CHAPTER 3

Chapter 3

Generalized π-compactness and generalized π-connectedness

3.1 Introduction

The notions of compactness and connectedness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Many researchers have investigated the basic properties of compactness and connectedness. The productivity and fruitfulness of these notions of compactness and connectedness motivated mathematicians to generalize these notions. In the course of these attempts many stronger and weaker forms of compactness and connectedness have been introduced and investigated. The notion homeomorphism plays a very important role in topology. By definition, a homeomorphism between two topological spaces X and Y is a bijective map $f : X \rightarrow Y$ where both f and f^{-1} are continuous. It is well known that as Janich (1980) says homeomorphisms play the same role in topology as linear isomorphisms play in linear algebra, or that biholomorphic maps play in function theory, or group isomorphisms in group theory, or isometries in Riemannian geometry. In the course of generalizations of the notion of homeomorphism, Maki et al. (1991) introduced g-homeomorphisms and gc-homeomorphisms in topological spaces. Devi et al. (1995) studied semi-generalized homeomorphisms and generalized semi homeomorphisms.

In this chapter, we have introduced the concepts of gπ-closed map, gπ-open map, gπ*-closed map, gπ*-open map, gπ-homeomorphism, gπ*-homeomorphism, gπ-compact spaces, gπ-connected spaces, gπ-US spaces, Sequentially gπ-closed set, and Sequentially GπO-compact set and highlighted their properties.

3.2. Generalized π -Closed Maps

Defnition 3.2.1. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called gπ-closed if the image of every closed set in (X, τ) is gπ-closed in (Y, σ).

Example 3.2.2 Let X = Y = {a, b, c}, τ = {X, φ, {a}} and σ = {Y, φ, {b} }. Define a map $f : (X, \tau) \rightarrow (Y, \sigma)$ by f(a) = a, f(b) = b and f(c) = c . Then f is a gπ-closed map.

Theorem 3.2.3. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is gπ-closed if and only if for each subset S of (Y, σ) and for each open set U containing $f^{-1}(S)$ there exists a gπ-open set V of (Y, σ) such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof: Suppose that f is a gπ-closed map. Let $S \subset Y$ and U be an open subset of (X, τ) such that $f^{-1}(S) \subset U$. Then $V = (f(U^c))^c$ is a gπ-open set containing S such that $f^{-1}(V) \subset U$.

Conversely, let S be a closed set of (X, τ). Then $f^{-1}((f(S))^c) \subset S^c$ and S^c is open. By assumption, there exists a gπ-open set V of (Y, σ) such that $(f(S))^c \subset V$ and $f^{-1}(V) \subset S^c$ and so $S \subset (f^{-1}(V))^c$. Hence $V^c \subset f(S) \subset f((f^{-1}(V))^c) \subset V^c$ which implies $f(S) = V^c$. Since V^c is gπ-closed, f(S) is gπ-closed and therefore f is gπ-closed.

The following example shows that the composition of two gπ-closed maps need not be gπ-closed.

Example 3.2.4. Let X = {a, b, c, d},

τ = {X, φ, {b}, {c}, {a, b}, {b, c}, {a, b, c}, {a, b, d}}, σ = {X, φ, {b}}, and μ = {X, φ, {b}, {c}, {b, c}}. Define $f : (X, \tau) \rightarrow (X, \sigma)$ by f(a) = a, f(b) = b, f(c) = c, f(d) = d and $g : (X, \sigma) \rightarrow (X, \mu)$ by g(a) = a, g(b) = b, g(c) = c, g(d) = d. Then both f and g are gπ-closed maps but their composition

$g \circ f: (X, \tau) \to (X, \mu)$ is not a gπ-closed map. Since for the closed set {c} in (X, τ), $(g \circ f)(\{c\}) = \{c\}$, is not a gπ-closed set in (X, μ).

Theorem 3.2.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a closed map and $g : (Y, \sigma) \to (Z, \eta)$ be a gπ-closed map, then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is gπ-closed.

Proof: Obvious.

Remark 3.2.6 If $f : (X, \tau) \to (Y, \sigma)$ is gπ-closed and $g : (Y, \sigma) \to (Z, \eta)$ is closed, then their composition need not be a gπ-closed map as seen from the following example.

Example 3.2.7 Let X = {a, b, c, d},

$\tau = \{X, \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$, $\sigma = \{X, \varphi, \{b\}\}$, and $\mu = \{X, \varphi, \{b\}, \{c\}, \{b, c\}\}$. Define $f : (X, \tau) \to (X, \sigma)$ by f(a) = a, f(b) = b, f(c) = c, f(d) = d and $g : (X, \sigma) \to (X, \mu)$ by g(a) = a, g(b) = b, g(c) = c, g(d) = d. Then f is a gπ-closed map and g is a closed map but their composition $g \circ f: (X, \tau) \to (X, \mu)$ is not a gπ-closed map. Since for the closed set {c} in (X, τ), $(g \circ f)(\{c\}) = \{c\}$, is not a gπ-closed set in (X, μ).

Theorem 3.2.8 Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two mappings such that their composition $g \circ f : (X, \tau) \to (Z, \eta)$ be a gπ-closed mapping. Then the following statements are true.

(i) If f is continuous and surjective, then g is gπ-closed.

(ii) If g is gπ-irresolute and injective, then f is gπ-closed.

Proof: (i). Let A be a closed set of (Y, σ). Since f is continuous, $f^{-1}(A)$ is closed in (X, τ) and since $g \circ f$ is gπ-closed, $(g \circ f)(f^{-1}(A))$ is gπ-closed in (Z, η). i.e., g(A) is gπ-closed in (Z, η), since f is surjective. Therefore, g is a gπ-closed map.

(ii). Let B be a closed set of (X, τ). Since $g \circ f$ is gπ-closed, $(g \circ f)(B)$ is gπ-closed in (Z, η). Since g is gπ-irresolute, $g^{-1}((g \circ f)(B))$ is gπ-closed in (Y, σ) . i.e., f(B) is gπ-closed in (Y, σ) , since g is injective. Thus, f is a gπ-closed map.

Theorem 3.2.9 If a map $f : (X, \tau) \rightarrow (Y, \sigma)$ is gπ-closed and A is a closed subset of (X, τ), then $f_A : (A, \tau_A) \rightarrow (Y, \sigma)$ is gπ-closed, where f_A is restriction map of f on A.

Proof: Let B be a closed set of A. Then $B = A \cap F$ for some closed set F of (X, τ) and so B is closed in (X, τ). Since f is gπ-closed, f(B) is gπ-closed in (Y, σ). But $f(B) = f_A(B)$ and therefore f_A is a gπ-closed map.

Definition 3.2.10 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to a gπ-open map if the image f(A) is gπ-open in (Y, σ) for each open set A in (X, τ).

Example 3.2.10 (a) Let X = {a, b, c, d},

τ = {X, φ, {b}, {c}, {a, b}, {b, c}, {a, b, c}, {a, b, d}}, σ = {X, φ, {b}}, Define $f : (X, \tau) \rightarrow (X, \sigma)$ by f(a) = a, f(b) = b, f(c) = c, f(d) =d. Then f is a gπ-open map.

Theorem 3.2.11 For any bijection $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:

(i) $f^{-1} : (Y, \sigma) \rightarrow (X, \tau)$ is gπ-continuous,

(ii) f is a gπ-open map and

(iii) f is a gπ-closed map.

Proof: (i) ⇒ (ii): Let U be an open set of (X, τ) . By assumption $(f^{-1})^{-1}(U) = f(U)$ is gπ-open in (Y, σ) and so f is gπ-open.

(ii) $\Rightarrow$ (iii): Let F be a closed set of (X, τ). Then F^c is open in (X, τ). By assumption, $f(F^c)$ is gπ-open in (Y,σ). i.e., $f(F^c) = (f(F))^c$ is gπ-open in (Y,σ) and therefore f(F) is gπ-closed in (Y,σ). Hence f is gπ-closed.

(iii) $\Rightarrow$ (i): Let F be a closed set in (X, τ) By assumption f(F) is gπ-closed in (Y,σ). But $f(F) = (f^{-1})^{-1}(F)$ and therefore f^{-1} is gπ-continuous on Y.

Definition 3.2.12 Let x be a point of (X, τ) and V be a subset of X. Then V is called a gπ-neighbourhood of x in (X, τ) if there exists a gπ-open set U of (X, τ) such that $x \in U \subset V$.

Theorem 3.2.13 Let f : (X, τ) $\rightarrow$ (Y, σ) be a mapping. Then the following statements are equivalent:

(i) f is a gπ-open mapping.

(ii) For a subset A of (X, τ), f(int(A) $\subset$ gπ-Int(f(A))).

(iii) For each $x \in$ X and for each neighborhood U of x in (X,τ), there exists a gπ-neighbourhood W of f(x) in (Y, σ) such that W $\subset$ f(U).

Proof: (i) $\Rightarrow$ (ii): Suppose f is gπ-open. Let A $\subset$ X. Since int(A) is open in (X,τ), f(int(A)) is gπ-open in (Y, σ). Hence f(int(A)) $\subset$ f(A) and, f(int(A)) $\subset$ gπ-Int(f(A)).

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. Let $x \in X$ and U be an arbitrary neighbourhood of x in (X,τ). Then there exists an open set G such that $x \in G \subset U$. By assumption, f(G) = f(int(G)) $\subset$ gπ-Int(f(G)). This implies f(G) = gπ-Int(f(G)). Therefore, f(G) is gπ-open in (Y,σ). Further, f(x) $\in$ f(G) $\subset$ f(U) and so (iii) holds, by taking W = f(G).

(iii) $\Rightarrow$ (i): Suppose (iii) holds. Let U be any open set in (X, τ), x $\in$ U and f(x)= y. Then for each x $\in$ U, y $\in$ f(U). By assumption there exists a gπ-neighbourhood W_y of y in (Y,σ) such that $W_y \subset$ f(U). Since W_y is a gπ-neighbourhood of y, there exists a gπ-open set V_y in (Y,σ) such that

$y \in V_y \subset W_y$. Therefore, $f(U) = \{V_y : y \in f(U)\}$. Since any union of gπ-open sets is an gπ-open set, f(U) is a gπ-open set of (Y, σ). Thus, f is a gπ-open mapping.

Theorem 3.2.14 A map $f : (X, \tau) \to (Y, \sigma)$ is gπ-open if and only if for any subset B of (Y, σ) and for any closed set S containing $f^{-1}(B)$, there exists a gπ-closed set A of (Y, σ) containing B such that $f^{-1}(A) \subset S$.

Proof: Suppose f is a gπ-open map. Let $B \subset Y$ and S be a closed set of (X, τ) such that $f^{-1}(B) \subset S$. Now $X - S$ is an open set in (X, τ). Since f is gπ-open map, $f(X - S)$ is gπ-open set in (Y, σ). Then $A = Y - f(X - S)$ is a gπ-closed set in (Y, σ). $f^{-1}(B) \subset S$ implies $B \subset A$ and $f^{-1}(A) = X - f^{-1}(f(X - S)) \subset X - (X - S) = S$, That is $f^{-1}(A) \subset S$.

Conversely, let U be an open set of (X, τ). Then $f^{-1}((f(U))^c) \subset U^c$ and U^c is a closed set in (X, τ). By hypothesis, there exists a gπ-closed set A of (Y, σ) such that $(f(U))^c \subset A$ and $f^{-1}(A) \subset U^c$ and so $U \subset (f^{-1}(A))^c$. Hence $A^c \subset f(U) \subset f((f^{-1}(A)))^c \subset A^c$ which implies $f(U) = A^c$. Since A^c is a gπ-open, f(U) is gπ-open in (Y, σ) and therefore f is gπ-open map.

Theorem 3.2.15 A map $f : (X, \tau) \to (Y, \sigma)$ is gπ-open if and only if $f^{-1}(g\pi Cl(B)) \subset Cl(f^{-1}(B))$ for every subset B of (Y, σ).

Proof: Let $f : (X, \tau) \to (Y, \sigma)$ be a gπ-open map and B be any subset of (Y, σ). Then $f^{-1}(B) \subset cl(f^{-1}(B))$ and $cl(f^{-1}(B))$ is a closed set in (X, τ). By Theorem 3.2.14, there exists a gπ-closed set A of (Y, σ) such that $B \subset A$ and $f^{-1}(A) \subset cl(f^{-1}(B))$. Therefore, $f^{-1}(g\pi Cl(B)) \subset f^{-1}(A) \subset cl(f^{-1}(B))$, since A is a gπ-closed set in (Y, σ).

Conversely, let S be any subset of (Y, σ) and F be any closed set containing $f^{-1}(S)$. Put $A = g\pi Cl(S)$. Then A is a gπ-closed set and $S \subset A$. By assumption, $f^{-1}(A) = f^{-1}(g\pi Cl(S)) \subset cl(f^{-1}(S)) \subset F$ and therefore by Theorem 3.2.14, f is gπ-open.

Definition 3.2.16 A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a gπ*-closed map if the image f(A) is gπ-closed in (Y, σ) for every gπ-closed set A in (X, τ).

Example 3.2.17 Let X = {a, b, c, d},

$\tau = \{X, \varphi, \{a, b, d\}\} = \sigma$. Define $f : (X, \tau) \to (X, \sigma)$ by f(a) = a, f(b) = b, f(c) = c, f(d) = d. Then f is a gπ*-closed map.

Theorem 3.2.18 If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are gπ*-closed maps, then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is also gπ*-closed.

Proof: Let F be a gπ-closed set in (X, τ). Since f is a gπ*-closed map, f(F) is a gπ-closed set in (Y, σ). Since g is a gπ*-closed map, g(f(F)) is a gπ-closed set in (Z, η). Therefore $g \circ f$ is gπ*-closed map.

Definition 3.2.19 A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a gπ*-open map if the image f(A) is gπ-open in (Y, σ) for every gπ-open set A in (X, τ).

Theorem 3.2.20 If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are gπ*-open maps, then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is also gπ*-open

Theorem 3.2.21 For any bijection $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:

(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is gπ-irresolute

(ii) f is a gπ*-open and

(iii) f is a gπ*-closed map.

Lemma 3.2.22 Let A be a subset of X. Then $p \in g\pi Cl(A)$ if and only if for any gπ-neighborhood N of p in X, $A \cap N \neq \varphi$

Definition 3.2.23 Let A be a subset of X. A mapping $r : X \to A$ is called a gπ-continuous retraction if r is gπ-continuous and the restriction r_A is the identity mapping on A.

Theorem 3.2.24 Let A be a subset of X and $r : X \to A$ be a gπ-continuous retraction. If X is gπ-Hausdorff, then A is a gπ-closed set of X.

Proof: Suppose that A is not gπ-closed. Then there exists a point x in X such that $x \in g\pi Cl(A)$ but $x \notin A$. Since r is a gπ-continuous retraction $r(x) \neq x$. Since X is gπ-Hausdorff, there exists disjoint gπ-open sets U and V in X such that $x \in U$ and $r(x) \in V$. Now let W be an arbitrary gπ-neighborhood of x. Then $W \cap U$ is a gπ-neighbourhood of x. Since $x \in g\pi Cl(A)$, by Lemma 3.2.21, $(W \cap U) \cap A \neq \varphi$. Therefore there exists a point y in $W \cap U \cap A$. Since $y \in A$, $r(y) = y \in U$ and hence $r(y) \notin V$. This implies that $r(W) \not\subset V$ because $y \in W$. This is contrary to the gπ-continuity of r. Consequently, A is a gπ-closed set of X.

Theorem 3.2.25. Let $\{X_i | i \in I\}$ be any family of topological spaces. If $f : X \to \prod X_i$ is a gπ-continuous mapping, then $Pr_i \circ f : X \to Xi$ is gπ-continuous for each $i \in I$, where Pr_i is the projection of $\prod X_j$ on X_i.

Proof: We shall consider a fixed $i \in I$. Suppose U_i is an arbitrary open set in X_i. Then $Pr_i^{-1}(U_i)$ is open in $\prod X_i$. Since f is gπ-continuous, $f^{-1}(Pr_i^{-1}(U_i)) = (Pr_i \circ f)^{-1}(U_i)$ gπ-open in X. Therefore, $Pr_i \circ f$ is gπ-continuous.

3.3. Generalized π -homeomorphisms

Definition 3.3.1 A bijection $f : (X, \tau) \to (y, \sigma)$ is called generalized π-homeomorphism (gπ -homeomorphism) if f and f^{-1} are gπ -continuous.

Example 3.3.1 (a) Let X = {a, b, c, d},

τ = {X, φ, {b}, {c}, {a, b}, {b, c}, {a, b, c}, {a, b, d}} and σ = {X, φ, {b}}, Define f : (X, τ) → (X, σ) by f(a) = a, f(b) = b, f(c) = c, f(d) = d. Then f is a gπ-homeomorphism.

Theorem 3.3.2 A bijection f : (X, τ) → (y, σ) is gπ-homeomorphism then it is a g-homeomorphism

Proof: Since f is gπ-homeomorphism, both f and f^{-1} are gπ-continuous. As every gπ-continuous function is g-continuous, f and f^{-1} are g-continuous and f is a g-homeomorphism.

The converse of the theorem need not be true as seen from the following example.

Example 3.3.3 Let X = {a, b, c, d}, with

τ = {X, φ, {b}, {c}, {b, c}, {a, b}, {a, b, c}, {a, b, d}}= σ Define a map f : (X, τ) → (X, σ) be the identity map. Then f is g -homeomorphism but not a gπ-homeomorphism, since the inverse image of the closed set { c} in X is { c} which is not gπ-closed set in Y.

Theorem 3.3.4 For any bijection f : (X, τ) → (Y, σ) , the following are equivalent:

(i) f is gπ-open map,

(ii) f is a gπ-homeomorphism

(iii) f is a gπ-closed map

Remark 3.3.5 The composition of two gπ-homeomorphism maps need not be a gπ-homeomorphism

As can be seen from the following example

Example 3.3.6 Let X = {a, b, c, d},

τ = {X, φ, {b}, {c}, {a, b}, {b, c}, {a, b, c}, {a, b, d}}, σ = {X, φ, {b}}, μ = {X, φ, {b}, {c}, {b, c}}. Define f : (X, τ) → (X, σ) by f(a) = a, f(b) = b, f(c) = c, f(d) = d and g : (X, σ) → (X, μ) by g(a) = a, g(b) = b, g(c) = c, g(d) = d. Then both f and g are gπ- homeomorphism but their composition g ∘ f: (X, τ) → (X, μ) is not a gπ- homeomorphism, because for the closed set {c} in (X, τ), (g ∘ f)({ c}) = { c}, which is not a gπ-closed set in (X, μ). Therefore g ∘ f is not a gπ-closed map and so g ∘ f is not a gπ- homeomorphism.

Definition 3.3.7 A bijection f : (X, τ) → (y, σ) is said to be gπ*-homeomorphism if both f and f^{-1} are gπ -irresolute.

We denote the family of all gπ*-homeomorphism of a topological space (X , τ) onto itself by gπ*h(X, τ).

Theorem 3.3.8 If f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η) are gπ*-homeomorphisms, then their composition g ∘f : (X, τ) → (Z, η) is also gπ*-homeomorphism.

Proof: Let U be a gπ-open set in (Z, η). Now, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$. By hypothesis, V is gπ-open in (Y, σ) and f is a gπ*-homeomorphisms , $f^{-1}(V)$ is gπ-open in (X, τ). Therefore, g ∘ f is gπ-irresolute. For any gπ-open set G in (X, τ), (g ∘f)(G) = g(f(G)) = g(W), where W = f(G). Since f and g are gπ*-homeomorphisms f(G) is gπ-open in (Y, σ) and g(f(G)) is gπ-open in (Z, η). i.e., (g∘f)(G) is gπ-open in (Z, η) and therefore $(g \circ f)^{-1}$ is gπ-irresolute. Hence g ∘ f is a gπ*-homeomorphism.

Theorem 3.3.9 The set gπ*-h(X, τ) is a group under the composition of maps.

Proof: Let f, g $\in$ gπ*h(X, τ). Then g $\circ$ f $\in$ gπ*h(X, τ) and so gπ*h(X, τ) is closed under the composition of maps. Composition of maps is always associative. The identity map I : (X, τ) → (X, τ) is a gπ* -homeomorphisms and so I $\in$ gπ*h(X, τ) . Also f $\circ$ I = I $\circ$ f = f for every f $\in$ gπ*h(X, τ). If f $\in$ gπ*h(X, τ) then $f^{-1} \in$ gπ*h(X, τ) and $f \circ f^{-1} = f^{-1} \circ f$ = I. Hence gπ*h(X, τ) is a group under the composition of maps.

Theorem 3.3.10 Let f : (X, τ) → (Y, σ) be a gπ*-homeomorphism. Then f induces an isomorphism from the group gπ*h(X, τ) onto the group gπ*h(Y, σ).

Proof: Let θ_f : gπ*h(X, τ) → gπ*(Y, σ) by defined as $\theta_f(h) = f \circ h \circ f^{-1}$ for every h$\in$ gπ*h(X, τ) Then θ_f is a bijection. Further,for all h_1, $h_2 \in$ gπ*h(X, τ), $\theta_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta_f(h_1) \circ \theta_f(h_2)$. Therefore, θ_f is a homeomorphism and so it is an isomorphism induced by f.

Theorem 3.3.11. gπ*-homeomorphism is an equivalence relation in the collection of all topological spaces.

Theorem 3.3.12. If f : (X, τ) → (Y, σ) is a gπ*-homeomorphism, then gπCl(f^{-1}(B)) = f^{-1}(gπCl(B)) for all B $\subset$ Y .

Proof: Since f is a gπ*-homeomorphism, f is gπ-irresolute. Since gπCl(f(B)) is a gπ-closed set in (Y, σ) , f^{-1}(gπCl(f(B))) is gπ-closed in (X, τ) . As, f^{-1}(B)$\subset f^{-1}$(gπCl(B)), gπCl(f^{-1}(B))$\subset f^{-1}$(gπCl(B)). Since f is a gπ*-homeomorphism, f^{-1} is a gπ-irresolute. Since gπCl(f^{-1}(B)) is gπ-closed in (X, τ) , $(f^{-1})^{-1}$(gπCl(f^{-1}(B))) = f(gπCl(f^{-1}(B))) is gπ-closed in (Y, σ). As, B $\subset (f^{-1})^{-1}(f^{-1}$(B)) $\subset (f^{-1})^{-1}$ (gπCl(f^{-1} (B))) = f(gπCl(f^{-1} (B))) and so gπCl(B) $\subset$ f(gπCl(f^{-1} (B))). Therefore,

f^{-1} (gπCl(B)) $\subset f^{-1}$ (f(gπCl(f^{-1} (B)))) $\subset$ gπCl(f^{-1} (B)) and hence the equality holds.

Theorem 3.3.13. If f : $(X, \tau) \to (Y, \sigma)$ is a gπ*-homeomorphism, then gπCl(f(B)) =f(gπCl(B)) for all $B \subset X$.

Proof: Since f : $(X,\tau) \to (Y,\sigma)$ is a gπ*-homeomorphism, $f^{-1}: (Y, \sigma) \to (X, \tau)$ is also a gπ*-homeomorphism. Therefore, by Theorem 3.3.12, gπCl($(f^{-1})^{-1}$(B)) =$(f^{-1})^{-1}$(gπ-Cl(B)) for all $B \subset X$. i.e., gπCl(f(B)) = f(gπCl(B)).

Theorem 3.3.14. If f : $(X, \tau) \to (Y, \sigma)$ is a gπ*-homeomorphism, then f(gπInt(B)) = gπInt(f(B)) for all $B \subset X$.

Proof: For any set $B \subset X$, gπInt(B) = (gπCl(B^c))^c. Thus,

f(gπ Int(B))= f((gπCl(B^c))^c) = (f(gπCl(B^c)))^c = (gπ Cl(f(B^c)))^c, by Theorem 3.3.13

= (gπCl((f(B))^c))^c = gπInt(f(B)).

Theorem 3.3.15. If f : $(X, \tau) \to (Y,\sigma)$ is a gπ*-homeomorphism, then f^{-1}(gπInt(B))= gπInt(f^{-1}(B)) for all $B \subset Y$.

Proof: Since $f^{-1}: (Y,\sigma) \to (X, \tau)$ is a gπ*-homeomorphism, f^{-1}(gπInt(B))= gπInt(f^{-1}(B))

3.4 Generalized π -compactness

Definition 3.4.1 A collection $\{A_i : i \in \Lambda\}$ of gπ-open sets in a topological space X is called a gπ-open cover of a subset B of X if $B \subset \cup\{A_i : i \in \Lambda\}$ holds.

Definition 3.4.2 A topological space X is gπ-compact if every gπ-open cover of X has a finite sub-cover.

Definition 3.4.3 A subset B of a topological space X is said to be gπ-compact relative to X if, for every collection $\{A_i : i \in \Lambda\}$ of gπ-open subsets of X such that $B \subset \cup \{A_i : i \in \Lambda\}$ there exists a finite subset Λ_0 of Λ such that $B \subseteq \cup \{A_i : i \in \Lambda_0\}$

Definition 3.4.4 A subset B of a topological space X is said to be gπ-compact if B is gπ-compact as a subspace of X.

Theorem 3.4.5 Every gπ-closed subset of a gπ-compact space is gπ-compact relative to X.

Proof: Let A be gπ-closed subset of gπ-compact space X. Then A^C is gπ-open in X. Let $M = \{G_\alpha : \alpha \in \wedge\}$ be a cover of A by gπ-open sets in X. Then $M* = M \cup A^c$ is a gπ-open cover of X. Since X is gπ-compact, $M*$ is reducible to a finite subcover of X, say $X = G_{\alpha 1} \cup G_{\alpha 2} \cup \cdot\cdot\cdot \cup G_{\alpha m} \cup A^C$, $G_{\alpha k} \in M$. But A and A^C are disjoint hence $A \subset G_{\alpha 1} \cup \cdot\cdot\cdot \cup G_{\alpha m}$, $G_{\alpha k} \in M$, which implies that any gπ-open cover M of A contains a finite sub-cover. Therefore A is gπ-compact relative to X. Thus every gπ-closed subset of a gπ-compact space X is gπ-compact.

Theorem 3.4.6 A gπ-continuous image of a gπ-compact space is compact.

Proof: Let $f : X \rightarrow Y$ be a gπ-continuous map from a gπ-compact space X onto a topological space Y . Let $\{A_i : i \in \wedge\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in \wedge\}$ is a gπ-open cover of X. Since X is gπ-compact it has a finite sub-cover say $\{f^{-1}(A_1), f^{-1}(A_2), \cdot\cdot\cdot, f^{-1}(A_n)\}$. Since f is onto $\{A_1, \cdot\cdot\cdot, A_n\}$ is a cover of Y, which is finite. Therefore Y is compact.

Theorem 3.4.7 If a map $f : X \rightarrow Y$ is gπ-irresolute and a subset B of X is gπ-compact relative to X, then the image f(B) is gπ-compact relative to Y.

Proof: Let $\{A_\alpha : \alpha \in \wedge\}$ be any collection of gπ-open subsets of Y such that $f(B) \subset \cup\{A_\alpha : \alpha \in \wedge\}$. Then $B \subset \cup\{f^{-1}(A_\alpha) : \alpha \in \wedge\}$ holds. Since by hypothesis B is gπ-compact relative to X there exists a finite subset $\wedge_0$ of $\wedge$ such that $B \subset \cup \{f^{-1}(A_\alpha) : \alpha \in \wedge_0\}$. Therefore $f(B) \subset \cup \{A_\alpha : \alpha \in \wedge_0\}$, which shows that f(B) is gπ-compact relative to Y.

Definition 3.4.8 A space X is said to be

(1) countably gπ- compact if every countable cover of X by gπ-open sets has a finite sub-cover;

(2) gπ-Lindelof if every gπ-open cover of X has a countable sub-cover.

Theorem 3.4.9 Let f : X→Y be an almost contra-gπ-continuous surjection. The following statements hold:

(1) if X is gπ-compact, then Y is S-closed

(2) if X is gπ-Lindelof, then Y is S-Lindelof

(3) if X is countably gπ-compact, then Y is countably S-closed.

Proof: (1) Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y . Since f is almost contra- gπ- continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is gπ - open cover of X. Since X is gπ -compact, there exists a finite subset I_0 of I such that $X \subseteq \cup\{ f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y \subseteq \cup\{ (V_\alpha) : \alpha \in I_0\}$ is finite sub-cover for Y. Therefore, Y is S-closed.

(2) Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y . Since f is almost contra- gπ -continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is gπ -open cover of X. Since X is gπ -Lindelof, there exists a countable subset I_0 of I such that $X \subseteq \cup\{ f^{-1}(V_\alpha) : \alpha \in I_0\}$ Since f is surjective, $Y \subseteq \cup\{ (V_\alpha) : \alpha \in I_0\}$ is finite sub-cover for Y . Therefore, Y is S-Lindelof.

(3) Let $\{V_\alpha : \alpha \in I\}$ be any countable regular closed cover of Y . Since f is almost contra- gπ -continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable gπ -open cover of X. Since X is countably gπ -compact, there exists a finite subset I_0 of I such that $X \subseteq \cup\{ f^{-1}(V_\alpha) : \alpha \in I_0\}$ Since f is surjective, $Y \subseteq \cup\{ (V_\alpha) : \alpha \in I_0\}$ is finite sub-cover for Y . Therefore, Y is countably S-closed.

Definition 3.4.10 A space X is said to be

(1) gπ-closed compact if every gπ-closed cover of X has a finite sub-cover.

(2) countably gπ-closed compact if every countable cover of X by gπ-closed sets has a finite sub-cover.

(3) gπ-closed-Lindelof if every cover of X by gπ-closed sets has a countable sub-cover.

Theorem 3.4.11. Let f : X→Y be an almost contra-gπ-continuous surjection. The following statements hold:

(1) if X is gπ-closed compact, then Y is nearly compact;

(2) if X is gπ-closed-Lindelof , then Y nearly Lindelof;

(3) if X is countably gπ-closed compact, then Y is nearly countably compact.

Proof: (1). Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y. Since f is almost contra-gπ-continuous, then $\{ f^{-1} (V_\alpha) : \alpha \in I\}$ is an gπ-closed cover of X. Since X is gπ-closed compact, there exists a finite subset I_0 of I such that $X \subseteq \cup\{f^{-1} (V_\alpha) : \alpha \in I_0\}$. Thus, $Y \subseteq \cup\{V_\alpha : \alpha \in I_0\}$ and Y is nearly compact.

Similarly we can prove (2) and (3).

Theorem 3.4.12. If f : X→Y is contra-gπ-continuous and A is gπ-compact relative to X, then f (A) is strongly S-closed in Y.

Proof: Let $\{V_i : i \in I\}$ be any cover of f (A), by closed sets of the subspace f (A). For $i \in I$, there exists a closed set A_i of Y such that $V_i = A_i \cap f (A)$. For each $x \in A$, there exists $i(x) \in I$ such that $f (x) \in A_{i(x)}$ and there exists $U_x \in g\pi O(X,x)$ such that $f (U_x) \subseteq A_{i(x)}$. Since the family $\{U_x : x \in A\}$ is a cover of A by gπ-open sets of X. Since A is gπ-compact, there exists a finite subset A_0 of A such that $A \subseteq \cup\{U_x : x \in A_0\}$. Therefore,$f (A) \subseteq \cup\{f (U_x) : x \in A_0\}$. which is a subset of $\cup\{A_{i(x)} : x \in A_0\}$. Thus $f (A)= \cup\{V_{i(x)} : x \in A_0\}$ and hence f (A) is strongly S-closed.

Corollary 3.4.13 If f : X→Y is contra-gπ-continuous surjection and X is gπ-compact, then Y is strongly S-closed.

3.5 Generalized π -connectedness

Definition 3.5.1 A topological space X is said to be gπ-connected if X cannot be expressed as a disjoint union of two non-empty gπ-open sets. A subset of X is gπ-connected if it is gπ-connected as a subspace.

Example 3.5.2 Let X = {a, b} and let τ = {X, φ, {a}}. Then it is gπ-connected.

Remark 3.5.3 Every gπ-connected space is connected but the converse need not be true in general, which follows from the following example.

Example 3.5.4 Let X = {a, b} and let τ = {X, φ}. Clearly (X, τ) is connected. The gπ-open sets of X are {X, φ, {a}, {b}}. Therefore (X, τ) is not a gπ-connected space, because X = {a} ∪{b} where {a} and {b} are non-empty gπ-open sets.

Theorem 3.5.5 For a topological space X the following are equivalent.

(i) X is gπ-connected.

(ii) X and φ are the only subsets of X which are both gπ-open and gπ-closed.

(iii) Each gπ-continuous map of X into a discrete space Y with at least two points is a constant map.

Proof: (i) ⇒ (ii) : Let O be any gπ-open and gπ-closed subset of X. Then O^C is both gπ-open and gπ-closed. Since X is disjoint union of the gπ-open sets O and O^C, from (i) either O = φ or O = X.

(ii) ⇒ (i) : Suppose that X = A ∪ B where A and B are disjoint non-empty gπ-open subsets of X. Then A is both gπ-open and gπ-closed. By assumption A = φ or X.Therefore X is gπ-connected.

(ii) ⇒ (iii) : Let f : X → Y be a gπ-continuous map. Then X can be covered by gπ-open and gπ-closed covering {f^{-1}(Y) : y ∈ (Y)}. By assumption f^{-1}(y) = φ or X for each y ∈ Y. If f^{-1}(y) = φ for all y ∈ Y , then f fails to be a map. Then there exists only one point $y \in Y$ such that $f^{-1}(y) \neq \varphi$ and hence $f^{-1}(y) = X$. This shows that f is a constant map.

(iii) ⇒ (ii): Let O be both gπ-open and gπ-closed in X. Suppose $O \neq \varphi$. Let $f : X \rightarrow Y$ be a gπ-continuous map defined by $f(O) = y$ and $f(O^C) = \{w\}$ for some distinct points y and w in Y. By assumption f is constant. Therefore $O = X$.

Theorem 3.5.6 If $f : X \rightarrow Y$ is a gπ-continuous and X is gπ-connected, then Y is connected.

Proof: Suppose that Y is not connected. Let $Y = A \cup B$ where A and B are disjoint non-empty open sets in Y . Since f is gπ-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty gπ-open sets in X. This contradicts the fact that X is gπ-connected. Hence Y is connected.

Theorem 3.5.7 If $f : X \rightarrow Y$ is a gπ-irresolute surjection and X is gπ-connected, then Y is gπ-connected.

Proof: Suppose that Y is not gπ-connected. Let $Y = A \cup B$ where A and B are disjoint non-empty gπ-open sets in Y. Since f is gπ-irresolute and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty gπ-open sets in X. This contradicts the fact that X is gπ-connected. Hence Y is gπ -connected.

Theorem 3.5.8 If X is a $T_{g\pi}$-space then X is connected if and only if it is gπ-connected.

Proof: Suppose that X is connected. Then X cannot be expressed as disjoint union of two non-empty proper subsets of X. Suppose X is not a gπ-connected space. Let A and B be any two gπ-open subsets of X such that $X = A \cup B$, where $A \cap B = \varphi$ and $A \subset X$, $B \subset X$. Since X is $T_{g\pi}$-space and A, B are gπ-open, A, B are open subsets of X, which contradicts that X is connected. Therefore X is gπ-connected.

Conversely, every open set is gπ-open. Therefore every gπ-connected space is connected.

Theorem 3.5.9 If the gπ-open sets C and D form a separation of X and if Y is gπ-connected subspace of X, then Y lies entirely within C or D.

Proof: Since C and D are both gπ-open in X the sets $C \cap Y$ and $D \cap Y$ are gπ-open in Y. These two sets are disjoint and their union is Y. If they were both non-empty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely in C or in D.

Theorem 3.5.10 Let A be a gπ-connected subspace of X. If $A \subset B \subset g\pi Cl(A)$ then B is also gπ-connected.

Proof: Let A be gπ-connected and let $A \subset B \subset g\pi Cl(A)$. Let $B = C \cup D$ is a separation of B by gπ-open sets. By Theorem 3.5.9 A must lie entirely in C or in D. Let $A \subset C$. Then $g\pi Cl(A) \subseteq g\pi Cl(C)$. Since $g\pi Cl(C)$ and D are disjoint, B cannot intersect D. This contradicts the fact that D is a non-empty subset of B. So $D = \varphi$ which implies B is gπ-connected.

Theorem 3.5.11 If $f : X \to Y$ is a contra gπ-continuous function from an gπ-connected space X onto any space Y, then Y is not a discrete space.

Proof: Let Y be a discrete space. Let A be a proper nonempty open and closed subset of Y. Then $f^{-1}(A)$ is a proper nonempty gπ-clopen subset of X, which is a contradiction to the fact that X is gπ-connected.

Theorem 3.5.12 If $f : X \to Y$ is a contra-gπ-continuous surjection and X is gπ-connected, then Y is connected.

Proof: Suppose that Y is not a connected space. Then there exist nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen in Y. Since f is contra-gπ-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are gπ-open in X. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. Hence X is not gπ-connected. This is a contradiction. Therefore Y is connected.

Theorem 3.5.13 If f : X→Y is almost contra-gπ-continuous surjection and X is gπ-connected, then Y is connected.

Proof: Suppose that Y is not a connected space. Then there exist nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen sets. Thus they are regular open in Y. Since f is almost contra gπ-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are gπ-open in X. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not gπ-connected. This is a contradiction. Hence Y is connected.

3.6 Generalized π - US spaces

Definition 3.6.1 A sequence $\{ x_n \}$ in a space X, gπ-converges to a point $x \in X$ if $\{ x_n \}$ is eventually in every gπ-open set containing x.

Definition 3.6.2 A space X is said to be gπ-US if every sequence in X, gπ - converges to a point of X.

Theorem.3.6.3 Every gπ-US space is gπ-T_1 .

Proof: Let X be a gπ-US space and x, y be two distinct points of X. Consider the sequence $\{ x_n \}$, where $x_n = x$ for any $n \in N$. Clearly, $\{ x_n \}$ gπ converges to x. Since $x \neq y$ and X is gπ-US, $\{ x_n \}$ does not gπ-converge to y, i.e., there exists a gπ-open set U containing x but not y. Similarly, there exists a gπ-open set V containing y but not x. Thus, X is gπ -T_1.

Theorem 3.6.4 Every gπ-T_2 space is gπ- US.

Proof: Let X be a gπ-T_2 space and $\{ x_n \}$ a sequence in X. Assume that $\{ x_n \}$ gπ-converges to two distinct points x and y. Then $\{ x_n \}$ is eventually in every gπ-open set containing x and also in every gπ-open set containing y. Since X is gπ-T_2, { x $_n$} is eventually in two disjoint gπ-open sets. A contradiction. Therefore, X is gπ-US.

Definition 3.6.5 A subset A of a space X is said to be:

1.Sequentially gπ-closed if every sequence in A, gπ-converges to a point in A,

2.Sequentially GπO-compact if every sequence in A has a subsequence which gπ- converges to a point in A.

Theorem 3.6.6 A space is gπ-US if and only if the diagonal set Δ is a sequentially gπ-closed subset of the product space $X \times X$.

Proof: Suppose that X is a gπ-US space and $\{(x_n, x_n)\}$ is a sequence in the diagonal Δ. Then { x $_n$} is a sequence in X. Since X is gπ-US, the sequence { x $_n$} gπ-converges to a unique point, $x \in X$. This implies that the sequence $\{(x_n, x_n)\}$ gπ-converges to (x, x) which clearly belongs to Δ. Therefore, Δ is a sequentially gπ-closed subset of $X \times X$.

Conversely, suppose that the diagonal Δ is a sequentially gπ-closed subset of $X \times X$. Assume that a sequence $\{x_n\}$ is gπ-converging to x and y. Then it follows that $\{(x_n, x_n)\}$ gπ-converges to (x, y). By hypothesis, since Δ is sequentially gπ-closed, $(x, y) \in \Delta$. Thus, x= y. Therefore, X is a gπ-US space.

Theorem 3.6.7 If a space X is gπ-US and a subset M of X is sequentially GπO compact, then M is sequentially gπ-closed.

Proof: Assume that $\{ x_n \}$ is any sequence in M which gπ-converges to a point $x \in X$. Since M is sequentially GπO-compact, there exists a subsequence $\{x_{nk}\}$ of $\{ x_n\}$ such that $\{ x_{nk} \}$ gπ-converges to $m \in M$. Since X is gπ-US, x=m. This shows that M is sequentially gπ-closed.

Theorem 3.6.8 The product space of an arbitrary family of gπ-US topological spaces is a gπ-US topological space.

Proof: Let $\{X_\lambda: \lambda \in \wedge\}$ be a family of gπ-US topological spaces with the index set $\wedge$. The product space of $\{X_\lambda: \lambda \in \wedge\}$ be denoted by ΠX_λ. Let $\{x_n(\lambda)\}$ be a sequence in ΠX_λ. Suppose that $\{x_n(\lambda)\}$ gπ-converges to two distinct points x and y in ΠX_λ. Then there exists a $\lambda_0 \in \wedge$ such that $x(\lambda_0) \neq y(\lambda_0)$. Then $\{x_n(\lambda_0)\}$ is a sequence in $X_{\lambda 0}$. Let $V_{\lambda 0}$ be any gπ-open set in $X_{\lambda 0}$ containing $x(\lambda_0)$. Then $V = V_{\lambda 0} \times \Pi_{\lambda \neq \lambda 0} X_\lambda$ is a gπ-open set of ΠX_λ containing x. Therefore, $\{x_n(\lambda)\}$ is eventually in V. Thus, $\{x_n(\lambda_0)\}$is eventually in $V_{\lambda 0}$ and it gπ-converges to $x(\lambda_0)$. Similarly, the sequence $\{x_n(\lambda_0)\}$ gπ-converges to $y(\lambda_0)$. This is a contradiction as $X_{\lambda 0}$ is a gπ-US space. Therefore, the product space ΠX_λ is gπ-US.

3.7 Sequentially GπO-compact preserving functions

Definition 3.7.1 A function $f : X \to Y$ is said to be:

(1) Sequentially gπ-continuous at $x \in X$ if the sequence $\{f(x_n)\}$ gπ- converges to f (x) whenever a sequence $\{x_n\}$ gπ-converges to x. If f is sequentially gπ-continuous at each $x \in X$, then it is said to be sequentially gπ-continuous.

(2) Sequentially nearly gπ-continuous, if for each sequence $\{x_n\}$ in X that gπ-converges to $x \in X$, there exists subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that the sequence $\{f(x_{nk})\}$ gπ-converges to $\{f(x_n)\}$.

(3) Sequentially sub gπ-continuous if for each point $x \in X$ and each sequence $\{x_n\}$ in X gπ-converging to x, there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ and a point $y \in Y$ such that the sequence $\{f(x_{nk})\}$ gπ-converges to y.

(4) Sequentially GπO-compact preserving if the image f (M) of every sequentially GπO-compact set M of X is a sequentially GπO-compact subset of Y.

Theorem 3.7.2 Let $f_1 : X \to Y$ and $f_2 : X \to Y$ be two sequentially gπ- continuous functions. If Y is gπ-US, then the set $E = \{x \in X: f_1(x) = f_2(x)\}$ is sequentially gπ-closed.

Proof: Suppose that Y is gπ-US and $\{ x_n \}$ is any sequence in E that f_1 gπ-converges to $x \in X$. Since f_1 and f_2 are sequentially gπ-continuous functions, the sequence $\{ f_1(x_n) \}$ gπ- converges to $f_1(x)$ (respectively, $\{ f_2 (x_n)\}$) gπ-converges to $f_2(x)$). Since $x_n \in E$ for each $n \in N$ and Y is gπ-US, $f_1(x) = f_2(x)$ and hence $x \in E$. This shows that E is sequentially gπ-closed.

Lemma 3.7.3 Every function $f : X \to Y$ is sequentially sub-gπ-continuous if Y is sequentially GπO-compact.

Proof: Let $\{x_n\}$ be a sequence in X that gπ converges to $x \in X$. Therefore $\{f(x_n)\}$ is a sequence in Y. Since Y is sequentially GπO compact, there exists a subsequence $\{ f(x_{nk})\}$ of $\{f(x_n)\}$ that gπ-converges to a point $y \in Y$. Therefore, $f: X \to Y$ is sequentially sub-gπ-continuous.

Theorem 3.7.4 Every sequentially nearly gπ-continuous function is sequentially GπO-compact preserving.

Proof: Let $f : X \to Y$ be a sequentially nearly gπ-continuous function and M be any sequentially GπO-compact subset of X. We show that f (M) is a sequentially GπO-compact subset of Y. Let, $\{ y_n \}$ be any sequence in f (M). Then for each positive integer n, there exists a point x_n in M such that $f(x_n) = y_n$. Since $\{ x_n \}$ is sequentially GπO-compact set M. There exists a subsequence $\{ x_{nk} \}$ of $\{ x_n \}$ gπ-converging to a point x in M . Since f is sequentially nearly gπ-continuous, there exists a subsequence $\{ x_j \}$ of $\{ x_{nk} \}$ such that $\{ f(x_j)\}$ gπ-converges to f (x). Therefore, there exists a subsequence $\{y_i\}$ of $\{ y_n \}$ gπ-converging to f (x) in f(M). This implies that f (M) is a sequentially GπO-compact set of Y.

Theorem 3.7.5 Every sequentially GπO-compact preserving function is sequentially sub-gπ-continuous.

Proof: Suppose that $f : X \to Y$ is a sequentially GπO-compact preserving function. Let x be any point of X and $\{ x_n \}$ a sequence that gπ converges to x.

Let $\{ x_n : n \in N\}$ be A and $M = A \cup \{x\}$. Since $\{ x_n \}$ gπ-converges to x, M is sequentially GπO-compact. By hypothesis, f is sequentially GπO-compact preserving and hence f (M) is a sequentially GπO-compact subset of Y. Now in f (M) there exists a subsequence $\{f(x_{nk})\}$ of $\{f(x_n)\}$ that gπ-converges to a point $y \in f(M)$. This implies that f is sequentially sub-gπ-continuous.

Theorem 3.7.6 A function $f : X \to Y$ is sequentially GπO-compact preserving if and only if $f|_M : M \to f(M)$ is sequentially sub gπ-continuous for each sequentially GπO-compact set M of X.

Proof: **Necessity:** suppose that $f : X \to Y$ is a sequentially GπO-compact preserving function. Then f (M) is sequentially GπO-compact in Y for each sequentially GπO-compact subset M of X. Therefore, by Theorem.3.7.5 $f|_M : M \to f(M)$ is sequentially sub-gπ-continuous.

Sufficiency: Let M be any sequentially GπO-compact set of X. We will show that f (M) is sequentially GπO-compact subset of Y. Let $\{ y_n \}$ be any sequence in f (M). Then for each $n \in N$, there exists a point $x_n \in M$ such that $f(x_n) = y_n$. Since $\{ x_n \}$ is a sequence in the sequentially GπO-compact set M there exists a subsequence $\{ x_{nk} \}$ of $\{ x_n \}$ that gπ-converges to a point in M. By hypothesis $f|_M : M \to f(M)$ is sequentially sub-gπ-continuous hence there exists a subsequence $\{ y_{nk} \}$ of $\{ y_n \}$ that gπ-converges to $y \in f(M)$. This implies that f (M) is sequentially GπO-compact in Y.

Corollary 3.7.7 If a function $f : X \to Y$ is sequentially sub-gπ-continuous and f (M) is sequentially gπ-closed in Y for each sequentially GπO-compact set M of X, then f is sequentially GπO-compact preserving.

Proof: It will suffice to show that $f|_M : M \to f(M)$ is sequentially sub gπ-continuous for each sequentially GπO-compact set M of X, and by Lemma.3.7.3. So, let $\{ x_n \}$ be any sequence in M that gπ-converges to a point $x \in M$. Then, since f is sequentially sub-gπ-continuous there exists a subsequence $\{ x_{nk} \}$ of $\{ x_n \}$ and a point $y \in Y$ such that $\{f(x_{nk})\}$ gπ converges to y. Since $\{f(x_{nk})\}$ is a sequence in the sequentially gπ-closed set f (M) of Y, we obtain $y \in f(M)$. This implies that $f|_M : M \to f(M)$ is sequentially sub-gπ-continuous. Hence f is sequentially GπO-compact preserving map.