-open set containing D.Since η*-openness is openness and Y is a Partition space in which every open set is closed,M is closed. Thus Cl(D) $\subseteq$ Cl(M) =M. Hence every subset of (Y,ζ) is J-closed.

(b) Let D be an open set in (Y,ζ).By criteria,every subset of Y is J-closed. If (Y,ζ) is $T_{1/2}$ and semi regular, then J-closed sets and closed sets coincide [By **Corollary 2.3.65.**]. Hence D is closed in (Y,ζ). Thus (Y,ζ) is a partition space.

Note 2.3.72. The converse of Theorem 2.3.71.(a) is not true which is seen in the following Counter Example.

Counter Example 2.3.73. Let Y={p,q,r},ζ ={ϕ, Y, {p,q}}.Here JC(Y,ζ) = P(Y).In this space every subset is J-closed but it is not a partition space.

J-Open Sets

In this section we introduce the concept of J-open sets in topological spaces and study some of their properties.

Definition 2.3.74. A subset D of a topological space (Y,ζ) is called **J-open** if its complement D^c is J-closed in (Y,ζ). The collection of all J-open sets in (Y,ζ) is denoted by JO(Y,ζ).

Theorem 2.3.75. If a subset D of a topological space (Y,ζ) is open, then it is J-open in Y.

Proof Let D be a open set in a topological space (Y,ζ). Then D^c is closed in Y. By **Proposition 2.3.2.**, D^c is J-closed in (Y,ζ). Hence D is J-open in (Y,ζ).

Remark 2.3.76. The converse of the above theorem need not be true as seen in the following Counter Example.

Counter Example 2.3.77. Let Y={p,q,r,s}, ζ={ϕ, X, {p}}. Then the subset {r} is J-open but not open in (Y,ζ).

Theorem 2.3.78. If a subset D of a topological space (Y,ζ) is δ-open, then it is J-open in Y.

Proof Let D be a δ-open set in a topological space (Y,ζ). Then D^c is δ-closed in Y. By **Proposition 2.3.4.**,D^c is J-closed in (Y,ζ). Hence D is J-open in (Y,ζ).

Remark 2.3.79. The converse of the above theorem need not be true as seen in the following Counter Example.

Counter Example 2.3.80. Let Y={p,q,r,s}, ζ={ϕ, Y, {p}}. Then the subset {r} is J-open but not δ-open in (Y,ζ).

Proposition 2.3.81. Every clopen set is a J-open set.

Proof Let D be a clopen set in a topological space (Y,ζ).Then Cl(D) = D and int(D) = D. Thus D is regular open and therefore D is δ-open which is J-open from **Theorem 2.3.78.**

Remark 2.3.82. A J-open set need not be clopen as seen from the following Counter Example.

Counter Example 2.3.83. Let Y={p,q,r,s}, ζ={ϕ, X, {p}}. Then the subset {r} is J-open but not clopen in (Y,ζ).

The following Proposition can be proved similar to Theorem 2.3.75. and Theorem 2.3.78.

Proposition 2.3.84. Every δg*-open,δg-open,g-open set,g*-open set and *g-open set, ĝ-open sets is a J-open set and every J-open is gδ-open (respectively gpr-open, rwg-open, gspr-open, *π*g-open, *π*gp-open, *π*gsp -open, *π*gs-open, *π*g*α* -open).

Result 2.3.85. For a subset D of (Y,ζ), Cl(Y – *D*) = Y – int(*D*).

Proof By **Result 1.1.13.,**it is obvious.

Theorem 2.3.86. A subset D of a topological space (Y,ζ) is J-open if and only if G ⊆ int(D) whenever G ⊆ D and G is η*-closed.

Proof Assume that D is J-open. Let G be a η*-closed set in (Y,ζ) contained in D. Then D^c is J-closed. Then G^c is a η*-open set in (Y,ζ) containing D^c.Since D^c is J-closed,$Cl(D^c) \subseteq G^c$ equivalently $G \subseteq int(D)$.

Conversely assume that $G \subseteq int(D)$ whenever G is contained in D and G is η*-closed in (Y,ζ). (i) $G \subseteq D \Rightarrow D^c \subseteq G^c$, where G^c is η*-open. (ii) $G \subseteq int(D) \Rightarrow G^c \supseteq (int\ D)^c = Cl(D^c)$.Now (i) and (ii) $\Rightarrow D^c$ is J-closed. Hence D is J-open.

Proposition 2.3.87. If int(D)$\subseteq$B $\subseteq$ D and D is J-open in (Y,ζ), then B is J-open in (Y,ζ).

Proof Follows from **Proposition 2.3.59.** and **Result 2.3.85.**

Theorem 2.3.88. If A and B are J -open sets in (Y,ζ), then A $\cap$B is J-open in (Y,ζ).

Proof Let A and B be J-open sets in Y. Then Y – A and Y – B are J-closed sets and $(Y - A) \cup (Y - B) = Y - (A \cap B)$ is J-closed. Hence $A \cap B$ is J-open.

Theorem 2.3.89. If D is J-open in Y then the only η*-open set containing int(D) $\cup$ D^c is Y.

Proof Let D be a J-open set and G be η*-open and $int(D) \cup D^c \subseteq G$. This gives $G^c \subseteq (int(D) \cup D^c)^c = (int(D))^c \cap D = Cl(D^c) - D^c$.Since D^c is J-closed and G^c is η*-closed. From **Theorem 2.3.55.**, it follows that $G^c = \emptyset$. Therefore $G = Y$.

Theorem 2.3.90. Every singleton set is either η*-closed or J-open in (Y,ζ).

Proof If {a} is η*-closed,then there is nothing to prove. Suppose that {a} is not η*-closed in Y, then $\{a\}^c$ is not η*-open and the only η*-open set containing $\{a\}^c$ is the space Y itself.That is $\{a\}^c \subseteq Y$.Therefore $Cl(\{a\}^c) \subseteq Y$. and so $\{a\}^c$ is J-closed and hence {a} is J-open in (Y,ζ).

Definition 2.3.91. The intersection of all η*-open subsets of Y containing D is called the η*- **kernel of D** and is denoted by η*-ker(D).

[η*-ker(D) = ∩{ U/ U is η*-open in (Y,ζ) and D⊆U }].

Note 2.3.92. Theorem 2.3.90. gives a decomposition for (Y,ζ) as $Y = Y_1 \cup Y_2$ where $Y_1 = \{y \in Y / \{y\}$ is η*-closed} and $Y_2 = \{y \in Y / \{y\}$ is J-open}.

Theorem 2.3.93. For a subset D of (Y,ζ), the following properties are equivalent :

- **(a) D is J-closed**
- **(b) $Cl(D) \subseteq$η*-ker(D) holds**
- **(c) (i)$Cl(D) \cap Y_1 \subseteq D$**
  - **(ii) $Cl(D) \cap Y_2 \subseteq$ η*-ker(D).**

Proof (a) ⇒(b) Let x ∉ η*-ker(D).Then there exists a set U ∈η*O(Y,ζ) such that D⊆ U and x ∉ U.Since D is J-closed, Cl(D) ⊆ U and so x ∉ Cl(D). Hence Cl(D) ⊆η*-ker(D) holds.

(b) ⇒ (a) Let U ∈η*O(Y,ζ) such that D⊆ U. Then we have that η*-ker(D) ⊆ U and so by (b)Cl(D) ⊆ U.Therefore, D is J-closed.

(b) ⇒(c) (i) Let $x \in Cl(D) \cap Y_1$ ------ (1)

x ∈ Cl(D)then by **(b)** x ∈η*-ker(D) ------ (2)

(1)⇒ $x \in Y_1$, {y} is η*-closed ------ (3)

If x ∉ D and say that U = Y – {x} is a η*-open set and D⊆ Y – {x}, η*-ker(D) ⊆ Y – {x}. Then by (2) we have that x ∈ Y – {x}, which is a contradiction.Therefore x ∈ D.

(ii) $Cl(D) \cap Y_2 \subseteq Cl(D)$ ------ (4)

(b) ⇒ Cl(D) ⊆ η*-ker(D)holds. ------ (5)

From (4) and (5) ⇒$Cl(D) \cap Y_2 \subseteq$ η*-ker(D).

(c)⇒(b) Cl(D)= Cl(D) ∩ Y

$= Cl(D) \cap (Y_1 \cup Y_2)$

$= (Cl(D) \cap Y_1) \cup (Cl(D) \cap Y_2)$

⊆ D ∪η*-ker(D) (using **(c)(i) and(ii)**)

Therefore Cl(D) ⊆ η*-ker(D).

Corollary 2.3.94. Let $\mathcal{P} = \{D \subseteq Y/Cl(D) \cap Y_2 \subseteq \eta^*\text{-}ker(D)\}$.Then

(a) If $\bigcap_{i\in\varepsilon} D_i \in \mathcal{P}$ and D_i is J-closed in (Y,ζ) for each i,then $\cap D_i$ is J-closed

(b) If $\mathcal{P} = \mathcal{P}$ (Y) and D_i is J -closed in (Y,ζ) for each i $\in\varepsilon$, then $\bigcap_{i\in\varepsilon} D_i$ is J-closed in (Y,ζ)

(c) If $Cl(D_i) \cap Y_2 \subseteq \eta^*\text{-}ker(D_i)$and D_i is J-closed in (Y,ζ) for each i $\in\varepsilon$, then $\bigcap_{i\in\varepsilon} D_i$ is J-closed in (Y,ζ).

Proof (a) By **Theorem 2.3.93.**, $Cl(D_i) \cap Y_1 \subseteq D_i$ for each $i\in\varepsilon$. Then we have $Cl(\cap D_i) \cap Y_1 \subseteq \cap D_i$ using assumption and **Theorem 2.3.93.(c)(i)**, $\cap D_i$ is J-closed.

(b) and **(c)** follow from **(a)**.

Theorem 2.3.95. If a subset D is J-closed in (Y,ζ),then Cl(D) −D is J-open.

Proof Let D be J-closed in (Y,ζ).Let M ⊆ Cl(D) − D and M is η*-closed.Since D is J-closed, Cl(D)−D does not contain a non-empty η*-closed set (by **Theorem 2.3.55.**).Hence M = ∅.Thus M ⊆ int(Cl(D)−D).Therefore Cl(D)−D is J-open.

Remark 2.3.96. Finite union of J-open sets need not be a J-open set.It can be proved by the following Counter Example.

Counter Example 2.3.97. Let Y = {p,q,r,s }, ζ = {Y, ϕ,{p}}.Here {q},{r} and {s} are J-open sets.But their union {q,r,s} is not a J-open set.

§ 2.4. J*-Closed Sets in Topological Spaces

This section contains perception of J*-closed sets along with the analysis of this new concept with existing other closed sets.

Definition 2.4.1. Let D ⊆ Y. A subset D of (Y,ζ) is known as **J*-closed set** if η*-Cl(D) ⊆ M whenever D ⊆ M, M ∈ ζ**.**

Note 2.4.2. We represent the family of all J*-closed sets of (Y,ζ) by J*C(Y,ζ).

Proposition 2.4.3. Every η*-closed set is a J*-closed set.

Proof Let D be η*-closed. To prove D is J*-closed. Assume that M ∈ ζ containing D. Since D is η*-closed, η*-Cl(D) = D ⊆ M. Therefore D becomes J*-closed .

Proposition 2.4.4. Every J*-closed set is a g-closed set.

Proof Let D be J*-closed. To prove D is g-closed. Assume that M ∈ ζ containing D. Since D is J*-closed, η*-Cl(D) ⊆ M. By **Note 2.2.21.**,Cl(D) ⊆ η*-Cl(D).Therefore D becomes g-closed.

Proposition 2.4.5. Every J*-closed set is a J-closed set.

Proof Let D be a J*-closed set. To prove D is J-closed. Assume that M is a η*-open set containing D.By **Note 2.2.9.**, every η*-open is open and D is J*-closed , η*-Cl(D) ⊆ M. By **Note 2.2.21.**,Cl(D) ⊆ η*-Cl(D).Therefore D becomes J- closed.

Proposition 2.4.6. Every J*-closed set is a αg- closed set.

Proof Given D is a J*-closed set. To prove D is an αg- closed set. Assume that M ∈ ζ containing D.Since D is J*-closed,η*-Cl(D) ⊆ M. By **Note 2.2.21.**, αCl(D) ⊆ η*-Cl(D).Therefore D becomes αg- closed.

Proposition 2.4.7. Every J*-closed set is a gδ- closed set.

Proof Given D is a J*-closed set. To prove D is a gδ- closed set. Assume that M ⊇ D, δ-open. By **Note 2.2.9.**, every δ-open set is an open set and D is J*-closed, η*-Cl(D) ⊆ M. By **Note 2.2.21.**,Cl(D) ⊆ η*-Cl(D),Cl(D) ⊆ M. Hence D becomes gδ- closed.

Proposition 2.4.8. Every δ-closed set is a J*-closed set.

Proof Given is a δ- closed set. To prove D is a J*-closed set. Assume that M ∈ ζ containing D. Since δCl(D) = D ⊆ M. By **Note 2.2.21.**,we get η*-Cl(D) ⊆ δCl(D).This makes D, a J*-closed set.

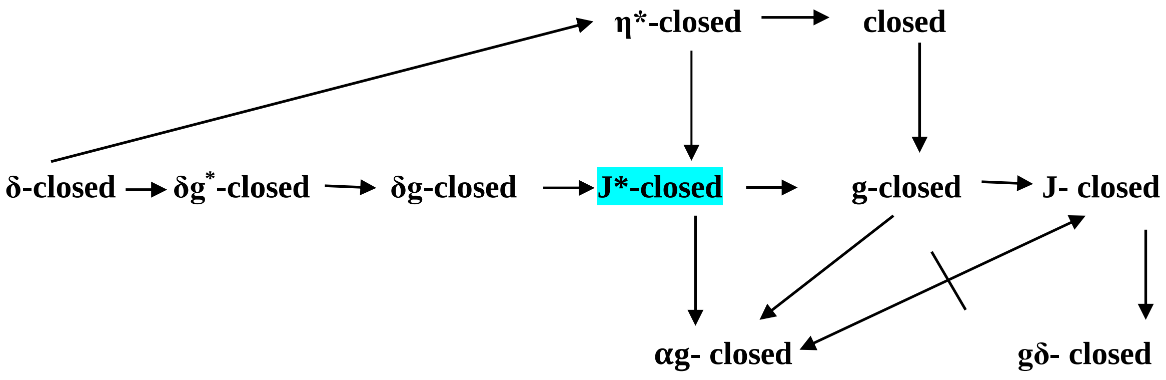
Proposition 2.4.9. Every δg^*-closed set is a J*-closed set.

Proof Given D is a δg^*- closed. To prove D is a J*-closed set. That is to prove η*-Cl(D) ⊆ M whenever D ⊆ M, M ∈ ζ. Assume that M ∈ ζ containing D.As every open set is a g-open set and D is δg^*- closed, $\delta Cl(D) \subseteq M$. As η^*-$Cl(D) \subseteq \delta Cl(D)$ [By **Note 2.2.21.**] which implies η^*-$Cl(D) \subseteq M$ implies D becomes J*-closed .

Proposition 2.4.10. Every δg-closed set is a J*-closed set.

Proof Given D is a δg- closed. To prove D is a J*-closed set. Assume that $M \in \zeta$ containing D.As D is δg- closed, $\delta Cl(D) \subseteq M$. As η^*-$Cl(D) \subseteq \delta Cl(D)$ [By **Note 2.2.21.**] which implies η^*-$Cl(D) \subseteq M$ implies D becomes J*-closed .

Remark 2.4.11. From the above Lemmas and Propositions, the following implications among various closed sets are obtained.



Proposition 2.4.12. Every J*-closed set is a rg- closed set.

Proof Given D is a J*-closed set. To prove D is a rg- closed set. Assume that $D \subseteq M$, M is regular -open. By **Note 2.2.9.**,every regular open is open and D is J*-closed, η^*-$Cl(D) \subseteq M$. By **Note 2.2.21.**,$Cl(D) \subseteq \eta^*$-$Cl(D)$,$Cl(D) \subseteq M$. Hence D becomes rg-closed.

Proposition 2.4.13. Every J*-closed set is a gpr- closed set.

Proof Given D is a J*-closed set, and $M \supseteq D$,regular open in Y. By **Note 2.2.9.**, every regular open set is an open set and D acts as J*-closed, η^*-$Cl(D) \subseteq M$. As $pCl(D) \subseteq \eta^*$-$Cl(D)$ [By **Note 2.2.21.**] which implies $pCl(D) \subseteq M$ implies D becomes gpr-closed.

Proposition 2.4.14. Every J*-closed set is a rwg- closed set.

Proof Given D is a J*-closed set. Consider M is a regular-open set containing D. By **Note 2.2.9.**,every regular open set is open and D is J*-closed , we get η^*-Cl(D)$\subseteq$M. As int(D)$\subseteq$D. we have η^*-Cl(int(D))$\subseteq\eta^*$-Cl(D)$\subseteq$M --- (1) (By **Properties 2.2.29.d.**). Moreover Cl(D)$\subseteq$ η^*-Cl(D) and hence Cl(int(D))$\subseteq$ η^*-Cl(int(D)) --- (2).From (1) and (2), Cl(int(D))$\subseteq$ M. Hence D is rwg-closed.

Proposition 2.4.15. Every J*-closed set is a gspr- closed set.

Proof Let D be J*-closed , and D $\subseteq$ M which is any regular open set in Y. By **Note 2.2.9.**, every regular open set is open and as D is J*-closed, η^*-Cl(D)$\subseteq$M. As spCl(D)$\subseteq$ η^*-Cl(D) [By **Note 2.2.21.**]. we get spCl(D)$\subseteq$M which implies D is gspr -closed.

Proposition 2.4.16. Every J*-closed set is a πg- closed set.

Proof Given D is a J*-closed set. Let M be a π-open set containing D. By **Note 2.2.9.**,every π-open set is open and as D is J*-closed,η^*-Cl(D)$\subseteq$M. By **Note 2.2.21.**,Cl(D)$\subseteq$ η^*-Cl(D),Cl(D)$\subseteq$M and hence D is πg- closed.

Proposition 2.4.17. Every J*-closed set is a πgp- closed set.

Proof Given D is a J*-closed set. Let M be a π-open set containing D. By **Note 2.2.9.**,every π-open set is open and as D is J*-closed, η^*-Cl(D)$\subseteq$M. As pCl(D)$\subseteq$ η^*-Cl(D) [By **Note 2.2.21.**] which implies pCl(D)$\subseteq$ M .Hence D is πgp- closed.

Proposition 2.4.18. Every J*-closed set is a πgsp- closed set.

Proof Given D is a J*-closed set. Let M be a π-open set containing D. By **Note 2.2.9.**,every π-open set is open and as D is J*-closed , η^*-Cl(D)$\subseteq$M. As spCl(D)$\subseteq$ η^*-Cl(D) [By **Note 2.2.21.**] which implies spCl(D)$\subseteq$M and hence D is πgsp- closed.

Proposition 2.4.19. Every J*-closed set is a πgs- closed set.

Proof Given D is a J*-closed set. Let M be a π-open set containing D. By **Note 2.2.9.**,every π-open set is open and as D is J*-closed, η^*-Cl(D)$\subseteq$M. As sCl(D)$\subseteq$ η^*-Cl(D) [By **Note 2.2.21.**] which implies sCl(D)$\subseteq$M and hence D is πgs- closed.

Proposition 2.4.20. Every J*-closed set is a πgα- closed set.

Proof Given D is a J*-closed set. Let M be a π-open set containing D. By **Note 2.2.9.**, every π-open set is open and as D is J*-closed, η*-Cl(D)⊆M. As αCl(D)⊆η*-Cl(D) [By **Note 2.2.21.**]. We get αCl(D)⊆M and hence D is πgα- closed.

Proposition 2.4.21. Every J*-closed set is a gs- closed set.

Proof Given D is a J*-closed set. Assume that M is an open set containing D.Since D is J*-closed , η*-Cl(D)⊆M. As sCl(D)⊆η*-Cl(D) [By **Note 2.2.21.**]. We get sCl(D)⊆M implies that D becomes gs-closed.

Proposition 2.4.22. Every J*-closed set is a gsp- closed set.

Proof Given D is a J*-closed set. Let M be an open set containing D.Since D is J*-closed , η*-Cl(D)⊆M. As spCl(D)⊆η*-Cl(D) [By **Note 2.2.21.**]. We get spCl(D)⊆M implies that D becomes gsp-closed.

Note 2.4.23. The following Counter Examples explain the converse of the above Propositions need not be true.

Counter Example 2.4.24. Consider Y={p,q,r}, ζ={ϕ, Y, {r,p},{p},{p,q}}.Analysing we get {q} ⊆ Y happens to be g-closed not satisfying the condition for J*-closed.

Counter Example 2.4.25. Consider Y={p,q,r}, ζ={ϕ, Y, {p,q}}. Analysing we get {q} ⊆ Y happens to be J-closed not satisfying the condition for J*-closed.

Counter Example 2.4.26. Consider Y={p,q,r}, ζ={ϕ, Y, {p},{p,q}}.Analysing we get {q} ⊆ Y happens to be αg-closed not satisfying the condition for J*-closed.

Counter Example 2.4.27. Consider Y={h,m,t},ζ={ϕ, Y,{h}, {h,m}}. Analysing we get {h} ⊆ Y happens to be gδ- closed not satisfying the condition for J*-closed.

Counter Example 2.4.28. Consider Y={p,q,r}, ζ ={ϕ,Y,{p}}.Analysing we get {q} ⊆ Y happens to be J*-closed not satisfying the condition for δ-closed.

Counter Example 2.4.29. Consider Y={p,q,r},ζ={ϕ, Y, {p}}.Analysing we get {q,r}⊆ Y happens to be J*-closed not satisfying the condition for δg^*-closed.

Counter Example 2.4.30. Consider $Y=\{j,k,l\}, \zeta =\{\phi, Y, \{j\},\{j,k\}\}$.Analysing we get $\{j\} \subseteq Y$ happens to be rg-closed not satisfying the condition for J*-closed in (Y,ζ).

Counter Example 2.4.31. Consider $Y=\{m,n,o\}, \zeta =\{\phi, Y,\{m\},\{n\},\{m,o\}\}$.Analysing we get $\{m,o\} \subseteq Y$ happens to be gpr- closed not satisfying the condition for J*-closed .

Counter Example 2.4.32. Consider $Y=\{p,q,r\}, \zeta =\{\phi, Y,\{q\},\{q,p\},\{p\}\}$. Analysing we get $\{q,p\} \subseteq Y$ happens to be rwg- closed not satisfying the condition J*-closed in (Y,ζ).

Counter Example 2.4.33. Consider $Y=\{p,q,r\}, \zeta =\{\phi, Y,\{p\},\{p,q\},\{q\}\}$. Analysing we get $\{q\} \subseteq Y$ happens to be gspr- closed not satisfying the condition for J*-closed.

Counter Example 2.4.34. Consider $Y=\{e,f,g\}, \zeta =\{\phi, Y,\{e\},\{e,f\}\}$. Analysing we get $\{e\} \subseteq Y$ happens to be πg- closed not satisfying the condition for J*-closed.

Counter Example 2.4.35. Consider $Y=\{p,q,r\}, \zeta =\{\phi, Y,\{p\},\{p,q\}\}$.Analysing we get $\{q\} \subseteq Y$ happens to be πgp- closed(resp. gsp-closed) not satisfying the condition for J*-closed.

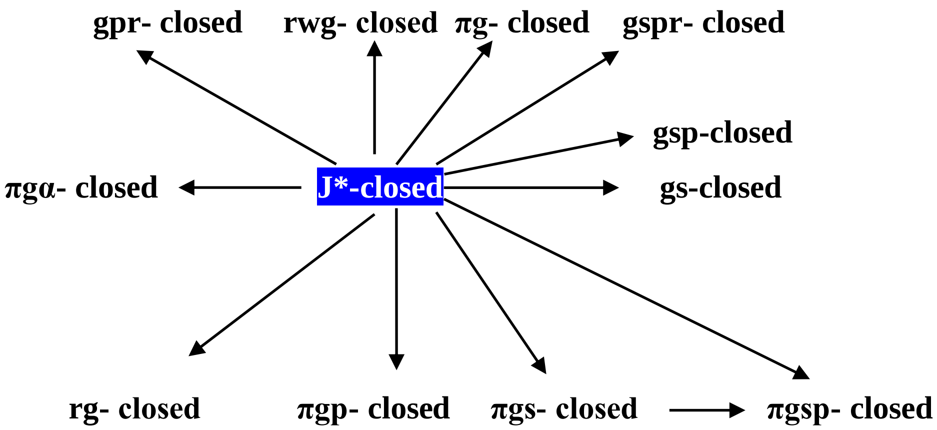
Counter Example 2.4.36. Consider $Y=\{r,s,t\}, \zeta =\{\phi, Y,\{r,s\},\{s\},\{r\}\}$. Analysing we get $\{r\} \subseteq Y$ happens to be πgsp- closed not satisfying the condition for J*-closed.

Counter Example 2.4.37. Consider $Y=\{d,e,f\}, \zeta =\{\phi, Y,\{e\},\{d\},\{d,e\}\}$. Analysing we get $\{e\} \subseteq Y$ happens to be πgs- closed not satisfying the condition for J*-closed .

Counter Example 2.4.38. Consider $Y=\{p,q,r\}, \zeta =\{\phi, Y, \{p\},\{p,q\}\}$. Analysing we get $\{p,q\} \subseteq Y$ happens to be πgα- closed not satisfying the condition for J*-closed.

Counter Example 2.4.39. Consider $Y=\{p,q,r\}, \zeta =\{\phi, Y, \{p\},\{p,q\}\}$.Analysing we get $\{q\} \subseteq Y$ happens to be gs- closed not satisfying the condition for J*-closed.

Remark 2.4.40. The above results are given in the following diagram.



Remark 2.4.41. The following counter-examples explain that J*-closed set is not equivalent of closed,#gs- closed and g*s- closed sets.

Counter Example 2.4.42. Consider Y={a,m,p},ζ ={ϕ, Y, {a},{a,m}}.Analysing we get {a,p} ⊆ Y happens to be J*-closed not satisfying the condition for closed.

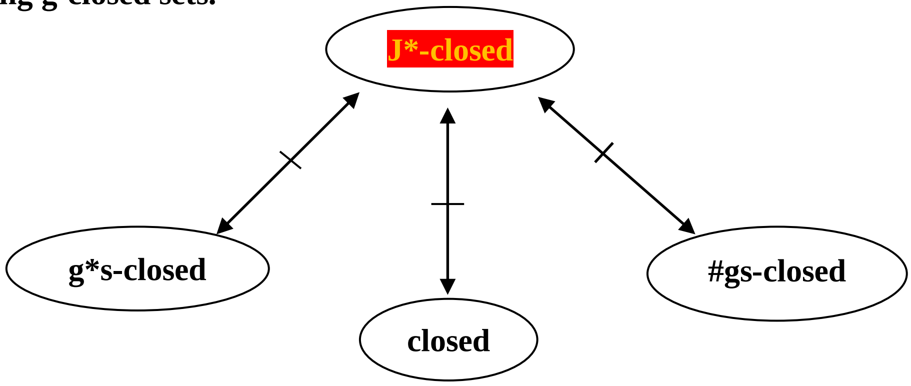
Counter Example 2.4.43. Consider Y={r,s,t},ζ ={ϕ,Y,{r},{r,s},{r,t}}.Analysing we get {s} ⊆ Y happens to be closed not satisfying the condition for J*-closed .

Counter Example 2.4.44. Consider Y={k,l,m},ζ={ϕ, Y,{k},{l},{k,l}}.Analysing we get {k} ⊆ Y happens to be not satisfying the condition for J*-closed but it is #gs- closed and g*s- closed.

Counter Example 2.4.45. Consider Y={u,v,w},ζ ={ϕ, Y, {u},{v,w}}. Analysing we get {u,w} ⊆ Y happens to be J*-closed not satisfying the condition for g*s- closed.

Counter Example 2.4.46. Consider Y={p,q,r,s},ζ ={ϕ, Y,{p}}.Analysing we get {p,q,r} ⊆ Y happens to be J*-closed not satisfying the condition for #gs- closed.

Remark 2.4.47. From the above discussions, we get J*-closed sets is independent with the following g-closed sets.



Properties of J*-Closed Sets

Theorem 2.4.48. The finite union of J*-closed sets is J*-closed.

Proof Let $\{D_i\}_{i=1}^{n}$ be a finite class of J*-closed sets of (Y,ζ). Let $D = \cup_{i=1}^{n} D_i$. Let M be an open set containing D. This implies $D_i \subseteq$ M for every i. By assumption η*-Cl(D_i) ⊆ M for every i. This implies $\cup_{i=1}^{n} \eta * -\mathrm{Cl}(D_i)$ ⊆ M. Then η*-Cl($\cup_{i=1}^{n} D_i$) ⊆ M (By

Properties 2.2.29.f.).Thus η^*-Cl$(D) \subseteq$ M. Hence finite union of J*-closed sets is J*-closed in (Y,ζ).

Result 2.4.49. The following Counter Example shows that finite intersection of any two J*-closed sets in Y need not be J*-closed.

Counter Example 2.4.50. Let Y = {p,q,r,s }, ζ = {Y, ϕ,{p}}. Consider D and E are subsets of Y. Then the set {p,q} and {p,r} are J*-closed but the intersection D$\cap$E = {p} is not J*-closed in (Y,ζ).

Note 2.4.51. The next Counter Example explains the difference of two J*-closed sets in Y need not be J*-closed.

Counter Example 2.4.52. Take Y = {t,u,v,w }, ζ = {Y, ϕ,{t,u}}. Then the set {t,v} and {t,w} are J*-closed but their difference {t} is not J*-closed in (Y,ζ).

Theorem 2.4.53. If D is a J*-closed and F is η^*-closed, then, D$\cap F$ becomes J*-closed.

Proof Assume D is J*-closed and F be η^*-closed. Let V = D$\cap F$.Let M be open such that V $\subseteq$ M. Then , D$\cap F \subseteq$M which implies D $\subseteq$M$\cup F^c$. Here F^c is η^*-open. So F^c is open (By **Theorem 2.2.5.**) and by assumption D is J*-closed which implies η^*-Cl(D)$\subseteq$ M$\cup F^c$ that is η^*-Cl(D) $\cap F \subseteq$M ----- (1).Hence η^*-Cl(V) = η^*-Cl(D$\cap F$) = η^*-Cl(D) $\cap$ η^*-Cl(F) [By **Properties 2.2.29. g**] = η^*-Cl(D)$\cap$ F$\subseteq$M from (1).Therefore η^*-Cl(V)$\subseteq$M. Hence D$\cap F$ becomes J*-closed .

Theorem 2.4.54. Every J*-closed set, then η^*-Cl(D) – D $\nsubseteq$ M, where M is a non-empty closed set.

Proof Consider D is J*-closed , and M acts as a closed set. Then M $\subseteq \eta^*$-Cl(D) – D. Now M $\in \zeta$ such that D$\subseteq$M^c. Since D is J*-closed, η^*-Cl(D) $\subseteq$M^c.Thus M $\subseteq$ (η^*-Cl(D))^c. Also M $\subseteq \eta^*$-Cl(D) – D.Therefore M $\subseteq$ (η^*-Cl(D))$^c\cap$ η^*-Cl(D) =ϕ. Hence M = ϕ.

Proposition 2.4.55. If D is an open set and a J*-closed set of (Y,ζ), is a η^*-closed set of Y.

Proof Since D is open, and J*-closed , η^*-Cl(D)$\subseteq$D. Always D$\subseteq\eta^*$-Cl(D). Hence D is η^*- closed in (Y,ζ).

Theorem 2.4.56. If D is a J*-closed and open and F is η^*-closed in (Y,ζ), then D∩F becomes η^*- closed.

Proof Given D is J*-closed and open, D is η^*-closed by **Proposition 2.4.55.** By assumption F is η^*-closed in Y, D∩F is η^*-closed in Y (By **Theorem 2.2.10.(c)**).

Proposition 2.4.57. If D is a J*-closed set in (Y,ζ) and D⊆B⊆η^*-Cl(D) then B becomes also a J*-closed set.

Proof Assume D is a J*-closed set in (Y,ζ). Consider M is an open set of Y such that B $\subseteq$M implies D$\subseteq$M. Since D is J*-closed set, η^*-Cl(D)$\subseteq$ M. Since B$\subseteq$ η^*-Cl(D), η^*-Cl(B) $\subseteq$ η^*-Cl(η^*-Cl(D)) = η^*-Cl(D) [by **Remark 2.2.30.(i)**] Hence η^*-Cl(B) $\subseteq$ M. Therefore B is also a J*-closed set.

Definition 2.4.58. Let O⊆N⊆Y. Then O is **J*-closed relative to N** if η^*-Cl$_N$(O)$\subseteq$M whenever O $\subseteq$ M, M ∈ ζ in N.

Theorem 2.4.59. Let O⊆N⊆Y and consider that O is J*-closed in Y, then O is J*-closed relative to N. The converse is true if N is η^*-closed Y.

Proof Suppose that O is J*-closed in Y. Let O⊆M,M∈ ζ in N. Since M∈ζ in N, M = V∩N where V∈ζ in Y. Hence O⊆M⊆V. Since O is J*-closed in Y, η^*-Cl(O)$\subseteq$ V.Hence η^*-Cl(O) ∩N $\subseteq$V∩N which gives η^*-Cl$_N$(O)$\subseteq$ V∩N = M.Hence O is J*-closed relative to N.

Now to prove the reverse, assume that O⊆N⊆Y where N is η^*-closed in Y and O is J*-closed relative to N.Let O⊆M, M∈ ζ in Y. Then N ∩ M∈ζ in N by subspace topology definition. Since O⊆N and O⊆M, O⊆N ∩ M. Since O is J*-closed relative to N, η^*-Cl$_N$(O)$\subseteq$ N ∩ M.Since O⊆N, η^*-Cl(O)$\subseteq$ η^*-Cl(N). Hence η^*-Cl(O)$\subseteq$N. Therefore η^*-Cl(O) ∩ N = η^*-Cl(O) which implies η^*-Cl$_N$(O) = η^*-Cl(O).Hence η^*-Cl(O)$\subseteq$ N ∩ M = M . Thus O is J*-closed in Y.

Theorem 2.4.60. If D acts as a J*-closed set of (Y,ζ),then D is η^*- closed iff η^*-Cl(D) – D is closed.

Proof (Necessity) Consider D is a η^*-closed subset of (Y,ζ).Then η^*-Cl(D) = D and therefore η^*-Cl(D) – D=ϕ which is closed.

(**Suffficiency)** Let η^*-Cl(D) −D be closed set. Since D is J*-closed, by **Theorem 2.4.54.**, η^*-Cl(D) −D $\not\subseteq$ a non-empty closed set which implies η^*-Cl(D)−D=ϕ. That is η^*-Cl(D)=D. Hence D is η^*- closed.

Characterizations of J*-Closed Sets

Proposition 2.4.61. In a $T_{1/2}$ -space, J*-closed sets coincide with δg-closed sets.

Proof By **Proposition 2.4.10.**,δg-closed set is a J*-closed set. To prove the other way, in a $T_{1/2}$-space,η^*-closed sets coincide with δ-closed sets (By **Lemma 2.2.23.**).Hence η^*-Cl(D) =δCl(D) $\subseteq$ M which implies J*-closed set is a δg-closed set. Hence J*-closed sets coincide with δg- closed sets.

Example 2.4.62. Consider $Y=\{p,q,r\}$,$\zeta=\{\phi, Y,\{p\},\{q,r\}\}$.Here $J^*C(Y,\zeta) = \delta gC(Y,\zeta) = P(Y)$.It is also a $T_{1/2}$ -space.

Theorem 2.4.63. If a space is a semi-regular space, then J*-closed sets and J-closed sets coincide with with g-closed sets, δg-closed sets, gδ-closed sets and $\delta g^{\ddagger}$ -closed sets.

Proof The proof follows from **Lemma 2.2.26.,Theorem 2.3.64.** and **Corollary 2.3.67.**

Example 2.4.64. Consider $Y=\{p,q,r\}$,$\zeta=\{\phi, Y,\{p\},\{q,r\}\}$.Here $\delta C(Y,\zeta) = \eta^*C(Y,\zeta) =\zeta^c$ and $JC(Y,\zeta) = J^*C(Y,\zeta) = gC(Y,\zeta) = \delta gC(Y,\zeta) =\delta g^{\ddagger}C(Y,\zeta) = g\delta C(Y,\zeta) = P(Y)$.

Remark 2.4.65. In general, a J*-closed set is independent with a closed set. But for a space which is T_b-space and αT_b -space, every J*-closed set is a closed set. This can be seen in the following Propositions.

Proposition 2.4.66. If D is J*-closed, then it is closed when (Y,ζ) is T_b-space.

Proof By **Proposition 2.4.21.,**every J*-closed is gs-closed. In T_b space, every gs- closed is closed. Hence J*-closed is closed.

Proposition 2.4.67. If D is J*-closed, then it is closed when (Y,ζ) is αT_b- space.

Proof By **Proposition 2.4.6.,** every J*-closed is αg-closed. In αT_b -space, every αg-closed is closed.Hence J*-closed is closed.

Theorem 2.4.68. In an almost weakly Hausdorff space (Y,ζ),the g- closed sets of (Y,ζ_S) are δ-closed sets in (Y,ζ) and thus they are δg*-closed sets, δg-closed sets, J*-closed sets respectively in (Y,ζ).

Proof By **Proposition 2.4.8.,** every δ-closed set is J*-closed . Hence to prove the given condition it is enough to prove every g- closed sets of(Y,ζ_S) are δ-closed in (Y,ζ). Let D $\subseteq$ Y be a g- closed subset of (Y,ζ_S).Consider $x \in \delta Cl(D)$.If $\{x\}$ is δ-open, then $x \in D$.If not, thenY \{ x } is δ-open, since Y is almost weakly Hausdorff .Let $x \notin D$.By the given condition D is g-closed in (Y,ζ_S) then δCl(D) ⊆Y \{ x },(i. e) $x \notin \delta Cl(D)$.By contradicition $x \in D$.Then δCl(D)=D.Hence D is δ-closed hence by Proposition 2.2.2.(Sudha,2014),Theorem 3.1.(i)(Dontchev,1996), **Proposition 2.4.8.** and D is δg*-closed sets, δg-closed sets, J*-closed sets respectively in (Y,ζ).

Theorem 2.4.69. Let D be a pre-open subset of a topological space (Y,ζ).Then the following conditions are equivalent:

(a) D is δg-closed
(b) D is J*-closed
(c) D is g-closed.

Proof **(a) ⟹ (b),(b) ⟹(c)** are true for any subset D by **Proposition 2.4.10., Proposition 2.4.4.**If D is pre-open in (Y,ζ), then by a result of (Jankovic,1985),Cl(D) = δCl(D) and so **(c) ⟹ (a)**.

Concerning partition spaces, the following characterization via J*-closed sets is obtained.

Corollary 2.4.70. Let D be a subset of the partition space (Y,ζ). Then the following conditions are equivalent:

(a) D is δg-closed
(b) D is J*-closed
(c) D is g-closed.

Proof A topological space is a partition space if and only if every subset is pre-open.Thus the claim follows straight from **Theorem 2.4.69.**

Example 2.4.71. Let Y={p,q,r},ζ ={ϕ, Y, {p},{q,r}}.This is a partition space.It is seen that δgC(Y,ζ) = J*C(Y,ζ) =gC(Y,ζ) =P(Y).

Corollary 2.4.70.is modified by replacing the partition space by T_δ-space as follows.

Proposition 2.4.72. If a topological space (Y,ζ) is a T_δ-space, then the following conditions are equivalent:

(a) D is δg-closed
(b) D is J*-closed
(c) D is g-closed
(d) D is J-closed
(e) D is gδ-closed.

Proof **(a)** ⇒ **(b)** and **(b)** ⇒**(c)**, **(c)** ⇒ **(d)** by **Proposition 2.4.10, Proposition 2.4.4., Proposition 2.3.10., Proposition 2.3.12.**In T_δ-space,gδ-closed sets coincide with δ-closed sets and every δ-closed set is a δg-closed set by Theorem 3.1(i) of (Dontchev,1996). Hence **(d)** ⇒ **(a)**.

Theorem 2.4.73. If every subset of a T_b-space (resp.a αT_b- space) of (Y,ζ) is J*-closed, then (Y,ζ) is a partition space.

Proof Let D be an open set.Since every subset is J*-closed,D is J*-closed.By **Proposition 2.4.66.(**resp. **Proposition 2.4.67.)**,D is closed. Hence (Y,ζ) is a partition space.

Theorem 2.4.74. For a compact subset D of a R_1 -topological space (Y,ζ), every J*-closed set is a $\delta g^{\ddagger}$-closed set. The converse is true if (Y,ζ) is semi-regular.

Proof Let D be J*-closed and D be a compact subset of R_1 -topological space (Y,ζ). Let $D \subseteq M$, where M is δ-open. By **Theorem 2.2.3.**, every δ-open set is η*-open ---- (1). In R_1-spaces the concepts of closure and δ-closure coincide for compact sets (By Theorem 3.6 of (Jankovic,1980)).Hence for a compact subset D,δCl(D) =Cl(D) ----- (2).Generally δ-closed ⟶ η*- closed ⟶ closed.Therefore $Cl(D) \subseteq \eta^*Cl(D) \subseteq \delta Cl(D)$ -----

(A).Hence (2) implies Cl(D) = η*Cl(D)= δCl(D) ---- (3). From (1) and (3) and since D is J*-closed, we get D is a δ$g^{\ddagger}$-closed set.

Conversely let (Y,ζ) be semi-regular and D be δ$g^{\ddagger}$-closed.Let D⊆ M, where M is open. By **Lemma 2.2.26.**, η*-open sets coincide with δ-open sets and open sets in semi-regular spaces. From (A), η*Cl(D) ⊆ δCl(D) ⊆ M,then D becomes J*-closed.

Corollary 2.4.75. In Hausdroff spaces, a finite set is J*-closed if and only if it is δ$g^{\ddagger}$-closed.

§ 2.5. J**-Closed Sets in Topological Spaces

In this section a new class of generalized closed sets, called J-closed sets are introduced. The relations between J**-closed sets and various existing closed sets are analysed.**

Definition 2.5.1. A subset D of a topological space (Y,ζ) is said to be **J**-closed set** if η*-Cl(D)⊆M whenever D⊆M, M is η*-open in (Y,ζ). The class of all J**-closed sets of (Y,ζ) is denoted by J**C(Y,ζ).

Proposition 2.5.2. Every η*-closed set is J-closed but not conversely.**

Proof Let D be a η*-closed set and M be any η*-open set containing D. Since D is η*-closed, η*-Cl(D) =D (by **Remark 2.2.30.(ii)**). Therefore η*-Cl(D) =D⊆M. Hence D is J**-closed.

Counter Example 2.5.3. Let Y = {p,q,r}, ζ = {ϕ,Y, {p},{p,q}},η*O(Y,ζ) = {ϕ,Y,{p}}, The subset {q} is J**-closed but not η*-closed.

Proposition 2.5.4. Every δ-closed set is J-closed but not conversely.**

Proof Let D be a δ- closed set and M be any η*-open set containing D. Since D is δ-closed, δCl(D)=D. Therefore δCl(D)=D⊆M.By **Theorem 2.2.3.**, δ-open ⟶ η*-open, we get η*-Cl(D)⊆δCl(D) and hence D is J**-closed.

Counter Example 2.5.5. In the above **Counter Example 2.5.3.**, the subset {r} is J**-closed but not δ-closed.

Proposition 2.5.6. Every δg*-closed set is J-closed but not conversely.**

Proof The proof follows from the facts that (i) η*-open set $\longrightarrow$ g-open set [By **Note 2.2.9.].** (ii) η*-Cl(D)$\subseteq$δCl(D) [By **Note 2.2.21.**] and from the definitions of δg*-closed sets and J**-closed sets.

Counter Example 2.5.7. Let Y={p,q,r},ζ={ϕ, Y, {p}, {p,q}, {p,r}}. Then the subset {q} is J**-closed but not δg*-closed in (Y,ζ).

Proposition 2.5.8. Every δg-closed set is J-closed but not conversely.**

Proof The proof follows from the facts that (i) η*-open set $\longrightarrow$ open set [By **Theorem 2.2.5.].** (ii) η*-Cl(D)$\subseteq$δCl(D) [By **Note 2.2.21.**] and from the definitions of δg-closed sets and J**-closed sets.

Counter Example 2.5.9. In the above **Counter Example 2.5.7.,**the subset {q} is J**-closed but not δg-closed in (Y,ζ).

Proposition 2.5.10. Every J*-closed set is J-closed but not conversely.**

Proof Let D be a J*-closed and M be any η*-open set containing D in Y. By **Note 2.2.9.**, every η*-open set is an open set and since D is J*-closed, η*-Cl(D)$\subseteq$M which implies D is J**-closed.

Counter Example 2.5.11. Let Y = {p,q,r,s},ζ = {ϕ,Y,{p}, {q},{p,q},{p,q,r}, {p,q,s}}, η*O(Y,ζ) ={ϕ,Y,{p},{q},{p,q}}. The subset {q,r} is J**-closed but not J*-closed.

Proposition 2.5.12. Every J-closed set is J-closed but not conversely.**

Proof Let D be a J**-closed set. To prove D is J-closed. Let M be a η*-open set containing D.Since D is J**-closed, η*-Cl(D)$\subseteq$M. By **Note 2.2.21.**,Cl(D)$\subseteq$ η*-Cl(D).Therefore D is J-closed.

Counter Example 2.5.13. Let Y = {p,q,r,s},ζ = {ϕ,Y,{p}, {r},{p,q},{p,r},{p,q,r}, {p,r,s}} ,η*O(Y,ζ) = {ϕ,Y,{p,q}}, The subset {q} is J-closed but not J**-closed.

Remark 2.5.14.

$$\begin{tikzcd}
 & & & \eta^*\text{-closed} \arrow[r] \arrow[d] & \text{closed} \arrow[d] \arrow[dr] & \\
\delta\text{-closed} \arrow[r] \arrow[urrr] & \delta g^*\text{-closed} \arrow[r] & \delta g\text{-closed} \arrow[r] \arrow[dr] & J^{**}\text{-closed} \arrow[r] & J\text{-closed} \arrow[r] & g\delta\text{-closed} \\
 & & & J^*\text{-closed} \arrow[u] \arrow[r] & g\text{-closed} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Proposition 2.5.15. Every J-closed set is gδ-closed but not conversely.**

Proof The proof follows from **Proposition 2.5.12.** and **Proposition 2.3.12.**

Counter Example 2.5.16. Let Y = {p,q,r,s},ζ = {ϕ,Y,{p}},η*O(Y,ζ) = ζ. The subset {p} is gδ-closed but not J**-closed.

Proposition 2.5.17. Every J-closed set is gpr-closed but not conversely.**

Proof The proof follows from the facts that (i) regular open set ⟶ η*-open set [By **Note 2.2.9.].** (ii) pCl(D)⊆ η*-Cl(D) [By **Note 2.2.21.**] and from the definitions of J**-closed sets and gpr-closed sets.

Counter Example 2.5.18. Let Y={p,q,r},ζ ={ϕ, Y,{p},{q},{p,q}}.Then the subset {p,q} is gpr-closed but not J**-closed in (Y, ζ).

Proposition 2.5.19. Every J-closed set is rwg-closed but not conversely.**

Proof By **Proposition 2.5.12.** and **Proposition 2.3.25.,**we get the proof.

Counter Example 2.5.20. In the above **Counter Example 2.5.18.**,the subset {p,q} is rwg -closed but not J**-closed in (Y, ζ).

Proposition 2.5.21. Every J-closed set is gspr-closed but not conversely.**

Proof The proof follows from the facts that (i) regular open set ⟶ η*-open set [By **Note 2.2.9.].** (ii) spCl(D)⊆ η*-Cl(D) [By **Note 2.2.21.**] and from the definitions of J**-closed sets and gspr-closed sets.

Counter Example 2.5.22. In the above **Counter Example 2.5.18.**, the subset {q} is gspr-closed but not J**-closed in (Y, ζ).

Proposition 2.5.23. Every J-closed set is π g-closed but not conversely.**

Proof The proof follows from the facts that (i) π-open set ⟶ η*-open set [By **Note 2.2.9.].** (ii) Cl(D) ⊆ η*-Cl(D) [By **Note 2.2.21.**] and from the definitions of J**-closed sets and πg-closed sets.

Counter Example 2.5.24. In the **Counter Example 2.5.3.**, the subset {p} is πg-closed but not J**-closed in(Y,ζ).

Proposition 2.5.25. Every J-closed set is πgp-closed but not conversely.**

Proof The proof follows from the facts that (i) π-open set ⟶ η*-open set [By **Note 2.2.9.].** (ii) pCl(D) ⊆ η*-Cl(D) [By **Note 2.2.21.**] and from the definitions of J**-closed sets and πgp-closed sets.

Counter Example 2.5.26. In the above **Counter Example 2.5.3.,**the subset {p} is πgp-closed but not J**-closed in (Y,ζ).

Proposition 2.5.27. Every J-closed set is πgsp-closed but not conversely.**

Proof The proof follows from the facts that (i) π-open set ⟶ η*-open set [By **Note 2.2.9.].** (ii) spCl(D) ⊆ η*-Cl(D) [By **Note 2.2.21.**] and from the definitions of J**-closed sets and πgsp-closed sets.

Counter Example 2.5.28. In the **Counter Example 2.5.18.,**the subset {p} is πgsp-closed but not J**-closed in (Y,ζ).

Proposition 2.5.29. Every J-closed set is πgs-closed but not conversely.**

Proof The proof follows from the facts that (i) π-open set ⟶ η*-open set [By **Note 2.2.9.].** (ii) sCl(D) ⊆ η*-Cl(D) [By **Note 2.2.21.**] and from the definitions of J**-closed sets and πgs-closed sets.

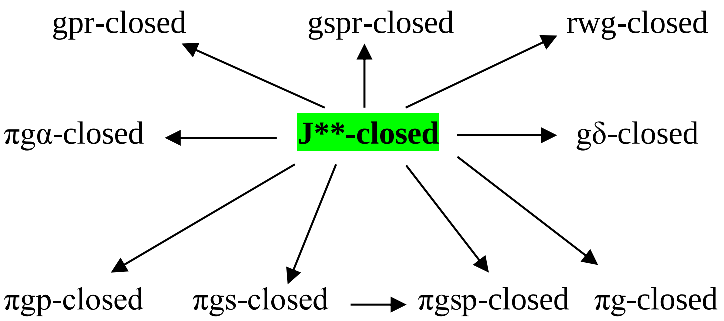
Counter Example 2.5.30. Consider the above **Counter Example 2.5.18.,** the subset {p} is πgs-closed but not J**-closed in (Y,ζ).

Proposition 2.5.31. Every J-closed set is πgα-closed but not conversely.**

Proof The proof follows from the facts that (i) π-open set $\longrightarrow$ η^*-open set [By **Note 2.2.9.]**. (ii) $\alpha Cl(D) \subseteq \eta^*\text{-}Cl(D)$ [By **Note 2.2.21.**] and from the definitions of J**-closed sets and $\pi g\alpha$-closed sets.

Counter Example 2.5.32. In the above **Counter Example 2.5.3.,**the subset {p} is $\pi g\alpha$-closed but not J**-closed in (Y,ζ).

Remark 2.5.33. From the above discussions, we get J-closed set is related with other existing g-closed sets in the following manner.**



Remark 2.5.34. The following Counter examples show that J-closed set is independent from gs-closed, #gs-closed, g*s-closed and $\delta g^{\ddagger}$-closed sets.**

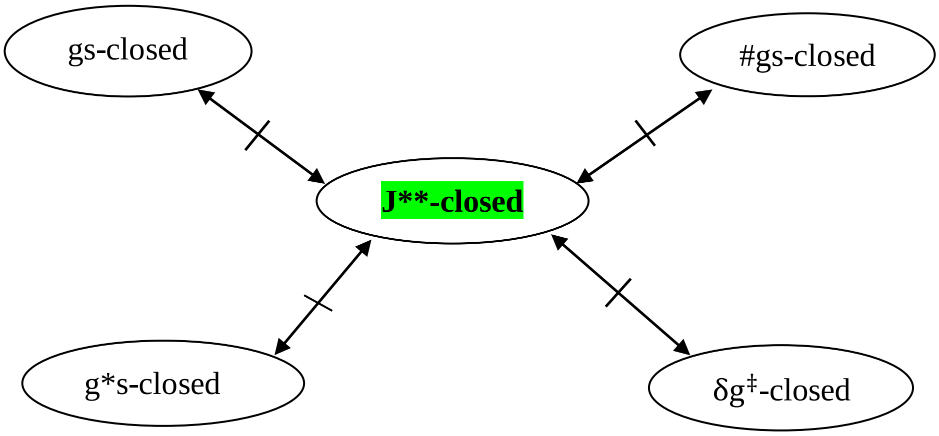
Counter Example 2.5.35. In the above **Counter Example 2.5.18.**, the subset {p} is gs-closed, #gs-closed, g*s-closed but not J**-closed.

Counter Example 2.5.36. Let $Y=\{p,q,r\}, \zeta=\{\phi, Y, \{p,q\}\}$ The subset {p} is J**-closed but not gs-closed, #gs-closed, g*s-closed.

Counter Example 2.5.37. Let $Y=\{p,q,r,s\}, \zeta=\{\phi, Y, \{p\}\}$. The subset {p} is not J**-closed but it is $\delta g^{\ddagger}$-closed.

Counter Example 2.5.38. Let $Y=\{p,q,r,s\}, \zeta=\{\phi, Y, \{p\},\{r\},\{p,q\},\{p,r\},\{p,q,r\},\{p,r,s\}\}$ The subset {r} is J**-closed but not $\delta g^{\ddagger}$-closed.

Remark 2.5.39. From the above discussions, we get J-closed sets is isolated from the following g-closed sets.**



Properties of J**-Closed Sets in Topological Spaces

Theorem 2.5.40. The finite union of J-closed sets is J**-closed.**

Proof Let $\{D_i\}_{i=1}^{n}$ be a finite class of J**-closed sets of (Y,ζ). Let $D = \cup_{i=1}^{n} D_i$. Let M be aη*-open set containing D. This implies $D_i \subseteq$ M for every i. By assumption η*-Cl(D_i) ⊆ M for every i.This implies $\cup_{i=1}^{n} \eta^* - \mathrm{Cl}(D_i) \subseteq$ M.Then η*-Cl($\cup_{i=1}^{n} D_i$)⊆M (By **Properties 2.2.29.f.**).Thus η*-Cl(D)⊆M. Hence finite union of J**-closed sets is J**-closed in (Y,ζ).

Result 2.5.41. The following Counter Example shows that finite intersection of any two J-closed sets in Y need not be J**-closed.**

Counter Example 2.5.42. Let Y = {p,q,r }, ζ = {Y, ϕ,{p}}. Then the sets D = {p,q} and E ={p,r} are J**-closed but the intersection D∩E = {p} is not J**- closed in (Y,ζ).

Theorem 2.5.43. The intersection of a J-closed set and a δ-closed set is always J**-closed.**

Proof Let D be J**-closed and F be δ-closed. Let V = D ∩ F.Take M to be η*-open such that V ⊆ M. Then D ∩ F ⊆ M which implies D ⊆M∪ F^c. Since F^c is δ-open by **Note 2.2.9.**,F^c is η*-open.Hence M∪ F^c is η*-open (By **Theorem 2.2.10.(b)**) and by assumption D ⊆M∪ F^c since D is J**-closed, η*-Cl(D)⊆ M∪ F^c and hence η*-Cl(D) ∩ F⊆M. Now η*-Cl(V) = η*-Cl(D∩ F) = η*-Cl(D) ∩ η*-Cl(F)⊆ η*-Cl(D) ∩ δ-Cl(F) =

η^*-Cl(D) ∩ F $\subseteq$ M (By **Properties 2.2.29.g** and **Theorem 2.2.19.**). Thus η^*-Cl(V) $\subseteq$ M. Hence D ∩ F is J**-closed.

Remark 2.5.44. The following Counter Example shows that the difference of any two J-closed sets in Y need not be J**-closed.**

Counter Example 2.5.45. Let Y = {p,q,r }, ζ = {Y, ϕ,{p}}. Consider D and E are subsets of Y.Then the subset D={p,q} and E ={q,r} are J**-closed but the difference D – E = {p} is not J**- closed in (Y,ζ).

Theorem 2.5.46. Let D be a J-closed set of (Y,ζ). Then η^*-Cl(D) – D does not contain a non- empty η^*-closed set.**

Proof Suppose that D is J**-closed, let M be a η^*-closed set contained in η^*-Cl(D) – D. Now M^c is a η^*-open set in Y such that D $\subseteq M^c$. Since D is J**-closed, η^*-Cl(D) $\subseteq M^c$.Thus M $\subseteq (\eta^*\text{-Cl(D)})^c$.Also M $\subseteq$ η^*-Cl(D)–D. Therefore M $\subseteq (\eta^*\text{-Cl(D)})^c$ ∩ η^*-Cl(D) which is ϕ.Hence M = ϕ.

Proposition 2.5.47. If D is a η^*-open set and a J-closed set of (Y,ζ), then D is a η^*-closed set of Y.**

Proof Since D is η^*-open and J**-closed, η^*-Cl(D) $\subseteq$ D. Obviously, D $\subseteq$ η^*-Cl(D). Hence D is η^*-closed in (Y,ζ).

Theorem 2.5.48. If D is J-closed and η^*-open and F is η^*-closed in (Y,ζ), then D ∩ F is η^*-closed.**

Proof Since D is J**-closed and η^*-open, D is η^*-closed by **Proposition 2.5.47.**,Since F is η^*-closed in Y, D ∩ F is η^*-closed in Y (by **Properties 2.2.29.g.**).

Proposition 2.5.49. If D is a J-closed set in a space (Y,ζ) and D $\subseteq$ B $\subseteq$ η^*-Cl(D) then B is also a J**-closed set.**

Proof Let M be η^*-open set of Y such that B $\subseteq$ M. Then D $\subseteq$ M. Since D is J**-closed set, η^*-Cl(D) $\subseteq$ M. Since B $\subseteq$ η^*-Cl(D), η^*-Cl(B) $\subseteq$ η^*-Cl(η^*-Cl(D)) = η^*-Cl(D) (By **Remark 2.2.30.(i)**). Hence η^*-Cl(B) $\subseteq$ M. Therefore B is also a J**-closed set .

Theorem 2.5.50. Let D be a J-closed set of (Y,ζ).Then D is η*-closed iff η*-Cl(D) – D is η*-closed.**

Proof (Necessity) Let D be a η*-closed subset of (Y,ζ).Then η*-Cl(D) = D and therefore η*-Cl(D) – D=ϕ which is η*-closed.

(**Suffficiency)** Let η*-Cl(D) −D be η*-closed set. Since D is J**-closed, by **Theorem 2.5.46.**,η*-Cl(D)−D does not contain a non-empty η*-closed set which implies η*-Cl(D)−D=ϕ. That is η*-Cl(D)=D. Hence D is η*-closed.

Definition 2.5.51. Let $B \subseteq A \subseteq Y$.Then B is **J**-closed relative to A** if η*-$Cl_A(B) \subseteq M$, whenever $B \subseteq M$, M is η*-open in A as in **Definition 2.2.34.**

Theorem 2.5.52. Let $B \subseteq A \subseteq Y$ and suppose that B is J-closed in Y, then B is J**-closed relative to A. The converse is true if A is η*-closed in Y.**

Proof Suppose that B is J**-closed in Y. Let $B \subseteq M$, M is η*-open in A. Since M is η*-open in A,M = V ∩ A for some η*-open set V in Y. Hence $B \subseteq M \subseteq V$. Since B is J**-closed in Y,η*-Cl(B)$\subseteq$V. Hence η*-Cl(B) ∩ A$\subseteq$V∩A which in turn and hence η*-$Cl_A(B) \subseteq$ V ∩ A = M. Therefore B is J**-closed relative to A.

Now to prove the converse, assume that $B \subseteq A \subseteq Y$ where A is η*-closed in Y and B is J**-closed relative to A. Let $B \subseteq M$, M is η*-open in Y. Then A ∩ M is η*-open in A by the definition of subspace topology. Since $B \subseteq A$ and $B \subseteq M$, $B \subseteq A \cap M$. Since B is J**-closed relative to A, η*-$Cl_A(B) \subseteq A \cap M$. Since $B \subseteq A$, η*-Cl(B)$\subseteq$ η*-Cl(A) (By **Properties 2.2.29.d**). Hence η*-Cl(B)$\subseteq$A,since A is η*-closed.Therefore η*-Cl(B) ∩ A = η*-Cl(B) which implies η*-$Cl_A(B)$ = η*-Cl(B). Hence η*-Cl(B)$\subseteq A \cap M \subseteq M$. Thus B is J**-closed in Y.

Characterization Theorems of J**-Closed Sets

Proposition 2.5.53. In a $T_{1/2}$ -space for a subset D the following are equivalent:

- **(i) D is J**-closed**
- **(ii) D is $\delta g^{\ddagger}$ -closed.**

Proof (i) ⇒(ii) Let D be J**-closed, D⊆ M where M is δ-open. In a $T_{1/2}$-space, η*-open sets coincide with δ-open sets (By **Lemma 2.2.23.**).Since D is J**-closed, η*Cl(D)⊆ M,then by **Lemma 2.2.23.,**δCl(D)⊆ M ⇒ D is $\delta g^{\ddagger}$ -closed.

(ii) ⇒(i) Let D be $\delta g^{\ddagger}$ -closed, D⊆ M where M is η*-open. In a $T_{1/2}$-space, η*-open sets coincide with δ-open sets (By **Lemma 2.2.23.**).Since D is $\delta g^{\ddagger}$-closed, δCl(D)⊆ M,then η*Cl(D)⊆δCl(D)⊆ M ⇒ D is J**-closed.

Proposition 2.5.54. If a space is a semi-regular space, then J-closed sets,J*-closed sets, J-closed sets ,g-closed sets, δg-closed sets, gδ-closed sets and $\delta g^{\ddagger}$ -closed sets coincide.**

Proof In a semi-regular space, η*-closed sets coincide with closed sets by **Lemma 2.2.26.** By **Theorem 2.4.63.**,in a semi-regular space,J*-closed sets and J-closed sets coincide with g-closed sets,δg-closed sets, gδ-closed sets and $\delta g^{\ddagger}$ -closed sets.Hence the Proof.

Example 2.5.55. Let Y = {p,q,r}, ζ = {Y, ϕ,{p},{q,r}}.Here Y is a semi-regular space,where JC(Y,ζ) =J*C(Y,ζ) = J**C(Y,ζ) = gC(Y,ζ) = δgC(Y,ζ) =$\delta g^{\ddagger}$C(Y,ζ) = gδC(Y,ζ) =P(Y).

Note 2.5.56. The converse of above Proposition 2.5.54. is not correct.It follows from the Counter Example.

Counter Example 2.5.57. Let Y = {p,q,r}, ζ = {Y, ϕ,{p},{p,q}}.Here JC(Y,ζ) =P(Y) −{p} = J**C(Y,ζ) , J*C(Y,ζ) = gC(Y,ζ) = δg*C(Y,ζ) = δgC(Y,ζ) ={Y, ϕ,{r},{p,r},{q,r}} and $\delta g^{\ddagger}$C(Y,ζ) = gδC(Y,ζ) =P(Y) ,but Y is not a semi-regular space.

Theorem 2.5.58. Let D be a subset of a semi-regular space (Y,ζ) then D is J-closed if and only if D is gδ-closed.**

Proof By **Proposition 2.5.15.**,every J**-closed set is gδ-closed. In a semi-regular space (Y,ζ),every gδ-closed set is J**-closed.

Note 2.5.59. If (Y,ζ) is a $T_{1/2}$ -space, then the above Theorem need not be true.But in the case of J-closed sets the result is true for $T_{1/2}$ -spaces by Theorem 2.3.62.

Counter Example 2.5.60. Let Y = {p,q,r,s}, ζ = {Y, ϕ,{p},{r},{p,q}, {p,r},{p,q,r}, {p,r,s}}. HereJ**C(Y,ζ) =P(Y) −{{p},{q},{p,q}} ≠ gδC(Y, ζ) =P(Y)−{{p},{r},{p,q}, {q,r},{p,r},{p,q,r}},but Y is a $T_{1/2}$ -space.

Theorem 2.5.61. If a topological space (Y,ζ) is a T_δ-space, then the following conditions are equivalent:

- **(a) D is δg-closed**
- **(b) D is J*-closed**
- **(c) D is J**-closed**
- **(d) D is J-closed**
- **(e) D is gδ-closed.**

Proof (a) ⇒ (b), (b) ⇒ (c) , (c) ⇒ (d) and **(d) ⇒(e)** by **Propositon 2.4.10., Propositon 2.5.10., Propositon 2.5.12. and Proposition 2.3.12.**In T_δ-space, gδ-closed is δ-closed and δ-closed set is a δg-closed set by Theorem 3.1(i)(Dontchev,1996).Hence **(e) ⇒ (a)**.

Remark 2.5.62. In general J-closed sets are independent with g-closed sets. But the following Theorem explains in a semi-regular space, they are coincident.**

Theorem 2.5.63. Let D be a subset of a semi-regular space (Y,ζ) then D is J-closed if and only if D is g-closed.**

Proof By Lemma 2.2.26., η*-closed sets coincide with δ-closed sets and closed sets. We reached the proof.

Theorem 2.5.64. In an almost weakly Hausdorff space (Y,ζ),the g-closed sets of (Y,ζ_S) are δ-closed sets in (Y,ζ) and thus δg*-closed sets, δg-closed sets, J-closed sets, J*-closed sets respectively in (Y,ζ).**

Proof Let D ⊆ Y be a g-closed subset of (Y,ζ_S). Let $x \in$δCl(D).If $\{x\}$ is δ-open, then $x \in$D.If not, then Y \\{ x } is δ-open, since Y is almost weakly Hausdorff. Assume that $x \notin D$.Since D is g-closed in (Y,ζ_S) then δCl(D) ⊆Y \\{x},that is $x \notin$δCl(D). By contradicition $x \in$D.Then Cl(D) = D in (Y,ζ). Hence D is δ-closed. Since D is δ-closed and hence by Proposition 2.2.2.(Sudha,2014),Theorem 3.1.(i) (Dontchev,1996),

Proposition 2.5.4.and **Proposition 2.4.8.**, D is δg*-closed, δg-closed, J**-closed and J*-closed respectively in (Y,ζ).

Remark 2.5.65. In general the concepts of J-closed sets and δg‡-closed sets are not equivalent. But for a compact subset D of a space which is $T_{1/2}$ and R_1, the concepts of J**-closed sets and δg‡-closed sets coincide. This can be seen in the following Theorem.**

Theorem 2.5.66. For a compact subset D of a R_1 -topological space (Y,ζ),every J-closed set is a δg‡-closed set. The converse is true if (Y,ζ) is $T_{1/2}$ (resp.semi-regular).**

Proof Let D be J**-closed and D be a compact subset of R_1 -topological space (Y,ζ). Let D ⊆ M, where M is δ-open. By **Theorem 2.2.3.**, every δ-open set is η*-open -----(1). In R_1-spaces the concepts of closure and δ-closure coincide for compact sets (By Theorem 3.6 of (Jankovic,1980)).Hence for a compact subset D,δCl(D) =Cl(D) -----(2).Generally, δ-closed ⟶ η*- closed ⟶ closed.Therefore Cl(D) ⊆ η*Cl(D) ⊆ δCl(D) -----(A).Hence (2) implies Cl(D) = η*Cl(D)= δCl(D) ----(3). From (1) and (3) and since D is J**-closed, we get D is a δg‡-closed set.

Conversely let (Y,ζ) be $T_{1/2}$ and D be δg‡-closed.Let D⊆ M, where M is η*-open. By **Lemma 2.2.23..(resp.Lemma 2.2.26.)**, η*-open sets coincide with δ-open sets in $T_{1/2}$-spaces(resp.semi-regular spaces).From (A), η*Cl(D) ⊆ δCl(D) ⊆M. Then D becomes J**-closed.

Corollary 2.5.67. In Hausdroff spaces, a finite set is J-closed if and only if it is δg‡-closed.**

Conclusion

1. In a $T_{1/2}$-space, η*-closed sets coincide with δ-closed sets. Moreover in this space, the family of J-closed sets is equivalent with that of g*δ*-closed sets and the family of J*-closed sets coincide with the family of *δ*g-closed sets and the family of J**-closed sets coincide with the family of δg‡-closed sets.

2. In a semi-regular space, δ-closed sets, η^*-closed sets and closed sets coincide. So δ-closure, η^*-closure, closure of any subset coincide. Therefore from definitions J**-closed sets, J*-closed sets,J-closed sets,g-closed sets, δg-closed sets, $g\delta$-closed sets and $\delta g^{\ddagger}$-closed sets coincide.
3. (a) In a $T_{1/2}$-space and a semi-regular space, J-closed sets are closed.
   (b) In T_b (resp. αT_b) -spaces,J*-closed sets are closed.
4. Regarding partition spaces we have obtained following results.
   - (a) Every subset of a partition space is J-closed. The result fails for J*-closed sets and J**-closed sets.
   - (b) If every subset of a space Y which is both $T_{1/2}$ and semi-regular is J-closed, then Y is also a partition space.
   - (c) In a T_b-space (resp. αT_b-space) Y if every subset is J*-closed, then Y is a partition space.
   - (d) Let D be a subset of the partition space (Y,ζ). Then the following conditions are equivalent: (i) D is δg-closed (ii) D is J*-closed (iii) D is g-closed.
5. (a) If a compact subset D of a R_1-topological space (Y,ζ) is J-closed(resp.J*-closed, J**-closed), then D is a $\delta g^{\ddagger}$-closed set.
   (b) The converse is true only when
      - (i) Y is $T_{1/2}$ in the case of J-closed sets.
      - (ii) Y is semi-regular in the case of J*-closed sets.
      - (iii) Y is $T_{1/2}$ (resp.semi-regular) in the case of J**-closed sets.
6. In Hausdroff spaces, a finite set D is one of these three types namely J-closed set, J*-closed set or J**-closed set,then D is $\delta g^{\ddagger}$-closed sets.
7. In a T_δ-space, δg-closed, J**-closed, J*-closed, g-closed, J-closed,$g\delta$-closed sets are equivalent.
8. In an almost weakly Hausdorff space (Y,ζ),the g-closed sets of (Y,ζ_S) are δ-closed sets in (Y,ζ) and thus δg^*-closed sets, δg-closed sets,J**-closed sets,J*-closed sets respectively.
9. If D is a pre-open subset of a topological space (Y,ζ).Then the following conditions are equivalent:
   (a) D is δg-closed (b) D is J*-closed (c) D is g-closed.